ΠΑΝΕΠΙΣΤΗΜΙΟ ΠΕΙΡΑΙΩΣ

ΣΧΟΛΗ ΧΡΗΜΑΤΟΟΙΚΟΝΟΜΙΚΗΣ ΚΑΙ ΣΤΑΤΙΣΤΙΚΗΣ

Τμήμα Στατιστικής και Ασφαλιστικής Επιστήμης

Π.Μ.Σ. στην “Αναλογιστική Επιστήμη και Διοικητική Κινδύνου “

“Μοντέλα Τιμολόγησης Παραγώγων Καιρικών Φαινομένων”

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Ιούλιος 2016

UNIVERSITY OF PIRAEUS

SCHOOL OF FINANCE AND STATISTICS

Department of Statistics and Insurance Science

M.Sc. in "Actuarial Science and Risk Management"

"On Modeling and Pricing Weather Derivatives"

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Piraeus,

July 2016

In Memory of my

Beloved Mother,

Maria

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to Dr. Vassilios Sevroglou, Assistant Professor in the Department of Statistics and Insurance Science at the University of Piraeus, for his highly significant support, guidance, assistance and patience throughout the realization of this thesis.

In addition, I would like to thank Dr. Michael Glezakos, Professor in the Department of Statistics and Insurance Science at the University of Piraeus, as well as Dr. Michael Boutsikas, Assistant Professor in the same Department, for their participation in the three-member advisory committee.

ΠΕΡΙΛΗΨΗ

Η εργασία αυτή διαπραγματεύεται μοντέλο τιμολόγησης παραγώγων καιρού θεωρώντας ως υποκείμενη μεταβλητή τη θερμοκρασία. Χρησιμοποιούνται ιστορικά δεδομένα με τη βοήθεια των οποίων προτείνεται κατάλληλη στοχαστική διαδικασία και ειδικότερα η στοχαστική διαδικασία Ornstein-Uhlebeck, η οποία περιγράφει την εξέλιξη της θερμικρασίας. Αρχικά αναλύεται η έννοια των παραγώγων και οι κατηγορίες στις οποίες αυτά χωρίζονται, καθώς επίσης παρουσιάζονται οι ανάγκες των παραγώγων καιρού στις χρηματοοικονομικές αγορές. Παρουσιάζονται βασικοί ορισμοί της θεωρίας πιθανοτήτων, η θεωρία των στοχαστικών διαδικασιών, της κίνησης Brown, των διαδικασιών martingales, καθώς και αυτή των στοχαστικών διαφορικών εξισώσεων αφού είναι άμεσα συνυφασμένες με το μοντέλο τιμολόγησης που θέλουμε να παρουσιάσουμε. Τέλος, θα θεωρήσουμε κατάλληλο στοχαστικό μοντέλο τιμολόγησης παραγώγων με τη θερμοκρασία ως υποκείμενη μεταβλητή, καθώς και εφαρμογές του.

ABSTRACT

In this thesis a weather derivatives pricing model, considering temperature as the underlying variable will be studied. The use of historic data enables us suggest a stochastic process and, specifically, the Ornstein-Uhlebeck stochastic process, which describes the fluctuation of temperature. In the first place, the concept of derivatives and the categories into which are divided, are analyzed and the needs of derivatives in the financial markets are indicated. Moreover, basic definitions of the probability and stochastic processes theory and martingales, as well as Brownian motion and stochastic differential equations are introduced, since all these theories are closely related to our pricing model. In conclusion, both a proper weather derivatives stochastic pricing model and its applications, which take temperature as an underlying variable will be considered .

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INTRODUCTION

The basic idea of this work is to study a proper weather derivatives pricing model, setting temperature as the underlying variable.

Modeling securities for new commodities has become a common practice over the past few years. This practice intends to cover the risks of securities. This way, we led to several types of derivatives models, such as oil (1983), electricity (1993), weather derivatives (1999), CO_2 (2005) derivatives market etc. Given the abundance of financial products, there is also the need for the corresponding derivatives.

Weather derivatives were first negotiated back in 1999, at the Chicago Mercantile Exchange (CME), based on an Index that provided data according to temperature's fluctuations. When the average daily temperature is higher than the set temperature, the Index is called "Heating Degree Day"(HDD). While, if the daily temperature is lower than the set temperature, then it is called "Cooling Degree Day"(CDD). Generally, this Index calculates the risk related to several financial activities in the fields of agricultural economy, energy and even department stores.

Since 2003, the CME has begun to offer weather derivatives to 5 north European cities (Amsterdam, Berlin, Paris, Stockholm and London). The indices that we mentioned earlier are accumulating average seasonal temperature indices, expressed in Celsius degrees and they consist underlying securities in futures and options. The innovative aspect of introducing weather derivatives into the financial market was that it was the first time that a weather Index was used to hedge the risk and protect the interests of parties from severe weather phenomena.

We organize our work as follows: Chapter 1 of this work traces back to the origins of the first ancient derivatives, which were used in order to provide the necessary amount of goods and commodities, at a fixed price, known to the transactors in advance. Also, the definition of derivatives is given, while the categories (forwards, futures, options, swaps) into which are divided are also mentioned. After describing the evolution of derivatives, we examine the new types of derivatives which are used to hedge the risks, caused by the overgrowth of industry and commercial nowadays. One of the new types is called weather derivatives and are reported in more details in the next chapter. We emphasize at the difference between a weather derivatives contract and an insurance policy. Thus, in chapter 2, we mention the mathematical methods which will be used for describing the weather derivatives pricing model. Basic definitions from the probability theory, the stochastic processes and martingales method are given, while the Brownian motion and the Ornstein-Uhlenbeck stochastic process, which indicates the evolution of temperature that we consider as an underlying variable in the pricing model are also described. At last, in chapter 3, we introduce the categories of weather derivatives contracts. Also, we represent an appropriate stochastic model of derivatives pricing, as well as its multiple applications, setting temperature as the underlying variable.

CHAPTER 1

An Introduction to Derivatives.
The Weather Derivatives

This chapter provides a historical overview of the creation and use of derivatives in trade since ancient times. In these days derivatives assisted people in trading commodities at a delivery price and delivery date . This type of trade that took place in ancient Greece, Rome and Egypt was the ancestor of modern day financial derivatives, which were developed and became very popular among investors after 1970. Thus, we are going to analyze the categories in which financial derivatives are divided, stating definitions and examples and emphasizing on why an investor could use them. Afterwards, we are going to investigate the evolution of the derivatives. In modern societies, the increased energy demand, the environmental pollution as well as several disastrous phenomena resulted in the creation of new derivatives such as petroleum products derivatives, electrical energy derivatives, pollutant and destruction derivatives and finally weather derivatives, which we are going to examine in the final part of the chapter. Finally, we are going to mention when and how the derivatives were created, which needs they serve and then briefly analyze their connection with pricing models.

1.1 THE NEED OF USING DERIVATIVES

In modern economies, the markets for derivatives play a significant role in money and capital markets, giving traders the ability to effectively manage their positions in these markets. In derivatives markets, which have made explosive growth from the 1970s until today, offered exchange traded and OTC (Over The Counter) futures and options with a large number of raw materials and financial instruments covered almost all business activities.

The main concern of people from the beginning of the history is to ensure the goods, that allows them a normal living. This constant effort made people living in groups, to cooperate in forage for food, but also to protect from elements of nature even from the aggression of other people. They were cultivating plants that provided food, which endures to ensure nourishment in cases where crops were not possible for various reasons or was inadequate. The production of goods that could be sustained over time and the abundance of some of them in conjunction with the absence of any others created possibility of trading and commercial relations between groups and people.

This trade processing came up against some troubles. Except difficulties in transport, the greatest difficulty in commercial transactions was to obtain the necessary quantities of merchandise, especially when there was tight supply and also the continuous price changes when super producers and when to commercial resulting in abnormal functioning of commerce. Ensuring a necessary quantity of goods, such as wheat or olive oil, at a fixed price prior known to traders was a first approach to better management of resources but also to the normal operation of city-states. This need impelled people in ancient Greece and Rome to create the first commodity futures. These are agreements between traders and producers, that producers would deliver certain quantity of their production, at a specified price, at a specific place. These agreements became the ancestors of the derivatives we know today [6].

1.2 ESSENTIAL GOODS - SECURITIES

The production of goods originated from the survival needs of human being. Manual equipment and technology assisted in producing goods, which were at times excessive and, therefore, people would exchange them with other goods of similar value. This created local exchange product markets. Gradually, returned merchandise was replaced by a new type of exchange, which included medium of exchange such as silver or gold. For the first time in history the price of goods was expressed as monetary units. The excess of a commodity due to overproduction or low demand leads to decrease in value while low demand or lack of product leads to increase in price. The continuous lack of a product results in price increase, while the lack of numerous products leads to inflation.

In addition to the products that the production of goods originated from the survival needs of human being. Manual equipment and technology assisted in producing goods, which were at times excessive and therefore, people would exchange them with other goods of similar value. This created local exchange product markets. Gradually, returned merchandise was replaced by a new type of exchange, which included medium of exchange such as n be found in retail stores, there are also valuable essential goods such as arable land and residence. The value of immovable property is depicted in title deeds. The titles that refer to essential goods are called "essential titles" and they are bargained in organized markets. Another type of "essential title" relates to commodities transported by ships or cars and constitutes evidence of ownership.

These documents can be used by the owner of the assets as collateral, in order to borrow money from a bank. Besides them, there are elementary titles which pertain to several goods and services, grouped together under a single form of organization. The total value of these goods is represented in title deeds, known as shares, and they refer to the units of ownership of a shareholder with regard to business assets and distributed profits. After the Industrial Revolution, complex business units, which required investment funds, were formed. In order to make this possible, several individuals had to participate as co-owners and to share both the risks and profits of a business unit. The existence of complex business units, apart from the stocks, conduced to the development of many essential titles, such as bonds, convertible bonds and commercial bonds.

The essential securities and the integration of the value of goods or services in them has more advantages in comparison with the essential goods [6].

1. The essential securities can be traded on the secondary market and thus the investor is not required to hold the investment until the timeout. The investor can negotiate an essential security as desired and with minimum cost.

2. The existence of secondary markets, leading to a continuous assessment of the value of securities according to the value of assets and how they are used. Thus, the investor can predict the evolution of an investment as he has daily information, while there is a control on the actions and decisions of managing business.

3. The essential and debt securities invalidated after their expiry and repayment. The essential assets that "survive" after the repayment of securities unshackle and they can be used as collateral for the creation of a new security.

The connection of assets with essential securities is not only the result of creating business units. States borrow in order to survive and be required to pay amounts to borrowing Maturity Date. These loans are essential securities. International audit organizations such as Standard & Poor's or Moody's, continuously evaluated the creditworthiness of the state based on the kind of management of the economy of each state and the policies implemented. This assessment results in a rating, which offers information for investors for the risk of their investment. The existence of secondary markets allows the assessing of this information on an ongoing basis, helping the valuation of essential securities.

1.3 DERIVATIVES

The existence of essential securities allows the operation of numerous commodity markets where traders are taking positions according to their needs and expectations. Thus, we have sellers and buyers, bankers and depositors, individual and professional investors. The above positions require immediate payment of the sum of investment, credit, deposit, value of shares etc. and the success is judged in the future by the development in supply and demand in the market of essential goods, securities or services [13], [14].

Despite the orderly functioning and liquidity in the money market with immediate delivery of the title or commodity and the simultaneous payment, the liberalization of commodity markets, money and capital as well as increased competition in these markets to ensure the goods or titles at fixed prices, created the need for negotiation today with title delivery and simultaneous payment in future. The derivatives market is active for numerous products, such as wheat, oil, sugar, coffee, gold etc. We now give the following definition [3], [9].

Definition 1.1 *Derivative is a security, which price depends on delivered from one or more underlying assets. It is a contract between two or more parties and its value is determined by fluctuations in the underlying asset. The underlying assets can be stocks, bonds, commodities, market indexes, currencies or interest rates.*

The basic kinds of derivatives are:

a. Forwards

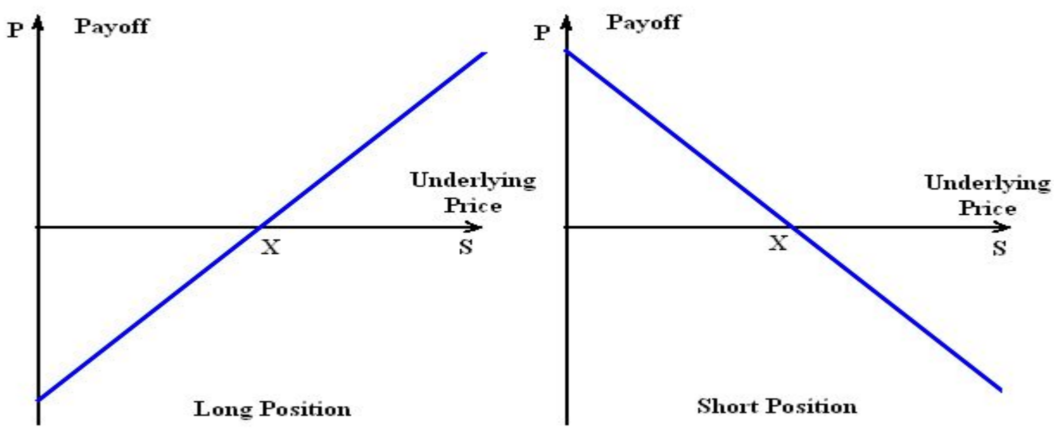
b. Futures

c. Options (Put and Call Options)

d. Swaps

1.3.1 Forwards and Future

Forward contracts are agreements between a buyer and a seller for the purchase or sale of certain units of an essential commodity or security at a specified time in the future at a predetermined price. For example, someone agrees to receive 10,000 shares after 25 days and pay 14.3 euros per share, regardless of their price at the moment of the delivery. If the stock price after 25 days is over 14.3 euros, the investor has profit otherwise he has loss. In this transaction, the buyer expects rise in the price of the shares, while the seller expects reduction of the share price. This is shown in the graphs below:



(Figure 1.1)

The use of forward contracts emerged from the uncertainty that the production cycles and the business cycles created. Such kind of contracts seems to be used in Ancient Greece in olive trade, also in Rome in wheat trading with Egypt. In the last quarter of the 20th century, derivatives were imported in financial instruments and services.

Future contracts have the same characteristics as forwards. The only difference is that forwards negotiate on regulated market while futures are not offered for secondary trading.

1.3.2. Options

Options are agreements between a buyer and a seller where the buyer has the privilege but not the obligation to buy (call option) or sell (put option) certain units of an essential commodity or security at a specific time in the future at a predetermined price. The seller has the reverse obligation. The core elements of the contracts are [3]:

- **The Underlying Asset**, refers to the asset for which the buyer has the call option and the seller has the put option. The Underlying Asset can be stock, title, stock index or any asset under which contracts the option.

- **The size of the contract**, includes the number of shares which cover each option.

- **Maturity**, the time period until the expiry of the contract..

- **Strike/Exercise Price**, refers to the price that a call or put option owner can purchase or sell a security. Exercise price is determined and it doesn't change.

- **Premium**, is the sum that the buyer of the option is asked to pay to the seller, regardless of exercising or not the option. Premium is formed according to market's supply and demand.

Options may belong to two categories depending on the way the transactions occur:

- Exchange-traded Options

- Over-the-counter Options

Exchange-traded options is the most common category of options, also known as “listed options”. These options traded on a regulated exchange where the terms of each option are determined by the exchange. The contract is customized so that underlying asset, expiry date, quantity, and strike price are known in advance. Stock options, commodity options, bond and other interest rate options, index options and options on future contracts are the main exchange-traded options. Over-the-counter options are not traded on exchanges and they can be customized according to the terms of the option contract.

As regards the time at which the options are used, we have the following categories [3]:

- **European Options**, that can be exercised at their maturity.
- **American Options**, that can be exercised anytime until their maturity, and they give investors the opportunity to exercise the contract any time until the expiry date.
- **Bermudan Options**, that can be exercised on specific dates until their maturity.
- **Barrier Options**, that can be exercised only if the price of the underlying asset exceeds a predetermined threshold.
- **Exotic Options**, which are complex since they offer investors many choices to exercise the contract. They are usually Over-the-counter Options.

