VALUE BASED AND NETWORK DATA ENVELOPMENT ANALYSIS: NEW MODELS AND APPLICATIONS

A dissertation submitted in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

in the Department of Informatics
in the School of Information and Communication Technologies
at the University of Piraeus

by

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Under the Supervision of

Professor
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University of Piraeus

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Περίληψη

Η μέτρηση της αποδοτικότητας των συστημάτων παραγωγής αποτελεί καθοριστικό παράγοντα για την βελτίωση τους. Η αποτίμηση της αποδοτικότητας επιτυγχάνεται με τη χρήση παραμετρικών ή μη-παραμετρικών τεχνικών. Η Περιβάλλουσα Ανάλυση Δεδομένων - ΠΑΔ (Data Envelopment Analysis - DEA) αποτελεί μια από τις δημοφιλέστερες μη-παραμετρικές τεχνικές για την εκτίμηση της αποδοτικότητας ομοειδών μονάδων ενός συστήματος, επί τη βάσει πολλαπλών εισροών και εκροών και στηρίζεται στο γραμμικό προγραμματισμό. Σύγχρονες επεκτάσεις της ΠΑΔ αποτελούν η Περιβάλλουσα Ανάλυση Αξιών (value based DEA) και η πολυσταδιακή Περιβάλλουσα Ανάλυση Δεδομένων (network DEA).

Η Περιβάλλουσα Ανάλυση Αξιών στηρίζεται σε τεχνικές της Πολυκριτηριακής Ανάλυσης Αποφάσεων (Multiple Criteria Decision Analysis - MCDA) για το μετασχηματισμό των επιπέδων των εισροών και εκροών σε αξίες. Ο μετασχηματισμός αυτός εξυπηρετεί στην ενσωμάτωση των προτιμήσεων του εκάστοτε αναλυτή, σχετικά με τις συναρτήσεις αξιών των εισροών/εκροών, στην εκτίμηση της αποδοτικότητας των μονάδων. Παρόλο που τα μοντέλα της Περιβάλλουσας Ανάλυσης Αξιών που προτείνονται στη βιβλιογραφία παρέχουν ευκολία στην εισαγωγή προτιμήσεων σχετικά με τις συναρτήσεις αξιών των εισροών/εκροών, διαχωρίζουν μόνο τις αποδοτικές από τις μη-αποδοτικές μονάδες και αδυνατούν να εκτιμήσουν την αποδοτικότητα κάθε αποτιμώμενης μονάδας.

Η πολυσταδιακή Περιβάλλουσα Ανάλυση Δεδομένων αποτελεί μια από τις σημαντικότερες επεκτάσεις της κλασσικής Περιβάλλουσας Ανάλυσης Δεδομένων. Τα συμβατικά μοντέλα της ΠΑΔ θεωρούν ότι ο μηχανισμός παραγωγής συνίσταται σε ένα στάδιο. Παρόλα αυτά, υπάρχουν περιπτώσεις όπου ο αναλυτής γνωρίζει ότι ο μηχανισμός παραγωγής αποτελείται από υποδιαδικασίες (υποστάδια). Η πληροφορία αυτή επηρεάζει την εκτίμηση αποδοτικότητας και τα κλασσικά μοντέλα της ΠΑΔ αδυνατούν να αφομοιώσουν την πληροφορία αυτή. Η πολυσταδιακή Περιβάλλουσα Ανάλυση Δεδομένων αντιλαμβάνεται το μηχανισμό παραγωγής ως ένα δίκτυο από υποστάδια, τα οποία συνδέονται μεταξύ τους με ενδιάμεσους παράγοντες

(intermediate measures). Όμως, τα μοντέλα της Πολυσταδιακής Περιβάλλουσας Ανάλυσης Δεδομένων που παρουσιάζονται στη βιβλιογραφία, δεν προσδιορίζουν ένα μοναδικό δείκτη αποδοτικότητας στα επιμέρους στάδια (υποστάδια) για κάθε αποτιμώμενη μονάδα. Επίσης, η εκτίμηση της αποδοτικότητας επιτυγχάνεται εισάγοντας άγνωστη στάθμιση στα υποστάδια αυτά. Τα ζητήματα αυτά θέτουν σε αμφισβήτηση την εγκυρότητα των αποτελεσμάτων καθώς, για κάθε αποτιμώμενη μονάδα, δίνεται διαφορετική και άγνωστη στάθμιση στα επιμέρους στάδια για τη διαμόρφωση του δείκτη αποδοτικότητας.

Στο πρώτο μέρος της παρούσας διδακτορικής διατριβής, επικεντρωνόμαστε στην Περιβάλλουσα Ανάλυση Αξιών. Στόχος είναι η ανάπτυξη ενός νέου μοντέλου το οποίο, σε αντίθεση με τα μοντέλα της Περιβάλλουσας Ανάλυσης Αξιών που προτείνονται στη βιβλιογραφία, εκτιμά την αποδοτικότητα κάθε αποτιμώμενης μονάδας. Συγκεκριμένα, εισάγουμε ένα μετασχηματισμό των δεδομένων και των μεταβλητών των γραμμικών μοντέλων της ΠΑΔ, με τον οποίο οι νέες μεταβλητές εκφράζουν πλέον αξία αντί για βάρη. Δείχνουμε ότι ο μετασχηματισμός αυτός ενισχύει τα κλασσικά μοντέλα της ΠΑΔ με επιπλέον ιδιότητες και επίσης, ότι επιλύει το θέμα ασυνέχειας που παρουσιάζουν οι συναρτήσεις αξίας στην ΠΑΔ με μη γραμμικές εικονικές εισροές/εκροές. Τα ευρήματα αυτά, μας οδηγούν στη δημιουργία ενός νέου μοντέλου της Περιβάλλουσας Ανάλυσης Αξιών, το οποίο εκτιμά την αποδοτικότητα κάθε αποτιμώμενης μονάδας. Επίσης, εισάγουμε μια νέα μεθοδολογία για την εισαγωγή προτιμήσεων στα πλαίσια της ΠΑΔ μέσω Μονότονης Παλινδρόμησης (Ordinal Regression). Η αποτελεσματικότητα του νέου μοντέλου της Περιβάλλουσας Ανάλυσης Αξιών που αναπτύξαμε αναδεικνύεται μέσω της εφαρμογής του σε μια μελέτη περίπτωσης που έχει ήδη παρουσιασθεί στη βιβλιογραφία, καθώς και μέσα από την ανάπτυξη μιας νέας εφαρμογής που σχετίζεται με την αξιολόγηση της ερευνητικής δραστηριότητας ακαδημαϊκών καθηγητών, η οποία λαμβάνει υπόψη της την ποιοτική και ποσοτική διάσταση των δημοσιεύσεων των καθηγητών.

Στο δεύτερο μέρος της παρούσας διδακτορικής διατριβής, επικεντρωνόμαστε στην πολυσταδιακή Περιβάλλουσα Ανάλυση Δεδομένων, όπου και αναπτύσσουμε μια νέα μεθοδολογία με στόχο την επίλυση των μειονεκτημάτων που χαρακτηρίζουν

τα μοντέλα της πολυσταδιακής Περιβάλλουσας Ανάλυσης Δεδομένων που παρουσιάζονται στη βιβλιογραφία. Συγκεκριμένα, εισάγουμε μια προσέγγιση πολυκριτηριακού προγραμματισμού η οποία χρησιμοποιεί την L_∞ μετρική, ώστε να υπολογίσει τις αποδοτικότητες των μονάδων στα υποστάδια όσο πιο κοντά γίνεται στα ιδανικά τους επίπεδα. Η προσέγγιση αυτή, σε αντίθεση με τα υπάρχοντα μοντέλα που προτείνονται στη βιβλιογραφία, παρέχει μοναδικό δείκτη αποδοτικότητας για κάθε μονάδα σε κάθε υποστάδιο και επίσης διαχειρίζεται τα επιμέρους υποστάδια με την ίδια βαρύτητα. Τα πλεονεκτήματα της νέας αυτής μεθοδολογίας γίνονται ιδιαίτερα σαφή όταν συγκρίνουμε τα αποτελέσματα της μεθοδολογίας που αναπτύξαμε με τα αποτελέσματα άλλων μοντέλων, που παρουσιάζονται στη βιβλιογραφία, πάνω σε συνθετικά δεδομένα και σε δεδομένα που έχουν ήδη μελετηθεί στη βιβλιογραφία. Η πρακτικότητα της παραπάνω μεθόδου αναδεικνύεται περαιτέρω μέσω της εφαρμογής της σε μια καινοτόμο μελέτη περίπτωσης, που σχετίζεται με την ερευνητική δραστηριότητα ακαδημαϊκών καθηγητών αναλυόμενη σε δύο στάδια συνδεδεμένα σειριακά.

Λέξεις κλειδιά: *Περιβάλλουσα Ανάλυση Δεδομένων, Περιβάλλουσα Ανάλυση Αξιών, κανονικοποίηση των δεδομένων, πολυσταδιακή Περιβάλλουσα Ανάλυση Δεδομένων, ερευνητική δραστηριότητα ακαδημαϊκών στην Ανώτατη Εκπαίδευση.*

Abstract

Performance measurement of production units is a critical aspect for their improvement. Performance assessment can be achieved either by parametric approaches, when specific parametric functional forms that transform particular inputs to outputs are assumed or by non-parametric approaches, when no assumptions on the production functions are made. Data Envelopment Analysis (DEA) is a non-parametric technique for measuring the performance of Decision Making Units that use multiple inputs to produce multiple outputs and has been established as the leading technique in performance measurement. Recent extensions of DEA, among others, include value based DEA and network DEA.

Value based DEA is a recent development that resorts to value assessment protocols from Multiple Criteria Decision Analysis (MCDA) to transform the original input/output data to a value scale so as to incorporate individual prior views according to the value functions of the inputs and/or outputs in the efficiency assessment. Although the existing value based DEA models are flexible, they fail to provide a measure of efficiency.

Network DEA is one of the major extensions of the conventional DEA. Specifically, conventional DEA models assume one stage production processes. However, there are cases where the internal flow of the production process is known and it plays a crucial role in the efficiency assessment. Network DEA conceives the production process that characterizes the DMUs as a network of sub-processes (stages, divisions), which are linked with intermediate measures. However, the proposed models in network DEA do not necessarily provide unique divisional efficiency scores. In addition, the estimation of the overall and the divisional efficiency scores is achieved by unduly and implicitly assigning different priority to the sub-processes. These issues question the neutrality of the results, which generally can be biased and to lead to erroneous interpretation.

In this dissertation, we provide critical reviews on the value based and network DEA models proposed in the literature and we develop new models which deal with

the aforementioned defects. Specifically, in the first part of this dissertation we introduce a data transformation – variable alteration technique as a means to transform the original input/output weights into values. We show that this transformation enhances the conventional DEA models with additional properties and that it treats successfully the discontinuity issue of the value functions in DEA, when non-linear value functions for the inputs and the outputs are assumed (non-linear virtual inputs/outputs). These findings allow us to develop a novel value based DEA model, which unlikely the value based DEA models proposed in the literature, provides a measure of efficiency for the evaluated units. Moreover, we develop a two-phase approach to incorporate individual preferences in a DEA assessment framework by means of Ordinal Regression. The effectiveness and the applicability of the novel value based DEA model is further illustrated by revisiting a case study drawn from the literature and by providing an application concerning the assessment of the research performance of academics which takes into account both the quantity as well as the quality of the research output. In the second part of this dissertation, we deal with network DEA. Specifically, we introduce a multi-objective programming approach for general series multi-stage processes, which employs the L_∞ norm as a distance measure to locate the stage efficiency scores as close as possible to their ideal values that are obtained independently through standard DEA models. Our new approach overcomes the defects of the basic network DEA models as it provides unique and unbiased stage efficiency scores. When data are available in the literature, the effectiveness of our approach is illustrated by comparing the results obtained by our method with those obtained by other methods presented in the literature. When data are not available in the literature, synthetic data are used for testing and validation. The effectiveness and the applicability of our approach, is further illustrated by providing an application for the assessment of the academic research activity in higher education viewed as a two stage network process.

Keywords: *Data Envelopment Analysis (DEA), value judgments, value based DEA, max column normalization, network DEA, composition paradigm in network DEA, academic research activity in Higher Education.*

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Sotiros Dimitrios – Georgios,

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Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

The last decades, globalization and the soaring competition in the world market have forced firms to increase their productivity and performance. Generally, the improvement of performance requires a constant evaluation of the services, production and sales of products that the firm is related to. To this end, performance measurement and benchmarking can be viewed as supplementary fundamental aspects for the longevity of a firm.

Measures such as sales per worker hour or sales per employee can be easily established and provide partial information over the firm's productivity. However, such indicators are based on single measures and they provide limited information. They neglect any interrelation with other performance measures and they can generally lead to misleading results. Evaluating the firm's performance on the basis of multiple factors is not an easy task. When specific functional forms that transform particular inputs to outputs are assumed and the assessment task is based on the estimation of the parameters that fit the performance data, parametric approaches are utilized. Contrarily, in non-parametric approaches no assumption is made on the production functions. The production functions are empirically estimated on the basis of the best practice units. Thus, benchmarking is achieved by comparing the evaluated firm with other similar firms.

Data Envelopment Analysis (DEA) is a non-parametric technique for measuring the relative efficiency of decision making units (DMUs) that use multiple inputs to produce multiple outputs. The underlying mathematical instrument for performing the analysis is linear programming. The two milestone DEA models, namely the CCR (Charnes et el., 1978) and the BCC (Banker et al., 1984) models have become standards in the literature of performance measurement, under the assumption of constant (CRS) and variable (VRS) returns to scale respectively. Both are stated and solved either in the multiplier forms or their duals, the envelopment forms. In terms of the multiplier form, the efficiency of a DMU is explicitly defined on a bounded scale by the ratio of a weighted sum of its outputs to a weighted sum of its inputs. The weights assigned to the input and the output data are the variables of the corresponding linear program utilized in the efficiency assessment and they are estimated in favor of the evaluated unit, so as to maximize its relative efficiency. The units that achieve the highest efficiency score (equal to one) and they are not dominated, they form the efficient frontier and they are used as benchmarks for the inefficient units. The envelopment form, along with the efficiency scores, provides the projections of the inefficient units on the efficient frontier, by assuming either an input or an output orientation. Since the seminal paper of Charnes et al. (1978), DEA has been established as the leading technique in performance measurement. Recent extensions of DEA include, among others, the value based DEA and the network DEA.

Several authors have spotted the relation between DEA and Multiple Criteria Decision Analysis (MCDA). For example, Joro et al. (1998) and Halme et al. (1999) related DEA to multi-objective programming. Bouyssou (1999) and Stewart (1996) related DEA to MCDA ranking problems. Athanassopoulos and Podinovski (1997) related linear programming formulations used in DEA to those used in MCDA with partial information on weights. MCDA has developed many concepts and protocols to elicit and use the preferences of the analyst providing so a broad methodological framework to incorporate value judgements in DEA assessments, i.e. to incorporate individual prior views according to the relative importance of the inputs and/or outputs in the efficiency assessment. Within this framework, value based DEA is a recent development that resorts to value assessment protocols from MCDA to transform the original input/output data to a value scale on the basis of the analyst's preferences. Gouveia et al. (2008) and Almeida and Dias (2012) were the first to use concepts form multi-attribute utility/value theory in DEA assessments. A limitation of these works however, which in fact is attributed to the DEA model used, is that no direct measure of efficiency is provided. In the first part of this dissertation, we develop a novel value based DEA model which treats the aforementioned defect.

Specifically, we introduce a data transformation – variable alteration technique as a means to transform the original input/output weights into values. We show that this transformation enhances the conventional DEA models with additional properties. Moreover, we provide a critical review on DEA with non-linear virtual inputs and non-linear virtual outputs that spots the discontinuity issue of the value functions. We show that by extending the data transformation – variable alteration technique to DEA with non-linear virtual inputs/outputs, the aforementioned discontinuity issue is effectively treated. These findings and extensions allow us to develop a novel value based DEA model which, unlikely the value based DEA models proposed in the literature, provides a measure of efficiency for the evaluated units. Then, we revisit a case study drawn from the literature. By assimilating the preferential information given in the original work, the assessment results show that our approach successfully locates the efficient DMUs and unlike the assessment method used in the original work that discriminates only between efficient and inefficient units, it provides a measure of efficiency. Additionally, apart from using direct preferential information for the desired levels of the inputs and the outputs to estimate the value functions, we develop an alternative indirect approach, based on Ordinal Regression analysis, to assess a prototype of the value functions. To this end, we develop a two-phase approach that bridges UTASTAR (Siskos and Yannacopoulos, 1985) with DEA. Finally, we further illustrate the effectiveness and the applicability of the novel value based DEA model by presenting an application concerning the assessment of the research performance of academics which takes into account both the quantity as well as the quality of the research output.

Network DEA is one of the major extensions of the conventional DEA. The conventional DEA models assume one stage production processes and they generally treat the production processes as "*black box*"; only the levels of the external inputs that the system uses and the levels of the final outputs that the systems produces are known. However, there are cases where the internal flow of the production process is known and it plays a crucial role in the efficiency assessment. The conventional DEA models fail to incorporate this information in the efficiency assessment. Network DEA conceives the production process that characterizes the DMUs as a network of sub-processes (stages, divisions), which are linked with intermediate measures. The most well-known approaches to network DEA are the multiplicative efficiency decomposition introduced by Kao and Hwang (2008), the additive efficiency decomposition introduced by Chen, Cook, Li and Zhu (2009) and the network SBM approach introduced by Tone and Tsutsui (2009). However, these approaches do not necessarily provide unique stage efficiency scores and moreover, the estimation of the overall and the divisional efficiency scores is achieved by unduly and implicitly assigning different priorities to the sub-processes. Recently, Despotis et al. (2016) introduced an alternative approach to network DEA, which provides unique and unbiased results. However, their modeling approach can be applied only in a two-stage series network structure and cannot be extended in multi-stage series structures. In the second part of this dissertation, we deal with network DEA and we develop a novel network DEA approach for general series multi-stage processes which treats the aforementioned defects of the network DEA models presented in the literature. Particularly, we introduce a multi-objective programming approach, which employs the L_∞ norm as a distance measure to locate the stage efficiency scores as close as possible to their ideal values that are obtained independently through standard DEA models. Our new approach overcomes the defects of the basic network DEA models as it provides unique and unbiased stage efficiency scores. When data are available in the literature, the advantages of our approach are illustrated by comparing the results obtained by our method with those obtained by other methods presented in the literature. When data are not available in the literature, synthetic data are used for testing and validation. The effectiveness and the applicability of our approach, is further illustrated by providing an application for the assessment of the academic research activity in higher education viewed as a two-stage network process.

Summarizing, in this dissertation we provide critical reviews on the value based and network DEA models proposed in the literature and we develop new models that overcome their defects. The effectiveness and the applicability of our new models are further illustrated by providing new applications.

1.1 Motivation and objectives of this research

In DEA, driving the efficiency assessments in line with individual preferences is of major importance. The recent developed value based DEA models, although they provide the analyst with the ability to incorporate individual preferences into the assessment process, they do not provide an index of efficiency as they discriminate only between efficient and non-efficient units.

Network DEA is a recent extension of conventional DEA for the efficiency assessment of DMUs, where their internal structure is taken into account. Particularly, the entire production process of a DMU is analyzed into sub-processes (stages, divisions) whose linkage is represented by series or parallel network structures. The currently developed network DEA models do not provide unique divisional efficiencies, whereas the efficiency scores are derived by implicitly assuming unknown and different priorities to different stages, which bias the efficiency assessment.

This dissertation deals with the above observed defects and it introduces novel models that overcome the aforementioned drawbacks. Thus, the objectives of this dissertation are:

- To present a review on methods utilized for incorporating value judgements in DEA and to develop a novel value based DEA model which overcomes the aforementioned defects.
- To present the current state of network DEA and to develop new network DEA models, which provide unique and unbiased divisional stage efficiencies.

1.2 Contribution of this research

The contribution of this research is summarized as follows:

- Provides a thorough interpretation of the weight variables in DEA when max-column normalization is applied on the data.
- Introduces a data transformation – variable alteration technique, which enhances the conventional DEA models with additional properties and deals effectively with the discontinuity issue of the value functions in piece-wise linear DEA.
- Introduces a novel value based DEA model, which provides efficiency scores for the evaluated units.
- The proposed value based DEA model builds the bridge between the value based DEA and the ordinal regression analysis MCDA approach.
- Develops a framework for the assessment of the research activity of academics via the value based piece-wise linear DEA approach.
- Presents an application of the proposed value based DEA model to a case study drawn from the literature and compares the results obtained.
- Introduces a novel DEA approach for general multi-stage processes that provides neutral and unbiased efficiency scores.
- Presents an application of the proposed network DEA approach to the assessment of the academic research activity in Higher Education.

1.3 Organization of this dissertation

This dissertation is organized as follows:

Chapter 2: The chapter begins with an introduction to the basic assumptions, concepts and definitions in DEA. Section 2.3 presents the fundamental DEA models, namely, the CCR, the BCC and the Additive DEA models. Section 2.4 presents techniques and approaches utilized for post DEA analysis. Section 2.5 discusses some basic extensions of the conventional DEA that are developed to meet different assumptions on the input/output data analyzed.

Chapter 3: Reviews different methods to incorporate individual preferences in DEA assessments. Section 3.2 analyzes the meaning of the input/output weight variables in DEA. Section 3.3 discusses cases where individual preferences need to be incorporated in DEA assessments. Section 3.4 reviews two broad classes of methods to incorporate value judgment in DEA namely, the *assurance region approach* that introduces restrictions on the input/output weights and methods that incorporate external preference information by transforming the dataset or by adding fictitious units. Then, the pros and the cons of these methods are discussed.

Chapter 4: Introduces a novel value based DEA model. In section 4.2 we introduce a data transformation – variable alteration technique as a means to transform the original input/output weights into values. In this way, value functions are introduced in the DEA assessments that enhance the conventional DEA models with additional properties. In section 4.3 we provide a critical review of DEA with non-linear virtual inputs and outputs that spots the discontinuity issue of the value functions. Then, we extend the data transformation – variable alteration technique to DEA models with non-linear virtual inputs and outputs by employing piece-wise linear value functions. This extension effectively treats the aforementioned discontinuity issue. In section 4.4, we develop a novel value based DEA model which, unlikely the value based DEA models proposed in the literature, provides a measure of efficiency for the evaluated units. Then, we revisit a case drawn from the literature that concerns the assessment of a Portuguese retail chain in the pharmacy-cosmetics-hygiene sector to apply our novel approach. For comparison purposes, we assimilate the preferential information given in the original work. The assessment results obtained show that our approach successfully locates the efficient DMUs and unlike the assessment method used in the original work that discriminates only between efficient and inefficient units, it provides a thorough efficiency measure. Finally, in section 4.5, we introduce a novel hybrid approach to incorporate individual preferences in a DEA assessment framework by means of ordinal regression.

Chapter 5: Illustrates the effectiveness and the applicability of the novel value based DEA model presented in chapter 4 with an application. Particularly, we develop a framework for assessing the research performance of academics by taking into account both the quantity as well as the quality of the research output. The effectiveness of our approach is justified by comparing our results with those obtained by standard DEA models.

Chapter 6: Deals with network DEA. Section 6.2 provides a general literature review on network DEA. The basic network DEA approaches are presented in sections 6.3, 6.4, 6.5 and 6.6. Section 6.7 concludes the chapter by spotting specific shortcomings of the basic network DEA approaches.

Chapter 7: Develops a novel network DEA approach for general series multi-stage processes. We introduce a multi-objective programming approach, which employs the L_∞ norm as a distance measure to locate the stage efficiency scores as close as possible to their ideal values that are obtained independently through standard DEA models. Our new approach overcomes the defects of the basic network DEA models spotted in the previous chapter by providing unbiased and unique stage efficiency scores. When data are available in the literature, the superiority of our approach is illustrated by comparing the results obtained by our method with those obtained by other methods presented in the literature. Synthetic data are used for testing and validation in general network structures where the existing methods are not applicable. In section 7.2 we categorize two-stage processes in four types and we develop our models and solution procedures for each one of them. In section 7.3 we extend our formulations to general multi-stage processes. The chapter ends with our main conclusions.

Chapter 8: Develops a framework to assess the academic research activity in higher education viewed as a two-stage network process. The first stage represents productivity and the second stage represents the recognition of the research outputs and the achievements of the academic staff. Measures of the volume and the quality of the research work are both taken into account in the assessment process, which is carried out by employing the network DEA approach developed in chapter 7. The assessment framework and the factors that were included in the analysis are presented in section 8.2. Sections 8.3 and 8.4 present the results and our main conclusions respectively.

Chapter 9: Concludes the dissertation by summarizing the main research findings presented in the dissertation and provides future research directions.

1.4 Publications based on this disseration

International journals

- Despotis D.K., Sotiros D. & Koronakos G. (2016). A network DEA approach for series multi-stage processes. *Omega*, 61, 35-48.
- Despotis D.K. & Sotiros D. (2014). Value-based Data Envelopment Analysis: A piece-wise linear programming approach. *International Journal of Multicriteria Decision Making*, 4 (1), 47-68.

Book chapters

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Analysis approach. 2nd International Symposium & 24th National Conference on Operational Research, September 26-28, Greece, Athens.

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Chapter 2

Data Envelopment Analysis

2.1 Introduction

The competiveness in the world market has lead science to focus on how plants (production units) can improve their performance. To this end, the measurement of the efficiency of these units is a necessity. Efficiency is a measure of the ability of a production unit to transform inputs to outputs. Performance measurement is not a straightforward task. The classical approach to performance measurement assumes specific forms of the production functions such as Cobb and Douglas (1928) and the assessment of the units is made by estimating the parameters of the production functions (parametric techniques) that fit the performance data. Farrell (1957) introduced a non-parametric approach in the field of efficiency measurement where no assumption is made for the production functions, that is the mechanism that transforms inputs to outputs is assumed unknown and the assessment of the units is based solely on the performance data.

Charnes et al. (1978), based on the innovative work of Farrell (1957), introduced the Data Envelopment Analysis (DEA), a non-parametric technique based on linear programming, to assess the relative efficiency of production units (Decision Making Units - DMUs). DEA assumes that all DMUs are comparable, homogeneous and that they consume the same inputs, to produce the same outputs; only the level of inputs/outputs differs.

The choice of factors (inputs/outputs) in an assessment framework depends on the case study and the availability of the data. Inputs are traditionally factors that are consumed and their level should be decreased, while outputs are factors that are produced and their level should be increased.

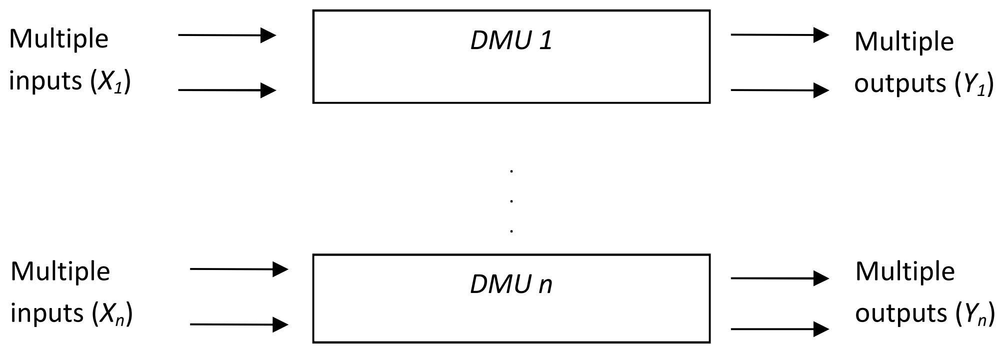


Figure 2.1: Example of DMUs

Figure 2.1 presents a generic system consisting of *n* DMUs where each unit consumes multiple inputs (vector X_j) to produce multiple outputs (vector Y_j). Generally, DMUs belong to a production environment with an unknown technology (T). The aim of DEA is to create an envelopment technology (T^{env}) from the observed DMUs. The creation of the envelopment technology is based on the minimal extrapolation principle e.g. to define the smallest convex set, which envelops all observed DMUs and it is based on the following assumptions:

- All DMUs consume the same inputs to produce the same outputs. Only the levels of the inputs/outputs that DMUs consume/produce differentiate from one DMU to another. This assumption sets all DMUs comparable.
- The DMUs are observed entities that originate from the same unknown technology *(T)*.
- The envelopment technology is technically achievable ($T^{env} \subseteq T$).
- The input/output data are non-negative scalars.

DEA is built on the following axioms:

- *Convexity*: Any convex combination of DMUs that belong to T^{env}, belong to T^{env}, i.e. $\sum_{j=1}^{n} \lambda_j \begin{bmatrix} X_j \\ Y_j \end{bmatrix} \in T^{env}$, $\sum_{j=1}^{n} \lambda_j = 1;\ \lambda_j \geq 0;\ j = 1,...,n$

- *Monotonicity*: if $\begin{bmatrix} X_j \\ Y_j \end{bmatrix} \in T^{env}$ and $X_0^+ \geq X_j$, $Y_0^- \leq Y_j$ then $\begin{bmatrix} X_0^+ \\ Y_0^- \end{bmatrix} \in T^{env}$.
- *Ray unboundedness*: $\begin{bmatrix} X_j \\ Y_j \end{bmatrix} \in T^{env} \Rightarrow k \begin{bmatrix} X_j \\ Y_j \end{bmatrix} \in T^{env}$, $\forall k \in \Re_0^+; j = 1,\ldots,n$.

A major advantage of DEA over the parametric approaches is that it does not require any a priori assumption of the production function that transforms inputs to outputs. DEA is also referred to as a benchmarking technique, as it assess the efficiency of decision making units relatively to the best practice units that are located on the boundary of the production possibility set (efficient frontier). Since the the innovative work of Charnes et al. (1978), more than 4000 articles have been published, on DEA applications and extensions (Emrouznejad et al., 2008).

2.2 Basic concepts and definitions

In order to illustrate the basic concepts in DEA, three numerical examples are presented below.

2.2.1 The one input – one output case

In this example, 6 stores (A-F) are presented and compared. Each store uses one input (Employees) to produce a single output (Sales). The 2nd and 3rd rows of Table 2.1 present, for each store, the number of employees and the sales achieved in one month, measured in units of tens of thousands, respectively.

Table 2.1: The one input – one output case

Store	A	B	C	D	E	F
Employees	2	7	5	8	4	10
Sales	1	6	3	2	6	7
Sales/Employee	0.5	0.857	0.6	0.25	1.5	0.7
Efficiency	0.333	0.571	0.4	0.167	1	0.467

The forth row of Table 2.1, depicts the sales per employee for each store. The latter index is commonly used in management as a measure of productivity. The store E is the most productive among the stores. The relative efficiency (productivity) of the stores is obtained by comparing the stores against the most productive one (store E) as follows:

$$\frac{\text{sales per employee of store i}}{\text{sales per employee of store E}} \leq 1, \ \ \text{i} = (\text{A, B, C, D, E, F})$$

The last row of Table 2.1 presents the efficiency scores of the stores. Thus, efficient are the units whose relative efficiency score is 1 (store E). The units with relative efficiency score less than one are inefficient (A, B, C, D, F). Figure 2.2 exhibits the input/output data of the six stores presented in Table 2.1.

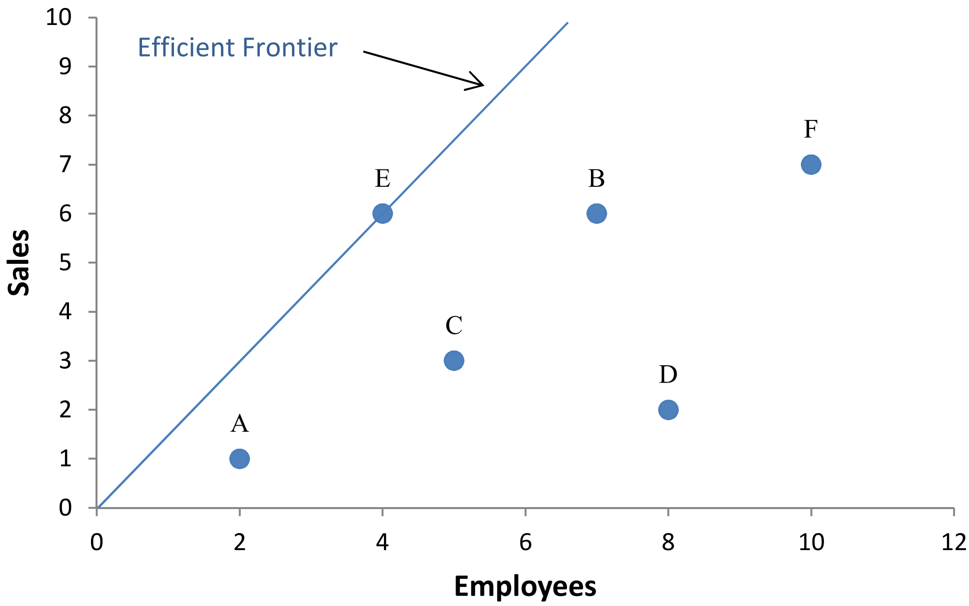


Figure 2.2: The production possibility set and the efficient frontier

The slope of the rays, which pass through the origin of the axes and the observations, represent the sales per employee index for each store. The ray with the largest slope that passes through the point E is called *efficient frontier*. The units A, B, C, D and F are inefficient as they are below the efficient frontier. The efficient frontier *envelops* all the observed units in the convex hull defined by the horizontal axes and the efficient frontier. This convex hull is the *production possibility set*.

Besides the relative efficiency of the units, DEA provides also prescriptions for their improvement so as to be deemed efficient. For instance, it yields sufficient and adequate information on how an inefficient DMU can be projected onto the efficient frontier.

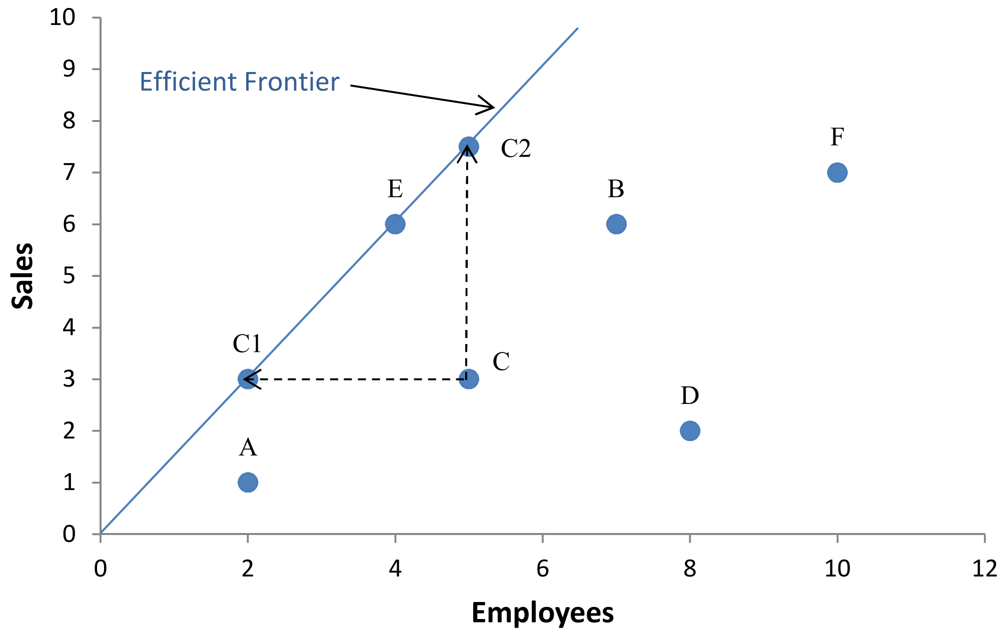


Figure 2.3: Projections of unit C on the efficient frontier

As depicted in Figure 2.3, the inefficient unit C can be projected on the efficient frontier in various ways. Every projection of DMU C to any point of the line segment C_1C_2 renders the unit efficient. Generally, there are two types of DEA models. The *input-oriented model,* which aim to reduce the level of the inputs while satisfying at least the given output levels (projection to C_1) and the *output-oriented model***,** which attempt to increase the level of the outputs without requiring more of any of the observed input (projection to C_2). Non-oriented models have been also presented in the literature (such models are illustrated in the next sub-section).

2.2.2 The two inputs – one output case

In this example we present a case of five stores, which use two inputs (full time equivalent employees and floor area) to produce one output (sales). Table 2.2 shows the input/output data for the five stores.

Table 2.2: The two inputs – one output case

Store	A	B	C	D	E
Employees	1	9	2	5	3
Floor Area	2	6	8	75	6
Sales	1	12	4	25	6
Employees/Sales	1	0.75	0.5	0.2	0.5
Area/Sales	2	0.5	2	3	1

To make the presentation of the data on a two-dimensional space possible, we get for each store the level of inputs per unit of the output, as depicted in the last two rows of Table 2.2. The data is graphically illustrated in Figure 2.4.

It is clear that the units which consume less input to produce 1 unit of output are more efficient. Thus, the units B, D and E are efficient and they define the efficiency frontier. The efficiency score of the inefficient units (C and A) can be obtained by referring to the frontier. For example, the efficiency score of unit C is given by the ratio $\frac{OC_1}{OC} = 0.8125$, where the point C_1 is the intersection of the ray OC with the line segment DE, which is part of the efficient frontier. This ratio is always less than one for the inefficient units and denotes the level of inputs at which they should be decreased proportionally so as the inefficient units to be deemed efficient. For instance, the levels of both inputs of DMU C should be decreased by 18.75% (0.8125*2=1.625, 0.8125*8=6.5) i.e. the full time equivalent employees should be reduced at the level 1.625 and the area should be decreased to 6.5.

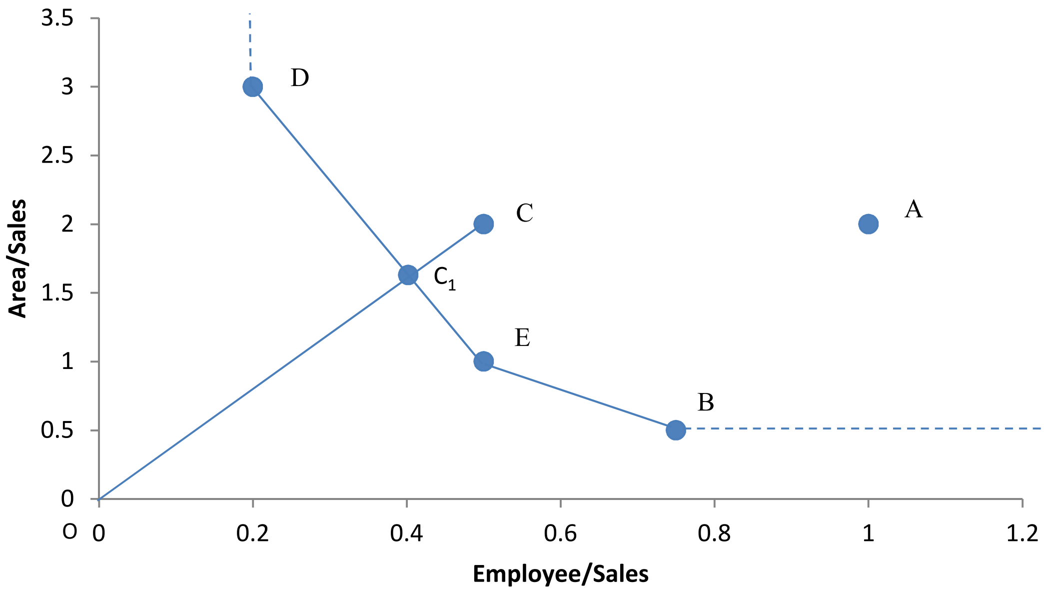


Figure 2.4: The production possibility set and the efficient frontier

2.2.3 Single input – two outputs

In this example we present a case of five stores with one input (employees) and two outputs (customers and sales) as shown in Table 2.3.