There are also four basic positions at the option market. From these, we can make much more complicated positions [3].

- **Long call**: One buys the call option to purchase a determined quantity of a commodity at an appointed date and at a fixed price.
- **Short call**: One sells call options. In this case,the investor is required to sell a determined quantity of a commodity, at a fixed price and at an appointed date in the future.

- **Long put**: One buys the put option to sell a determined quantity of a commodity at a fixed price and at an appointed date.
- **Short put**: One sells the put option to sell a determined quantity of a commodity at a fixed price and an appointed date. In this position, the investor is required to buy the quantity of the commodity according to the terms of the contract.

An investor can follow various strategies in order to hedge the portfolio risk, combining the aforementioned positions.

Example 1.3.1 Suppose an investor expects the price of a particular share to be increased in the future, but does not wish to invest a large amount in shares.
He decides to buy a call option to purchase a share in the current price of 20 euros. This option costs 1 euro per share. If the share price increased to 25 euros, the buyer will exercise his right, that will buy the stock at 20 euros which soon will sell on derivative market for 25 euros. He will win 25 - 20 - 1 = 4 euros .

1.3.3. Swaps

A swap contract is defined as an agreement between two parties,through which an exchange of cash flow sequences will occur.This exchange will take place in the future with predetermined terms and conditions. This agreement includes the swap date, the method of calculating cash flows and cash flow payment. Swaps are typically divided into [3]:

- **Currency Swaps**, which are calculated for different currencies with flat interest rate.
- **Interest Rate Swaps**, which are Fixed-for-floating rate swaps with same currency
- **Synthetic Swaps**, which is a combination of currency and interest rate swaps.

The main types of swaps are:

- **Interest Rate Swaps**
- **FX Swaps**

- **Equity and Total return Swaps**

In the case of Interest Rate Swaps, the two counterparties agree to exchange interest rate cash flows, which are calculated with flat and floating rate multiplied by a notional principal amount. Therefore, the one counterparty agrees to pay the interest rates which correspond to the aforementioned capital on agreed/fixed interest rate, while the other counterparty agrees to pay the interest rates which correspond to the capital, based on the agreed floating rate.

A requirement for the realization of interest rate swaps is the constant determined value of interest-rate for consecutive periods of time (month, trimester, year etc.). For that purpose, indices which specialize by currency, have been created in order to inform interested parties about the prices of interest rates anytime. The main indices are the LIBOR and the EURIBOR.

LIBOR Index (London Interbank Offered Rate) is formed for several currencies and reflects the cost of borrowing between banks. Since LIBOR reflects the low-risk investment return, in case of high risk, the floating interest rate which is used for the swap is the risk premium LIBOR e.g. LIBOR + 1,5%.

EURIBOR index (Euro Interbank Offered Rate) is formed only for EURO and reflects the interest rate set by Eurozone banks for lending other Eurozone banks. As regads risk premium, the LIBOR rules apply to EURIBOR as well.

When investors or enterprises desire to enter into an interest rate swap, they do not communicate directly with the interested counterparty, but they contact a pleader, who either intermediates as a swap broker or a swap dealer. Swap brokers are paid on commission, while swap dealers receive salary plus an amount of money for the risk they bear.

Example 1.3.2 We have two companies A and B: A want to borrow money at floating rate while B wants to borrow at flat rate. Given that both companies want to raise funds which amount to 10 million euros through borrowing, with a loan duration period of 5 years. The chart below shows the interest rates at which the 2 companies can borrow money:

	Flat Interest-rate	Floating Interest-rate
Company A	4,0%	Semester LIBOR - 0.1%
Company B	5,2%	Semester LIBOR + 0,6%

As we can see from the chart above, the spread for the flat rate is 1,2% in favor of the A company, while the floating rate is 0,7%. That means that the A company has a comparative advantage, when borrowing at flat rate, over the B company which borrows at floating rate.

These two companies contract a swap deal with the X bank (4% commission) as shown:

A company agrees to pay a semester LIBOR and to receive 4,33% flat rate (4,35% - 0,02% commission). Hence, A company :

1. Pays a 4% rate for its 10 million loan.

2. Pays LIBOR, due to the swap deal with its 10 million euros fictitious capital.

3. Earns 4,33% due to the swap deal with its 10 million euros fictitious capital.

Taking into consideration the three aforementioned cashflows, we become aware of the fact that A company pays LIBOR - 0,33% for its loan, while if it had not contracted the swap deal, it would borrow at a semester LIBOR - 0,1%. B company contracts a swap deal with the bank and it agrees to pay 4,37% flat rate (4,35% + 0,02% commission) and it gets a semester LIBOR. Therefore, the B company:

1. Pays LIBOR + 0,6% for its 10 million loan.

2. Gets LIBOR, due to the swap deal with its 10 million euros fictitious capital.

3. Pays 4,37%, due to the swap deal with its 10 million euros fictitious capital.

Taking into consideration the three cashflows mentioned above, we can see that B company pays a 4,97% flat rate for its loan, while if it had not contracted the swap deal, it would borrow with 5,2% rate.

The swaps that the two companies have contracted with the X bank are shown below:

Figure 1.2

The above agreed swaps have given the companies the ability to borrow, with the type of rate they desired and profiting 0,23% more. Obviously, this fact does not mean that every investor or company that contracts a swap deal, will always profit from it. In our example, in order for the B company to maintain its paying rate at 4,97%, in the six month interval that the rate gets readjusted, the rate must remain at LIBOR + 0,6% of the initial loan. Thus, if the company's assessment is reduced and at the next revaluation of the interest rate the company is asked to pay for the initial loan LIBOR + 1,2%, then, due to the swap, it will be asked to pay 5,57% interest rate, which is higher than the interest rate at which it would originally borrow without revaluation. A company has locked its payments for the next 5 years at LIBOR+0,33%. Still, the risk of the counterparty's (A company) bankruptcy is reminent. This risk would not occur, if the company had borrowed in LIBOR and didn't proceed in contracting swap. A swap deal's value is initially null. However, as time goes by, it can take positive or negative prices. These methods are involved in their pricing:

1. Swap pricing as a difference of two bonds: Swap in this case is divided in a flat rate bond and a floating rate bond. Despite the capital being considered fictitious, in this case we consider that they are exchanged at the start and the end of contract to make calculations simpler. Hence, for the counterparty that pays at floating interest rate, which means that they buy a flat rate bond and issue a floating interest rate bond, it appears that [3]:

$$\mathbf{V_{swap} = B_{fix} - B_{float}} \quad (1.1)$$

While the counterparty that pays at flat interest rate, which means that they buy a floating interest rate bond and issues a flat rate bond, it appears that:

$$V_{swap} = B_{float} - B_{fix} \quad (1.2)$$

2. Swap pricing as FRA (Forward Rate Agreement).

1.4 The Evolution and Use of Derivatives

The derivatives can be used by investors for the purpose of hedging, which means to proceed to trading, highly reducing their exposure to several investment risks. Moreover, an investor can take advantage of the market variability in order to proceed to transactions aiming at making money with minimum or zero risk which may occur in the future as a consequence of prices variability. The derivatives are also used for arbitrage or speculation, therefore, the risk adoption from a transactor through derivatives is possible either because there are simultaneous transactors with exactly opposite perception of the particular good or title, or because there are individuals who aspire to profit from the investment risk. In both cases, we have risk distribution.

As we mentioned before, in early times, the main activity of humans was to search for or to produce the basic goods (food,clothes) for their basic needs. The acquirement of such goods enabled them to overcome adverse circumstances and to develop civilization and culture. On the contrary, the lack of basic goods led to mass destruction of entire communities. Trade, which refers to the exchange of goods, either with remuneration or other goods, resulted from the abundance of commodities. Subsequently, financial and commercial centers were established and later emerged the need to reach accords about the delivery of commodities at a particular time and a delivery price at the time of the agreement. As a result, the first organized derivatives market, which meant to play a major role in the development of the derivatives, was created.

Forward Contracts can be traced back in 1960's in Japan and 1970's in Europe. Tulip futures skyrocketed in the Netherlands, while rice futures flourished in Japan. Later, the financial strengthening of the American economy contributed to the development of derivative market and, thus, in 1865 the General Rules of the board of trade which regulate futures transaction in CBOT (Chicago Board of Trade) were instituted. The General Rules originate from the rules of Japanese rice market and assist in managing product stock efficiently along with price stability. The first products for which Future Contracts were created were agricultural products, such as wheat and corn, since the need for price stability and delivery of the agreed quantity, was huge. The successful function of commodity markets assisted in creating the Chicago Mercantile Exchange which made the transaction of various commodities and animals possible. Meanwhile, the New York Cotton Exchange and the New York Mercantile Exchange were established in New York and traded new goods such as cotton, coffee, sugar, gold, copper etc. The development of stock markets showed an upward trend, resulting in the the birth of new stock markets, as well as the addition of new products in Future Contracts. This is how currency futures, crude oil futures and electricity futures eventuated.

The proliferation of Future Contracts was followed by a parallel development of Option Financial Markets in the financial assets that were already in effect. The creation of derivatives on new products is an incessant activity which intends to cover the risks of the corresponding products. Since there is abundance of financial products, there is also need for the corresponding derivatives. Such an instance is the introduction of Future Contracts and Options in insurance policy in 1994 at Chicago Board of Trade. Future Contracts have as an underlying asset a profitability index of insurance policy which originate from several regions of USA. The utility of this type of derivatives is very significant for the users and especially for insurance companies and reinsurers. These factors of insurance market intend to limit natural disasters risks, such as hurricanes, tornados, blizzards and flood.

As a consequence, we realize that the ingenuity of derivatives markets does not confine itself in producing commodities which are intimately related to consumerism or financial assets of several types. Recently, the Chicago Mercantile Exchange started negotiating on derivatives which are based on weather conditions. These derivatives are called weather derivatives. Since 3 October 2003, the Chicago Mercantile Exchange has begun offering weather derivatives to five northern European cities (London, Amsterdam, Berlin, Paris and Stockholm). Temperature is used for the making of numerous indices (monthly or seasonal heat wave index, monthly or seasonal cumulative average temperature index) expressed in Celcius degrees. These indices are underlying assets of Future Contracts and Options [6].

1.5 Introduction to Weather Derivatives

Nowadays, economy highly depends on weather since severe weather phenomena have a huge financial impact. As a fact, humans can't affect or change weather condition. Also, we can't even forecast weather accurately and for an extended period of time. On the contrary, weather conditions can affect basic economic sectors, such as agriculture, forestry, fishing, construction, transport, mining, retail, property insurance and funding and many more services. Weather risk is defined as the chance of injury or material and financial damage due to severe weather phenomena such as heatwave and drought, flood and blizzard, as well as unusual seasonal variance.

Companies program their production taking weather into consideration. As a result, in case of an extreme weather phenomenon, the size and the cost of production will be affected. According to a research conducted among 205 risk managers, companies in USA, and especially the ones who belong to energy industry, realize the impact of weather on their industry. Specifically [7]:

- 59% of risk managers claimed that their companies are exposed to weather variability and they need protection from this risk.
- 43% of companies which appertain to energy and agriculture is aware of the increased weather variability over the last few years..
- 74% of energy companies attempts to quantify the weather variability impact on their enterprises.
- 51% of the respondents claimed that their companies were not ready to confront the consequences of severe weather conditions.
- 82% admits possible future changes of long-term business models in order to adjust to weather variability increase.

- Among the 10% of companies which have already used weather risk management instruments, 86% claims they were useful.

1.5.1. Financial Impact of Weather

The impact that weather conditions might have in an enterprise ranges from a slight reduction on earnings on a rainy day due to a lower customer attendance to a total disaster of a factory due to a tornado. Apart from tornados, weather phenomena include tropical cyclones, thunderstorms, hailstorms, ice storms, all of which can lead to material damage or even deaths. In such cases, In such cases, companies that wish to protect themselves from the financial impact of the aforementioned damages, should insure in order to be compensated depending on the extent of the damage.

However, weather derivatives are designed to assist companies protect themselves against non-catastrophic phenomena. Non-catastrophic weather variation include hot or cold seasons, rainy or dry seasons etc. They can happen quite often and they cause significant difficulties in enterprises because their profits partially depend on weather conditions. These enterprises wish to hedge using weather derivatives, for the purpose of gradually decreasing the volatility of their profits. This method is strongly beneficial because a low profit volatility reduces interest rates by which they borrow money, their stocks present lower fluctuation and thus, higher valuation and finally, there are fewer chances of bankruptcy.

1.6 The Weather Derivatives

In accordance with what we have already mentioned, a typical derivative contract should include [7]:

- Date of issue and expiration date, in order to make its duration viewable.
- A weather station and a weather variable for which meteorological data will be obtained throughout the contract.

- A payment function which converts ratio into cash flow regulated by the derivative after the expiration date.
- In particular cases, a premium which is disbursed to the seller at the beggining of the contract.

The development of derivatives market started in the USA in 1997 as a result of El Niño which was one of the most disastrous phenomena. [1] El Niño became widely known by the media and thus, many companies agreed to hedge seasonal risk in order not to face a decrease in profit. It is estimated that more than 3000 contracts of approximately 5.5 billion dollars worth have been signed, while over 100 contracts worth of 30 million euros have been signed in Europe. European Market is not as advanced as American Market, because the Energy Industry is not completely deregulated, however there is growth potential. According to USA Department of Commerce, about 22% out of 9 trillion dollars of Gross Domestic Product (GDP) is exposed to weather conditions. With respect to a WRMA research in 2010-2011, it emerges that [25]:

- Customized weather derivatives market increased by 30%.
- The total nominal value of over-the-counter (OTC) contracts amounted to 2,4 billion dollars.
- The total market amounted to 11,8 billion dollars (20% increase).

Weather derivatives give companies the opportunity to actively manage weather conditions risks. Nowadays, not only over-the-counter (OTC) markets negotiate on them, but they are also negotiated at Chicago Mercantile Exchange (CME), which offers contracts based on temperature for various cities in the USA. Back in September 1999, CME introduced electronic weather derivatives trading with the intention of enlarging market and reducing credit risk.

Despite the data of stochastic researches mentioned above, it is a fact that there are not many who trade contracts at CME. As a result derivatives market is not that liquid and spreads are larger. This is because several companies have yet to realize that they are really susceptible to adverse weather conditions or they haven't adopted policy to hedge against adverse weather risk [1], [11].

One could easily suppose that weather derivatives and insurance policy which would compensate an enterprise in case of financial damage caused by severe weather phenomena are the same thing. However, this is not true since the difference between them is that the insurance policy covers high risk and low probability events, while enterprises can profit from weather derivatives due to low risk and high probability events. In addition, insurance policy holders have to prove that they incurred damage in order to compensate them,something that is not applied to the case of weather derivatives.

Compensations of weather derivatives contracts considerate only the actual weather outcome and not the effect of it on the holder of the contract. Since weather derivatives can be used just for speculation, a buyer does not need to own a production that is sensitive to weather conditions. Weather derivatives can be designed to offer payouts in each and every weather condition while insurance contracts protect the holder only from extreme weather phenomena.

Another significant benefit of derivatives is that on the market there may be an actor who profits from warm winter and an actor that takes advantage of cold winter. In derivatives, there's the potential of entering a contract between the two parties in order to hedge each others risks, something which is not possible in plain insurance contracts.

Example 1.6.1 Here is an example taken from the real world. In the summer of 2009, the London-based chain of wine bars Corney & Barrow bought coverage to protect itself against bad weather. The deal was, if the temperature fell below 24 ^{0}C on Thursdays or Fridays between June and September the company received a payment. The payments were fixed at £15 000 per day and up to £100 000 for the whole period.

In Chapter 2 we will mention the mathematical tools which will be used for describing the weather derivatives pricing model (see later Chapter 3). In particular, we will give basic definitions from the probability theory, the stochastic processes and martingales method and we will describe the Brownian motion and the Ornstein-Uhlenbeck stochastic process.