Table 2.3: The 1 input - 2 outputs case

Store	A	B	C	D	E
Employees	1	2	2	4	2
Customers	2	6	8	4	6
Sales	7	12	4	24	6
Customers/ Employees	2	3	4	1	3
Sales/ Employees	7	6	2	6	3

To present the data on a two-dimensional space, we get for each store the level of outputs per unit of the input, as shown in the last two rows of Table 2.3. The data is graphically illustrated in Figure 2.5.

It is clear that the units with higher levels of outputs per unit of input are more efficient. Thus, the units A, B and C are efficient and they define the efficiency frontier whereas the units D and E are inefficient. The efficiency score of the inefficient units (D, E) can be obtained by referring to the frontier. For example, the efficiency of unit E is defined by the ratio $\frac{OE}{OP_2}$ where point P_2 is the intersection of the ray OE with the line segment BC, which is part of the efficient frontier. The unit E is projected on the efficient frontier between the efficient units B and C. Thus, the latter units are the *reference units* for E. In order the unit E to be deemed efficient both outputs must be increased proportionally by $1\Big/\frac{OE}{OP_2}$. This kind of inefficiency which can be eliminated by a proportional improvement of the outputs is called *technical inefficiency*. However, there are cases where the proportional improvement of the outputs is not sufficient to restore the efficiency of an inefficient unit. For instance, the efficiency of unit D is $\frac{OD}{OP_1}$. However, the virtual unit P_1, although it lies on the boundary of the production possibility set, it is not efficient. Compared to unit A, P_1 is inefficient because both exhibit the same level of sales per employee but unit P_1 exhibits a lower level of customers per employee than A. Thus an extra non-radial increase to the customers per employee is required (shift to point A) to restore the efficiency of unit D. This kind of inefficiency is called as *mix inefficiency.*

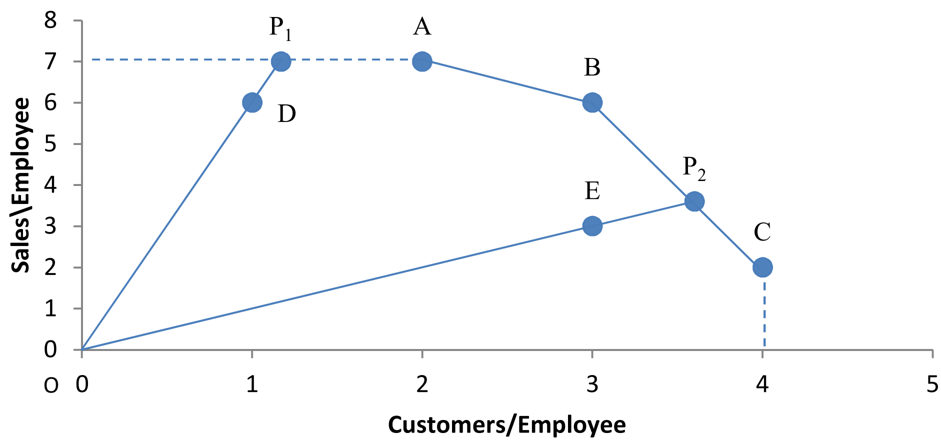


Figure 2.5: The production possibility set and the efficient frontier

2.3 The basic DEA models

This section presents the basic DEA models introduced by Charnes et al. (1978) and Banker et al. (1984) namely, the CCR and the BCC models respectively.

2.3.1 The CCR model

The CCR model was introduced by Charnes et al. (1978) and it is based on the assumption of *constant returns to scale (CRS)*. According to this assumption, a proportional change to the inputs αX, $\alpha \in \Re^{+}$, will lead to the same proportional change of the outputs αY.

Consider now a system that consists of *n* DMUs where each unit consumes *m* inputs to produce *s* outputs. We denote by y_{rj} the level of the output *r* (*r* =1,…,*s*) produced by DMU *j* (*j*=1,…,*n*) and by x_{ij} the level of the input *i* (*i*=1,…, *m*) consumed by DMU *j*. $X_j = \left(x_{1j},\ldots,x_{mj}\right)^T$ represents the vector of the inputs that DMU *j* consumes and $Y_j = \left(y_{1j},\ldots,y_{sj}\right)^T$ the vector of outputs that DMU *j* produces. The relative efficiency of the evaluated unit j_0 is estimated by the following *fractional model* (2.1).

$$
\begin{aligned}
&\max\ e_{j_0} = \frac{\hat{u}Y_{j_0}}{\hat{v}X_{j_0}} \\
&s.t. \\
&\frac{\hat{u}Y_j}{\hat{v}X_j} \le 1,\ j = 1,2,\ldots,n \\
&\hat{u},\hat{v} \ge 0
\end{aligned}
\tag{2.1}
$$

where the variables $\hat{u} = \left(\hat{u}_1,\ldots,\hat{u}_s\right)$ and $\hat{v} = \left(\hat{v}_1,\ldots,\hat{v}_m\right)$ are the weights assigned to the outputs and the inputs respectively. The ratio $\hat{u}Y_{j_0} / \hat{v}X_{j_0}$ in the objective function denotes the efficiency of the evaluated unit j_0, which is to be maximized, whereas the constraints $\hat{u}Y_j / \hat{v}X_j \le 1,\ j = 1,\ldots,n$ and $\hat{u},\hat{v} \ge 0$ bound the efficiency scores of all the units, including the evaluated unit, in the interval $(0,1]$. The model (2.1) is solved for each unit at a time. The terms $\hat{u}_r y_{rj}$ and $\hat{v}_i x_{ij}$ are called *virtual output* and *virtual input* respectively whereas $\hat{u}Y_j$ is called *total virtual output* and $\hat{v}X_j$ is called *total virtual input*.

Let $\hat{u}^* = \left(\hat{u}_1^*, \hat{u}_2^*, ..., \hat{u}_s^*\right)$ and $\hat{v}^* = \left(\hat{v}_1^*, \hat{v}_2^*, ..., \hat{v}_m^*\right)$ be an optimal solution of the model (2.1) and $e_{j_0}^*$ the optimal value of the objective function when DMU j_0 is evaluated. DMU j_0 is *CCR-efficient* if and only if $e_{j_0}^* = 1$ and there exists at least one optimal solution $\left(\hat{u}^*, \hat{v}^*\right)$ with $\hat{u}^* > 0$ and $\hat{v}^* > 0$. Otherwise, DMU j_0 is *CCR-inefficient.* Often the non-negativity constraints $\hat{u}, \hat{v} \geq 0$ are replaced by $\hat{u}, \hat{v} \geq \varepsilon$ where, ε is a non-Archimedean infinitesimal number, in order to deal only with non-zero weights. When a unit is evaluated and $e_{j_0}^* < 1$ then, there will be at least one constraint for which the equality holds (binding constraint) when applying the optimal multipliers $\hat{u}^*, \hat{v}^*$. The set $E_0^{'} = \left\{ j : \hat{u}^* Y_j = \hat{v}^* X_j \right\}$ is composed of efficient units and it is called the *reference set* or the *peer group* of DMU j_0.

Model (2.1) is in fractional form and thus non-linear. However, it can be transformed to an equivalent linear model by applying the Charnes-Cooper (Charnes and Cooper, 1962) transformation (C-C transformation hereafter). Consider a scalar $t \in \Re^+$ such as $t = 1/\hat{v}X_j$. By multiplying all terms of model (2.1) with $t > 0$ and by setting $v = t\hat{v}$ and $u = t\hat{u}$ model (2.1) is transformed to the following linear equivalent:

$$
\begin{aligned}
&\max\ e_{j_0} = uY_{j_0} \\
&s.t. \\
&vX_{j_0} = 1 \\
&uY_j - vX_j \leq 0,\ j = 1, 2, ..., n \\
&u, v \geq 0
\end{aligned}
\qquad (2.2)
$$

Model (2.2) is the CCR input oriented DEA model. Indeed, the objective function of this model aims to maximize the total virtual output of the evaluated unit. If $e_{j_0}^* < 1$, the unit is inefficient and the total virtual output cannot achieve a higher level. Thus, in order to be rendered efficient, reduction of the input levels is required.

The dual of model (2.2) is as follows:

$$\begin{aligned} &\min \ \theta \\ &s.t. \\ &\theta X_{j_0} - X\lambda \geq 0 \\ &Y\lambda - Y_{j_0} \geq 0 \\ &\lambda \geq 0 \end{aligned} \tag{2.3}$$

where $\lambda = (\lambda_1, ..., \lambda_n)^T$. In the context of DEA, the model (2.2) is referred as the *multiplier model* whereas the model (2.3) as the *envelopment model*. The correspondence of these models is illustrated on Table 2.4.

Table 2.4: Correspondence between multiplier and envelopment models

Constraint (model 2.2)	Dual variable model (2.3)	Constraint model (2.3)	Primal variable model (2.2)
$vX_{j_0} = 1$	θ	$\theta X_{j_0} - X\lambda \geq 0$	$v \geq 0$
$uY_j - vX_j \leq 0,\ j = 1,2,...,n$	$\lambda \geq 0$	$Y\lambda - Y_{j_0} \geq 0$	$u \geq 0$

The model (2.3) can be expressed in its standard (augmented) form by introducing the non-negative slack variables $s^- = (s_1^-, ..., s_m^-)^T$ and $s^+ = (s_1^+, ..., s_r^+)^T$ as presented in the model (2.4).

$$\begin{aligned} &\min \ \theta \\ &s.t. \\ &\theta X_{j_0} - X\lambda - s^- = 0 \\ &Y\lambda - Y_{j_0} - s^+ = 0 \\ &\lambda, s^-, s^+ \geq 0 \end{aligned} \tag{2.4}$$

The variables $s^- = \theta X_{j_0} - X\lambda$ and $s^+ = Y\lambda - Y_{j_0}$ are called *input excesses* and *output shortfalls* respectively.

The envelopment form estimates the efficiency scores (θ) of the evaluated units and provides the projections of the inefficient units on the efficiency frontier. This can be achieved by solving the linear program (2.4) in two phases. In the first phase model (2.4) is solved so as to acquire the optimal value of the objective function θ^*. Then, in order to discover possible input excesses and/or output shortfalls, the model (2.5) is solved.

$$\begin{aligned} &\max\ e^m s^- + e^s s^+ \\ &s.t. \\ &s^- = \theta^* X_{j_0} - X\lambda \\ &s^+ = Y\lambda - Y_{j_0} \\ &\lambda, s^-, s^+ \geq 0 \end{aligned} \tag{2.5}$$

where $e^m \in \Re^{1\ x\ m}, e^s \in \Re^{1\ x\ s}$ are vectors of appropriate dimensions, whose all elements are equal to one. In model (2.5), the objective function aims to maximize the summation of the slack variables while the optimal value of the objective function obtained in the first phase is maintained.

If $\theta^* = 1$ and all the slack variables are zero then, the evaluated unit is efficient otherwise, it is inefficient. According to the complementary slackness theorem in linear programming, it holds that $s^{-*}v^* = 0$ and $s^{+*}u^* = 0$. Hence, if the optimal value of the objective function in model (2.5) is greater than zero, i.e. there are non-zero slacks then, there will be at least one element of u^*, v^* such as $u_r^* = 0$ or $v_i^* = 0$ and the unit is characterized as inefficient. On the other hand, if the optimal value of the objective function in the model (2.5) is zero, then, according to the strong complementary slackness theorem, a positive optimal solution is assured ($u^* > 0$ and $v^* > 0$) in terms of the multiplier form and the unit is efficient. To conclude, if an optimal solution of the linear program (2.5) satisfies that $\theta^* = 1$ and $s^{-*} = 0$, $s^{+*} = 0$ (max slack solution is zero), the unit is *CCR efficient*.

From the fractional model (2.1) an output oriented linear equivalent model can be derived. Table (2.5) presents the input and output oriented CCR DEA models in their multiplier and envelopment forms.

It is worth noting that the optimal solution of the output oriented CCR DEA model can be obtained from the optimal solution of the input oriented CCR DEA model according to the following linear transformations:

- $\eta^* = 1/\theta^*$
- $\mu_j^* = \lambda_j^* / \theta^*,\ j = 1,2,...,n$
- $t_i^{-*} = s_i^{-*} / \theta^*,\ i = 1,2,...,m$ (2.6)
- $t_r^{+*} = s_r^{+*} / \theta^*,\ r = 1,2,...s$

Table 2.5: Input and output oriented CCR DEA models

	Multiplier model	Envelopment model
Input oriented	$\max\ e_{j_0} = uY_{j_0}$ $s.t.$ $vX_{j_0} = 1$ (2.2) $uY_j - vX_j \le 0,\ j = 1,2,...,n$ $u, v \ge 0$	$\min\ \theta$ $s.t.$ $\theta X_{j_0} - X\lambda - s^- = 0$ (2.4) $Y\lambda - Y_{j_0} - s^+ = 0$ $\lambda, s^-, s^+ \ge 0$
Output oriented	$\min\ vX_{j_0}$ $s.t.$ $uY_{j_0} = 1$ $uY_j - vX_j \le 0,\ j = 1,2,...,n$ (2.7) $u, v \ge 0$	$\max\ \eta$ $s.t.$ $X_{j_0} - X\mu - t^- = 0$ (2.8) $Y\mu - \eta Y_{j_0} - t^+ = 0$ $\mu, t^-, t^+ \ge 0$

The optimal solution of the input oriented CCR DEA model can be derived from the output oriented CCR DEA model by the following linear transformations

- $\theta^* = 1/\eta^*$

- $\lambda_j^* = \mu_j^* / \eta^*,\ j = 1,2,\dots,n$
- $s_i^{-*} = t_i^{-*} / \eta^*,\ i = 1,2,\dots,m$
- $s_r^{+*} = t_r^{+*} / \eta^*,\ r = 1,2,\dots,s$

To conclude, an efficient unit according to the input oriented CCR DEA model will be efficient according to the output oriented CCR DEA model as well. Concerning the inefficient units, both models provide the same efficiency scores $\eta^* = 1/\theta^*$.

2.3.2 The BCC model

The BCC DEA model was introduced by Banker et al. (1984). It is considered as an extension of the CCR model to cases where a proportional change to the inputs may lead to a change to the outputs with different proportion. Thus, the BCC model is based on the assumption of *variable returns to scale* (VRS). Specifically, when a proportional change to the inputs by α (αX) lead to a proportional change to the outputs by $\beta(\beta Y)$ then:

- If $\beta > \alpha$ the returns to scale is increasing
- If $\beta = \alpha$ the returns to scale is constant
- If $\beta < \alpha$ the returns to scale is decreasing

Figure 2.6 depicts the efficiency frontier for the data illustrated in Table 2.1 under the variable returns to scale assumption.

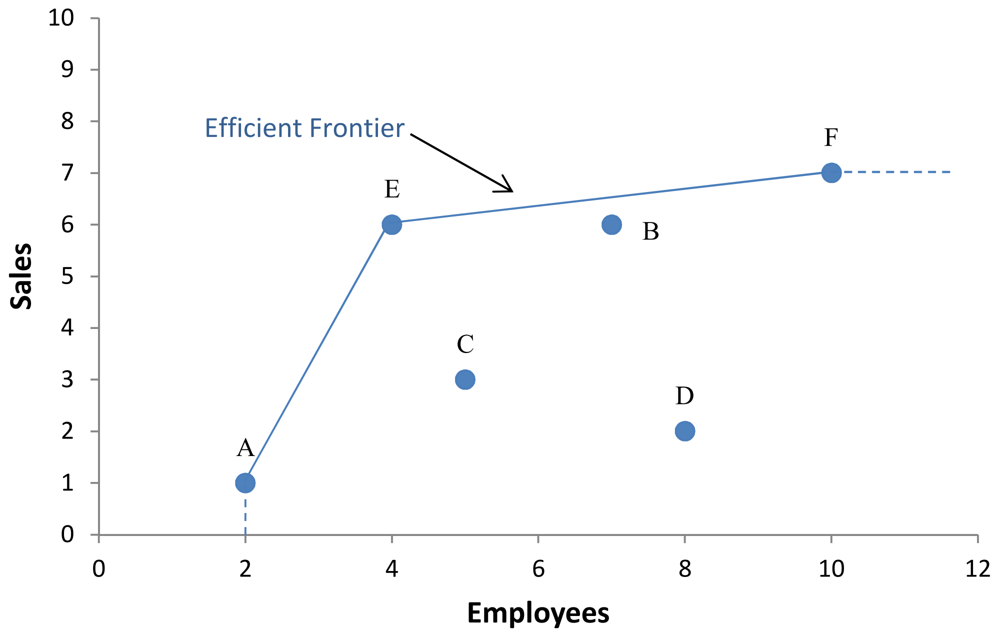


Figure 2.6: Efficient frontier under the assumption of Variable Returns to Scale

The efficient frontier is piece-wise linear and it is defined by units A, E and F. Table 2.6 presents the multiplier and the envelopment forms of the input oriented and the output oriented BCC models. In terms of the multiplier form, the main structural difference of the BCC model (2.9) compared to the CCR model (2.2) is the free of sign variable $d \in \Re$, which is the supportive hyper plane that defines locally the efficient frontier. This new variable is associated with the additional convexity constraint $e^n \lambda = 1$ in the dual (envelopment) form where $e^n \in \Re^{1 \times n}$ is a vector whose all elements are equal to one.

Table 2.6: Input and output oriented BCC DEA models

	Multiplier model	Envelopment model
Input oriented	$$\begin{aligned} &\max\ uY_{j_0} - d \\ &s.t. \\ &vX_{j_0} = 1 \\ &uY_j - d - vX_j \le 0,\ j = 1,2,...,n \\ &u, v \ge 0 \\ &d \in \Re \end{aligned} \quad (2.9)$$	$$\begin{aligned} &\min\ \theta \\ &s.t. \\ &\theta X_{j_0} - X\lambda - s^- = 0 \\ &Y\lambda - Y_{j_0} - s^+ = 0 \\ &e^n \lambda = 1 \\ &\lambda, s^-, s^+ \ge 0 \end{aligned} \quad (2.10)$$
Output oriented	$$\begin{aligned} &\min\ vX_{j_0} - p \\ &s.t. \\ &uY_{j_0} = 1 \\ &uY_j - vX_j + p \le 0,\ j = 1,2,...,n \\ &u, v \ge 0 \\ &p \in \Re \end{aligned} \quad (2.11)$$	$$\begin{aligned} &\max\ \eta \\ &s.t. \\ &X_{j_0} - X\mu - t^- = 0 \\ &Y\mu - \eta Y_{j_0} - t^+ = 0 \\ &e^n \mu = 1 \\ &\mu,\ t^-, t^+ \ge 0 \end{aligned} \quad (2.12)$$

The definitions of efficiency and reference (peer) sets in the BCC models are the same with the definitions of the CCR models. Concerning the returns to scale, in terms of the multiplier form, the following Theorem holds:

Theorem: Assuming that (x_0, y_0) is on the efficient frontier, the following conditions identify the situation for returns to scale at this point.

1. Increasing returns to scale prevails at (x_0, y_0) if and only if $d < 0$ for all optimal solutions.
2. Decreasing returns to scale prevails at (x_0, y_0) if and only if $d > 0$ for all optimal solutions.
3. Constant returns to scale prevails at (x_0, y_0) if and only if $d = 0$ in any optimal solution.

However, checking all optimal solutions can be laborious. This can be avoided by applying a procedure introduced by Banker et al. (1996). Suppose an optimal solution

$\left(\theta^*, \lambda^*, s^{+*}, s^{-*}\right)$ of model (2.10) and that $d^* < 0$ is the value of the variable d in the optimal solution of the dual model (2.9). To verify whether condition 1 or 3 applies from the above theorem, the following model is solved.

$$\begin{aligned}
&\max\ d \\
&\text{s.t.} \\
&v\overline{X}_{j_0} = 1 \\
&u\overline{Y}_{j_0} - d = 1 \\
&uY_j - d - vX_j \leq 0,\ j = 1, 2, \ldots, n;\ j \neq j_0 \\
&d \leq 0 \\
&u, v \geq 0
\end{aligned} \tag{2.13}$$

Where $\overline{X}_{j_0} = \theta^* X_{j_0} - s^{-*}$ and $\overline{Y}_{j_0} = Y_{j_0} + s^{+*}$. The constraint $d \leq 0$ restricts the variable to be non-positive. If its maximum value reaches zero, then the returns to scale is constant. Otherwise, the returns to scale is increasing. A similar procedure can be applied when the optimal value of the free variable is positive $\left(d^* > 0\right)$.

When the BCC model is compared to the CCR, the following hold:

- When a DMU achieves the minimum level in at least one input (column minimum) or the highest level in at least one output (column maximum) then, it is BCC efficient.
- The efficiency scores according to the BCC model are greater or equal to the efficiency scores obtained by the CCR model.
- The set of the efficient units according to the CCR model is a subset of the BCC efficient units.

2.3.3 The additive model

The basic additive model, introduced by Charnes et al. (1985), is a non-oriented DEA model. Its main characteristic is that it does not provide a direct measure of efficiency but it only discriminates the efficient and the inefficient DMUs. The mathematical formulations of the additive models are presented in Table 2.7.

According to the Additive model, a DMU is characterized efficient if and only if the optimal value of the objection function is zero. Concerning its relation with the CCR and the BCC models, the following hold:

- A DMU is additive-efficient under the assumption of constant returns to scale if and only if it is CCR-efficient.
- A DMU is additive-efficient under the assumption of variable returns to scale if and only if it is BCC-efficient.

Table 2.7: Additive models and their duals

	Multiplier form	Envelopment form
CRS assumption	$\begin{aligned}&\min\ vX_{j_0} - uY_{j_0}\\&s.t.\\&vX_j - uY_j \geq 0,\ j=1,2,...,n\\&u,v \geq 1\end{aligned}$ (2.14)	$\begin{aligned}&\max\ e^m s^- + e^s s^+\\&s.t.\\&X\lambda + s^- = X_{j_0}\\&Y\lambda - s^+ = Y_{j_0}\\&\lambda, s^-, s^+ \geq 0\end{aligned}$ (2.15)
VRS assumption	$\begin{aligned}&\min\ vX_{j_0} - uY_{j_0} + d\\&s.t.\\&vX_j - uY_j + d \geq 0,\ j=1,2,...,n\\&u,v \geq 1\\&d \in \Re\end{aligned}$ (2.16)	$\begin{aligned}&\max\ e^m s^- + e^s s^+\\&s.t.\\&X\lambda + s^- = X_{j_0}\\&Y\lambda - s^+ = Y_{j_0}\\&e^n\lambda = 1\\&\lambda, s^-, s^+ \geq 0\end{aligned}$ (2.17)

As already mentioned, the additive model does not provide a direct measure of efficiency. To this end, a slacks-based measure (SBM) of efficiency has been introduced by Tone (1997, 2001). The SBM model is presented, in analytical form, in the model (2.18).

$$
\begin{aligned}
& \min \quad \frac{1-\frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{ij_0}}{1+\frac{1}{s}\sum_{r=1}^{s} s_r^+ / y_{rj_0}} \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{ij_0},\ i=1,2,\ldots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rj_0},\ r=1,2,\ldots,s \\
& \lambda_j \geq 0,\ j=1,2,\ldots,n \\
& s_i^- \geq 0,\ i=1,2,\ldots,m \\
& s_r^+ \geq 0,\ r=1,2,\ldots,s
\end{aligned}
\tag{2.18}
$$

The model (2.18) is in fractional form and thus non-linear. However, it can be transformed to an equivalent linear model by applying the C-C transformation.

2.4 Post DEA Analysis

DEA discriminates the efficient and the inefficient units and it provides their efficiency scores. In addition, concerning the inefficient units, it yields sufficient information on how they can be projected onto the efficiency frontier so as to be rendered efficient. However, it cannot discriminate the efficient DMUs among them and the results may be sensitive to changes on the data or on the weights that are applied to the factors. To this end, several post-analysis techniques have been developed.

2.4.1 Super efficiency

Super efficiency was introduced by Andersen and Petersen (1993) and it has been used to discriminate the efficient units. In the super efficiency model (2.19) the inequality constraint corresponding to the evaluated unit is omitted from the constraint set and thus, the efficiency of the evaluated unit is not bounded in the interval $(0,1]$. In this way, efficient DMUs are allowed to attain an efficiency score higher than unity and consequently they can be ranked.

$$\begin{aligned}&\max\ uY_{j_0}\\&s.t.\\&vX_{j_0}=1\\&uY_j-vX_j\le 0,\ j=1,2,...,n\ ;\ j\ne j_0\\&u,v\ge\varepsilon\end{aligned}\tag{2.19}$$

The model (2.19) is the input oriented super efficiency model under the assumption of constant returns to scale. Notice that the constraint $uY_j - vX_j \le 0$ does not hold for the evaluated unit j_0. However, it is notable to mention that according to Banker and Chang (2006) super efficiency procedure should be used for the detection of outliers and not for ranking the efficient DMUs.

2.4.2 Cross efficiency

In DEA, the units are free to select their optimal weights so as to maximize their relative efficiency. This is considered as one of the main advantages of DEA. However, as the efficiency score of the evaluated unit is strictly related to the choice of the (optimal) weights, it would be appealing to check how the efficiency score of a unit is affected when applying the optimal weights of the other units (Cross-efficiency). Doyle and Green (1994) emphasized the importance of cross efficiency and developed benevolent and aggressive models to assess the cross efficiency.

Cross-efficiency provides a $n \times n$ matrix where $E_{k,l}$ declares the efficiency score of unit l attained by using the optimal weights of unit k. From the above definition, it is clear that the elements $E_{k,k}$, $k = 1, 2, ..., n$ are the DEA efficient scores of the units $k = 1, 2, ..., n$ respectively. Table 2.8, indicatively portrays the cross-efficiency matrix of 4 units. Column-wise, the entries of the cross efficiency Table 2.8 provide the efficiency scores of a unit as occurred from the scope of other units (peer appraisal). For example, $E_{2,1}$ represents the efficiency score of unit 1 when the unit 2 is assessed. The average of the entries column-wise provides the cross efficiency score of the units. The elements of a row, present the score of the other units, under the scope of the evaluated DMU (appraisal of peers). The peer appraisal can be used so as to rank the efficient units while the averaged appraisal of peers can be used in a further analysis where the aim is to maximize (benevolent approach) or to minimize (aggressive approach) the average efficiency score of the other units while maintaining the efficiency of the evaluated unit.

Table 2.8: Cross efficiency Table

Rating DMU	Rated DMU				Averaged appraisal of peers
	1	2	3	4	
1	$\mathbf{E_{1,1}}$	$E_{1,2}$	$E_{1,3}$	$E_{1,4}$	A_1
2	$E_{2,1}$	$\mathbf{E_{2,2}}$	$E_{2,3}$	$E_{2,4}$	A_2
3	$E_{3,1}$	$E_{3,2}$	$\mathbf{E_{3,3}}$	$E_{3,4}$	A_3
4	$E_{4,1}$	$E_{4,2}$	$E_{4,3}$	$\mathbf{E_{4,4}}$	A_4
	e_1	e_2	e_3	e_4	
	Averaged appraisal by peers (peer appraisal)				

The model (2.20) presents the benevolent approach (when the objective function is maximized) and the aggressive approach (when the objective function is minimized) while maintaining the efficiency score of the evaluated unit j_0.

$$
\begin{aligned}
& \min/\max \ \sum_{\substack{j=1 \\ j\neq j_0}}^{n} \left(uY_j - vX_j \right) \\
& s.t. \\
& uY_j - vX_j \le 0,\ j = 1,2,\dots,n;\ j \ne j_0 \\
& vX_{j_0} = 1 \\
& uY_{j_0} = E_{j_0,j_0} \\
& v,u \ge 0
\end{aligned}
\tag{2.20}
$$

2.4.3 Ranking intervals

Cross-efficiency has been widely used as a post analysis method as it estimates how the optimal weights of the rest units affect the efficiency score of a unit. However, it does not take into consideration all the feasible set of weights. To this end, Salo and Punkka (2011) developed mixed integer linear programs so as to estimate the best and the worst ranking a DMU can achieve considering all feasible sets of weights.

$$
\begin{aligned}
& r_{j_0}^{best} = 1 + \min \sum_{\substack{j=1, \\ j \neq j_0}}^{n} z_j \\
& s.t. \\
& uY_j \leq vX_j + Cz_j,\ j = 1,2,\ldots,n;\ j \neq j_0 \\
& uY_{j_0} = 1 \\
& vX_{j_0} = 1 \\
& u, v \geq 0 \\
& z_j \in \{0,1\},\ j \neq j_0 \\
& C \ is\ a\ large\ positive\ number
\end{aligned}
\qquad (2.21)
$$

$$
\begin{aligned}
& r_{j_0}^{worst} = 1 + \max \sum_{\substack{j=1, \\ j \neq j_0}}^{n} z_j \\
& s.t. \\
& vX_j \leq uY_j + C(1 - z_j),\ j = 1,2,\ldots,n;\ j \neq j_0 \\
& uY_{j_0} = 1 \\
& vX_{j_0} = 1 \\
& u, v \geq 0 \\
& z_j \in \{0,1\},\ j \neq j_0 \\
& C \ is\ a\ large\ positive\ number
\end{aligned}
\qquad (2.22)
$$

Model (2.21) estimates the best ranking $\left(r_{j_0}^{best}\right)$ unit j_0 can achieve whereas model (2.22) the worst one $\left(r_{j_0}^{worst}\right)$. Thus, the rank of unit j_0, having selected any feasible solution, will lie in the interval $\left[r_{j_0}^{best}, r_{j_0}^{worst}\right]$. DMUs that have wider ranking interval are more sensitive to changes over the weights whereas those who have narrower ranking interval are more robust. This method, provides also dominance relations over the DMUs e.g. DMU k dominates DMU l if $r_l^{best} > r_k^{worst}$ and it can also discriminate efficient DMUs according to their ranking interval.

2.4.4 Weight restrictions

Restrictions on weights are commonly used in order to incorporate individual preferences in terms of tradeoffs among inputs and outputs. In this way, the flexibility of the evaluated unit to select its optimal weights is restricted and in effect the discriminative power of DEA is improved. The incorporation of weight restrictions in DEA assessments is further discussed in chapter 3.

2.5 Extensions of DEA

Conventional DEA models have been extended to deal with situations under different assumptions on the input/output data. In this section, we provide a short literature review on extensions of the standard DEA models.

Non-discretionary inputs

There are cases where some of the inputs are exogenous or generally fixed and their levels cannot be reduced (uncontrollable/non-discretionary inputs). Banker and Morey (1986) were the first who pointed out this issue and developed an approach to deal with such cases. Other similar methods to deal with non-discretionary inputs can be found in Ray (1991) and Ruggiero (1996). A review of these approaches and some improvements are presented in Ruggiero (1998).

Imprecise data

Conventional DEA models assume that the input/output data are fixed scalars. However, there are cases where some of the data are imprecise. For instance, they may be given in terms of bounded intervals or measured in an ordinal scale (ordinal data) Cooper et al. (1999) and Despotis and Smirlis (2002) were the first who developed approaches to deal with imprecise data in DEA (IDEA). A review of these approaches is provided in Zhu (2003).

Negative data

One of the assumptions that permeate DEA is that data are non-negative. Nevertheless, there are cases where negative values are meaningful. Scheel (2001) was among the first who addressed the issue of negative data in DEA. Portela et al. (2004), Sharp et al. (2007) and Emrouznejad et al. (2010) further explored the development of new models to deal with negative data. Matin and Azizi (2011) provided a review on these approaches and proposed an alternative approach for setting targets in the context of DEA with negative data.

Undesirable outputs

Often, a production process produces, besides its ordinary outputs, some bad outputs (for example pollutants). Such outputs are characterized as undesirable outputs and unlikely the ordinary outputs that are to be maximized, these outputs should be minimized. Seiford and Zhu (2002) and Fare and Grosskopf (2004), among others, introduced DEA variations to deal with undesirable outputs. A thorough review of these methods can be found in Liu et al. (2010).

Value based DEA

Value based DEA is a recent development that resorts to value assessment protocols from multiple criteria decision analysis (MCDA) to transform the original input/output data to a value scale. Gouveia et al. (2008) were among the first who linked DEA with MCDA by incorporating concepts from multi-attribute utility/value theory in the additive DEA model. Later, Almeida and Dias (2012) based on the seminal ideas presented in Gouveia et al. (2008), extended their methodology in the context of a real-world application.

Network DEA

The conventional DEA models assume one stage production processes e.g. only the levels of the external inputs that the system consumes and the levels of the final outputs that the systems produces are known. However, there are cases where the internal flow of the production process is known and it plays a crucial role in the efficiency assessment. Network DEA is a recent extension of DEA to measure the efficiency of DMUs when the production process is analyzed in sub-processes that produce intermediate products (measures). Fare and Whittaker (1995) where among the first who extended DEA to evaluate the efficiency in such processes. Several network DEA approaches have been proposed in the literature. Reviews of these approaches can be spotted in Castelli et al. (2010) and Kao (2014a).

The rest of this dissertation focuses on value based and network DEA models providing new developments and applications.

Part A

VALUE BASED DATA ENVELOPMENT ANALYSIS: NEW MODELS AND APPLICATIONS

Chapter 3

Value Judgments in DEA

3.1 Introduction

In DEA the efficiency assessment of units can be based either on a multiplier or an envelopment model. In the multiplier model, the efficiency is defined as the ratio of the weighted sum of outputs to the weighted sum of inputs. The optimal weights assigned to the inputs and the outputs are computed separately for each DMU, by solving a linear program which aims to maximize its relative efficiency. Consequently, the choice of the weights is made without any a priori knowledge about the relative importance of the factors and it is free of any assumptions. However, in real world problems external information on the relative importance of the inputs/outputs, as provided by the analysts, might be crucial. In such cases, although the flexibility privileged to the evaluated unit in selecting its own weights is one of the major advantages of DEA in locating inefficiencies, the weights assigned to the inputs and the outputs may not be necessarily in line with the analysts' individual preferences. A DMU, for instance, can be rendered efficient by assigning a zero weight to an output whose performance is at a very low level. In such a case, the DMU might be deemed efficient by implicitly neglecting a factor that may be determinant in the analysis framework. Thus, in such a situation, the flexibility in the selection of weights may unduly favor a unit contrarily to the analysts' preferences and to provide unreliable results.

To address this issue, various methods to incorporate value judgments in DEA efficiency assessments have been arisen. The necessity to intervene in the way the weights are assigned to the inputs and the outputs originates from a variety of reasons, such as to improve the discriminative power of DEA, to restrain the diversity of the weights assigned to the same factor by different DMUs and to incorporate individual preferences and trade-offs over the inputs and the outputs.

In this chapter, a short review is presented on the most important methodologies that have been proposed in the literature to incorporate value judgements in the context of DEA. The advantages and disadvantages of each method are also discussed.

3.2 The meaning of weights in DEA

As discussed in the previous chapter, the multiplier and the envelopment models are duals of each other and they differ in their structure and in their interpretation. The envelopment models are defined on the production space where the production possibility set (PPS) is specified by a linear combination of the observed levels of inputs and outputs of the DMUs. The efficiency in these models is defined as the maximum proportional expansion of outputs (output-oriented) or the minimum proportional reduction of inputs (input oriented) required to achieve the frontier of the PPS. On the other hand, the multiplier models are represented on a value space where, the weights are interpreted as imputed marginal values of outputs/inputs and the overall efficiency is defined as the ratio of the total imputed value of the outputs to the total imputed value of the inputs. To this end, weights in DEA are closely related to value and tradeoffs among the factors.

Table 3.1 illustrates the data presented in Table 2.3 where the last three columns provide the optimal weights when the CCR input oriented DEA model (2.2) is employed.

Table 3.1: The 1 input - 2 outputs case and the optimal weights

Store	Input 1 Employees	Output 1 Customers	Output 2 Sales	Efficiency	v_1^*	u_1^*	u_2^*
A	1	2	7	1	1	0	0.1429
B	2	6	12	1	0.5	0.0556	0.0556
C	2	8	4	1	0.5	0.1250	0
D	4	4	24	0.8571	0.25	0	0.0357
E	2	6	6	0.8333	0.5	0.1111	0.0278

When a DMU is assessed, there will be a least one binding constraint in model (2.2), which defines a hyperplane on the efficient frontier. For example, concerning the efficient DMU B, which lies on the efficient frontier, its hyperplane is given by the equation $0.5x_{1B} - 0.0556y_{1B} - 0.0556y_{2B} = 0$. Such equations provide information concerning the marginal rates of substitution among inputs/outputs. Specifically, for DMU B, the marginal rate of substitution between output 1 (customers) and output 2 (sales) is defined by the ratio:

$$\frac{dy_1}{dy_2} = -\frac{\partial/\partial y_2}{\partial/\partial y_1} = -\frac{u_2^*}{u_1^*} = -\frac{0.0556}{0.0556} = -1$$

which means that increasing output 2 (sales) by 1 unit, will lead to the reduction of 1 unit of output 1 (customers) while DMU maintains its efficiency score. Thus, the ratio of two optimal weights represents the marginal rate of substitution between the factors that the weights are associated with.

3.3 Reasons to incorporate value judgments

The incorporation of the analyst's preferences in a DEA assessment framework has turned to be essential in real world applications. Allen et al. (1997) and Thanassoulis et al. (2004) were the first who provided a comprehensive discussion about the needs that value judgments respond to. These can be summarized as follows:

To improve discrimination among the efficient DMUs

The discriminative power of DEA depends on the number of the evaluated DMUs and the number of factors (inputs/outputs) that are included in the analysis. That is, in cases where the number of the evaluated DMUs is relatively small compared to the number of factors, the results from DEA do not provide the desired discrimination between efficient and inefficient units. Returns to scale assumption (constant or variable) is another parameter, which affects the discriminative power. Variable returns to scale assumption provides the evaluated units with a higher degree of flexibility with the aid of supportive hyperplane thus, identifying more DMUs as efficient. These issues make more difficult the discrimination the efficient DMUs from the inefficient ones. For example, Thompson et al. (1986), in an effort to site nuclear physics facilities in Texas, they faced a problem with discrimination as five out of six DMUs were estimated as relatively efficient. They dealt with this issue by setting ranges of acceptable weights (*assurance region*), which were used to identify the preferred efficient DMU. Cook et al. (1992) also highlighted the need of locating a "winning" DMU among the efficient ones and examined various types of assurance region constraints to deal with this issue. Other methods for improving the discrimination among the efficient DMUs are the super-efficiency approach (Andersen and Petersen, 1993), the cross efficiency approach (Green et al., 1996 and Anderson et al., 2002) and multi-objective programming (Li and Reeves, 1999). These approaches do not require any a priori external information on the importance of the inputs and the outputs (Meza and Lins, 2002). Despotis (2002) introduced a non-parametric global efficiency approach to improve the discriminating power of DEA by employing different metrics and the common weights assumption. Salo and Punkka (2011) developed mixed integer linear programs so as to estimate the best and the worst ranking a DMU can achieve considering all feasible sets of weights.

To reduce the diversity of weights assigned to a factor by different DMUs

As each DMU is free to select the optimal weights assigned to the factors, very small or extremely large weights may be assigned to particular inputs and/or outputs by each evaluated DMU. Thus, the analyst might be interested in reducing the diversity of weights assigned to a factor. Roll et al. (1991) developed an approach where *Common Set of Weights* (*CSW*) were used for all DMUs thus, removing any flexibility on the weights selection. As they mention in their paper, *"difference between the efficiency measured with an 'individual' set of weights and that obtained with a CSW may indicate the effects of special circumstances under which a DMU operates"*. Roll and Golany (1993) and Cook et al. (1991) developed further the approach of CSW.

To incorporate preference information on marginal rates of substitution among the factors

As mentioned in the previous section, the ratios of optimal weights assigned to the factors by an evaluated DMU are interpreted as marginal rates of substitution among the inputs/outputs. However, as some weights may be zero at optimality, the related marginal rates of substitution will be ill-defined. In addition, even in the case where they are well-defined, they may not reflect experts' preferences and the desired tradeoffs among the factors. To deal with such cases, additional information is required to be incorporated in the DEA analysis framework so the results to be in line with the analyst prior views.

To incorporate relative importance between the inputs and/or outputs

There are situations where additional value preferences need to be included in the analysis. For example, Thanassoulis et al. (1995), measuring the efficiency of perinatal care units in UK, imposed the weight of "babies at risk" (input) to be the same with the weight of "number of survivals" (output). Beasley (1990) and Ahn and Seiford (1993), measuring the performance of university departments in UK and

USA, respectively, mentioned that universities with emphasis on post-graduate students should be rewarded.