CHAPTER 2

Pricing Derivatives via Stochastic Processes

An economic system consists of a number of markets that interact with each other. Among them, the money market is the one that presents the biggest interest, because, apart from connecting with each market, it also interacts with individuals. It is what enterprises use to raise the necessary funding, but it is also used by investors for their own profit. Finance is the field of Economics that deals with the function of Financial Market. On the other hand, Financial Mathematics is the field of Mathematics which is concerned with the development of mathematical models, relevant to the function of the money market, the investment options etc. These models highly assist in understanding several procedures which play a significant role in numerous phenomena but they also form a forecasting method.

Looking through financial websites and newspapers, one can easily understand that the prices of quite a few assets are unstable and they seem to be random. Regarding this fact, we conclude that the basic tools of Financial Mathematics are techniques of the Probability Theory and of Stochastic Processes. Hence, we use concepts such as the concept of probability, mean, variance, but also more advanced concepts such as the Martingale Process, the Brownian Motion, the stochastic integral and Ito processes, the stochastic differential equations etc. The aforementioned concepts are widely applied in Financial Mathematics (e.g the Martingale concept relates to the efficiency of markets).

Techniques of Probability Theory and Stochastic Processes offer sufficient ways for the qualitative and quantitative problem solving. Such problems may refer to the most suitable portfolio choice or the valuation of derivative contracts. However, the emergence of new financial problems, requires new mathematical techniques to solve them. This procedure is essential for the progress of mathematics. The development of both the Theory of Stochastic Differential Equations and Black-Scholes Model originated from the need to solve new financial problems. In less than a decade, the Black-Scholes Model for determining the value of Derivatives has proven to be an essential tool for every stock company employee [2].

2.1 Basic Concepts from Probability Theory

We have a simple experiment, a coin tossing. The possible outcomes: “head” or “tail” which are not predictable in the sense that they appear according to a random mechanism determined by the physical properties of the coin. The scientific treatment of an experiment requires that we assign a number to each random outcome. When tossing a coin, we can write “1” for “head” and “0” for “tail”. Thus, we get a random variable $X = X(\omega) \in \{0,1\}$, where ω belongs to the outcome space $\Omega = \{\text{head, tail}\}$. The value of a share price of stock is also a random variable. These numbers $X(\omega)$ give us information about the experiment. At the space Ω we collect all the possible outcomes ω of the underlying experiment. In mathematics, the random variable $X = X(\omega)$ is a real-valued function defined on Ω [23] .

To approach these problems, one first collects “good” subsets of Ω, the events, in a class **F**, say. **F** is called a σ-algebra and it is a class that contains all interesting events.

Definition 2.1.1 *A σ-algebra on* Ω *is a collection of subsets of* Ω *satisfying the following conditions[2]:*

i. It is not empty: $\emptyset \in \mathbf{F}$ and $\Omega \in \mathbf{F}$.

ii. $A \in \mathbf{F} \Rightarrow A^c \in \mathbf{F}$.

iii. $A_1, A_2, \ldots \in \mathbf{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathbf{F}$ and $\bigcap_{i=1}^{\infty} A_i \in \mathbf{F}$.

Example 2.1.1 (Some elementary σ-algebras) [22]

We have the following collections of subsets of Ω:

$\mathbf{F}_1 = \{\emptyset, \Omega\}$,

$\mathbf{F}_2 = \{\emptyset, \Omega, A, A^c\}$ for some $A \neq \emptyset$ and $A \neq \Omega$,

$\mathbf{F}_3 = P(\Omega) = \{A: A \subset \Omega\}$.

$\mathbf{F}_1$ is the smallest σ-algebra on Ω, and $\mathbf{F}_3$, the power set of Ω, is the biggest one, as it contains all possible subsets of Ω.

If we consider a share price X, not only the events $\{\omega: X(\omega) = c\}$ should belong to **F**, but also $\{\omega: \alpha < X(\omega) \leq b\}$, $\{\omega: b < X(\omega)\}$, $\{\omega: X(\omega) < \alpha\}$, and many more events that are relevant in this situation.

When flipping a coin, we have two possible outcomes, "head" or "tail". Possibilities measure the likelihood that these events happen. We consider the coin "fair", so the probability is 0.5 to both events. The expected fraction of occurrences of an event A in a long series of experiments where A and A^c are observed, is a number $P(A) \in [0,1]$ [22].

Definition 2.1.2 *For events* $A, B \in \mathbf{F}$

$$P(A \cup B) = P(A) + P(B) + P(A \cap B)$$

and if A and B are disjoint, then

$$P(A \cup B) = P(A) + P(B).$$

It is also holds

$$P(A^c) = 1 - P(A), \; P(\Omega) = 1 \; \textit{and} \quad P(\emptyset) = 0.$$

Definition 2.1.3 *The collection of probabilities*

$$F_X(x) = P(X \le x) = P(\{\omega : X(\omega) \le x\}), \quad x \in R = (-\infty, +\infty),$$

is the distribution function F_X *of* X.

Considering an interval $(\alpha, b]$,

$$P(\{\alpha < X(\omega) \le b\}) = F_X(b) - F_X(\alpha), \quad \alpha < b.$$

Moreover, we also obtain the probability that X is equal to a number:

$$\begin{aligned} P(X = x) &= P(\{\omega : X(\omega) = x\}) \\ &= P(\{\omega : X(\omega) \le x\}) - P(\{\omega : X(\omega) < x\}) \\ &= P(\{\omega : X(\omega) \le x\}) - \lim_{h \to 0} P(\{\omega : X(\omega) \le x - h\}) \\ &= F_X(x) - \lim_{h \to 0} F_X(x - h). \end{aligned}$$

With these probabilities one can approximate the probability of the event

$\{\omega : X(\omega) \in \mathrm{B}\}$ for very complicated subsets of **R** [22].

Definition 2.1.4 *The collection of the probabilities*

$$P_X(B) = P(X \in B) = P(\{\omega : X(\omega) \in \mathrm{B}\}) \qquad (2.1)$$

for suitable subsets $B \subset R$ *is the distribution of X.*

These "suitable" subsets of ***R*** *are called Borel sets.*

Example 2.1.2 (The Borel sets). Take $\Omega = \mathbf{R}$

and $C^{(1)} = \{(\alpha, b] : -\infty < a < b < \infty\}$.

The σ-algebra $B_1 = \sigma(C^{(1)})$ contains a very general subsets of R and it is called the Borel σ-algebra which elements are the Borel sets. There is a large variety of Borel sets [22].

We can also introduce the σ-algebra of the n-dimensional Borel sets $B_n = \sigma(C^{(n)})$, where Ω = $\mathbf{R}^n$ and $C^{(n)} = \{(\alpha, b] : -\infty < \alpha_i < b_i < \infty, i = 1,2,...,n\}$. Now we will also give the following definitions [22]:

Definition 2.1.5 *If we have a discrete random variable, the distribution function is*

$$F_X(x) = \sum_{k: x_k \leq x} p_k, x \in R \quad , \tag{2.2}$$

where $0 \leq p_k \leq 1$ *for all k and* $\sum_{k=1}^{\infty} p_k = 1$.

Definition 2.1.6 *If we have a continuous random variable X with density f_x, the distribution function is*

$$F_X(x) = \int_{-\infty}^{x} f_x(y) dy, \quad x \in R, \tag{2.3}$$

where $f_x(x) \geq 0$ *for every* $x \in R$ *and* $\int_{-\infty}^{\infty} f_X(y) dy = 1$.

If the random variable X is discrete with probabilities $p_k = P(X = x_k)$ he main characteristics of a random variable X are:

- The *mean value,* $\mu_x = E(X) = \sum_{k=1}^{\infty} x_k p_k$,
- The variance of X, $\sigma_X^2 = Var(X) = \sum_{k=1}^{\infty} (x_k - \mu_X)^2 p_k$,
- The rth moment of X, $E(X^r) = \sum_{k=1}^{\infty} x_k^r p_k$, for $r \in N$,

The analogous characteristics for a continuous random variable X with density f_X are:

- The *mean value,* $\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$,

- The variance of X, $\sigma_X^2 = Var(X) = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx$,

- The rth moment of X, $E(X^r) = \int_{-\infty}^{\infty} x^r f_X(x)dx$, for $r \in N$.

If we have a real-valued function h the mean value of h(X) is given by

$E(h(X)) = \sum_{k=1}^{\infty} h(x_k)p_k$ and $E(h(X)) = \int_{-\infty}^{\infty} h(x) f_X(x)dx$ respectively.

Definition 2.1.7 $X = (X_1,...,X_n)$ *is an n-dimensional random vector if its components* $X_1,...,X_n$ *are one-dimensional real valued random variables.*

Definition 2.1.8 *If the distribution of a random vector* ***X*** *has density* f_x *, the distribution function* F_X *of* ***X*** *can be represented as*

$$F_X(x_1,...,x_n) = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} f_X(y_1,...,y_n)dy_1...dy_n, (x_1,...,x_n) \in R^n$$

where the density is a function satisfying $f_X(x) \geq 0$ *for every* $x \in R^n$ *and*

$$\int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} f_X(y_1,...,y_n)dy_1...dy_n, (x_1,...,x_n) = 1.$$

The mean value, and covariance of a random vector are given by:

- $\mu_x = E(X) = (E(X_1),...E(X_n))$,

- $\mathrm{cov}(X_i, X_j) = \sigma_{X_{i,j}}^2 = E[(X_i - \mu_{X_i})(X_j - \mu_{X_j})] = E(X_i X_j) - \mu x_i \mu x_j$.

Definition 2.1.9

a. *Two events A and B are independent if:*

$$P(A \cap B) = P(A)P(B)$$

b. *Two random variables X_1 and X_2 are independent if*

$$P(X_1 \in A, X_2 \in B) = P(X_1 \in A)P(X_2 \in B)$$

for all suitable subsets A and B of R.

Definition 2.1.10

a. *The events $A_1,\ldots, A_n$ are independent if*

$$P(A_{i_1} \cap \ldots \cap A_{i_k}) = P(A_{i_1})\ldots P(A_{i_n}) \ , \qquad (2.4)$$

for $1 \le i_1 < \ldots < i_k \le n$ *and* $1 \le k \le n$.

b. *The random variables $X_1, \ldots, X_n$ are independent if*

$$P(X_{i_1} \in B_{i_1}, \ldots, X_{i_k} \in B_{i_k}) = P(X_{i_1} \in B_{i_1})\ldots P(X_{i_k} \in B_{i_k}),$$

for $1 \le i_1 < \ldots < i_k \le n$, $1 \le k \le n$ *and* $B_1, \ldots B_n \in R$.

Definition 2.1.11 *If* $X_1, \ldots, X_n$ *are independent, then*

$$E[g_1(X_1)\ldots g_n(X_n)] = Eg_1(X_1)\ldots Eg_n(X_n), \qquad (2.5)$$

for any real-valued functions $g_1, \ldots, g_n$.

We also give the following definition and theorem.

Definition 2.1.12 *A random sample of size n from a given distribution is a set of n independent random variables* $X_1, X_2, ..., X_n$ *each having the given distribution with expectation μ and variance* σ^2 *. This set of random variables is called* ***independent, identically distributed (iid)*** [27].

The *sample sum* (S) of these random variables is $S = \sum_{i=1}^{n} X_i$ with

$$E(S) = n\mu \text{ and } Var(S) = n\sigma^2.$$

We also have the *sample mean* $(\overline{X})$ which is $\overline{X} = \sum_{i=1}^{n} X_i$ with

$$E(\overline{X}) = \mu \text{ and } Var(\overline{X}) = \sigma^2/n.$$

According to the ***Central Limit Theorem*** *(CLT)* the mean and the sum of a random sample of a large enough size $(n \to \infty)$ from an arbitrary distribution have approximately normal distribution. We consider a random sample $X_1, X_2, ..., X_n$ with expectation μ and variance σ^2:

The **sample sum** $S = \sum_{i=1}^{n} X_i$ is approximately normal $N(n\mu, n\sigma^2)$ and the standardized version of S, $S^* = \frac{S - n\mu}{\sigma\sqrt{n}}$ is approximately standard normal.

The **sample mean** $\overline{X} = \sum_{i=1}^{n} X_i$ is approximately normal $N(\mu, \sigma^2/n)$ and the standardized version of $\overline{X}$, $\overline{X}^* = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$ is approximately standard norma [27]l.

2.2 Stochastic Processes

Stochastic processes were studied in the 19th century to aid in understanding financial markets and Brownian motion (see [2], [4], [16], [19], [20], [21], [22]). In probability theory, a stochastic process is a collection of random variables representing the evolution of some system of random values over time. Let's now define the meaning of a stochastic process:

Definition 2.2.1 *A stochastic process is a collection of random variables defined on some space Ω.*

$$(X_t, t \in T) = (X_t(\omega), \quad t \in T, \omega \in \Omega). \tag{2.6}$$

A stochastic process has two variables, t and ω.

- For every $t \in T$ (that we consider as constant and ordered), we have a random variable:

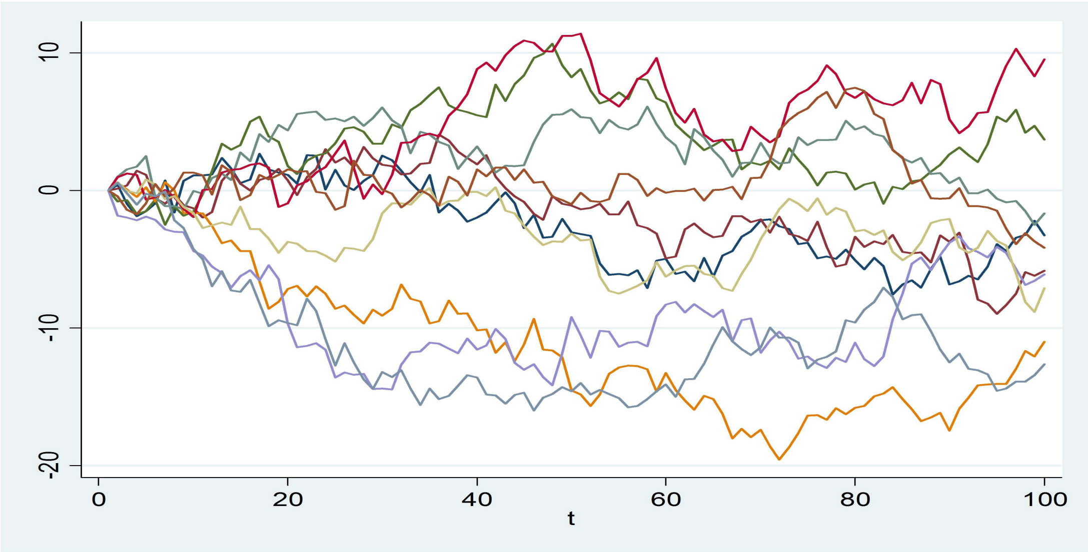
$$\omega \to X_t(\omega), \ \omega \in \Omega$$

- For a fixed random outcome $\omega \in \Omega$ we consider the function

$$t \to X_t(\omega), \ t \in T \ ,$$

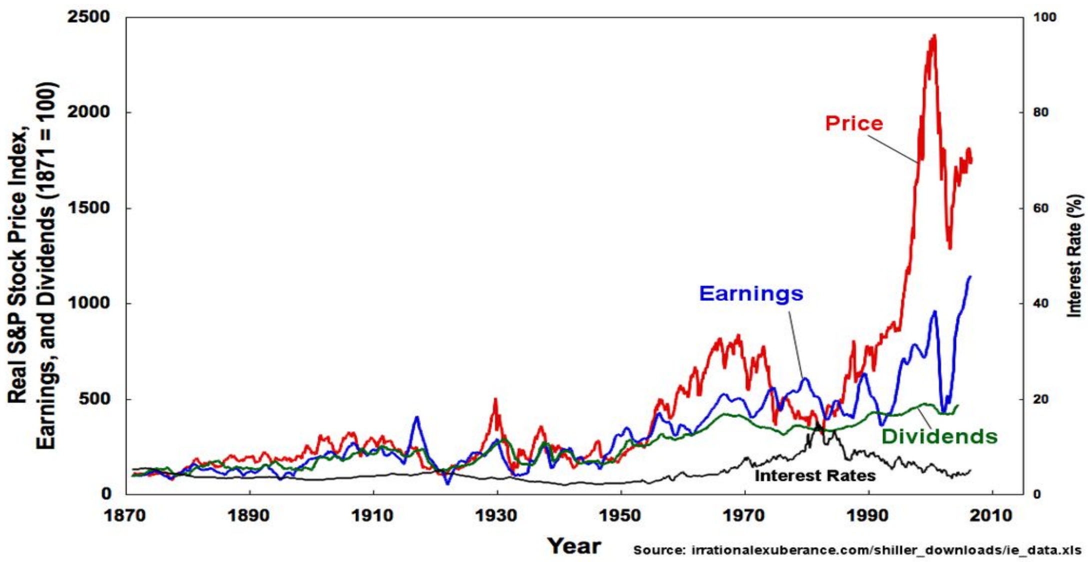
that is called a sample path of the process X_t.

We can consider t as the time, which may be continuous or discrete and ω is a particle or an experiment. A specific choice of ω is called a stochastic process. Then $X_t(\omega)$ is the position of particle ω the time t or the result of the experiment ω the time t.



Every path corresponds to a different $\omega \in \Omega$.

Figure 2.1



The scaled values of the S&P index over a period of 140 years. We consider the S&P time series as the sample path of a continuous-time process.

Figure 2.2

2.2.1 Distribution - Expectation - Covariance Function

In this section, we will introduce non-random characteristics of a stochastic process such as its distribution, expectation and covariance and we will describe its independent structure, which is more complicated than the description of a random vector. In order to avoid using the advanced mathematics that are required, we will find some simpler means [22].

Definition 2.2.2 *The finite-dimensional distributions (fidis) of the stochastic process X are the distributions of the finite-dimensional vectors*

$$(X_{t_1},\dots,X_{t_n}), \qquad t_1,\dots,t_n \in T, \quad n \geq 1. \tag{2.7}$$

The kind of fidis is one of the different criteria a stochastic process can be classified.

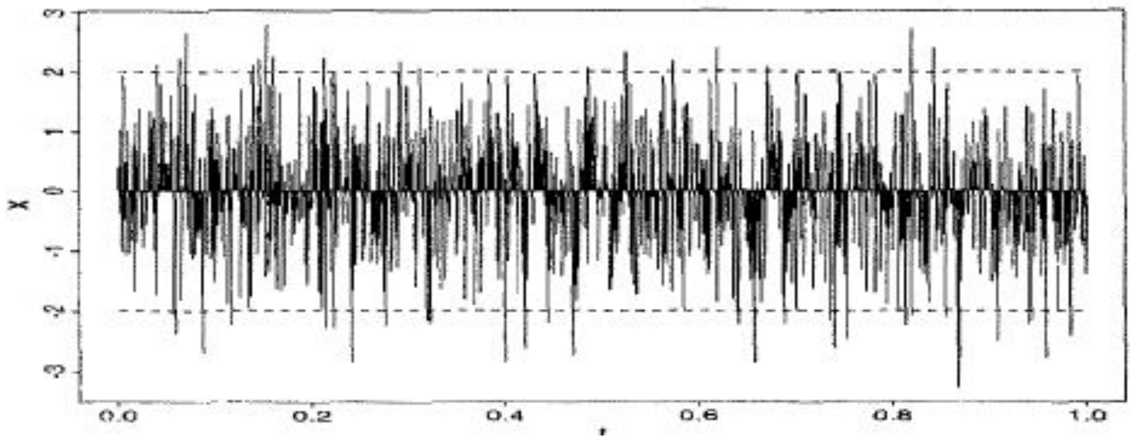
At the previous paragraph 2.1 we defined the *expectation* $\mu_x = E(X) = (E(X_1),\dots,E(X_n))$ and the *covariance* $\mathrm{Cov}(X_i X_j)$, $i, j = 1,2,\dots,n$ for a random vector $X = (X_1,\dots,X_n)$. We consider a stochastic process $X = (X_t,\ t \in T)$ as the collection of the random vectors

$(X_{t_1},\dots,X_{t_n})$, $t_1,\dots,t_n \in T$, $n \geq 1$.

Example 2.2.1 We write $N(\mu,\sigma)$ for the distribution of a n-dimensional Gaussian vector X with expectation μ and covariance matrix Σ. A stochastic process is called *Gaussian* if all its fidis are multivariate Gaussian. The distribution of a Gaussian stochastic process is determined by the collection of the expectations and covariance matrices of the *fidis*.

A simple Gaussian process on $T = [0,1]$ consists of iid $N(0,1)$ random variables and the *fidis* are characterized by the distribution functions

$$
\begin{aligned}
&P(X_{t_1} \le x_1, \ldots, X_{t_n} \le x_n) \\
&= P(X_{t_1} \le x_1) \ldots P(X_{t_n} \le x_n) \\
&= \Phi(x_1) \ldots \Phi(x_n), \\
&0 \le t_1 \le \ldots \le t_n \le 1, \quad (x_1, \ldots, x_n) \in R^n.
\end{aligned}
$$



A sample path of the Gaussian process $(X_t, t \in [0,1])$ *where the dist* X_t*s are iid N(0,1).*

The expectation is 0 and the dashed lines indicate the curves $\pm 2\sigma_x(t) = \pm 2$.

Figure 2.3

Further, we give the following definition

Definition 2.2.3

a. The *expectation function* of X is given by

$$\mu_x(t) = EX_t, t \in T.$$

b. The *covariance function* of X is given by

$$c_x(t,s) = \text{cov}(X_t, X_s) = E[(X_t - \mu_x(t))(X_s - \mu_x(s))], t, s \in T$$

c. The *variance function* of X is given by

$$\sigma_x^2(t) = c_x(t,t) = \text{var}(X_t), t \in T.$$

2.3 Martingales

Martingales is a specific class of stochastic processes and they are play an important role in probability theory and stochastic analysis (see [2], [22]). They considered as models for fair games where knowledge of past events doesn't help predict the meaning of the future winnings and they are also important for the understanding of the Ito stochastic integral. We will start defining the mean of filtration at first.

Definition 2.3.1 *Filtration is a collection of σ- fields F_t where*

$$s \le t \Rightarrow F_s \subset F_t . \tag{2.8}$$

A σ-field F_t can be considered as an information that is available until the time t. Thus the filtration can be considered as an increasing stream of information as time passes.

Definition 2.3.2 *The stochastic process $X=(X_t,\ t \ge 0)$ is said to be adapted to the filtration $(F_t,\ t\ge 0)$ if*

$$\sigma(X_t) \subset F_t \ \ \textit{for all}\ \ t \ge 0.$$

The stochastic process X is always adapted to the natural filtration generated by X:

$$F_t = \sigma(X_s, s \le t). \tag{2.9}$$

Thus adaptedness of a stochastic process X means that the X_ts do not carry more information than F_t . If we have a discrete-time process $X=(X_n,\ n=0,1,...)$ we define adaptedness in an analogous way: for a filtration $(F_n,\ n=0,1,...)$ we require that $\sigma(X_n) \subset F_n$.

Definition 2.3.3 *A stochastic process $X=(X_t,\ t \geq 0)$ is called martingale with respect to the filtration $(F_t,\ t\geq 0)$, if:*

- $E|X_t| < \infty$ *for all* $t \geq 0$.
- *X is adapted to (F_t).*
- $E(X_t \mid F_s) = X_s$ *for all* $0 \leq s < t$.

We write $(X,(F_t))$ *and X_s is the best prediction of X_t given F_s.*

Example 2.3.1 Suppose we have a coin tossing where the coin is unfair, with probability p of coming up heads and probability q=1-p of tails. Let $X_{n+1} = X_n \pm 1$ with "+" in case of "heads" and "-" in case of "tails". Let $Y_n = (q/p)^{X_n}$.

Then $\{Y_n : n = 1,2,3,...\}$ is a *martingale* with respect to $\{X_n : n = 1,2,3,...\}$.

Let's prove this:

$$\begin{aligned} E[Y_{n+1} \mid X_1,...,X_n] &= p(q/p)^{X_n+1} + q(q/p)^{X_n-1} \\ &= p(q/p)(q/p)^{X_n} + q(p/q)(q/p)^{X_n} \\ &= q(q/p)^{X_n} + p(q/p)^{X_n} \\ &= (q/p)^{X_n} \\ &= Y_n \end{aligned}$$

For a detail analysis we refer to [23].

2.4 Brownian Motion

Sometimes it's easier to use models when studying the evolution of stock prices. These models considerate the variable of time to be continual, since the transactions in a market take place very often, resulting to very short intervals between the transactions. Hence, we can turn to formulas concerning the prices of various derivatives contracts. These formulas are rather useful and they inform us about the approximate price range. In this paragraph, we will try to describe the stock prices as a stochastic process. The dynamic model that we will describe links the stock price at time period $t+\Delta t$ for $\Delta t \to 0$ and it is somehow the generalization of the binomial model in continuous time and state-space (see [2], [22]).

Brownian Motion plays an important role in probability theory, theory of stochastic processes, finance etc. We will start giving the definition of this process and then we will continue with some of its elementary properties.

2.4.1 Definition of Brownian Motion

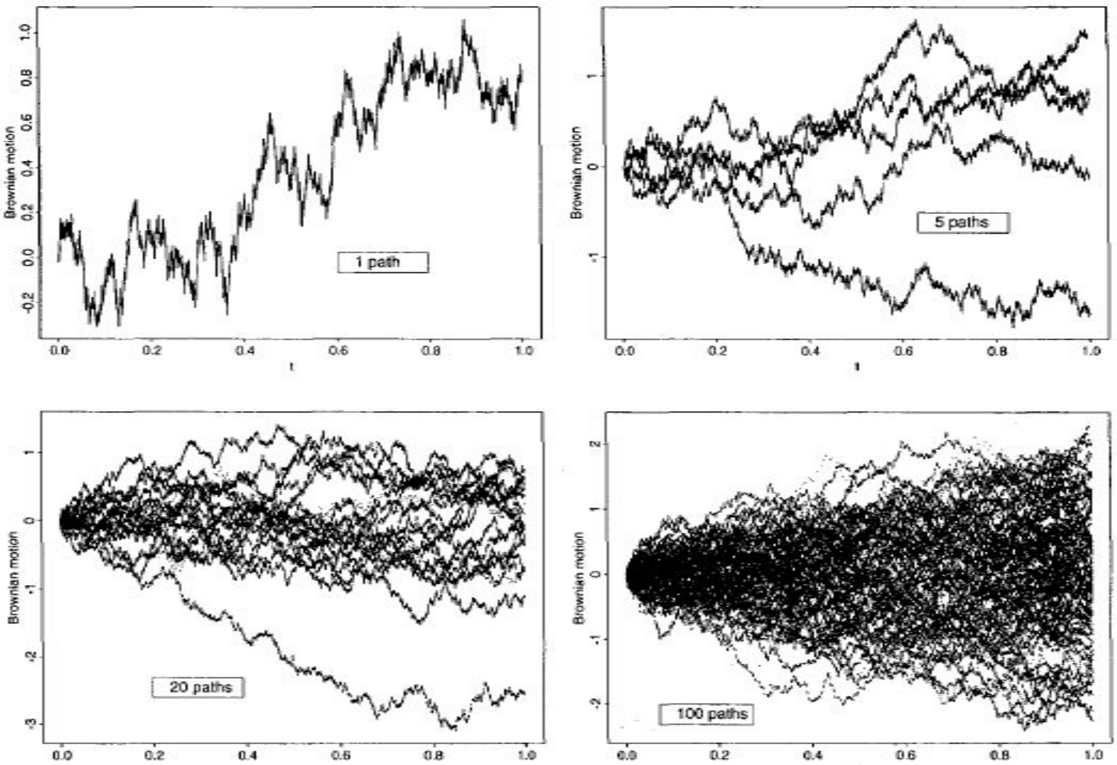
Definition 2.4.1 *Brownian Motion (or a Wiener process) is a stochastic process B_t, $t \in [0,\infty)$ that has the following properties:*

- *$B_0 = 0$. (Standard Brownian Motion also known as Wiener Motion)*
- *If $t_0 < t_1 < \ldots < t_n$ then the random variables $B_{t_0}, B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$ are* ***independent*** *increments.*
- *If s,t>0, then*

$$P(B_{s+t} - B_s \in A) = \int_A \frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{|x|^2}{2t} \right) dx,$$

where A is a Borel set. For every t>0, B_t has a normal N(0,t) distribution.

- *It has continuous sample paths.*



Sample paths of a Brownian Motion on [0,1]

Figure 2.4

Brownian Motion took its name from the biologist Robert Brown whose research dates to the 1820's. Louis Bachelier (1900), Albert Einstein (1905) and Norbert Wiener (1923) developed the mathematical theory of Brownian Motion later.

2.4.2 Distribution, Expectation and Covariance Function

The *fidis* of Brownian Motion are multivariate Gaussian process. Brownian Motion has independent Gaussian increments.

Definition 2.4.2 *The random variables* $B_t - B_s$ *and* B_{t-s} *have an* $N(0, t-s)$ *distribution for* $s < t$.

$B_t - B_s$ and B_{t-s} have the same distribution which is normal with expectation zero and variance t-s. Thus the variance belongs to the interval $[s, t]$, which means the larger the interval, the larger the fluctuations of Brownian motion on this interval.

The distribution identity $B_t - B_s = B_{t-s}$ does not entail pathwise identity. Generally

$$B_t(\omega) - B_s(\omega) \neq B_{t-s}(\omega)\,. \tag{2.10}$$

The Brownian motion has expectation function

$$\mu_B(t) = EB_t = 0\,, \qquad t \geq 0, \tag{2.11}$$

and since the increments $B_t - B_s$ and B_s are independent for $t > s$, the covariance function is

$$\begin{aligned} c_B(t,s) &= E[[(B_t - B_s) + B_s]B_s] \\ &= E[(B_t - B_s)B_s] + EB_s^2 \\ &= E(B_t - B_s)EB_s + s \\ &= 0 + s = s, \end{aligned} \tag{2.12}$$

for $0 \leq s < t$.

Definition 2.4.3 *Brownian motion is a Gaussian process with*

$$\mu_B(t) = 0 \;\; and \;\; c_B(t,s) = \min(s,t). \tag{2.13}$$

Example 2.4.1 We will prove that $E[B_{t+s} \mid B_t] = B_{t.}$

For any random variables X,Y and Z we know that $E(X+Y \mid Z) = E(X \mid Z) + E(Y \mid Z)$ and $E(X \mid X) = X$. Using these properties, we have

$$\begin{aligned} E[B_{t+s} \mid B_t] &= E[B_{t+s} - B_t + B_t \mid B_t] \\ &= E[B_{t+s} - B_t \mid B_t] + E[B_t \mid B_t] \\ &= 0 + B_t = B_t\,. \end{aligned}$$

We can easily understand that B_t is a **martingale**.

Example 2.4.2 Let B_t be a standard Brownian motion. We will prove that $E(B_t B_s) = \min\{t,s\}, \quad t,s \geq 0$.

Suppose that $t \geq s$. Since B_t is a standard Brownian motion, $B_0 = 0$ so we have

$$E[B_t] = E[B_t - B_0] = 0 \text{ and } Var[B_t] = Var[B_t - B_0] = t - 0 = t. \text{But}$$

$$Var[B_t] = E[B_t{}^2] - (E[B_t])^2 = E[B_t{}^2] - 0 = E[B_t{}^2] = t.$$

Now,

$$\begin{aligned} E[B_t\, B_s] &= E[(B_s + B_t - B_s\, B_s)] \\ &= E[B_s{}^2] + E[B_t - B_s B_s] \\ &= s + E[B_t - B_s]\, E[B_s] \\ &= s + 0 = \min\{t,s\} \end{aligned}$$

since $B_t - B_s \text{and} B_s$ are independent.