3.4 Incorporating value judgements in DEA

There is a considerable number of approaches in the literature to assimilate value judgements in DEA. These approaches can be categorized into two broad classes of methods:

- Introduction of weight restrictions
- Alteration of the data set

It is noteworthy to mention that other methods can be also employed to facilitate the analyst preferences. Indicative examples are the method introduced by Olesen and Petersen (1996), which restricts the hyperplanes where projections of the inefficient units can be driven, as well as the method coined by Bessent et al. (1988), which extends the efficient facets so that each inefficient DMU is fully enveloped by the efficient DMUs. Additional approaches which, are not included in the above two broad classes can be found in Halme et al. (1999), Podinovski (2004) and Cooper et al. (2000).

3.4.1 Introduction of weight restrictions

Weight restrictions can be applied either directly to the weights or by imposing constraints to the virtual inputs/outputs. These restrictions can be classified as follows:

Table 3.2: Types of weight restrictions

Absolute restrictions	$a_r \leq u_r \leq b_r$	*(r1)*
Assurance region Type I	$a_{kl} \leq \dfrac{u_k}{u_l} \leq b_{kl}$	*(r2)*
	$w_r u_r + w_k u_k \leq u_l$	*(r3)*
Assurance region Type II	$a_i v_i \geq u_r$	*(r4)*
Restrictions on virtual outputs	$a_r \leq \dfrac{y_{rj} u_r}{\sum_{r=1}^{s} y_{rj} u_r} \leq b_r$	*(r5)*

In Table 3.2 u_r, u_k and u_l represent the weights associated to the r^{th}, k^{th} and l^{th} output respectively with $r = 1,...s$, $k = 1,...s$, $l = 1,...,s$ and $r \neq k \neq l$. Analogously, v_i, $i = 1,...,m$, represent the weight assigned to the i^{th} input. The $a_r, a_i, a_{kl}, b_r, b_{kl}, w_r$ and w_k represent user defined constants (parameters), which denote the intensity of the analyst's preference. Restrictions *(r1)*- *(r3)* and *(r5)* are expressed in terms of the weights associated for the outputs. However, they can be also used to restrict the weights assigned to the inputs. Restriction *(r4)* relates the weights assigned among the inputs and the outputs. These types of constraints are further discussed below.

Absolute restrictions

Absolute weight restrictions are the most direct way for restricting the weight space. They were first introduced by Dyson and Thanassoulis (1988) and Cook et al. (1991, 1994). This type of constraints, restrict the weight variables to a continuous and closed interval whose bounds denote the lower and the higher level that the weights can achieve. These bounds can be viewed as thresholds of tolerance, which intend to avoid the overestimation or the underestimation of a factor e.g. to avoid a weight to attain a zero level and thus, the corresponding factor to be ignored in the analysis.

However, as the significance of the weights is on relative basis, defining absolute bounds is not an easy task. In addition, there is a strong linkage on the bounds in different weights. For example, setting an upper bound on one input weight imposes an upper bound on its virtual input and thus a lower bound to the summation of the virtual inputs of the remaining inputs. Furthermore, when absolute restrictions are employed under the constant returns to scales assumption, the input oriented models produce different relative efficiency scores from those obtained from the output oriented models. Thus, the selection of the absolute restrictions should be on the basis of the orientation. Finally, additional caution is needed when absolute restrictions are employed as they may lead to infeasibility.

Assurance region Type I (ARI)

These types of constraints associate the weights among inputs or outputs. They were first used by Thompson et al. (1986). Constraints of the form *(r2)* are more common in the literature and they are based on the economic notion of marginal rates of substitution. The bounds of such constraints are usually chosen on the basis of analyst's preferences in conjunction with prior price/cost information. In contrast to absolute restrictions, when ARI are introduced under the CRS assumption, both input and output oriented models provide the same efficiency scores.

Assurance region Type II (ARII)

This type of constraints links the weights of inputs with the weights of outputs. Thompson et al. (1990) was one of the first who discussed the introduction of such constraints. They noted that apart from the infeasibility issues that may occur when such types of constraints are employed, the relation of inputs and outputs may be not clear (see Thanassoulis et al. 2004). When ARII are introduced, under the constant returns to scale assumption, both input and output oriented models provide the same efficiency scores.

Restrictions on virtual inputs or virtual outputs

Restrictions on virtual inputs or virtual outputs constitute an alternative approach to incorporate individual preferences. Unlike the previous methods, which impose direct restrictions on the weights, this type of constraints restricts the contribution of a virtual input (output) in the total virtual input (output) to range within a bounded interval. Technically, the virtual input/output is dimensionless and represents the worth of the corresponding factor in the efficiency assessment. The ratio of the virtual input/output to the total virtual input/output denotes the relative importance of the input/output, which contributes in achieving the efficiency score. From a managerial aspect and application driven requisites, restraining of the relative importance of a factor may seem appealing. However, as the implied restrictions are DMU specific, the model may become computationally expensive due to a large number of additional constraints.

Wong and Beasley (1990) suggest the following approaches concerning the incorporation of constraints of type (*r5*) in the model:

I. Add restrictions of type *(r5)* only for the DMU being evaluated. The relative virtual values of the rest DMUs remain unbounded and thus, only two additional constraints are appended in the model.
II. Add restrictions of type *(r5)* to all the DMUs. Consequently, $2n$ constraints are included in the model, where n denotes the number of DMUs.
III. Add restrictions of type *(r5)* for the assessed DMU plus two additional constraints defined below *(r6)*.

$$L^r_{Average} \le \frac{u_r \sum_{j=1}^{n} \frac{y_{rj}}{n}}{\sum_{r=1}^{s} u_r \sum_{j=1}^{n} \frac{y_{rj}}{n}} \le U^r_{Average} \qquad (r6)$$

The numerator in the fraction of type *(r6)* denotes the virtual output *r* of a fictitious DMU, which produces the average level of output *r* across all DMUs. The denominator represents the total virtual output of the fictitious DMU.

The approach II restricts the relative importance of an output for all DMUs. However, this is accomplished at the cost of computation. Notice that, $2n$ additional constraints are included in the model each time the analyst wants to restrict the importance of a factor. In applications with many DMUs, restricting multiple factors will lead to the introduction of numerous constraints, which increase dramatically the complexity of the model.

The approach I, reduces the number of additional constraints to two per factor. Although, it is less computational expensive, it may lead to misinterpreted results. Since, the relative importance of output *r* for the rest DMUs remains unbounded when DMU j_o is evaluated, the optimal weights assigned to DMU j_o may be infeasible when another DMU is evaluated. This issue raises a question of validity of the estimated efficiency scores as they are calculated on a different basis.

The approach III, does not require the introduction of many additional constraints and it also takes into consideration the level of the outputs of the rest DMUs when unit j_o is evaluated. However, even in a less degree, it still suffers from the drawback of approach I.

The virtual inputs/outputs are dimensionless and not dependent on the units of measurement of the factors. This is an advantage of restricting the virtual inputs/outputs over restricting directly the weights. However, restrictions on virtual inputs/outputs are closely related to absolute restrictions. For example, in an output oriented model, since for the evaluated unit j_o holds that $\sum_{r=1}^{s} y_{rj_o} u_r = 1$, the constraint (*r5*) for this unit is reduced to $a_r \le y_{rj_o} u_r \le b_r$ or to $\frac{a_r}{y_{rj_o}} \le u_r \le \frac{b_r}{y_{rj_o}}$, i.e. to a constraint of Type 1. Notice, that the relation between absolute restrictions on the weights and restrictions on virtual measures holds when restrictions on virtual inputs are employed in an input oriented model or when restrictions on virtual outputs are introduced on an output oriented model. To this end, the efficiency scores are dependent on the selection of the model's orientation.

3.4.2 Alteration of the data set

The introduction of restrictions to the weights or to the virtual inputs/outputs constitutes a direct technique to incorporate value judgments in the DEA assessments. Another approach is the alteration of the data. This is accomplished by two methods:

- Transformation of the original data
- Introduction of artificial DMUs

Transformation of the original data

The *Cone Ratio (CR)* approach introduced by Charnes et al. (1989), constitutes one of the most well-known approaches to incorporate value judgments in DEA assessments by applying a transformation on the original performance data. The CR approach is quite close to AR constraints. For example, ARI restrictions can be also treated in the CR approach. However, the latter is more general.

Consider the following multiplier model (3.1)

$$\begin{aligned} &\max\ \mathrm{e}_{j_0} = uY_{j_0} \\ &s.t. \\ &vX_{j_0} = 1 \\ &uY - vX \leq 0 \\ &u \in U \\ &v \in V \end{aligned} \tag{3.1}$$

where $U \subseteq \Re_{+}^{s}$ and $V \subseteq \Re_{+}^{m}$ represent closed convex cones containing information of the weight restrictions. $\bar{U}$ and $\bar{V}$ represent their negative polar cones and $-\bar{U}, -\bar{V}$ contain information on how to convert the data set. Restrictions of ARI, like *(r2)*, can be expressed in a matrix form for inputs as $V = \left\{\mathrm{v} : Dv^{T} \geq 0, \mathrm{v} \geq 0\right\}$ and similarly for outputs as $U = \left\{u : Cu^{T} \geq 0, u \geq 0\right\}$. Charnes et al. (1989) proved that model (3.1) is equivalent to model (3.2).

$$\begin{aligned}
&\max\ \mathrm{e}_{j_0} = \bar{u}\left(BY_{j_0}\right) \\
&s.t. \\
&\bar{v}\left(AX_{j_0}\right) = 1 \\
&\bar{u}\left(BY\right) - \bar{v}\left(AX\right) \leq 0 \\
&\bar{u} \geq 0 \\
&\bar{v} \geq 0
\end{aligned} \tag{3.2}$$

where $A^T = \left(D^T D\right)^{-1} D^T$, $B^T = \left(C^T C\right)^{-1} C^T$ and $\bar{u}$, $\bar{v}$ are vectors representing the new weight variables.

In model (3.2) the data set is actually transformed so as to incorporate the desired weight restrictions. Notice, that the weights variables $\left(\bar{u},\bar{v}\right)$ in model (3.2) are only restricted to be non-negative.

The CR approach has the advantage that captures value judgments through the data transformation instead of restricting the weights directly as in AR approach. In addition, it can be used to associate any number of multipliers. Such links may not be translated in terms of Assurance Region. Thus, CR is considered more general than AR. The CR approach requires the computation of an inverse matrix. The problem that may rise is that the inverse of the matrix may not be defined and thus such data transformation is not always possible. Once the optimal solution of model (3.2) is obtained it must be transformed in terms of model (3.1) in order to be communicated.

Introduction of artificial DMUs

The *introduction of artificial DMUs* is another method to incorporate preference information in DEA. This approach does not change the structure of the constraints but, it enlarges the size of the PPS and changes the efficient frontier. Specifically, unlike the weight restrictions which aim to limit the feasible region of the weight variables directly, this method intends to reshape the efficient frontier by introducing new DMUs which are not in the original dataset. These DMUs are fictitious units which are designed to implement the desired best practice and to act as benchmarking units. Golany and Roll (1994) mention that there are two main advantages of the incorporation of artificial DMUs compared to the introduction of restrictions on the weights. As discussed in the previous sub-sections, the incorporation of analyst's preferences by imposing restrictions on the weight may lead to numerous additional constraints, which increase the computational load. However, in the case of adding a new DMU, the size of the constraints increases only by one in the multiplier form. Moreover, introducing an artificial DMU in the observations does not lead to infeasibility as the weight restrictions may do.

The incorporation of weight restrictions in the multiplier form leads to the addition of new variables in the envelopment form. These new variables can be treated as additional DMUs in the production technology. Roll et al. (1991) were the first who pointed out this connection by providing a geometric meaning of lower bounds to input weights. They showed that each weight restricted to be positive is equivalent to adding an unobserved unit in the data set.

Golany and Roll (1994), pointing out that *"the standards are used to determine both optimal output levels and the corresponding minimal inputs*", studied the effects of introducing "standard" DMUs in the DEA assessment. These DMUs can be interpreted as benchmarking practices which however are difficult to define. Nevertheless, they show that both weight restriction and "standard" DMUs affect the efficiency scores in the same direction. Thanassoulis and Allen (1998) generalized this finding by illustrating the equivalence of imposing AR weight restrictions of Type I and II with adding new DMUs. Under the CRS assumption, they developed a technique which produces a *Full Set of Unobserved DMUS (FSUD)* whose size is equal to the number of observed DMUs. They showed that the incorporation of the FSUD in the original data set is equal to imposing weight restrictions of type ARI and ARII. However, as the FSUD may include DMUs who are duplicated and/or whose input-output vectors are linear combinations of the input-output vectors of real/unobserved DMUs, they extended their approach, by employing the concept of super-efficiency (Andersen and Petersen, 1993), in order to get a subset of the FSUD, which is adequate to simulate the ARI and ARII. They called this subset as *Reduced*

Set of Unobserved DMUs (RSUD). Analogously, they extended their approach under the VRS assumption.

Allen and Thanassoulis (2004) incorporated *Unobserved DMUs (UDMUs)* in the dataset to capture value judgments. They proposed an approach similar to constrained facets approach (Bessent et al., 1988, Lang et al., 1995) in the sense that it operates directly on the PPS rather than on DEA weights. Nevertheless, their approach aims to project the inefficient DMUs on the efficient frontier in a manner that all the input/output weights of the evaluated unit to be non-zero at optimality. To this end they introduce the so called *Anchor DMUs (ADMUs)*, whose input-output levels are adjusted so as to reduce the inefficient part of the boundary of the PPS. This approach can be summarized in 5 steps as follows (Allen and Thanassoulis, 2004):

1. *Run an ordinary assessment by DEA to identify the DEA-efficient and non-enveloped DMUs. If all DEA-inefficient DMUs are properly enveloped, then stop.*
2. *If any non-enveloped DMUs exist, identify anchor DMUs (ADMUs) from which to construct Unobserved DMUs (UDMUs).*
3. *In respect of each ADMU identify, which output(s) to adjust in order to construct suitable UDMUs.*
4. *Using adjustments to the outputs identified in (3) and the analyst value judgments construct suitable UDMUs.*
5. *Re-assess the observed DMUs by DEA after adding the UDMUs constructed. The number of enveloped observed DMUs will generally increase, depending on the accuracy of the information supplied by the analyst and on any unenvelopment caused by the presence of non-full dimensional efficient facets (NFDEFs).*

This approach is limited to CRS DEA models with a single input and multiple outputs or a single output and multiple inputs.

3.4.3 Incorporation of weight restrictions versus alteration of the dataset

As discussed above, there is equivalence between the introduction of weight restrictions and the alteration of the data set in the sense that they both produce the same efficiency scores. However, weight restrictions affect directly the weight space whereas the alteration of the data set absorbs value judgements by changing the efficient frontier. The choice whether to include weight restrictions or to alter the dataset mainly depends on the information of value judgements that the analyst desire to incorporate. However, the choice of one method does not exclude the other. They can be both utilized simultaneously to capture prior views regarding the efficiency assessment. Nevertheless, both methods have advantages and disadvantages. Some of them are discussed below:

Local vs global trade offs

In the UDMUs approach introduced by Allen and Thanassoulis (2004), the analyst is asked to provide tradeoffs between the inputs or outputs for specific DMUs. Thus, these tradeoffs are set on a local level and depend on the DMU that is chosen. On the other hand, weight restrictions affect all the evaluated DMUs and thus, can be viewed as global tradeoffs. From a managerial aspect, it may seem more convenient to set local tradeoffs rather than global ones which affect the whole dataset. Furthermore, weight restrictions may imply constant marginal rates of substitution among the factors, which turns to be very restrictive in VRS technology where the marginal rates of substitution alter in different parts of the efficient frontier. From this point of view, setting local tradeoffs may seem more appealing in assessment exercises where variable returns to scale are assumed.

Projections on the Efficient Frontier

Efficiency in DEA is a radial measure of reduction of the inputs (input-oriented model) or expansion of the outputs (output-oriented model). However, this does not hold when weight restrictions are added in the DEA model. In this case in order an inefficient DMU to reach its target in the efficient frontier might need to increase particular inputs or to reduce particular outputs. An example of this irregularity can be spotted in the results given in Chilingerian and Sherman (1997) where DEA was employed to evaluate practice patterns of primary care physicians. The UDMUs approach introduced by Allen and Thanassoulis (2004) does not suffer from this irregularity as it maintains the radial nature of efficiency. However, not fully-enveloped DMUs are projected on artificial units (Unobserved DMUs), whose introduction has altered the efficient frontier.

Chapter 4

A novel value based DEA approach

4.1 Introduction

The incorporation of value judgments in DEA has drawn significant attention by the scientific community. The aim is to develop robust techniques that successfully assimilate the analyst's preferences in the efficiency assessment. Although there are many techniques developed to address this issue in the context of DEA, eliciting user preferences and incorporating them in the analysis remains a challenging research area.

In this chapter, we develop a novel approach, which allows for a better expression and incorporation of individual preferences. The models developed within this approach are enhanced with additional properties compared to the standard DEA models remaining however in the ground of DEA. Specifically, we show that by applying a data transformation – variable alteration technique, the new variables obtain a meaningful interpretation for the analyst, allowing him/her to express tradeoffs among the inputs and the outputs in a more effective manner. The new approach can be easily extended to situations where the virtual inputs/outputs are treated as piece-wise linear value functions so as to implement intra- and inter-input/output value tradeoffs. The extensions of the conventional DEA models introduced in this chapter allow us also to bridge DEA with Multi Criteria Decision Analysis (MCDA).

The chapter unfolds as follows. In section 4.2, we present a data transformation – variable alteration technique which leads to the development of max-column normalized DEA models. Their additional properties are also discussed. In section 4.3, the data transformation – variable alteration technique is extended to deal with non-linear virtual inputs and outputs. Actually, piece-wise linear functions are assumed to model the non-linear value functions. In section 4.4 we develop a general value based DEA model where the individual preferences are extracted by means of MCDA protocols. In section 4.5, a hybrid approach is presented where the value functions are estimated by means of the ordinal regression MCDA method UTASTAR.

4.2 Max-column normalized DEA models

In this section, we introduce a data-transformation – variable alteration technique, which is based on the max column normalization. Although rescaling the data in DEA has already been used in the literature, the advantages of the max-column data transformation has not been explored. In 4.2.1 we discuss why rescaling the data is needed in the DEA. In 4.2.2 we provide a thorough insight on the meaning of the variables and the additional properties that the DEA models are enhanced with when this sort of rescaling is employed.

4.2.1 Unbalanced data, rescaling and DEA

Performing a typical DE Analysis means solving a series of linear programs, one for each DMU, either by a dedicated DEA software or by using generic standard LP software. Whatever the case, one of the problems faced by some LP implementations used to execute a DEA model is that of scaling. Indeed, unbalanced data may cause problems in the execution of the LP software and may lead to round-off errors. Unbalanced data often occur in DEA performance measurement due to different magnitudes of input/output measures. Thus, rescaling the data before executing the DEA models is a common practice, implicitly or explicitly considered for computational purposes in order to eliminate the unbalance in the raw input/output data caused by units of measurement of different order of magnitude. As a means to address this problem, Sarkis (2007), for example, suggest to rescale the observed raw data for the inputs and the outputs by dividing them by their means, column-wise. However, although this data transformation is technically correct and effective, rescaling on the means is impossible to interpret in a DEA context.

Rescaling the data on the column maximum is another alternative. Let us call this sort of rescaling *max-normalization*. In a different context and for a different purpose, rescaling on the column maximum, followed by a variable alteration, has been used by Cooper et al. (2001) to transform a non-linear imprecise DEA (IDEA) model to a linear one. Our concern, however, in this section is to highlight the meaning of max-normalization in DEA, which has not been stated explicitly elsewhere and then, to underline some properties of the max-normalized DEA models.