A standard Brownian motion B_t can be approximated by a sum of independent binomial random variables [17], [22]. B_t is continuous, so for a small time period h we can estimate the change B_t from t to t+h by the equation

$$B_{t+h} - B_t = X_{t+h}\sqrt{h},$$

where X_t is a random draw from a binomial distribution ($X_t = \pm 1$ with probability 0.5, expectation 0 and variance 1). Let's take naw the interval [0,T] and divide it into n equal subintervals each of length $h = T/n$. Then we have

$$\begin{aligned} B_T &= \sum_{i=1}^{n}[B_{ih} - B_{(i-1)h}] = \sum_{i=1}^{n} X_{ih}\sqrt{h} \\ &= \sqrt{T}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_{ih}\right] \end{aligned}$$

since $E(X_{ih}) = 0$,

$$E\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_{ih}\right] = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} E(X_{ih}) = 0.$$

And since $\text{Var}(X_{ih}) = 1$,

$$\text{Var}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_{ih}\right] = \frac{1}{n}\sum_{i=1}^{n} 1 = 1.$$

Hence, a standard Brownian motion is being approximately generated from the sum of independent binomial draws with expectation 0 and variance h.

By using the Central Limit Theorem (par. 2.1.1) we presume that the limit

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X(ih)$$

approaches a standard normal distribution, say W. Hence

$$B_T = \sqrt{B}\, W_T, \tag{2.14}$$

where B_T is approximated by a normal random variable which has mean 0 and variance T. In the end we will represent B_t in integral form

$$B_T = \int_0^T dB_t \ ,$$

where the integral is called a **stochastic integral**.

If we rename h as dt and change B as dB_t, we can write (2.3)

$$dB_t = W_t\sqrt{dt}\ . \tag{2.15}$$

From the last equation (2.4) we consume that over smaal periods of time changes in the value of the process are normally distributed with a variance that is analogical to the length of the time period.

2.4.3 Path Properties

We have now one sample path $B_t(\omega), t \geq 0$ and we consider its properties [17],[22]. We mentioned before that the sample paths of a Brownian motion are continuous. Observing the simulated Brownian paths we see that the functions of t are extremely irregular because they oscillate wildly. This happens because the increments of B are independent.

Definition 2.4.4 *A stochastic process* $(X_t, t \in [0,\infty))$ *is H-self-similar for some H>0 if its fidis satisfies the condition*

$$(T^H B_{t_1},...,T^H B_{t_n}) \underline{\underline{d}} (B_{Tt_1},...,B_{Tt_n}) \tag{2.16}$$

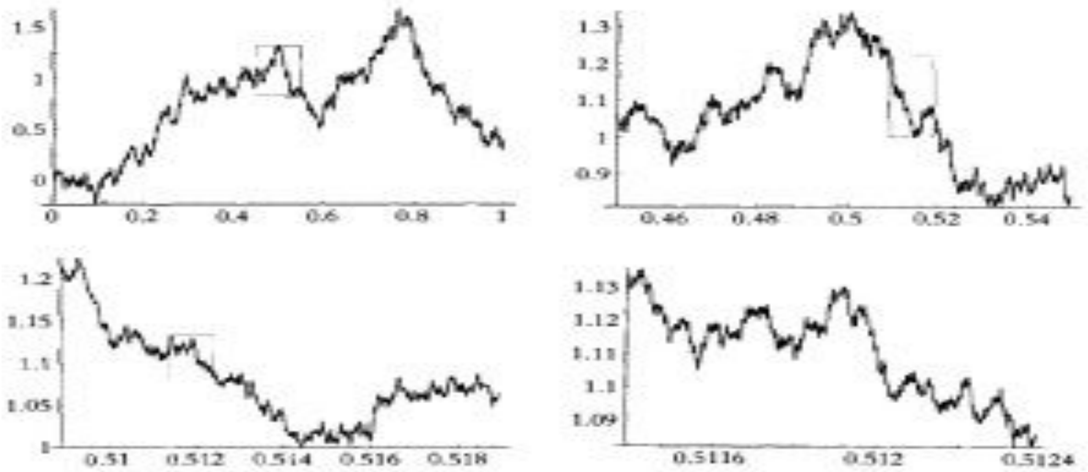
for $T > 0$, $t_i > 0$, $i=1,...,n$, *and* $n \geq 1$.

Self-similarity is not a pathwise property but a distributional.

Definition 2.4.5 *Brownian motion is 1/2-self-similar*

$$(T^{1/2} B_{t_1},...,T^{1/2} B_{t_n}) \underline{\underline{d}} (B_{Tt_1},...,B_{Tt_n}) \tag{2.17}$$

for $T > 0$, $t_i > 0$, $i=1,...,n$, *and* $n \geq 1$.



Self-similarity: the same Brownian sample path on different scales

Figure 2.5

A further indication of the irregularity of Brownian motion is given by the following definition:

Definition 2.4.6 *Brownian sample paths do not have bounded variation on any finite interval [0,T]. This means that*

$$\sup_{\tau}\sum_{i-1}^{n}|\,B_{t_i}(\omega)-B_{t_{i-1}}(\omega)\,|=\infty\,, \tag{2.18}$$

where the supremum (the supremum of a subset S of a partially ordered set T is the least element in T that is greater than or equal to all elements of S, if such an element exists. Consequently, the supremum is also referred to as the least upper bound) is taken over all possible partitions $\tau:\ 0=t_0<\ldots<t_n=T$ *of [0,T].*

The unbounded variation and non-differentiability of Brownian sample paths are the reasons for the failure of classical integration methods, when applied to these paths. Now, using binomial approximation to the Brownian process we have the following definition:

Definition 2.4.7 *Consider an interval [a,b] and divide it into n equal subintervals. The quadratic variation of a stochastic process* $B_t,\ a\le t\le b$ *is defined to be*

$$\lim_{n\to\infty}\sum_{i=1}^{n}[B_{t_i}-B_{t_{i-1}}]^2=\int_a^b[dB_t]^2$$

if the limit exists. If we have a standard Brownian motion the limit is

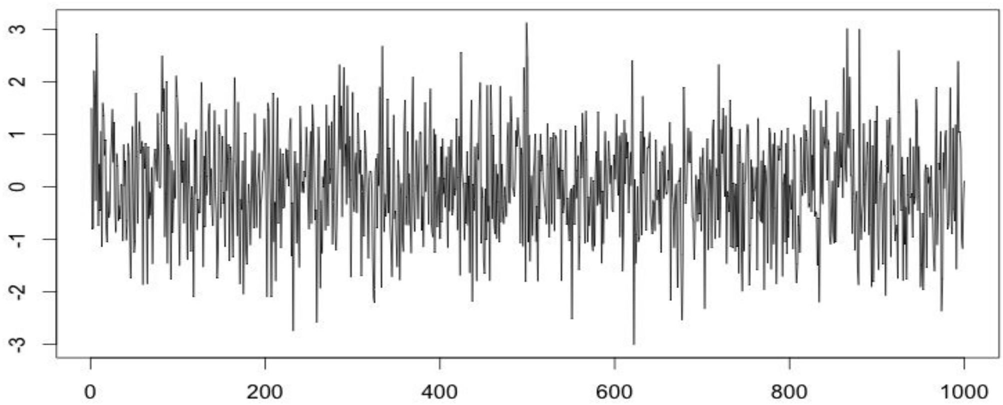
$$\lim_{n\to\infty}\sum_{i=1}^{n}[B_{ih}-B_{(i-1)h}]^2=\lim_{n\to\infty}\sum_{i=1}^{n}(X_{ih}\sqrt{h})^2$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}X_{ih}^2\,h=T<\infty.$$

Definition 2.4.8 *The total variation of the standard Brownian process is*

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}[B_{t_i}-B_{t_{i-1}}] &= \lim_{n\to\infty}\sum_{i=1}^{n}|X_{ih}|\sqrt{h} \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{h} = \sqrt{T}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n}} \\ &= \sqrt{T}\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{n} = \infty.\end{aligned}$$

We conclude that the Brownian path moves up and down very fast in the interval [0,T] having a starting point which is crossed an infinite number of times as we see in the Figure below.



Random sample path from N(0,1)

Figure 2.6

Example 2.4.3 Let B be a standard Brownian motion. We will prove that $\lim_{n\to\infty}\sum_{i=1}^{n}[B_{ih}-B_{(i-1)h}]^4 = 0.$

We have

$$| \sum_{i=1}^{n} [B_{ih} - B_{(i-1)h}]^4 | = | \sum_{i=1}^{n} (X_{ih} \sqrt{h})^4 |$$

$$= | \sum_{i=1}^{n} X_{ih}^4 h^2 |$$

$$\leq \sum_{i=1}^{n} h^2 = \frac{T^2}{n^2}.$$

So, $\lim_{n \to \infty} \sum_{i=1}^{n} [B_{ih} - B_{(i-1)h}]^4 = 0.$

2.4.4 Processes Derived from Brownian Motion

In this section we will examine the distributional and pathwise properties of Brownian motion[22]. Various Gaussian and non-Gaussian stochastic processes can be derived from Brownian motion. We will present some of them below.

Example 2.4.4 (Brownian Bridge)

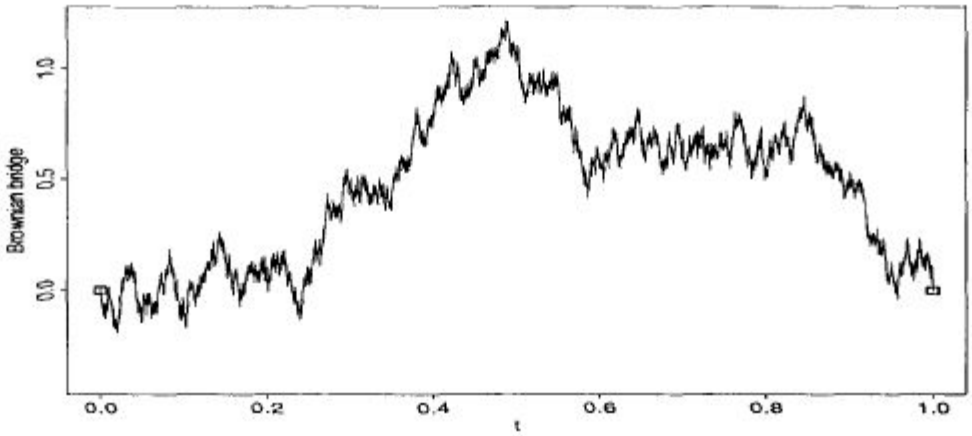
Consider the process

$$X_t = B_t - t B_1, \quad 0 \leq t \leq 1.$$

Obviously,

$$X_0 = B_0 - 0 B_1 = 0 \quad \text{and} \quad X_1 = B_1 - 1 B_1 = 0.$$

For this reason, the process X is called *Brownian Bridge.*



A sample path of a Brownian Bridge.

Figure 2.6

Observing the sample paths of this *"bridge"* at Figure 2.6, we may be convinced that this name is justified. We can show that the fidis of X are Gaussian, so X is a Gaussian process. Since X is Gaussian, we have these two functions for the Brownian bridge:

$$\mu_X(t) = 0 \quad \text{and} \quad c_X(t,s) = \min(t,s) - ts, \quad s,t \in [0,1].$$

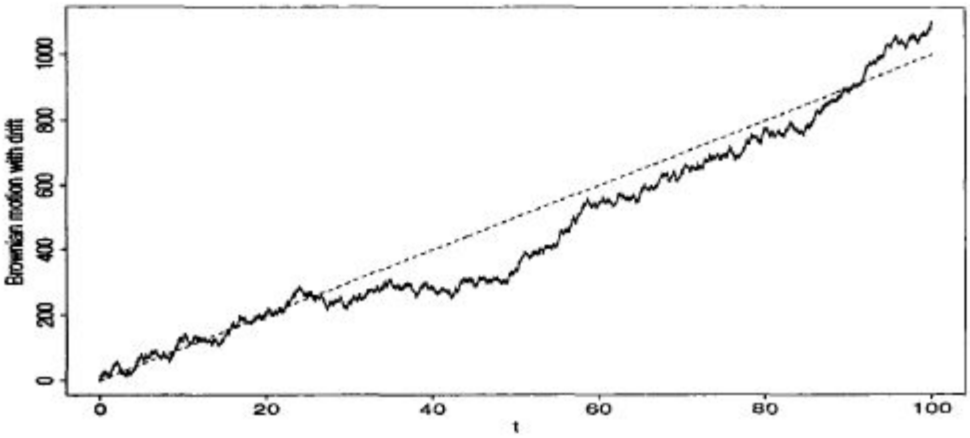
The Brownian bridge seems to be the limit process of the *normalized empirical distribution* function of a sample of iid uniform U(0,1) random variables.

Example 2.4.5 (Brownian motion with drift)

Consider the process $X_t = \mu t + \sigma B_t, \quad t \geq 0.$

for constants $\sigma > 0$ and $\mu \in R$. It is obvious that it is a Gaussian process. The expectation and covariance functions are:

$$\mu_X(t) = \mu t \quad \text{and} \quad c_X(t,s) = \sigma^2 \min(t,s), \quad t,s \geq 0 \quad \text{respectively.}$$



A sample path of Brownian motion with drift $X_t = 20B_t + 10t$ *on [0,100].*

Figure 2.7

As we can see in the Figure 2.7 the expectation function $\mu_X(t) = \mu t$ determines the characteristic shape of the sample paths. X is called *Brownian notion with drift*.

2.5 Arithmetic Brownian Motion. The Ornstein-Uhlenbeck Process

Considering the standard Brownian motion, dB_t has expectation 0 and variance 1 per unit time. In order to generalize this having nonzero mean and arbitrary variance, we give the following definition [17]

Definition 2.5.1 *We define* X_t *by*

$$X_{t+h} - X_t = \alpha h + \sigma Y_{t+h}\sqrt{h}\,,$$

Where Y_t *is a random draw from a binomial distribution,* αh *is called the* ***drift term*** *and* $\sigma\sqrt{h}$ *the* ***noise term****. We divide the interval [0,T], T > 0 into n subintervals each of length* $h = T/n$. *Then*

$$X_T - X_0 = \sum_{i=1}^{n}\left(\alpha\frac{T}{n} + \sigma Y_{ih}\sqrt{\frac{T}{n}}\right)$$
$$= \alpha T + \sigma\left(\sqrt{T}\sum_{i=1}^{n}\frac{Y_{ih}}{\sqrt{n}}\right) \qquad (2.19)$$

Recalling the Central Limit Theorem, $\sqrt{T}\sum_{i=1}^{n}\frac{Y_{ih}}{\sqrt{n}}$ *approaches a normal distribution with mean 0 and variance T. Hence, we can write (2.19)*

$$X_T - X_0 = \alpha T + \sigma B_t\ , \qquad (2.20)$$

where B is the standard Brownian motion. The stochastic differential form of (2.20) is

$$dX_t = \alpha\, dt + \sigma\, dB_t. \qquad (2.21)$$

Definition 2.5.2 *We call an arithmetic Brownian motion a stochastic process* $\{X_t\}_{t\geq 0}$, *that satisfies (2.21).*

We note that $E(X_t - X_0) = \alpha t$ and $Var\,(X_T - X_0 = (0)) = Var\,(\alpha t + \sigma B_t) = \sigma^2 t.$

We call α the instantaneous mean per unit time and σ^2 the instantaneous variance per unit time. $X_T - X_0$ follows normal distribution with mean αt and variance $\sigma^2 t.$ Finally, X_t is normally distributed with mean $X_0 + \alpha t$ and variance $\alpha^2 t$.

Example 2.5.1 Consider an arithmetic Brownian motion $\{X(t)\}_{t\geq 0}$ with drift factor α and volatility σ. Now we will prove that $X_t = X_\alpha + \alpha(t-\alpha) + \sigma\sqrt{t-\alpha}W_t$, where W is a standard normal random variable.

Actually,

$$X_t = X_\alpha + \alpha(t-\alpha) + \sigma\sqrt{t-\alpha}W_t \Leftrightarrow$$
$$X_t - X_\alpha = \alpha(t-\alpha) + \sigma\sqrt{t-\alpha}W_t.$$

X_t is normally distributed with mean $X_\alpha + \alpha(t-\alpha)$ and variance $\sigma^2(t-a)$.

Example 2.5.2 $\{X(t)\}_{t\geq 0}$ follows an arithmetic Brownian motion such that $X(20)=3$. The drift factor of this Brownian motion is given 0.345, and the volatility 0,65. What is $P(X(24)<0)$?

The mean of a normal distribution X(24) is $X(20)+\alpha(24-20)=3+0.345\times 4=4.38$.