4.2.2 On the meaning of max-column normalization

Consider the following couple of input-oriented CCR DEA models (*multiplier* and *envelopment* forms):

Multiplier form:

$$
\begin{aligned}
&\max E(u,v,j_0)=\sum_{r=1}^{s} y_{rj_0}u_r \\
&s.t. \\
&\sum_{i=1}^{m} x_{ij_0}v_i =1 \\
&\sum_{r=1}^{s} y_{rj}u_r-\sum_{i=1}^{m} x_{ij}v_i \le 0, \quad j=1,\ldots,n \\
&u_r, v_i \ge 0 \quad \forall r,i
\end{aligned}
\tag{4.1}
$$

Envelopment form:

$$
\begin{aligned}
&\min \theta \\
&s.t. \\
&\sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{rj_0} \qquad r=1,\ldots,s \\
&\theta x_{ij_0}-\sum_{j=1}^{n} x_{ij}\lambda_j - s_i^- =0 \quad i=1,\ldots,m \\
&\lambda_j \ge 0, s_r^+ \ge 0, s_i^- \ge 0 \quad \forall j,r,i
\end{aligned}
\tag{4.2}
$$

The units of measurement for the multipliers in model (4.1) are such that the virtual outputs $y_{rj}u_r$ and the virtual inputs $x_{ij}v_i$ are both dimensionless. In this manner outputs (inputs) with different units of measurement can be aggregated to a total virtual output (input), which is dimensionless as well. A typical interpretation of the multipliers u_r and v_i , as presented in section 3.2, is that they represent marginal values of output r and input i, respectively, with the efficiency measure E representing then the ratio of the total value of outputs to the total value of inputs, where the latter has been set to 1.

In the following, we illustrate that when the input/output data in the multiplier model (4.1) are normalized on the column maximum, the variables are altered in a manner that the derived DEA model, although structurally identical to the original one, does no longer make explicit reference to weights but it does make direct reference to worth instead. To facilitate the presentation, the following notations and transformations, related to the outputs first, are introduced. Let $l_r = \min_j \{y_{rj}\}$ and $h_r = \max_j \{y_{rj}\}$ be the lowest and the highest observed values for output r over the entire set of units, with $l_r > 0$ (strict positivity assumption). Then, $y_{rj} \in [l_r, h_r]$ for every unit j=1,…,n, with at least one unit having its output r at the level h_r. Let u_r be the optimal multiplier assigned in model (4.1) to the output measure Y_r by the evaluated unit j (represented by the slope of the line OA in Figure 4.1) and $p_{rj} = y_{rj}u_r$ the corresponding virtual output estimated in favor of unit j. When the optimal multiplier u_r is applied to the unit exhibiting the highest output h_r, it assigns to h_r the highest value $p_r = h_r u_r$. For these two value estimates the following holds (see Figure 4.1):

$$p_{rj} = \frac{y_{rj}}{h_r} p_r$$

or

$$p_{rj} = \hat{y}_{rj} p_r \text{, where } \hat{y}_{rj} = \frac{y_{rj}}{h_r} \tag{4.3}$$

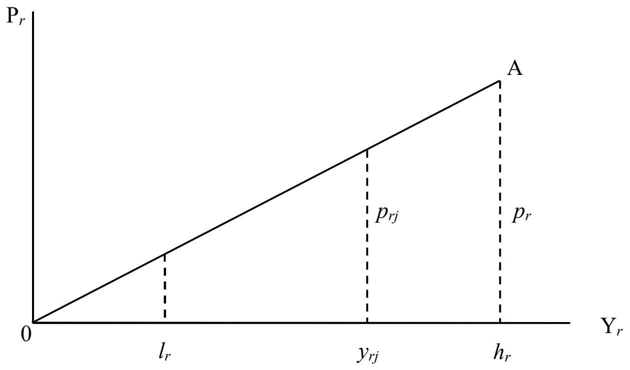


Figure 4.1: Value estimates for output measure *Y_r*

The treatment of inputs is analogous. Indeed, if $l_i = \min_j \{x_{ij}\} > 0$, $h_i = \max_j \{x_{ij}\}$, v_i is the optimal weight assigned to the input measure X_i by the evaluated unit *j*, and $q_{ij} = x_{ij} v_i$ is the associated virtual input for unit *j*, then the value assigned to the highest observed input h_i is $q_i = h_i v_i$ and

$$q_{ij} = \frac{x_{ij}}{h_i} q_i$$

or

$$q_{ij} = \hat{x}_{ij} q_i \text{, where } \hat{x}_{ij} = \frac{x_{ij}}{h_i} \tag{4.4}$$

Introducing the transformations (4.3) and (4.4) in models (4.1) and (4.2) the following couple of *max-normalized* DEA models is obtained:

Multiplier form:

$$\begin{aligned}
&\max E(p,q,j_0)=\sum_{r=1}^{s}\hat{y}_{rj_0}p_r\\
&s.t.\\
&\sum_{i=1}^{m}\hat{x}_{ij_0}q_i=1\\
&\sum_{r=1}^{s}\hat{y}_{rj}p_r-\sum_{i=1}^{m}\hat{x}_{ij}q_i\le 0,\quad j=1,\dots,n\\
&p_r,q_i\ge 0\quad \forall r,i
\end{aligned}\tag{4.5}$$

Envelopment form:

$$\begin{aligned}
&\min\theta\\
&s.t.\\
&\sum_{j=1}^{n}\hat{y}_{rj}\lambda_j-\frac{1}{h_r}s_r^{+}=\hat{y}_{rj_0}\qquad r=1,\dots,s\\
&\theta\hat{x}_{ij_0}-\sum_{j=1}^{n}\hat{x}_{ij}\lambda_j-\frac{1}{h_i}s_i^{-}=0\quad i=1,\dots,m\\
&\lambda_j\ge 0,s_r^{+}\ge 0,s_i^{-}\ge 0\quad \forall j,r,i
\end{aligned}\tag{4.6}$$

Typically, the model (4.5) is obtained by max-normalizing the raw data y_{rj} and x_{ij} and altering the variables from u_r and v_i to p_r and q_i respectively, according to the transformations (4.3) and (4.4). Structurally, the models (4.1) and (4.5) are identical, the meaning, however, of the coefficients and the variables are quite different. Indeed, $\hat{y}_{rj}$ is dimensionless and represents the performance of unit *j* on the output *r*, as a proportion of the maximum observed output *r*. The variable p_r represents the worth of the maximum observed output *r*. Thus, the term $\hat{y}_{rj}p_r$ represents the worth of the output y_{rj} as a proportion of p_r. The interpretations for $\hat{x}_{ij}$ and q_i are analogous. Thus the *weighting* variables v_i and u_r are altered to the *worth* variables q_i and p_r respectively. Applying a max-normalization without conceiving this alteration in the meaning of the variables will lead to erroneous interpretations of the results.

Lemma 4.1 and Theorem 4.1 below show the equivalence of the original DEA model (4.1) and the *max-normalized* model (4.5).

***Lemma 4.*1**

a) $p=(p_r, r=1,...,s)$, $q=(q_i, i=1,...,m)$ is a feasible solution to model (4.5) if and only if $u=(\frac{p_r}{h_r}=u_r, r=1,...,s)$, $v=\left(\frac{q_i}{h_i}=v_i, i=1,...,m\right)$ is a feasible solution to model (4.1);

b) $E(p,q,j)=E(u,v,j)$ for every feasible *p, q, u, v*.

Proof

a) Let $p=(p_r, r=1,...,s)$, $q=(q_i, i=1,...,m)$ be a feasible solution to the model (4.5). Setting $\tilde{x}_{ij}=\frac{x_{ij}}{h_i}$ and $\tilde{y}_{rj}=\frac{y_{rj}}{h_r}$, as in (4.3) and (4.4), the constraints of the model (4.5) become:

$$\sum_{i=1}^{m} x_{ij_0}\frac{q_i}{h_i}=1$$
$$\sum_{r=1}^{s} y_{rj}\frac{p_r}{h_r}-\sum_{i=1}^{m} x_{ij_0}\frac{q_i}{h_i}\le 0, \quad j=1,\dots,n$$
$$\frac{p_r}{h_r},\frac{q_i}{h_i}\ge 0 \quad \forall r,i$$

From the latter derives that the constraints of the model (4.1) are satisfied for $\frac{p_r}{h_r},\frac{q_i}{h_i}$, that is $u=\left(\frac{p_r}{h_r}=u_r, r=1,...,s\right)$, $v=\left(\frac{q_i}{h_i}=v_i, i=1,...,m\right)$ is a feasible solution to model (4.1). The proof of the inverse is straightforward.

b) $E(p,q,j)=\sum_{r=1}^{s}\tilde{y}_{rj}p_r=\sum_{r=1}^{s}\frac{y_{rj}}{h_r}p_r=\sum_{r=1}^{s}y_{rj}u_r=E(u,v,j)$, which completes the proof.▫

Theorem 4.1

$p^o = (p_r^o, r = 1,...,s)$, $q^o = (q_i^o, i = 1,...,m)$ is an optimal solution to model (4.5) if and only if $u^o = \left(\frac{p_r^o}{h_r} = u_r^o, r = 1,...,s\right)$, $v^o = \left(\frac{q_i^o}{h_i} = v_i^o, i = 1,...,m\right)$ is an optimal solution to model (4.1).

Proof

Let $p^o = (p_r^o, r = 1,...,s)$, $q^o = (q_i^o, i = 1,...,m)$ be an optimal solution to the model (4.5). Then

$$E(p^o, q^o, j_0) \geq E(p, q, j_0) \qquad \text{(A-1)}$$

for every feasible solution *p, q* of model (4.5) and, according to Lemma 4.1, $u^o = \left(\frac{p_r^o}{h_r} = u_r^o, r = 1,...,s\right), v^o = \left(\frac{q_i^o}{h_i} = v_i^o, i = 1,...,m\right)$ is a feasible solution to model (4.1) and

$$E(u^o, v^o, j_0) = E(p^o, q^o, j_0) \qquad \text{(A-2)}$$

Assume that $u^o = \left(\frac{p_r^o}{h_r} = u_r^o, r = 1,...,s\right)$, $v^o = \left(\frac{q_i^o}{h_i} = v_i^o, i = 1,...,m\right)$ is not an optimal solution to model (4.1). That is, there exists a feasible solution $u^* = \left(\frac{p_r^*}{h_r} = u_r^*, r = 1,...,s\right)$, $v^* = \left(\frac{q_i^*}{h_i} = v_i^*, i = 1,...,m\right)$ of the model (4.1) such that

$$E(u^*, v^*, j_0) > E(u^o, v^o, j_0) \qquad \text{(A-3)}$$

However, according to Lemma 4.1 $p^* = (p_r^*, r = 1,...,s), q^* = (q_i^*, i = 1,...,m)$ is a feasible solution to the model (4.1) and

$$E(p^*, q^*, j_0) = E(u^*, v^*, j_0) \qquad \text{(A-4)}$$

Then, from (A-2), (A-3) and (A-4) derives that $E(p^*, q^*, j_0) > E(p^o, q^o, j_0)$ which contradicts (A-1). The proof of the inverse is straightforward and thus omitted.▫

Theorem 4.1 shows that models (4.1) and (4.5) are equivalent, in the sense that they provide the same efficiency scores for the evaluated units and an optimal solution of model (4.1) is generated from an optimal solution of model (4.5) and vice versa. Thus, if $p_r^o, r=1,...,s$, $q_i^o, i=1,...,m$ is an optimal solution of model (4.5), optimal multipliers in terms of model (4.1) are recovered by the relations:

$$u_r^o = \frac{p_r^o}{h_r}, r=1,...,s$$
$$v_i^o = \frac{q_i^o}{h_i}, i=1,...,s$$

As in the standard DEA, the optimal solution $p_r^o, r=1,...,s$, $q_i^o, i=1,...,m$ of model (4.5) and thus the recovered optimal multipliers $u_r^o, r=1,...,s$, $v_i^o, i=1,...,m$ are not necessarily unique.

The equivalence of the envelopment forms (4.2) and (4.6) is straightforward. Indeed, they have the same objective function and the constraints related to the outputs r=1,…,s in model (4.6) derive by multiplying the terms in both sides of the corresponding constraints in model (4.2) with the positive value $1/h_r$. Similarly, the constraints of (4.6) that are related to the inputs i=1,…,m derive by multiplying the terms of the corresponding constraints of (4.2) with the positive number $1/h_i$. Thus, the models (4.2) and (4.6) have the same feasible and optimal solutions.

Given the transformations (4.3) and (4.4) the derivation of the BCC *max-normalized* models is straightforward as follows:

Multiplier form:

$$
\begin{aligned}
&\max E(p,q,p_0,j_0) = \sum_{r=1}^{s} \hat{y}_{rj_0} p_r + p_0 \\
&s.t. \\
&\sum_{i=1}^{m} \hat{x}_{ij_0} q_i = 1 \\
&\sum_{r=1}^{s} \hat{y}_{rj} p_r + p_0 - \sum_{i=1}^{m} \hat{x}_{ij} q_i \le 0, \quad j = 1,\dots,n \\
&p_r, q_i \ge 0 \quad \forall r,i \\
&p_0 \ \ free
\end{aligned}
\tag{4.7}
$$

Envelopment form:

$$
\begin{aligned}
&\min \theta \\
&s.t. \\
&\sum_{j=1}^{n} \hat{y}_{rj}\lambda_j - \frac{1}{h_r} s_r^+ = \hat{y}_{rj_0} \qquad r = 1,\dots,s \\
&\theta \hat{x}_{ij_0} - \sum_{j=1}^{n} \hat{x}_{ij}\lambda_j - \frac{1}{h_i} s_i^- = 0 \quad i = 1,\dots,m \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \ge 0, s_r^+ \ge 0, s_i^- \ge 0 \quad \forall j,r,i
\end{aligned}
\tag{4.8}
$$

4.2.2.1 Some properties of the max-normalized DEA models

Firstly, it is shown that one has nothing to lose by using the max-normalized DEA models instead of the original ones since all the information provided by the original models (4.1) and (4.2) can be recovered from the optimal solutions of models (4.5) and (4.6). The same applies for the max-normalized BCC models (4.7) and (4.8) as well. Then, the potential benefits of using the transformed models are discussed.

Recovery of optimal weights

As shown previously, the optimal weights in terms or model (4.1) can be easily recovered by the optimal solution of model (4.5).

Recovery of efficient projections

The efficient projections $(\hat{x}'_{ij_0}, \hat{y}'_{rj_0})$ in terms of model (4.6) are:

$$\hat{x}'_{ij_0} = \theta^* \hat{x}_{ij_0} - \frac{1}{h_i} s_i^{-*} \qquad \hat{y}'_{rj_0} = \hat{y}_{rj_0} + \frac{1}{h_r} s_r^{+*}$$

where θ^* is the optimal value of the objective function in (4.6) obtained in phase I of the two-phase procedure typically used to solve DEA models and s_i^{-*}, s_r^{*+} are the optimal slacks obtained from the max-slack solution of phase II. Multiplying the first equation with h_i and the second one with h_r we get the efficient projections in terms of the original model (4.2) as follows:

$$h_i \hat{x}'_{ij_0} = \theta^* x_{ij_0} - s_i^{-*} = x'_{ij_0} \qquad h_r \hat{y}'_{rj_0} = y_{rj_0} + s_r^{+*} = y'_{rj_0}$$

Restrictions on weights Vs restrictions on worth

Table 4.1 depicts various types of restrictions as stated in terms of weights in model (4.1) and how these constraints should be translated in terms of the max-normalized model (4.5), where $k, l, r \in \{1,...,s\}$, $i \in \{1,...,m\}$, $j \in \{1,...,n\}$ and a, b and w with the appropriate indices are user defined parameters.

Inversely, Table 4.2 shows how the restrictions stated originally in terms of the model (4.5) should be translated to apply in model (4.1).

Table 4.1: Translation of weight restrictions to worth restrictions:

	Stated in terms of weights in model (4.1)	Translated in terms of values in model (4.5)
Absolute restrictions	$a_r \leq u_r \leq b_r$	$a_r h_r \leq p_r \leq b_r h_r$
Assurance region Type I	$a_{kl} \leq \frac{u_k}{u_l} \leq b_{kl}$	$a_{kl}\frac{h_k}{h_l} \leq \frac{p_k}{p_l} \leq b_{kl}\frac{h_k}{h_l}$
	$w_r u_r + w_k u_k \leq u_l$	$\frac{w_r}{h_r} p_r + \frac{w_k}{h_k} p_k \leq \frac{1}{h_l} p_l$
Assurance region Type II	$a_i v_i \geq u_r$	$\frac{a_i}{h_i} q_i \geq \frac{1}{h_r} p_r$
Restrictions on virtual outputs	$a_r \leq \frac{y_{rj} u_r}{\sum_{r=1}^{s} y_{rj} u_r} \leq b_r$	$a_r \leq \frac{\hat{y}_{rj} p_r}{\sum_{r=1}^{s} \hat{y}_{rj} p_r} \leq b_r$

Table 4.2: Translation of worth restrictions to weight restrictions

	Stated in terms of values in model (4.5)	Translated in terms of weights in model (4.1)
Absolute restrictions	$a_r \leq p_r \leq b_r$	$\frac{a_r}{h_r} \leq u_r \leq \frac{b_r}{h_r}$
Assurance region Type I	$a_{kl} \leq \frac{p_k}{p_l} \leq b_{kl}$	$a_{kl}\frac{h_l}{h_k} \leq \frac{u_k}{u_l} \leq b_{kl}\frac{h_l}{h_k}$
	$w_r p_r + w_k p_k \leq p_l$	$w_r h_r u_r + w_k h_k u_k \leq h_l u_l$
Assurance region Type II	$a_i q_i \geq p_r$	$a_i h_i v_i \geq h_r u_r$
Restrictions on virtual outputs	$a_r \leq \frac{\hat{y}_{rj} p_r}{\sum_{r=1}^{s} \hat{y}_{rj} p_r} \leq b_r$	$a_r \leq \frac{y_{rj} u_r}{\sum_{r=1}^{s} y_{rj} u_r} \leq b_r$

Below, some other properties of the max-normalized DEA models are discussed, that can be regarded as advantages when using them instead of the original ones.

Units invariance

As mentioned in Lovell and Pastor (1995) the CCR and the BCC DEA models are not fully units invariant. The radial component of the efficiency scores obtained from these models is units invariant, whereas the slack component obtained by the max-slack solution is not. With their Theorem 4 and Corollary 2, Lovell and Pastor (1995) showed that the oriented weighted normalized CCR and BCC models are units invariant. They based their proof on the max-slack formulation of phase II by weighting the slacks with the inverse of the sample standard deviations of the output and the input variables, i.e. by normalizing the slacks on the sample standard deviations. Moreover, they pointed out that any other first order dispersion measures could be used as well to normalize the slacks. In the light of these findings, it is a direct implication that the oriented max-normalized CCR model (4.6) and its BCC counterpart (4.8) are units invariant.

Dimensionality

Both the data and the variables in the max-normalized DEA models (4.5)-(4.8) are dimensionless (units free). As the raw input/output data are normalized on the column maxima, any unbalance caused by units of measurement of different order of magnitude is eliminated.

Managerial implications

Concerning the multiplier model, the original model (4.1) makes explicit reference to *weights*, whereas the transformed model (4.5) makes reference to the worth of the column maxima. Thus the eventual difficulty in conceiving the nature and the meaning of the weights is bypassed when using the max-normalized DEA models and any preferential information originally stated in terms of weights (weight restrictions) can be equivalently and effectively be provided by the analyst in terms of worth.

4.3 Extension to DEA Models with non-linear virtual inputs and outputs

Piece-wise linear DEA (PL-DEA) is an extension of standard DEA dealing with cases where the partial value functions (virtual outputs/inputs) are assumed non-linear and are represented in a piece-wise linear form. In 4.3.1 a short review on PL-DEA is provided where we spot a discontinuity issue observed at the breakpoints of the value functions. In 4.3.2 we re-formulate the PL-DEA approach, in a manned that it provides a meaningful interpretation of the variables and eliminates the aforementioned discontinuity defect.

4.3.1 Piece-wise linear DEA

PL-DEA was first introduced by Cook and Zhu (2009) to handle Decreasing Marginal Values (DMV) and/or Increasing Marginal Values (IMV) in certain outputs in an application that measures the efficiency of maintenance patrols in the province of Ontario, Canada. Cook et al. (2009) further extended PL-DEA in the additive model for inputs with diminishing values. Despotis et al. (2010) provided a general CCR modeling approach for the efficiency assessment in the presence of non-linear virtual inputs and outputs in terms of assurance region constraints to implement concave output and convex input value functions. To illustrate their approach, they revisited a previous work of Despotis (2005) dealing with the assessment of the human development index in the light of DEA. Furthermore, Lofti et al. (2010) noticed that the PL-DEA model fails to produce acceptable targets so they revised the PL-DEA by proposing a two stage CCR modeling that handles the problem of setting the targets of the units precisely. PL-DEA has been also adapted to interval DEA (Smirlis and Despotis, 2013), i.e. to cases where the input/output data are only known to lie within intervals with given bounds. The authors defined appropriate interval segmentations to implement the piece-wise linear forms in conjunction with the interval bounds of the input/output data. PL-DEA has been also used as the background technique in Smirlis and Despotis (2012) to handle extreme observations (those that exhibit irregularly high values in some outputs and/or low values in some inputs) in DEA, instead of removing them from the analysis. Their modeling approach assumed that the contribution of output dimensions that show extreme values, to the efficiency score diminishes as the output increases beyond a pre-specified level. Using such pre-specified threshold levels as breakpoints, they applied the PL-DEA concept of diminishing returns to implement piece-wise concave value functions.

Let $Y_j = (y_{1j}, y_{2j}, ..., y_{sj})$ and $X_j = (x_{1j}, x_{2j}, ..., x_{mj})$ denote respectively the vectors of outputs and inputs for unit *j* in model (4.1). Then, $U_r(y_{rj}) = y_{rj}u_r$, $r = 1,...,s$ and $U_i(x_{ij}) = x_{ij}v_i$, $i = 1,...,m$ are the virtual outputs and inputs for unit *j* respectively, whereas the summations $\sum_{r=1}^{s} y_{rj}u_r = \sum_{r=1}^{s} U_r(y_{rj}) = U(Y_j)$ and $\sum_{i=1}^{m} x_{ij}v_i = \sum_{i=1}^{m} U_i(x_{ij}) = U(X_j)$ represent the total virtual output and input respectively for unit *j,* which are linear functions of the weights.

To deal with cases where the marginal value of an output diminishes as the output increases, Despotis et al. (2010) relaxed the linearity assumption in DEA by modeling the overall value of the output vector Y_j as an additive value function $U(Y_j) = U_1(y_{1j}) + U_2(y_{2j}) + ... + U_s(y_{sj})$ of piece-wise linear partial value functions. The interval $[l_r, h_r]$, where $l_r = \min_j \{y_{rj}\}$ and $h_r = \max_j \{y_{rj}\}$, is split into successive and non-overlapping segments by taking a number of breakpoints. Then, a different weight variable is assigned to each segment. Restrictions on the weights are then imposed to drive the concavity or the convexity of the value functions.

For the sake of simplicity, it is assumed here only one breakpoint b_r that splits the range of values of output *r* in two sub-intervals $[l_r, b_r]$ and $(b_r, h_r]$. On the basis of this segmentation, the output value $y_{rj} \in [l_r, h_r]$ of any unit *j* is decomposed in two parts and is expressed as $y_{rj} = \delta_{rj}^1 + \delta_{rj}^2$, where:

$$\delta_{rj}^1 = \begin{cases} y_{rj} & if \quad y_{rj} \le b_r \\ b_r & if \quad y_{rj} > b_r \end{cases} \qquad \delta_{rj}^2 = \begin{cases} 0 & if \quad y_{rj} \le b_r \\ y_{rj} - b_r & if \quad y_{rj} > b_r \end{cases} \tag{4.9}$$

In this manner, the partial value $U_r(y_{rj})$ is modeled in a piece-wise linear form as follows:

$$U_r(y_{rj}) = u_{r1}\delta^1_{rj} + u_{r2}\delta^2_{rj} \tag{4.10}$$

where u_{r1} and u_{r2} are the distinct weights associated with the two sub-intervals.

In general, the non-linearity assumption is applicable or desirable for particular outputs only (*non-linear outputs)*, with the rest of them complying with the linearity assumption. Without loss of generality, it can be assumed that the first *d* (*d*<*s*) outputs are linear and the rest of them (i.e. for $r = d+1,...,s$) are non-linear. Then, the total virtual output takes the following form:

$$U(Y_j) = \sum_{r=1}^{d} u_r y_{rj} + \sum_{r=d+1}^{s} (u_{r1}\delta^1_{rj} + u_{r2}\delta^2_{rj})$$

The virtual inputs are modeled analogously. Indeed, if $[l_i, h_i]$ is the interval defined by the minimum and the maximum values of input *i* and a_i is the breakpoint that splits this interval in two segments $[l_i\ ,\ a_i]$ and $(a_i\ ,\ h_i]$, the input value $x_{ij} \in [l_i, h_i]$ of any unit *j* is decomposed in two parts $x_{ij} = \gamma^1_{ij} + \gamma^2_{ij}$ where:

$$\gamma^1_{ij} = \begin{cases} x_{ij} \ \ if\ x_{ij} \le a_i \\ a_i \ \ if\ x_{ij} > a_i \end{cases} \qquad \gamma^2_{ij} = \begin{cases} 0 \qquad\quad if\ x_{ij} \le a_i \\ x_{ij} - a_i \ \ if\ x_{ij} > a_i \end{cases} \tag{4.11}$$

The virtual input $U_i(x_{ij})$ is then modeled as a piece-wise linear function:

$$U_i(x_{ij}) = v_{i1}\gamma^1_{ij} + v_{i2}\gamma^2_{ij} \tag{4.12}$$

where v_{i1} and v_{i2} are the input weights associated with the two sub-intervals. The total virtual input is then given by the following equation:

$$U(X_j) = \sum_{i=1}^{t} v_i x_{ij} + \sum_{i=t+1}^{m} (v_{i1}\gamma^1_{ij} + v_{i2}\gamma^2_{ij})$$

where the first *t* inputs are assumed linear and the rest of them non-linear. Imposing the homogeneous restrictions $u_{r1} - c_r u_{r2} \ge 0$ ($c_r \ge 1$) on the weights u_{r1} and u_{r2}, the value function (4.10) is restricted to be concave. Similarly, the relations $-v_{i1} + z_i v_{i2} \ge 0$

$(0 < z_i \leq 1)$, on the weights v_{i1} and v_{i2}, restrict the value function (4.12) to be convex.

Figure 4.2 presents the concave shape of the non-linear function U_r for a typical non-linear output y_r. Note that the function U_r shows discontinuity at the breakpoint value b_r. This is due to fact that the weights u_{r1} and u_{r2} applied on the successive sub-intervals $[0, b_r]$ and $(b_r, h_r]$ may be different. The same discontinuity issue also holds for the non-linear function U_i of the non-linear input x_i depicted in Figure 4.3. This is a defect of PL-DEA.

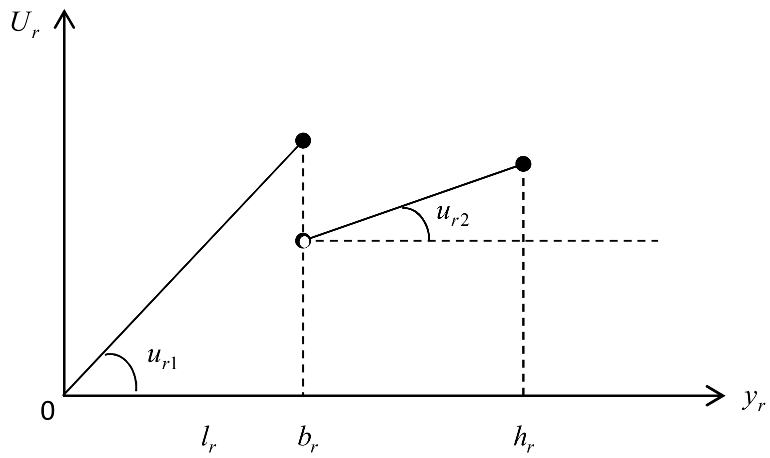


Figure 4.2: Concave form for the non-linear output *Y_r*

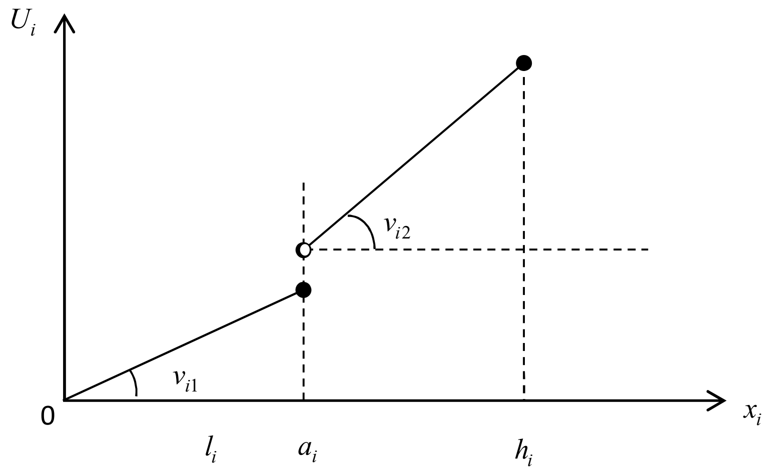


Figure 4.3: Convex form for the non-linear input *X_i*

The formulations presented above actually transform the original data set into an augmented data set by decomposing each one of the non-linear inputs and outputs in two auxiliary linear inputs and linear outputs respectively. This transformation allows performing the efficiency assessments without drawing away from the ground of the standard DEA methodology. Model (4.13) below is a piece-wise linear DEA model with weight restrictions imposing concave value functions for outputs and convex value functions for inputs. As the inputs are in the denominator of the efficiency ratio, convex value functions penalize the excess inputs.

$$
\begin{aligned}
&\max E(u,v,j_0)=\sum_{r=1}^{d} y_{rj_0}u_r+\sum_{r=d+1}^{s}(\delta^1_{rj_0}u_{r1}+\delta^2_{rj_0}u_{r2})\\
&s.t.\\
&\sum_{i=1}^{t} x_{ij_0}v_i+\sum_{i=t+1}^{m}(\gamma^1_{ij_0}v_{i1}+\gamma^2_{ij_0}v_{i2})=1\\
&\sum_{r=1}^{d} y_{rj}u_r+\sum_{r=d+1}^{s}(\delta^1_{rj}u_{r1}+\delta^2_{rj}u_{r2})-\sum_{i=1}^{t} x_{ij}v_i-\sum_{i=t+1}^{m}(\gamma^1_{ij}v_{i1}+\gamma^2_{ij}v_{i2})\le 0 \quad j=1,\dots,n\\
&u_{r1}-c_r u_{r2}\ge 0,\ r=d+1,\dots,s \quad (c_r\ge 1)\\
&-v_{i1}+z_i v_{i2}\ge 0,\ i=t+1,\dots,m \quad (0<z_i\le 1)\\
&u_r,v_i\ge 0 \quad r=1,\dots,d\ ;\ i=1,\dots,t\\
&u_{r1},u_{r2},v_{i1},v_{i2}\ge 0 \quad r=d+1,\dots,s\ ;i=t+1,\dots,m
\end{aligned}
\tag{4.13}
$$

4.3.2 Reformulation of Piece-wise linear DEA

We revisit, in the following, the work of Despotis et al. (2010) to provide an alternative, yet effective formulation of DEA models with non-linear partial value functions.

Applying the data rescaling-variable alteration technique presented in the previous section on (4.9) and (4.11) we get respectively

$$
\hat{\delta}^1_{rj}=\frac{\delta^1_{rj}}{b_r}=\begin{cases}\hat{y}_{rj}=\dfrac{y_{rj}}{b_r} & if\ y_{rj}\le b_r\\ 1 & if\ y_{rj}>b_r\end{cases}, \qquad \hat{\delta}^2_{rj}=\frac{\delta^2_{rj}}{h_r-b_r}=\begin{cases}0 & if\ \ y_{rj}\le b_r\\ \dfrac{y_{rj}-b_r}{h_r-b_r} & if\ \ y_{rj}>b_r\end{cases}
$$

and

$$\hat{\gamma}_{ij}^{1} = \frac{\gamma_{ij}^{1}}{a_i} = \begin{cases} \hat{x}_{ij} = \dfrac{x_{ij}}{a_i} & if\ x_{ij} \le a_i \\ 1 & if\ x_{ij} > a_i \end{cases}, \qquad \hat{\gamma}_{ij}^{2} = \frac{\gamma_{ij}^{2}}{h_i - a_i} = \begin{cases} 0 & if\ x_{ij} \le a_i \\ \dfrac{x_{ij} - a_i}{h_i - a_i} & if\ x_{ij} > a_i \end{cases}$$

Figure 4.4 depicts a piece-wise linear value function for a non-linear output measure U_r decomposed in two segments. With the above transformations, the weight variables u_{r1} and u_{r2}, which represent respectively the slopes of the line segments OA and AB, are replaced by the value variables p_{r1} and p_{r2}, which represent the value increments in the intervals $\left[0, b_r\right]$and $\left(b_r, \mathrm{h}_r\right]$ respectively.

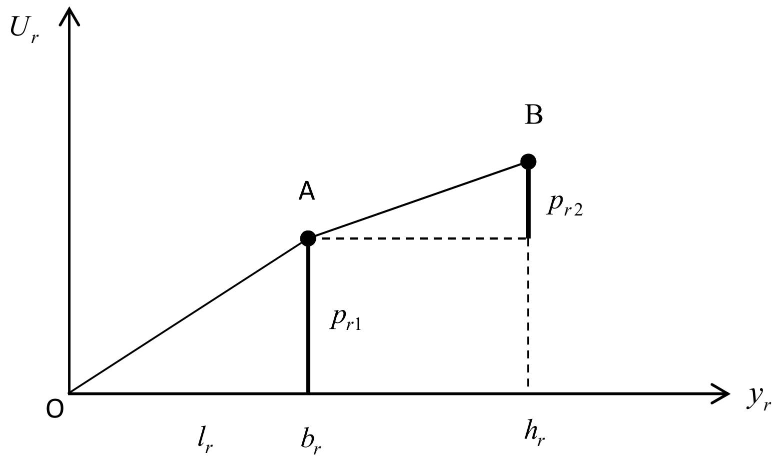


Figure 4.4: Value function for a non-linear output measure *Y_r*

Analogously, Figure 4.5 illustrates a piece-wise linear value function for a non-linear input measure x_i decomposed in two segments. With the above transformations, the weight variables v_{i1} and v_{i2}, which represent respectively the slopes of the line segments OE and EF, are replaced by the value variables q_{i1} and q_{i2}, which represent the value increments in the intervals $\left[0, \alpha_i\right]$and $\left(\alpha_i, \mathrm{h}_i\right]$ respectively.

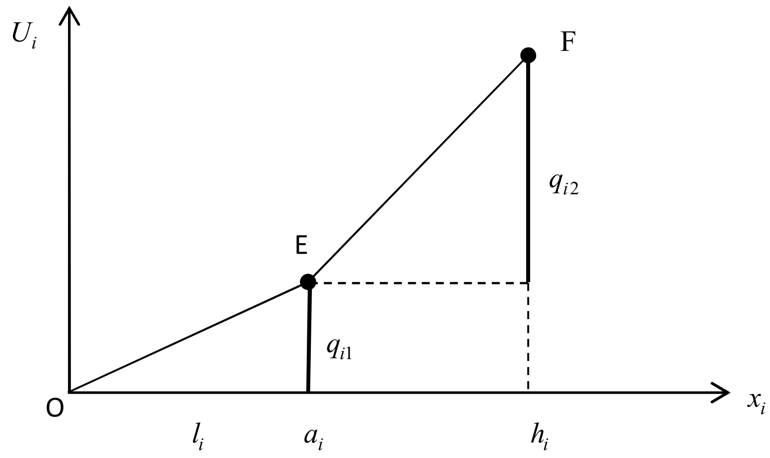


Figure 4.5: Value function for a non-linear input measure *X_i*

Model (4.13) is now transformed to the following model:

$$\begin{aligned}
&\max E(p,q,j_0)=\sum_{r=1}^{d}\hat{y}_{rj_0}p_r+\sum_{r=d+1}^{s}(\hat{\delta}^1_{rj_0}p_{r1}+\hat{\delta}^2_{rj_0}p_{r2})\\
&s.t.\\
&\sum_{i=1}^{t}\hat{x}_{ij_0}q_i+\sum_{i=t+1}^{m}(\hat{\gamma}^1_{ij_0}q_{i1}+\hat{\gamma}^2_{ij_0}q_{i2})=1\\
&\sum_{r=1}^{d}\hat{y}_{rj}p_r+\sum_{r=d+1}^{s}(\hat{\delta}^1_{rj}p_{r1}+\hat{\delta}^2_{rj}p_{r2})-\sum_{i=1}^{t}\hat{x}_{ij}q_i-\sum_{i=t+1}^{m}(\hat{\gamma}^1_{ij}q_{i1}+\hat{\gamma}^2_{ij}q_{i2})\le 0 \quad j=1,...,n\\
&(h_r-b_r)p_{r1}-b_rc_rp_{r2}\ge 0, r=d+1,...,s \quad (c_r\ge 1)\\
&-(h_i-a_i)q_{i1}+a_iz_iq_{i2}\ge 0\,, i=t+1,...,m \quad (0<z_i\le 1)\\
&p_r,q_i\ge 0, \quad r=1,...,d\ ; i=1,...,t\\
&p_{r1},p_{r2},q_{i1},q_{i2}\ge 0, \quad r=d+1,...,s\ ; i=t+1,...,m
\end{aligned} \tag{4.14}$$

In model (4.14) the new variables p_r, p_{r1} and p_{r2} for outputs and q_i, q_{i1} and q_{i2} for inputs represent worth as opposed to the variables u_r, u_{r1}, u_{r2}, v_i, v_{i1} and v_{i2} of model (4.13), which represent weights. Due to these variable transformations, the weight restrictions of (4.13) are transformed in (4.14) as well to impose concavity for the non-linear outputs and convexity for the non-linear inputs in terms of worth. Model (4.14) is a *max-normalized* DEA model with piece-wise linear value functions of inputs and outputs that is equivalent to model (4.13), in the sense that both provide the same efficiency scores and the optimal solution of the one can be generated from the optimal solution of the other. The equivalence is a direct implication of the Theorem 4.1 given in the previous section. Moreover, model (4.14) has all the additional properties discussed in the previous section concerning dimensionality and units invariance.

As spotted in 4.3.1, the augmentation of the dataset for non-linear outputs/inputs and the assignment of a distinct weight variable to each segment causes discontinuity in the value functions. However, applying the data transformation-variable alteration technique, introduced in 4.2.2, fixes this irregularity as illustrated in Figures 4.4 and 4.5.

4.4 Value based DEA

Incorporating value judgments in DEA is a broad methodological framework that facilitates driving the efficiency assessments in line with individual preferences. Value based DEA is a recent development that resorts to value assessment protocols from multiple criteria decision analysis (MCDA) to transform the original input/output data to a value scale. In this context, we introduce in this section a novel piece-wise linear programming approach to value based DEA, which employs a data transformation-variable alteration technique and assurance region constraints. In 4.4.1 we provide a brief review on the links between DEA and MCDA spotted in the literature and we highlight the motivations for the development of a new approach. The new approach is developed in section 4.4.2.

4.4.1 Links between MCDA and DEA

Multi-Criteria Decision Analysis (MCDA) has developed many concepts and protocols to elicit and utilize the analyst's preferences. Several authors have contributed in building bridges between DEA and MCDA. Joro et al. (1998) and Halme et al. (1999) related DEA with multi-objective linear programming. Bouyssou (1999), Doyle and Green (1993) and Stewart (1996) also connected DEA and discrete multiple criteria problems. Athanassopoulos and Podinovski (1997) spotted relations between DEA and MCDA with partial information on weights.

Gouveia et al. (2008) provided a link between DEA and MCDA. They treated DMUs as decision alternatives in terms of MCDA, which they are evaluated on criteria which correspond to the inputs and outputs in DEA models. In order to incorporate user's preferences in their hybrid assessment model, they employed the additive model using concepts from multi-attribute utility theory with imprecise information. Actually, they proposed the conversion of the input and output factors into utility functions, which were aggregated additively. Then, they minimize the loss of value of the evaluated unit relatively to the best unit, obtained for the evaluated DMU's optimal weights.