The standard deviation is $\sigma\sqrt{t-\alpha}=0.65\sqrt{24-20}=1.3$.

The probability is

$$P(X(24)<0)=P(Z<\frac{0-4.38}{1.3})=N(Z<-3.37)=0.9996.$$

2.5.1 The Ornstein-Uhlenbeck Process

Observing commodity prices, we can easily see that these prices have a tendency to revert to the mean. Hence, if a value departs from the mean, it will tend to return to the mean. Studying this concludes that a *mean-reversion model* is more logic than the arithmetic Brownian process we analyzed before. If we modify the drift term in (2.21) we have the following equation known as the **Ornstein-Uhlenbeck** process [17]

$$dX_t = \lambda(\alpha - X_t)\,dt + \sigma\,dB_t \tag{2.22}$$

where α is the long-run mean value which X_t tends to revert, λ is the speed of reversion, σ is the volatility factor and B_t is the standard Brownian motion. We will now solve the Eq. (2.22). At first we set

$$Y_t = X_t - \alpha \tag{2.23}$$

so we have

$$
\begin{aligned}
d\,Y_t &= -\lambda\,Y_t\,dt + \sigma\,dB_t \\
&\Leftrightarrow d\,Y_t + \lambda\,Y_t\,dt = \sigma\,dB_t \\
&\Leftrightarrow e^{\lambda t}(d\,Y_t + \lambda\,Y_t dt) = e^{\lambda t}\,\sigma\,dB_t \\
&\Leftrightarrow d\,[e^{\lambda t}\,Y_t] = e^{\lambda t}\sigma\,dB_t.
\end{aligned}
$$

We integrate from 0 to t

$$
\begin{aligned}
\int_0^t d\,[e^{\lambda s}\,Y_s] &= \int_0^t e^{\lambda s}\sigma\,dB_s \\
&\Leftrightarrow [e^{\lambda s}\,Y_s]_0^t = \sigma\int_o^t e^{\lambda s} dB_s \\
&\Leftrightarrow e^{\lambda t}\,Y_t - e^0\,Y_0 = \sigma\int_o^t e^{\lambda s}\,dB_s \\
&\Leftrightarrow e^{\lambda t}\,Y_t - Y_0 = \sigma\int_o^t e^{\lambda s}\,dB_s \\
&\Leftrightarrow Y_t = e^{-\lambda t}\,Y_0 + \sigma\int_0^t e^{-\lambda(t-s)}\,dB_s.
\end{aligned}
$$

Taking into consideration (2.23) we take the solution of the stochastic differential Eq. (2.22) as follows:

$$
X_t = X_0\,e^{-\lambda t} + \alpha\,(1 - e^{-\lambda t}) + \sigma\int_0^t e^{-\lambda(t-s)}\,dB_s. \qquad (2.24)
$$

2.5.2 Geometric Brownian motion - The Equivalent Martingale Measure - The Girsanov's Theorem

The arithmetic Brownian motion we presented before has many disadvantages. If X_t is negative, the arithmetic Brownian motion is not an appropriate model for stock pricing. Also, the stock price is independent from the mean and the variance of changes in dollar terms. For example, if the stock price doubles, we would expect to double both the dollar standard deviation and the dollar expected return. We can eliminate these disadvantages with geometric Brownian motion. Suppose that the drift factor α and the volatility σ of an arithmetic Brownian motion are functions of X_t, then we have the stochastic differential equation [17]

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dB_t \qquad (2.25)$$

We call (2.25) an Itô process. If now $\alpha(X_t) = \alpha X_t$ and $\sigma(X_t) = \sigma X_t$ we have from (2.25)

$$\begin{aligned} dX_t &= \alpha X_t dt + \sigma X_t dB_t \\ \Leftrightarrow \frac{dX_t}{X_t} &= \alpha dt + \sigma dB_t \end{aligned} \qquad (2.26)$$

which is called a **geometric Brownian motion**.

Example 2.6.1 The current price of a stock which follows a geometric Brownian motion with drift rate of 10% and variance rate of 9% per year is 100. What is the probability that the stock's price will exceed 200 after two years from now?

We have to calculate the probability

$$\Pr(S_2 > 200) = \Pr\left(\ln\left(\frac{S_2}{S_0}\right) > \ln 2\right).$$

Thus, $\ln\left(\frac{S_2}{S_0}\right)$ follows normal distribution with mean

$(\alpha - 0.5\sigma^2)t = (0.10 - 0.5(0.09) \times 2 = 0.11$ and variance $\sigma^2 t = 0.09 \times 2 = 0.18$.

The requested probability is

$$\Pr(S_2 > 200) = \Pr\left(Z > \frac{\ln 2 - 0.11}{\sqrt{0.18}}\right) = 1 - \Phi(1.37) = 1 - 0.915 = 0.085.$$

Definition 2.6.1 [17] *The* ***risk premium*** *of an asset is the excess return of the asset over the risk-free rate and is given by*

$$\text{Risk premium} = \alpha - r \qquad (2.27)$$

where a is the expected return on the asset and r is the risk-free rate.

The ***Sharpe ratio*** *of an asset is the risk premium divided by its volatility and is given by*

$$\text{Sharpe ratio} = \phi = \frac{\alpha - r}{\sigma} \qquad (2.28)$$

The Sharpe ratio characterizes how well the return of an asset will indemnify the investor for the risk that he has taken.

Assume now a stock that pays dividend at the compounded yield ε. The equation that gives the price process is

$$\frac{dS_t}{S_t} = (\alpha - \varepsilon)\,dt + \sigma\, dB_t \qquad (2.29)$$

where B_t is a martingale and dB_t is the unexpected portion of the stock return. In equivalent martingale pricing, we need an equivalent martingale version of (2.29). This process has a random part which involves a Brownian motion $\widetilde{B}$ that is a martingale under a transformed probability distribution which is referred as the **equivalent martingale measure**. We will call this random part Q.

We will find now $\widetilde{B}$ and Q. Let

$$\widetilde{B}_t = B_t + \frac{\alpha - r}{\sigma} t \qquad (2.30)$$

A result, known as *Girsanov theorem* asserts that a unique equivalent martingale measure Q exists, under which $\widetilde{B}_t$ is a standard Brownian motion and $\widetilde{B}$ is martingale under Q. If we differentiate (2.30) we have

$$d\widetilde{B}_t = dB_t - \frac{\alpha - r}{\sigma} dt \qquad (2.31)$$

Now if we put (2.31) in Eq. (2.29) we have

$$\frac{dS_t}{S_t} = (r - \varepsilon)\,dt + \sigma\, d\widetilde{B}_t \qquad (2.32)$$

Considering the volatility be the same for the true price process and the equivalent martingale price process, we refer to the Eq. (2.32) as the **equivalent martingale process**.

Example 2.6.2 Let have a stock paying dividends at the continuously compound yield ε. It's true price process is

$$\frac{dS_t}{S_t} = 0.08\,dt + \sigma\, dB_t.$$

The corresponding equivalent martingale price process is

$$\frac{dS_t}{S_t} = 0.03\,dt + \sigma\, d\widetilde{B}_t\,.$$

Find α and ε if the continuously compounded risk-free interest rate is 0.06.

Indeed, It is known that

$$\begin{aligned} & r - \varepsilon = 0.03 \\ & \Rightarrow 0.06 - \varepsilon = 0.03 \\ & \Rightarrow \varepsilon = 0.03 \end{aligned}$$

We also know that

$$\begin{aligned} & \alpha - \varepsilon = 0.08 \\ & \Rightarrow \alpha - 0.06 = 0.08 \\ & \Rightarrow \alpha = 0.14 \end{aligned}$$

Therefore, the *Girsanov theorem* describes how we can change the dynamics of a stochastic process when the original measure is changed to an equivalent probability measure. This theorem is very useful in the theory of financial mathematics as it tells how to convert from the physical measure which the probability that an underlying asset will take a particular value is described to the equivalent martingale measure which is an important tool for pricing derivatives.

In what follows (Chapter 3) we will present the categories of weather derivatives contracts emphasizing at the difference between a weather derivatives contract and an insurance policy. Also, we will find a pricing model for weather derivatives and we will suggest a stochastic process that describes the evolution of the temperature. Various applications setting temperature as the underlying variable will be given.

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CHAPTER 3

A Pricing Model for Weather Derivatives

Nowadays, our society is advanced and technology-based, but we still live at the mercy of the weather. Our lives and our choices many times are influenced by the weather. Also, the corporate revenues (including industry, agriculture, construction, energy, entertainment, travel and others) are depending on the weather too. For this reason, many companies would like to have some financial tools, using them in order to be protected against weather risk, which is very unique because it cannot be controlled and predicted consistently. The weather derivatives market offered them these tools, by making the weather a tradable commodity.

Until recently, insurance companies had undertook to protect all the aforementioned companies against unexpected weather phenomena. But insurance companies compensated only in case of catastrophic damages. In addition, compensations of weather derivatives contracts considerate only the actual weather outcome and not the effect of it on the holder of the contract, so they are designed to offer payouts in each and every weather condition. Since weather derivatives can be used just for speculation, a buyer does not need to own a production that is sensitive to weather conditions.

Another significant benefit of derivatives is that on the market there may be an actor who profits from warm winter and an actor that takes advantage of cold winter. In derivatives, there's the potential of entering a contract between the two parties in order to hedge each others risks, something which is not possible in plain insurance contracts [1], [6], [7], [15].

3.1 The Weather Derivatives Contracts

Weather derivatives are structured as futures, call or put options and swaps, based on several underlying weather indices such as humidity, snowfall, hot or cold days and rainfall. We consider degrees-days as underlying indices because they are widely used in the industry. Henceforth, by saying *"temperature"* we will refer to the definition underneath [1]:

Definition 3.1 *Let the daily temperature T_i, for day i defined as:*

$$T_i = \frac{T_i^{\max} + T_i^{\min}}{2}, \qquad (3.1)$$

where T_i^{max} and T_i^{min} the maximum and minimum daily temperature (Celcius Degrees), respectively, which we receive from a nonelocated weather station.

To distinguish between a hot day and a cold, we need a reference point. This point was set at 18^o C for the USA, as well as the terms heating, cooling degree - days (HDDs, CDDs). This was set as a reference point because it was observed that if the temperature is below 18^o C, more energy is consumed for heating homes, while if the temperature is over 18^o C, people use air conditioners to cool the air at their homes.

Definition 3.2 *We define the Heating Degree Day (HDD_i) and Cooling Degree Day (CDD_i) as follows:*

$$HDD_i := max\{18 - T_i, 0\} \text{ and } CDD_i := max\{T_i - 18, 0\}, \qquad (3.2)$$

given that T_i is the daily temperature.

3.1.1 The Chicago Mercantile Exchange Contracts (CME)

CME Degree Day Index provides futures which are necessary for the CME trading. This Index is a total of daily HDDs and CDDs, throughout a calendar month. The Index has already been specified for 11 US cities. HDD/CDD Index futures form agreements between a seller and a buyer to sell or purchase HDDs/CDDs value in the future. The target price of a contract is $100 times the Degree Day Index and they are measured in HDD/CDD Index points. In addition, they involve a daily marking-to-market, which is based to the Index, while the customer's account is updated with gain or loss. As regards call options, CME CDD/HDD contracts offer the owner the ability, though not the obligation, to purchase a HDD/CDD futures contract at a strike or exercise price. On the other hand, an HDD/CDD put option offers the owner the right, though not the obligation, to sell a HDD/CDD contract. The options on futures on the CME can only be exercised at the expiration date, since they are European style [1].

3.1.2 Weather Options

Apart from the CME, there are several different contracts traded on the OTC market, such as the option. Options are divided into two types, calls and puts. For instance, a person who buys an HDD contract must pay a premium to the seller at the beginning of the contract. As an exchange, given that the number of HDDs during the contract period is higher than the determined strike level, the owner of the contract will receive a payment in full. This payout depends on the strike and the tick size. The latter refers to the amount of money paid to the call holder for every degree-day above the strike level for a specific period. In many cases, the option includes a cap on the maximum payout, contrary to traditional options on stocks [1].

To create a general weather option, the following parameters must be specified:

- Type of contract (call/put)
- Period of contract (e.g February 2012)
- Underl44ying Index (CDD/HDD)
- A weather station to provide temperature data

- Tick size
- Maximum payout
- Strike level.

Let K be the strike level and α the tick size for a contract period consists of n days. We will try to find a formula for the payout of this option. At first, the number of HDDs and CDDs for this period are:

$$H_n = \sum_{i=1}^{n} HDD_i \quad \text{and} \quad C_n = \sum_{i=1}^{n} CDD_i \qquad (3.3)$$

respectively.

The payout X for an uncapped HDD call option can be written as:

$$X = \alpha \max\{H_n - K, 0\} \qquad (3.4)$$

In the same way we can define the payouts for HDD puts and CDD calls and puts.

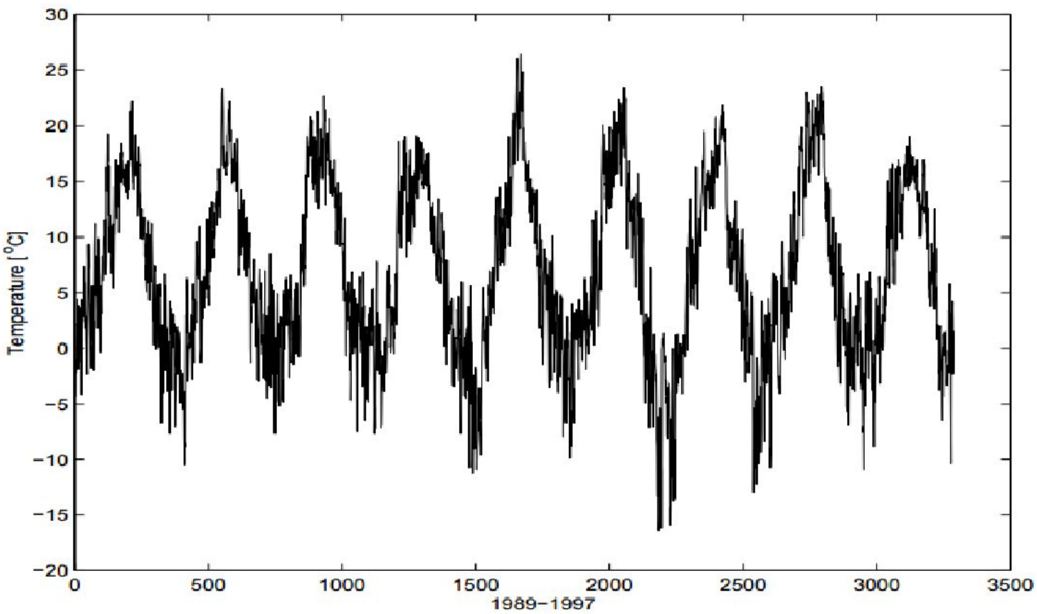
3.1.3 Weather Swaps

Swaps are contracts between two parties to exchange risks during a specific period of time. The majority of swaps involves payments between the two parties, in which one party pays a fixed price while the other pays a variable price. The type of weather swaps that is commonly used features only one date when the cash-flow swap takes place, in contrast to interest rate swaps which include several swap dates. These swaps that feature only one period can be considered as forward contracts. The contract periods are usually single calendar months or periods such as September-November. As regards standard HDD swaps, the two parties agree on a determined strike of HDDs for the period and the swapped amount is 10000€/HDD away from the strike. Also, there is often a maximum payout which corresponds to 200 degree-days [1].

3.2 Modeling Temperature

Considering temperature as the underlying variable of weather derivatives, we will now focus on finding a model to describe this variable, and in particular a stochastic process that can describe the temperature movements. This process offers great knowledge in the case of pricing weather derivatives (see subsection 3.3 later). To achieve this aim, we will consult a database with temperatures measured during the last 40 years in Sweden. This data consists of daily mean temperatures which have been computed based on Definition 3.1.

For the following analysis we use 40 years data series from Bromma Airport at Stockholm, and we consider the following Figure 3.1 in which the daily mean temperatures at Bromma Airport are presented (period of nine years) [1].



Period 1989-1997, Daily mean temperatures at Stockholm Bromma Airport

Figure 3.1

In Figure 3.1 we can see the plots of the daily mean temperatures taken from Bromma Airport (Stockholm) for nine years (1989-1997). We observe that the mean temperature is about -5° C in the winter and 20° C during the summers and we can see that a strong seasonal variation in the temperature is occurred. Considering the data of

Figure 3.1, we can propose a *sine-function* model for the seasonal dependence in the following form

$$\sin(\omega t + \phi), \tag{3.5}$$

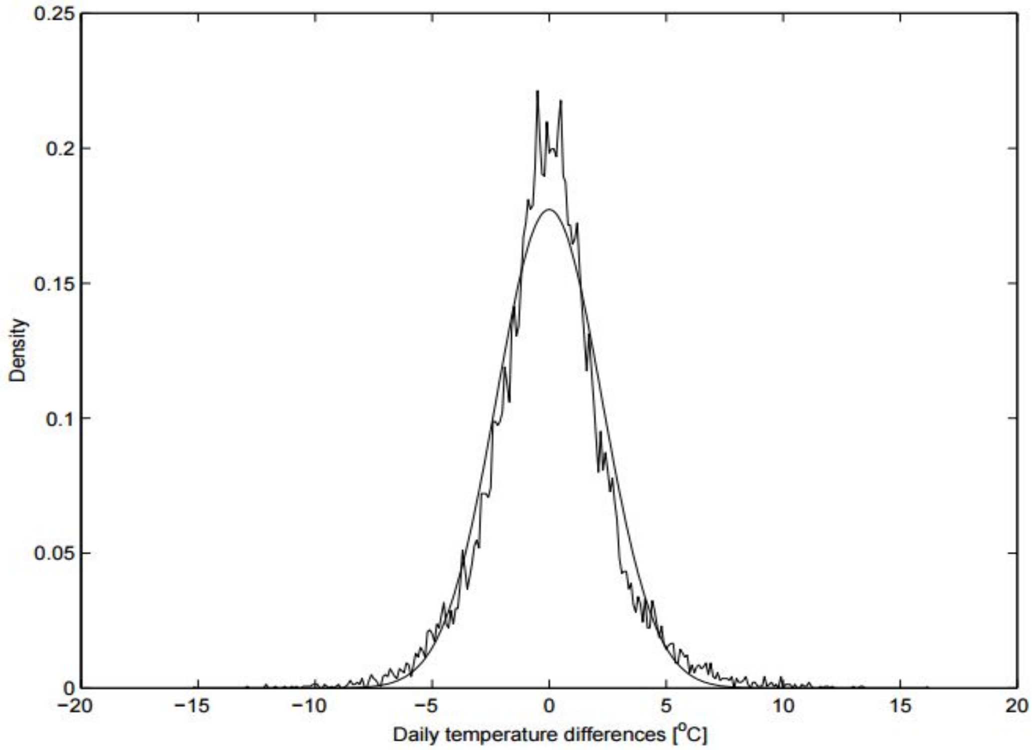
where t denotes the days, $t = 1,2,...$ where 1 is the first day of January, 2 is the second day of January etc. Considering that the period of oscillations is one year we have $\omega = 2\pi/365$. Also, we introduce a phase angle ϕ, because the yearly minimum and maximum mean temperatures do not always occur at January 1 and July 1. Observing the Figure 3.1 we see that the data have a weak positive trend, the mean temperature increases over the years. The global warming trend all over the world and the urban heating effect are responsible for this. Because of its weakness, we can approximate the warming trend as a linear function.

Hence, we propose a deterministic model T_t^m for the mean temperature at time t, which has the following form

$$T_t^m = K + Lt + M\sin(\omega t + \phi) \tag{3.6}$$

where K, L, M, ϕ are parameters that have to be chosen properly (these parameters will be estimated later).

In order to propose a more realistic model, because temperatures are not deterministic, we will add some sort of noise to the deterministic model (3.6). We will choose a standard Wiener process $(W_t, t \geq 0)$. In Figure 3.2 below, we see that the plotted daily temperature differences fit well with the corresponding normal distribution.



The density of the daily temperatures differences

Figure 3.2

Observing the data series, we see that the quadratic variation $\sigma_t^2 \in R_+$ of the temperature varies across the different months of the year, but is constant within each month. Also, it is much higher during the winter, than the rest of the year. Supposing that σ_t is a piecewise constant function, σ_t is specified as σ_1 during January, σ_2 during February, ..., σ_{12} during December, where $\{\sigma_i\}_{i=1}^{12}$ are positive constants. Furthermore, $(\sigma_t W_t,\ t \geq 0)$ could be a driving noise process temperature.

3.2.1 Reversing the mean

As a fact, temperature does not increase day after day for a considerable period of time. Given that, we can easily understand that a model must not let the temperature aberrate its main value for a long time. That means that the stochastic process which describes the temperature must be mean-reverting.

Considering all of the above, a stochastic process solution of the stochastic differential equation models temperature which is

$$dT_t = \alpha (T_t^m - T_t)\, dt + \sigma_t\, dB_t \tag{3.7}$$

where the speed of the mean-reversion is determined by $\alpha \in R$. The solution of this kind of equation is ordinary called an Ornstein-Uhlenbeck process.

However, we can see that the Eq. (3.7) is practically not reverting to T_t^m in the long term (e.g. Dornier and Queruel 2000). Hence, what we have to do in order to revert to the mean is to add the term:

$$\frac{dT_t^m}{dt} = L + \omega M \cos\,(\omega t + \phi) \tag{3.8}$$

to the drift in term (3.7). Since the mean temperature T^m is not standard, the aforementioned term will adjust the drift the stochastic differential equation solution will have the mean T_t^m in the long run.

Hence, the following model for the temperature occurs, i.e.,

$$dT_t = \left\{ \frac{dT_t^m}{dt} + \alpha (T_t^m - T_t) \right\} dt + \sigma_t\, dB_t \quad t > s \tag{3.9}$$

If we set

$$T_t = x_t + T_t^m \Rightarrow T_t^m - T_t = -x_t\,,$$

and starting at $T_s = x$, we solve Eq. (3.9) as follows

$$d(x_t + T_t^m) = \left\{ \frac{dT_t^m}{dt} + \alpha\,(T_t^m - x_t - T_t^m) \right\} dt + \sigma_t\, dB_t$$

$$\Leftrightarrow dx_t + dT_t^m = -\alpha\, x_t dt + dT_t^m + \sigma_t dB_t$$

$$\Leftrightarrow e^{\alpha t} dx_t + e^{\alpha t} dT_t^m = -e^{\alpha t} \alpha\, x_t dt + e^{\alpha t} dT_t^m + e^{\alpha t} \sigma_t\, dB_t$$

$$\Leftrightarrow e^{\alpha t} dx_t + e^{\alpha t} \alpha\, x_t dt = e^{\alpha t} \sigma_t\, dB_t$$

$$\Leftrightarrow d[e^{\alpha t} x_t] = e^{\alpha t} \sigma_t\, dB_t$$

$$\Leftrightarrow \int_s^t d[e^{\alpha\tau} x_\tau] = \int_s^t e^{\alpha\tau} \sigma_\tau\, dB_\tau$$

$$\Leftrightarrow [e^{\alpha\tau} x_\tau]_s^t = \int_s^t e^{\alpha\tau} \sigma_\tau\, dB_\tau$$

$$\Leftrightarrow e^{\alpha t} x_t - e^{\alpha s} x_s = \int_s^t e^{\alpha\tau} \sigma_\tau\, dB_\tau$$

$$\Leftrightarrow e^{\alpha t}\,[T_t - T_t^m] - e^{\alpha s}\,[T_s - T_s^m] = \int_s^t e^{\alpha\tau}\, \sigma_\tau\, dB_\tau$$

$$\Leftrightarrow e^{\alpha t}\, T_t - e^{\alpha t}\, T_t^m - e^{\alpha s}\, T_s + e^{\alpha s}\, T_s^m = \int_s^t e^{\alpha\tau}\, \sigma_\tau\, dB_\tau \Leftrightarrow$$

$$\Leftrightarrow T_t = T_t^m + x(e^{\alpha s} / e^{\alpha t}) - (e^{\alpha s} / e^{\alpha t}) T_s^m + \int_s^t (e^{\alpha\tau} / e^{\alpha t})\, \sigma_\tau\, dB_\tau$$

$$\Leftrightarrow T_t = (x - T_s^m)\, e^{-\alpha(t-s)} + T_t^m + \int_s^t e^{-\alpha(t-\tau)}\, \sigma_\tau\, dB_\tau \qquad (3.10)$$

where

$$T_t^m = K + Lt + M \sin(\omega t + \phi).$$

3.2.2 Parameter Estimation

In the previous paragraph we used the stochastic differential equation to model the temperature . Now we will estimate the unknown parameters K, L, M, ϕ, α and σ, data taken from Bromma Airport (Sweden) from the last 40 years.

In order to find numerical values for K, L, M, ϕ, α and σ we fit the function

$$Y_t = \alpha_1 + \alpha_2\, t + \alpha_3 \sin(\omega t) + \alpha_4 \cos(\omega t) \qquad (3.11)$$

to the temperature data with the method of least squares. So, we have to find the parameter vector $\xi = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ which is the solution of

$$\min_{\xi} \|\mathbf{Y} - \mathbf{X}\|^2 \tag{3.12}$$

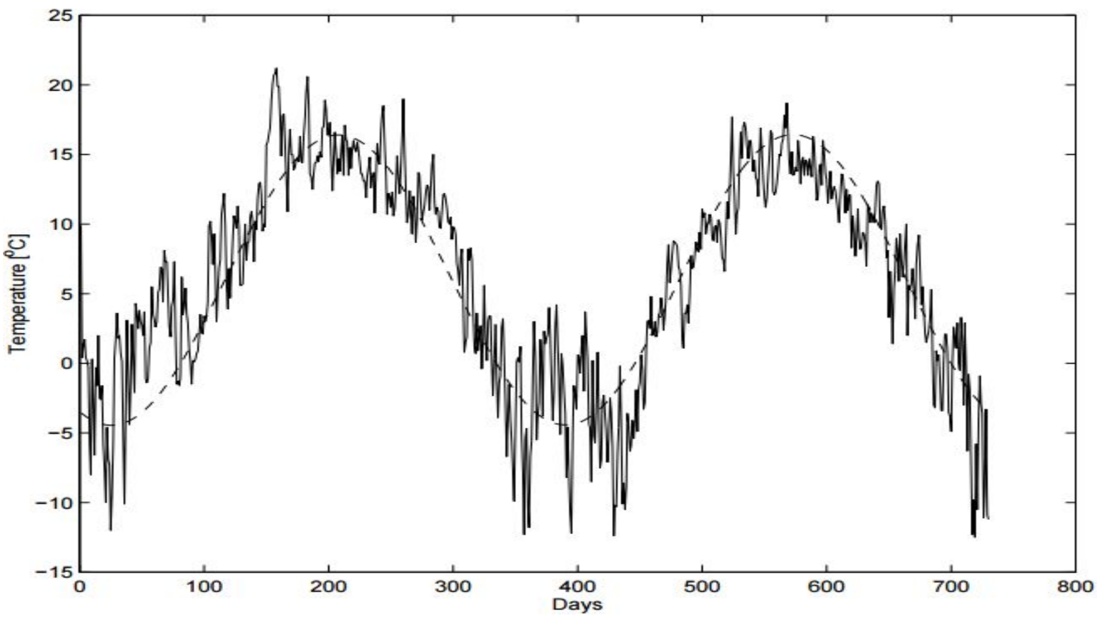
where Y is a vector with elements (3.6) and X is a data vector.

The constants in the model (3.11) are

$$\begin{aligned} K &= a_1 \\ L &= \alpha_2 \\ M &= \sqrt{\alpha_3^2 + \alpha_4^2} \\ \phi &= \arctan\left(\frac{\alpha_4}{\alpha_3}\right) - \pi \end{aligned} \tag{3.13}$$

If now we insert the numerical values of (3.13) into Equation (3.6), we have the following function for the mean temperature

$$T_t^m = 5.97 + 6.57 \times 10^{-5} t + 10.4 \sin\left(\frac{2\pi}{365} t - 2.01\right) \tag{3.14}$$



The mean temperature and the real temperature at Bromma Airport over two years

Figure 3.3

The sine function has a width about 10° C, so that the difference between a typical winter day and a summer day is about 20° C. Over 40 years the mean temperature is increased about 1° C, which is a very small transition.

Further, we will derive two estimators of σ from the data that we have collected for each month. We base the first estimator on the quadratic variation of T_t [1], [8]

$$\hat{\sigma}_{\mu}^{2} = \frac{1}{N_{\mu}} \sum_{j=0}^{N_{\mu}-1} (T_{j+1} - T_{j})^{2} \tag{3.12}$$

where μ is a specific month, N_{μ} are the days of month μ and $T_j, j = 1,2,..., N_{\mu}$ are the observed temperatures of month μ.

We now derive of σ, which is achieved by discretizing Eq. (3.9). Given a month μ, we can take the following regression equation

$$T_j = T_j^m - T_{j-1}^m + \alpha T_{j-1}^m + (1-\alpha) T_{j-1} + \sigma_{\mu} \varepsilon_{j-1} \qquad j-1,..., N_{\mu} \tag{3.13}$$

where $\{\varepsilon\}_{j=1}^{N_{\mu}-1}$ are independent random variables following normal distribution. Setting $\widetilde{T} \equiv T_j - (T_j^m - T_{j-1}^m)$, Eq. (3.13) can be written as

$$\widetilde{T} = \alpha T_{j-1}^m + (1-\alpha) T_{j-1} + \sigma_{\mu} \varepsilon_{j-1} \tag{3.14}$$

We can see (3.14) as a regression of today's temperature on yesterday's temperature, and hence we conclude to the second estimator of σ_{μ} (Brokwell and Davis, 1990)

$$\hat{\sigma}_{\mu}^{2} = \frac{1}{N_{\mu} - 2} \sum_{j=1}^{N_{\mu}} (\widetilde{T}_j - \hat{\alpha} T_{j-1}^m - (1-\hat{\alpha}) T_{j-1})^{2} \tag{3.15}$$

Now, an estimator of α is needed to be found, in order to find the estimator of σ_{μ} (see (3.15)). Since the time between observations of the temperature for one day is bounded away from zero, we will estimate the mean-reversion parameter α using the martingale estimation functions method [10]. This method is based on observations collected over n days. We will obtain an efficient estimator $\hat{\alpha}_n$ of α, by solving the equation

$$G_n(\hat{\alpha}_n) = 0 \tag{3.16}$$

where

$$G_n(\alpha) = \sum_{i=1}^{n} \frac{\dot{b}(T_{i-1};\alpha)}{\sigma_{i-1}^2} \{T_i - E\,[T_i \mid T_{i-1}]\}, \quad i = 1,2,\ldots,n \tag{3.17}$$

where $\dot{b}(T_t;\alpha)$ denotes the derivative with respect to α of the drift term

$$\dot{b}(T_t;\alpha) = \frac{dT_t^m}{dt} + \alpha\,(T_t^m - T_t) \tag{3.18}$$

Hence, we have to determine each of the terms in Eq. (3.17) in order to solve the Eq. (3.16). So, we have

$$T_t = (T_s - T_s^m)e^{-\alpha(t-s)} + T_t^m + \int_s^t e^{-\alpha(t-\tau)}\sigma_\tau\, dB_\tau \qquad t \geq s \tag{3.19}$$

which yields

$$E[T_i \mid T_{i-1}] = (T_{i-1} - T_{i-1}^m)\,e^{-\alpha} + T_i^m \tag{3.20}$$

where

$$T_t^m = K + Lt + M\sin(\omega t + \phi)$$

Therefore, (3.17) via (3.20) is written as

$$G_n(\alpha) = \sum_{i=1}^{n} \frac{T_{i-1}^m - T_{i-1}}{\sigma_{i-1}^2} \{T_i - (T_{i-1} - T_{i-1}^m)\,e^{-\alpha} - T_i^m\} \tag{3.21}$$

From (3.21) we can easily check that

$$\hat{\alpha}_n = -\log\left(\frac{\sum_{i=1}^{n} Y_{i-1}\{T_i - T_i^m\}}{\sum_{i=1}^{n} Y_{i-1}\{T_{i-1} - T_{i-1}^m\}} \right) \tag{3.22}$$

is the solution of the Eq. (3.16), where

$$Y_{i-1} \equiv \frac{T_{i-1}^m - T_{i-1}}{\sigma_{i-1}^2} \qquad i = 1,2,\ldots,n \tag{3.23}$$

If we insert into (3.12) and (3.15) the numerical values, we take estimations of σ for the different months, which are listed in Table 1. Taking σ from Table 1 we have $\hat{\alpha}=0.237$.

Let's now see how different is this value from an estimation, which is based only on the following discrete function [1]

$$\dot{\tilde{I}}_n(\alpha)=\sum_{i=1}^{n}\frac{\dot{b}(T_{i-1};\alpha)}{\sigma_{i-1}^2}(T_i-T_{i-1})-\sum_{i=1}^{n}\frac{\dot{b}(T_{i-1};\theta)\dot{b}(T_{i-1};\alpha)}{\sigma_{i-1}^2} \tag{3.24}$$

The solution of the Eq. (3.24) is given by

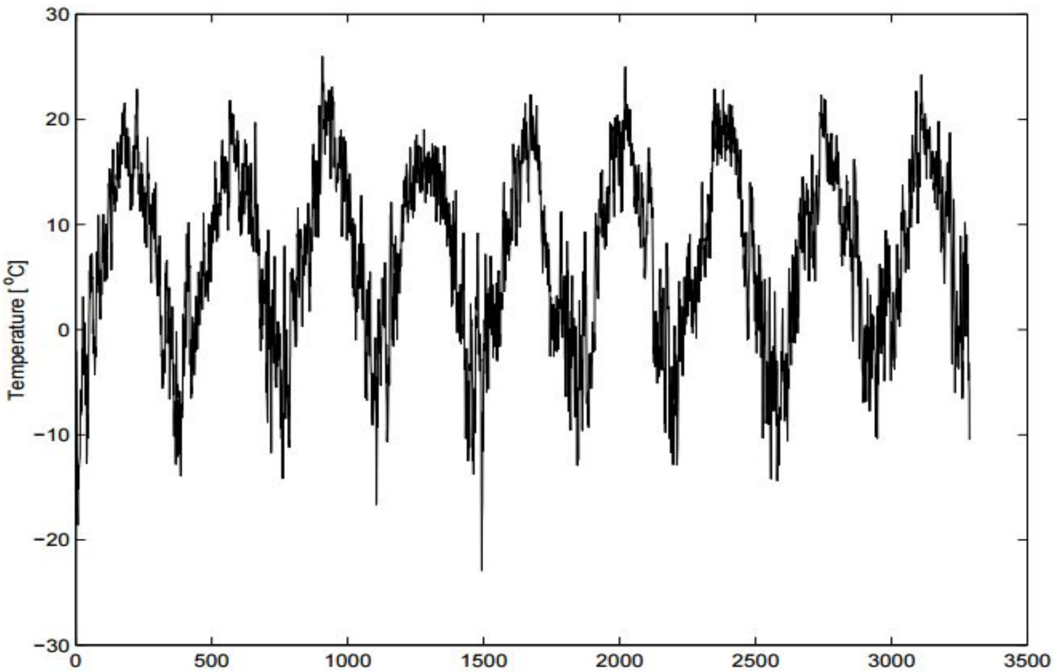
$$\hat{\alpha}_n' = \frac{\sum_{i=1}^{n} Y_{i-1}\{T_i - T_{i-1} - L - M\,\omega\cos(\omega(i-1)+\phi)\}}{\sum_{i=1}^{n} Y_{i-1}\{T_{i-1}^m - T_{i-1}\}} \tag{3.25}$$

where Y_{i-1} has defined in (3.23).

Month	Estimation 1	Estimation 2	Mean Value
January	3.46	3.37	3.41
February	2.96	2.98	2.97
March	2.32	2,27	2.29
April	2.00	1.95	1.98
May	2.01	1.99	2.00
June	1.98	1.95	1.96
July	1.70	1.68	1.69
December	3.36	3.25	3.30
August	1.61	1.58	1.60
September	1.86	1.83	1.85
October	2.42	2.33	2.38
November	2.66	2.58	2.62

(Table 1)

Inserting the numerical values at (3.25), we have $\hat{\alpha}'_n = 0.211$, which is 11% less than $\hat{\alpha}_n$. Thus, if we use the estimator $\hat{\alpha}'_n$ we will probably calculate erroneous price of a derivative. Finally, since we have estimated the parameters of the temperature model K, L, M and ϕ , we can simulate trajectories of the Ornstein-Uhlenbeck process. In Figure 3.4 below we can see a possible trajectory of the following years' temperature. If we compare this simulation with the real temperatures that plotted in Figure 3.1, we can observe that our model (3.9)-(3.10) for the temperature has the same properties as the observed temperature.



Trajectory of the Ornstein-Uhlenbeck process

Figure 3.4

3.3 Pricing Weather Derivatives

In this section, we will first briefly describe the meaning of incomplete markets. An incomplete market is one where most of the necessary conditions for the development of market exist. In the case of the incomplete markets, many entrepreneurs may enter the market in order to profit. However, the firms that do start-up will only correspond to a small part of demand. In these incomplete markets, it is not possible for the total supply to meet the needs of consumers. In such cases,a market may develop, but not completely and will be called an incomplete market [26].

The weather derivatives as we have said before, have as underlying variable the temperature, which is not tradable. Therefore, we have a typical example of an incomplete market. In order to gather unique prices for the weather derivatives contracts, we have to consider the market price of risk λ, which is assumed constant because there is not yet a real market to give us prices. Suppose we have a risk free asset with constant interest rate r and a contract that pays one unit of currency for each degree Celsius. We denote the price process by T_t and under a martingale measure Q and characterized by the market price of risk λ, we have the following equation [1]

$$dT_t = \left\{ \frac{dT_t^m}{dt} + \alpha (T_t^m - T_t) - \lambda \sigma_t \right\} dt + \sigma_t \, dV_t \tag{3.26}$$

where $(V_t,\ t \geq 0)$ is a Q-Wiener process. Since we express the price of a derivative as a discounted expected value under martingale measure Q, we will compute at first the expected value and the variance of T_t under the measure Q. The Girsanov transformation changes only the drift term, so the variance of T_t is the same under both measures. Therefore

$$Var\ [T_t | F_s] = \int_s^t \sigma_\mu^2 e^{-2\alpha(t-u)} \, du \tag{3.27}$$

$$E^P[T_t | F_s] = (T_s - T_s^m) e^{-\alpha(t-s)} + T_t^m \tag{3.28}$$

From the Equation (3.26) we have

$$E^{Q}[T_t|F_s]=E^{P}[T_t|F_s]-\int_s^t \lambda\sigma_{\mu}e^{-a(t-u)}\,du \tag{3.29}$$

We evaluate the integrals in one of the interval with constant σ

$$E^{Q}[T_t|F_s]=E^{P}[T_t|F_s]-\frac{\lambda\sigma_i}{\alpha}(1-e^{-\alpha(t-s)}) \tag{3.30}$$

We also have the variance

$$Var\,[T_t|F_s]=\frac{\sigma_i^2}{2\alpha}(1-e^{-2\alpha(t-s)}), \tag{3.31}$$

and the covariance of the temperature between two different days is given by

$$Cov\,[T_t,T_u|F_s]=e^{-\alpha(u-t)}\,Var\,[T_t|F_s], \qquad 0\le s\le t\le u. \tag{3.32}$$

We suppose now that t_1 is the first day of the month and t_n is the last day and we start the process at some time s from the previous month $[t_1,t_n]$. In this case, we will compute the expected value and variance of T_t, splitting the integrals in (3.27) and (3.29) into two integrals where σ is constant. Thus,

$$E^{Q}[T_t|F_s]=E^{P}[T_t|F_s]-\frac{\lambda}{\alpha}(\sigma_i-\sigma_j)e^{-\alpha(t-t_1)}+\frac{\lambda\sigma_i}{\alpha}e^{-\alpha(t-s)}-\frac{\lambda\sigma_j}{\alpha} \tag{3.33}$$

and

$$Var\,[T_t|F_s]=\frac{1}{2\alpha}(\sigma_i^2-\sigma_j^2)e^{-2\alpha(t-t_1)}-\frac{\sigma_i^2}{2\alpha}e^{-2\alpha(t-s)}+\frac{\sigma_j^2}{2\alpha} \tag{3.34}$$

The generalization to larger time internals is evident, and the latter relations will be useful as follows.

3.3.1 Pricing a Heating Degree Day (HDD) option

As we mentioned earlier, the majority of weather derivatives which involve the temperature depends on cooling degree days or heating degree days. In this part, we are going to indicate the way of pricing a standard heating degree day option, beginning first with the price of an **HDD call option**. To begin with, we should recollect that the payout of the option is of the form [1]

$$\aleph = \alpha \max\{H_n - K, 0\} \tag{3.35}$$

where, to simplify, considering without loss of generality that $\alpha = 1$, α stands for t unit of currency/HDD, we also have

$$H_n = \sum_{i=1}^{n} \max\{18 - T_t, 0\} \tag{3.36}$$

This contract represents a form of an arithmetic average Asian option. In case of a lognormal distribution of the underlying process, there is not a specific analytic formula for the option's price. We will now present a normally distributed process, however the maximum function makes it more difficult to find a pricing form. Therefore, we will try to find a more approximate way.

We know that, under martingale measure Q and information given us at time s

$$T_t \sim N(\mu_t, v_t) \tag{3.37}$$

where μ_t and v_t are given from (3.33) and (3.34), respectively.

Assume, now, that we wish to find the price of a contract which has a payout that depends on the HDD's accumulation during a specific period in the winter. For example, in Stockholm, the probability that max being $\{18 - T_{t_i}, 0\} = 0$ ought to be significantly small on a winter day. Hence, for this type of contract we can say

$$H_n = 18n - \sum_{i=1}^{n} T_{t_i} \tag{3.38}$$

This distribution can be easily determined. We already know that T_{t_i}, $i = 1,\ldots,n$ are samples taken from the Ornstein-Uhlenbeck (Gaussian) process.

Therefore, we can understand that the vector $(T_{t_1}, T_{t_2}, ..., T_{t_n})$ is Gaussian too. Considering the sum in (3.38) which is a linear combination of the components in this vector, H_n is also Gaussian. Having this new structure of H_n, we will now have to calculate the first and second moments. We have for $t < t_1$

$$E^Q[H_n | F_t] = E^Q\left[18n - \sum_{i=1}^{n} T_{t_1} | F_t\right] = 18n - \sum_{i=1}^{n} E^Q[T_{t_i} | F_t] \tag{3.39}$$

and

$$Var[H_n | F_t] = \sum_{i=1}^{n} Var[T_{t_i} | F_t] + 2\sum_{i<j}\sum Cov[T_{t_i}, T_{t_j} | F_t], \tag{3.40}$$

Therefore, after some calculations we can arrive to

$$E^Q[H_n | F_t] = \mu_n \quad \text{and} \quad Var[H_n | F_t] = \sigma_n^2 \tag{3.41}$$

where $H_n \sim N(\mu_n, \sigma_n)$. For $t < t_1$ we have the claim of (3.35)

$$\begin{aligned} c(t) &= e^{-r(t_n - t)} E^Q[\max\{H_n - K, 0\} | F_t] \\ &= e^{-r(t_n - t)} \int_K^{\infty} (x - K) f_{H_n}(x)\, dx \\ &= e^{-r(t_n - t)} \left((\mu_n - K)\Phi(-\alpha_n) + \frac{\sigma_n}{\sqrt{2\pi}} e^{-\frac{\alpha_n^2}{2}} \right) \end{aligned} \tag{3.42}$$

where $\alpha_n = (K - \mu_n)/\sigma_n$ and Φ represents the cumulative function of distribution for the basic normal distribution.

Following the same strategy as before, we can find a formula for the **Heating Degree Day (HDD) put option**, which is the claim

$$Y = \max\{K - H_n, 0\} \tag{3.43}$$

Working as before, its price is

$$
\begin{aligned}
p(t) &= e^{-r(t_n-t)}E^{Q}[\max\{K-H_n,0\}\,|\,F_t] \\
&= e^{-r(t_n-t)}\int_0^K (K-x)f_{H_n}(x)\,dx \qquad (3.44) \\
&= e^{-r(t_n-t)}\left[(K-\mu_n)\left(\Phi(\alpha_n)-\Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right)+\frac{\sigma_n}{\sqrt{2\pi}}\left(e^{-\frac{\alpha_n^2}{2}}-e^{-\frac{1}{2}\left(\frac{\mu_n}{\sigma_n}\right)^2}\right)\right]
\end{aligned}
$$

The aforementioned formulas hold mainly for contracts during winter, namely the period from November to March. These formulas cannot be used during the summer months without any restrictions, because if the mean temperatures are higher or closer to 18° C we cannot have $\max\{18-T_{t_i}\}\neq 0$. Instead, we can use the method of Monte Carlo simulations. The reference level of 18° C, originates from the market of the United States, but it is also around Europe.

On a more practical level, numerous options frequently have a cap on the maximum payout, in order to reduce the risks of severe weather conditions. Thus, an option with a maximum payout could be made from two options that do not have maximum payouts. Entering a long position in one option and a short position in another option which has a higher strike value, we could derive a pay function like the one in Figure 3.5. Hence, an option with a maximum payout can be used as a portfolio of two standard options which means that it is not necessary to find an explicit formula for the price of the capped option [1].

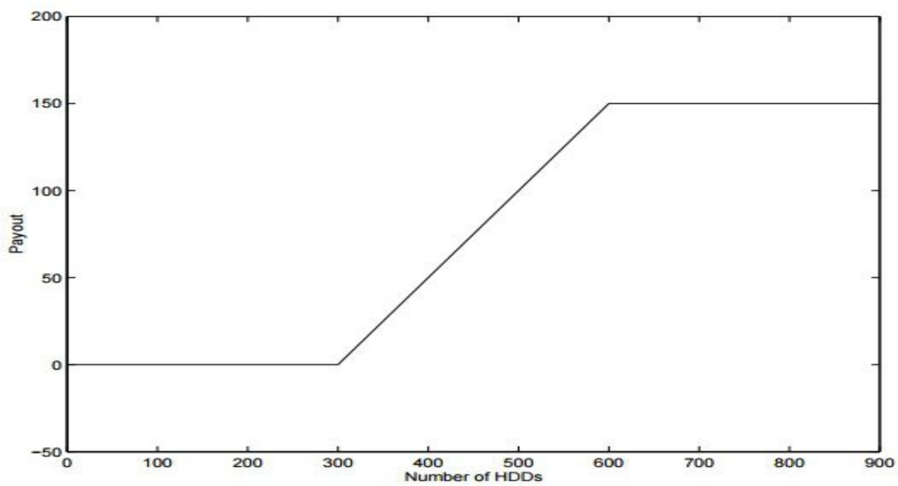


Figure 3.5

In order to find the price of the option during the contract period, supposedly at a time t_i, $t_1 < t_i < t_n$, we should write the variable H_n as

$$H_n = H_i + H_j \tag{3.45}$$

We know H_i at t_i, while H_j is stochastic and the HDD call option payout can be written again as

$$X = \max\{H_n - K, 0\} = \max\{H_i + H_j - K, 0\} = \max\{H_j - \widetilde{K}, 0\} \tag{3.46}$$

where $\widetilde{K} = K - H_i$.

Thus, an in-period option can be valued as an out-of-period option if we transform the variables as before. [1]

3.4 Conclusions

To improve the pricing model that we represented, various things can be done. Probably the most significant issue in terms of weather derivatives pricing is to acquire a good model for the weather. Here, we use a simplified temperature model, although it fits well the temperature data. To make this temperature model more realistic is to use a sophisticated model for the driving noise process. A model including stochastic volatility could possibly be more realistic.

We can also find a better temperature model considering larger models of the climate, in which the temperature constitutes one of numerous different variables. The contribution of better climate models and faster computers is very significant for the experts, because they are able to make more important long-term forecasts, which are valuable for the weather derivatives pricing.

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