Almeida and Dias (2012) developed the methodology of Gouveia et al. (2008) in the context of a real-world application. Similarly to Gouveia et al. (2008), they used preference elicitation protocols drawn from the MCDA in the frame of the weighted additive DEA model (Ali et al., 1995), as a mean to incorporate user preferences in the DEA efficiency assessments. Their approach unfolds in three phases as follows:

Phase 1:

The raw values of the observed inputs and outputs are mapped onto the value interval $[0,1]$. That is, the inputs and the outputs, as measured in their original scales, are converted into a value scale, by assuming either linear or non-linear value functions *V*. By this transformation, all factors are treated as outputs to be maximized: $V_j(X_j,Y_j)=(v_{1j},v_{2j},...,v_{mj},v_{m+1,j},...,v_{m+s,j})$. The overall value of unit j is given in the additive form

$$U_j[V_j(X_j,Y_j)]=\sum_{k=1}^{m+s} v_{kj}w_k$$

The weights $w_k,\ k=1,2,...,m+s$ are dimensionless scaling constants. Optimal weights are calculated for each individual unit j in phase 2.

Phase 2:

The following linear program is solved for a unit j_0 at a time:

$$\begin{aligned}
&\min\ d \\
&\text{s.t.} \\
&\sum_{k=1}^{m+s} w_k v_{kj} - \sum_{k=1}^{m+s} w_k v_{kj_0} \leq d \quad (j = 1,\ldots,n) \\
&\sum_{k=1}^{m+s} w_k = 1 \\
&(w_1, w_2, \ldots, w_{m+s}) \in W
\end{aligned} \tag{4.15}$$

where W denotes the set of intra-weight constraints reflecting the user's preferences. By convention, the weights are normalized so as to sum up to 1. Model (4.15) estimates for unit j_o an optimal vector of weights $(w_1^{j_0}, w_2^{j_0}, \ldots, w_{m+s}^{j_0})$ that minimizes, in the min-max sense, the loss of value to the best unit. Let d^{j_0} denote the optimal value of d in the optimal solution of (4.15). Then, if $d^{j_0} = 0$ and $w_k^{j_0} > 0, k = 1,\ldots,m+s$ for at least one optimal solution of (4.15), the unit j_0 is characterized as efficient. Otherwise, it is inefficient.

Phase 3:

The following linear program is solved for every inefficient unit j_0 to find its projection on the efficient frontier:

$$\begin{aligned}
&\max \quad z = \sum_{k=1}^{m+s} w_k^{j_0} s_k \\
&s.t. \\
&\sum_{j=1}^{n} v_{kj}\lambda_j - s_k = v_{kj_0} \quad (k = 1,\ldots,m+s) \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0 \ (j = 1,\ldots,n),\ s_k \geq 0 \quad (k = 1,\ldots,m+s)
\end{aligned} \tag{4.16}$$

Model (4.16) is the envelopment form of a weighted additive DEA model, where only outputs are considered. For the optimal values d^{j_0} and z^{j_0} of the objective functions of models (4.15) and (4.16) holds that $d^{j_0} = z^{j_0}$.

As mentioned above, the weights w_k, $k = 1,2,...,m+s$ are scaling constants, estimated for each unit at its best advantage in phase 2. In Almeida and Dias (2012) and Gouveia et al. (2008), these weights are generally interpreted as "*value trade-offs for the client*". To be exact, as long as each unit is left free to define its own (optimal) weights in phase 2, these value trade-offs differ from one unit to another and each time are estimated in favor of the evaluated unit. A limitation of this approach, which in fact is attributed to the choice of the additive DEA model, is that no direct measure of efficiency is provided. It only discriminates the efficient and the inefficient units. These issues motivated the development of a novel approach which can provide a measure of efficiency and in which the aforementioned weights acquire a particular meaning and are easily interpreted. This approach is presented in 4.4.2.

4.4.2 A piece-wise linear programming approach to value based DEA

The data transformation – variable alteration technique, introduced in the previous sections, allow for the development of a general value based DEA model. This modelling approach facilitates the incorporation of value judgments while the efficiency assessments remain on the ground of DEA.

Consider n DMUs that use m inputs $\left(X_1, X_2, ..., X_m\right)$ to produce s outputs $\left(Y_1, Y_2, ..., Y_s\right)$. Given the output vector $Y_j = (y_{1j}, y_{2j}, ..., y_{sj})$ of unit j, its overall value $U\left(Y_j\right)$ is given by the additive value function:

$$U(Y_j) = \sum_{r=1}^{s} U_r(y_{rj})$$

As long as the higher the levels of the outputs the greater their values, the partial value functions U_r, $r = 1,...,s$ are assumed non-decreasing. Notice that these partial value functions are generalizations of the so called *virtual outputs* in the DEA context, which are typically assumed linear, with the *total virtual output* given by

$$U(Y_j) = \sum_{r=1}^{s} y_{rj} u_r$$

where u_r, $r = 1,...,s$ are the weights assigned to the outputs. As concerns the inputs, we define the overall value $V\left(X_j\right)$ of the input vector $X_j = (x_{1j}, x_{2j}, ..., x_{mj})$ by

$$V(X_j) = \sum_{i=1}^{m} V_i(x_{ij})$$

As the less the input level the highest its value, the partial value functions V_i, $i = 1,...,m$ of the individual inputs are assumed non-increasing. Notice, again, that in the original DEA models, the partial value functions of the inputs (*virtual inputs*) are assumed linear and the *total virtual input* is given by

$$V(X_j) = \sum_{i=1}^{m} x_{ij} v_i$$

where v_i, $i = 1,...,m$ are the weights assigned to the inputs. However, as $V\left(X_j\right)$ forms the denominator of the efficiency ratio, the individual virtual inputs are considered non-decreasing value functions, so as excess inputs are penalized. Assuming, for the developments, non-increasing value functions for the inputs allows to treat the inputs as outputs. With such an arrangement, the value based relative efficiency E_{j_0} of the evaluated unit j_0 is estimated by the following general model:

$$\begin{aligned} &\max E_{j_0} = \sum_{r=1}^{s} U_r(y_{rj_0}) + \sum_{i=1}^{m} V_i(x_{ij_0}) \\ &s.t. \\ &\sum_{r=1}^{s} U_r(y_{rj}) + \sum_{i=1}^{m} V_i(x_{ij}) \leq 1 \quad (j = 1,...,n) \end{aligned} \tag{4.17}$$

Model (4.17) is equivalent to an input oriented DEA model with $m + s$ outputs and a dummy input, set at the level of 1 for all the units. In a different context, this sort of a DEA-like model was introduced by Despotis (2005) as an index-maximizing model for the reassessment of the human development index (HDI) via DEA. Unlike the basic assumption permeating the original DEA, that the virtual inputs and outputs are linear functions of the weights, in this general approach, non-linear value functions are allowed whenever necessary. Relaxing the linearity assumption, allows treating cases where, for example, the marginal value of an output diminishes as the output increases.

Modeling the value functions

In general, the non-linearity requirement is desirable for particular outputs (inputs) only, with the rest of them complying with the linearity assumption. To distinguish them, the former are called *non-linear (NL) outputs (inputs)* and the latter *linear (L) outputs (inputs)*. Without loss of generality, it can be assumed that the first d $\left(r=1,...,d\right)$ outputs are linear and the rest of them (i.e. for $r=d+1,...,s$) are non-linear. Analogously, the first t $\left(i=1,...,t\right)$ inputs are assumed linear and the rest of them $\left(i=t+1,...,m\right)$ are assumed non-linear.

Linear outputs

Let $l_r \le \min_j\{y_{rj}\}$ and $h_r \ge \max_j\{y_{rj}\}$ be fixed minimum and maximum values for output *r*, set so as the range $[l_r, h_r] \supseteq [\min_j\{y_{rj}\}, \max_j\{y_{rj}\}]$ covers the observed outputs of the entire set of units. By convention, it is set $U_r(l_r)=0$. Then, the value of any $y_{rj} \in [l_r, h_r]$ is given by:

$$U_r(y_{rj}) = (y_{rj} - l_r)u_r$$

Notice, that the analyst can choose any value $l_r^{'}$ such as $0 \le l_r^{'} < l_r$ and $U_r(l_r^{'}) = 0$. This flexibility allow to the DM to introduce additional information in the assessment framework whereas the standard DEA models cannot incorporate.

Applying the following transformation:

$$U_r(y_{rj}) = (h_r - l_r)u_r \frac{y_{rj} - l_r}{h_r - l_r} = \hat{y}_{rj} p_r$$

the value of $y_{rj} \in [l_r, h_r]$ is obtained as function of the new variable p_r as:

$$U_r(y_{rj}) = \hat{y}_{rj} p_r \tag{4.18}$$

with

$$\hat{y}_{rj} = \frac{y_{rj} - l_r}{h_r - l_r}$$

From the above transformation derives that for any two output observations y_{rj} and y_{rk} holds that

$$y_{rj} \geq y_{rk} \Leftrightarrow U_r(y_{rj}) \geq U_r(y_{rk})$$

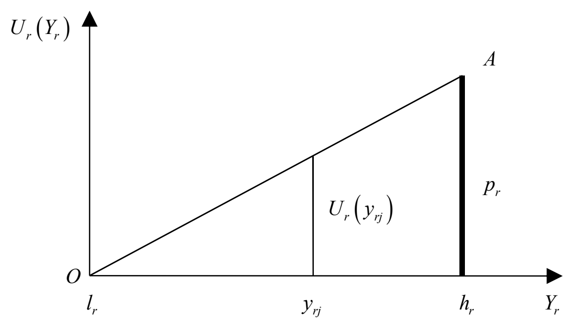


Figure 4.6: Value function for a linear output measure *Y*$_r$

As depicted in Figure 4.6, the above transformation alters the weight variable u_r, which represents the slope of the line OA, to the new variable p_r that represents the value of h_r. The coefficient $\hat{y}_{rj}$ is now dimensionless and the term $\hat{y}_{rj} p_r$ represents the value of the output y_{rj} as a proportion of p_r.

Non-linear outputs

The non-decreasing value functions for the non-linear outputs are modeled in a piece-wise linear form. To this end, k_r+1 breakpoints are assumed that split the range $[l_r, h_r]$ of the non-linear output r in k_r segments: $[b_r^1, b_r^2], [b_r^2, b_r^3], ..., [b_r^{k_r}, b_r^{k_r+1}]$, with $b_r^1 = l_r$ and $b_r^{k_r+1} = h_r$. By convention, it is set $U_r(l_r) = 0$. Then, any output $y_{rj} \in [l_r, h_r]$ can be decomposed as $y_{rj} = l_r + \delta_{rj}^1 + \delta_{rj}^2 + ... + \delta_{rj}^{k_r}$, where

$$\delta_{rj}^1 = \begin{cases} y_{rj} \text{-} b_r^1 & if \quad y_{rj} \le b_r^2 \\ b_r^2 \text{-} b_r^1 & if \quad y_{rj} > b_r^2 \end{cases}$$

$$\delta_{rj}^\mu = \begin{cases} 0 & if \quad y_{rj} \le b_r^\mu \\ y_{rj} - b_r^\mu & if \quad b_r^\mu < y_{rj} \le b_r^{\mu+1} \\ b_r^{\mu+1} - b_r^\mu & if \quad y_{rj} > b_r^{\mu+1} \end{cases} \quad , \ \mu = 2,3,...,k_r - 1 \tag{4.18a}$$

$$\delta_{rj}^{k_r} = \begin{cases} 0 & if \quad y_{rj} \le b_r^{k_r} \\ y_{rj} - b_r^{k_r} & if \quad b_r^{k_r} < y_{rj} \le b_r^{k_r+1} \end{cases}$$

Assuming that the value function is linear in each segment, a distinct weight variable $u_{r\mu}$ is assigned to each segment $\mu = 1,2,...,k_r$. Then, the partial value $U_r(y_{rj})$ is given in a piece-wise linear form as:

$$U_r(y_{rj}) = \delta_{rj}^1 u_{r1} + \delta_{rj}^2 u_{r2} + ... + \delta_{rj}^{k_r} u_{rk_r} = \sum_{\mu=1}^{k_r} \delta_{rj}^\mu u_{r\mu} \tag{4.19}$$

Applying to each segment the same transformation introduced for the linear outputs above, we get the value function (4.19) in terms of the new variables $p_{r1}, p_{r2}, ..., p_{rk_r}$ as follows:

$$U_r(y_{rj}) = \hat{\delta}_{rj}^1 p_{r1} + \hat{\delta}_{rj}^2 p_{r2} + ... + \hat{\delta}_{rj}^{k_r} p_{rk_r} = \sum_{\mu=1}^{k_r} \hat{\delta}_{rj}^\mu p_{r\mu} \tag{4.20}$$

with

$$\hat{\delta}_{rj}^{\mu} = \frac{\delta_{rj}^{\mu}}{b_r^{\mu+1} - b_r^{\mu}}, \; \mu = 1, 2, ..., k_r$$

It is straightforward from (4.20) that $U_r(h_r) = p_{r1} + p_{r1} + ... + p_{rk_r}$

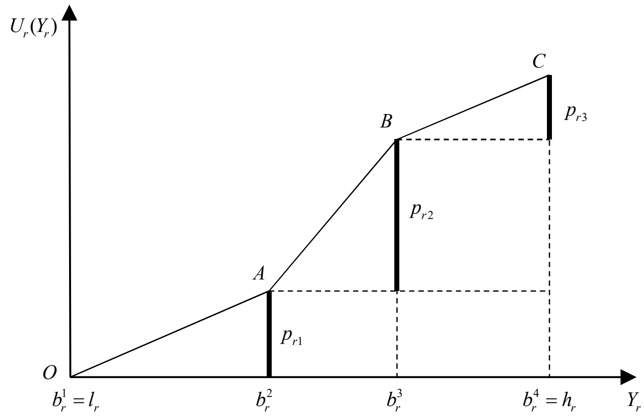


Figure 4.7: Value function for a non-linear output measure *Y_r*

Figure 4.7 depicts a piece-wise linear value function for a non-linear output measure Y_r decomposed in three segments. With the above transformations, the weight variables u_{r1}, u_{r2} and u_{r3}, which represent respectively the slopes of the line segments OA, AB and BC, are replaced by the value variables p_{r1}, p_{r2} and p_{r3}, which represent the value increments in the intervals $[b_r^1, b_r^2]$, $[b_r^2, b_r^3]$ and $[b_r^3, b_r^4]$ respectively.

Putting all together, i.e. the value functions of the linear and the non-linear outputs as given in (4.18) and (4.20) respectively, the value function (total virtual output) for the unit j is obtained, as follows:

$$U(Y_j) = \sum_{r=1}^{d} \hat{y}_{rj} p_r + \sum_{r=d+1}^{s} \sum_{\mu=1}^{k_r} \hat{\delta}_{rj}^{\mu} p_{r\mu} \qquad (4.21)$$

In equation (4.21), the first summation refers to linear outputs, whereas the second summation refers to non-linear outputs.

Linear inputs

As mentioned at the beginning of 4.4.2, the proposed value based modeling approach assumes non-increasing value functions for the inputs as a means to treat the inputs as outputs. Let $l_i \le \min_j\{x_{ij}\}$ and $h_i \ge \max_j\{x_{ij}\}$ be fixed minimum and maximum values for input *i*, set so as the range $[l_i, h_i] \supseteq [\min_j\{x_{ij}\}, \max_j\{x_{ij}\}]$ covers the observed inputs of the entire set of units. By convention, it is set $V_i(h_i) = 0$. Then, the value of any $x_{ij} \in [l_i, h_i]$ is given by:

$$V_i(x_{ij}) = (h_i - x_{ij})v_i$$

Applying the transformation:

$$V_i(x_{ij}) = (h_i - l_i)v_i \frac{h_i - x_{ij}}{h_i - l_i} = \hat{x}_{ij} q_i$$

the value of $x_{ij} \in [l_i, h_i]$ is obtained as function of the new variable q_i as:

$$V_i(x_{ij}) = \hat{x}_{ij} q_i \tag{4.22}$$

with

$$\hat{x}_{ij} = \frac{h_i - x_{ij}}{h_i - l_i}$$

From the above transformation derives that for any two input observations x_{ij} and x_{ik} holds that

$$x_{ij} \ge x_{ik} \Leftrightarrow V_i(x_{ij}) \le V_i(x_{ik})$$

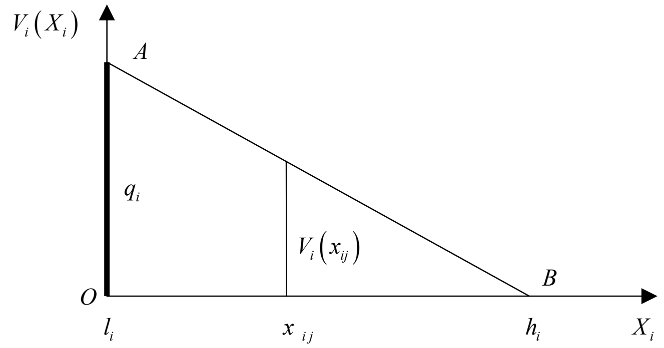


Figure 4.8: Value function for a linear input measure *X*$_i$

As depicted in Figure 4.8, the above transformation alters the weight variable v_i, which represents the tangent of the angle $O\hat{B}A$, to the new variable q_i that represents the value of the most preferred input level l_i. The coefficient $\hat{x}_{ij}$ is now dimensionless and the term $\hat{x}_{ij}q_i$ represents the value of the output x_{ij} as a proportion of q_i.

Non-linear inputs

The non-increasing value functions for the non-linear inputs are modeled analogously in a piece-wise linear form. Indeed, if k_i+1 is the number of breakpoints that split the interval $[l_i, h_i]$ in k_i segments $[a_i^1, a_i^2],[a_i^2, a_i^3],...,[a_i^{k_i}, a_i^{k_i+1}]$, with $a_i^1 = l_i$ and $a_i^{k_i+1} = h_i$, any input value $x_{ij} \in [l_i, h_i]$ is decomposed as $x_{ij} = h_i - (\gamma_{ij}^1 + \gamma_{ij}^2 + ... + \gamma_{ij}^{k_i})$ where:

$$\gamma_{ij}^1 = \begin{cases} 0 & if\ x_{ij} \geq a_i^2 \\ a_i^2 - x_{ij} & if\ a_i^1 \leq x_{ij} < a_i^2 \end{cases}$$

$$\gamma_{ij}^\mu = \begin{cases} 0 & if\ x_{ij} \geq a_i^{\mu+1} \\ a_i^{\mu+1} - x_{ij} & if\ a_i^\mu \leq x_{ij} < a_i^{\mu+1} \\ a_i^{\mu+1} - a_i^\mu & if\ x_{ij} < a_i^\mu \end{cases}, \mu = 2,3,...,k_i - 1 \tag{4.23}$$

$$\gamma_{ij}^{k_i} = \begin{cases} a_i^{k_i+1} - x_{ij} & if\ x_{ij} \geq a_i^{k_i} \\ a_i^{k_i+1} - a_i^{k_i} & if\ x_{ij} < a_i^{k_i} \end{cases}$$

Assigning a distinct weight variable $v_{i\mu}$ to each segment $\mu = 1, 2, ..., k_i$, the partial value $V_i(x_{ij})$ is given in a piece-wise linear form as:

$$V_i(x_{ij}) = \gamma_{ij}^1 v_{i1} + \gamma_{ij}^2 v_{i2} + ... + \gamma_{ij}^{k_i} v_{ik_i} = \sum_{\mu=1}^{k_i} \gamma_{ij}^{\mu} v_{i\mu} \tag{4.24}$$

Applying to each segment the same transformation introduced for the linear inputs, the value function (4.24) can be expressed in terms of the new variables $q_{i1}, q_{i2}, ..., q_{ik_i}$ as follows:

$$V_i(x_{ij}) = \hat{\gamma}_{ij}^1 q_{i1} + \hat{\gamma}_{ij}^2 q_{i2} + ... + \hat{\gamma}_{ij}^{k_i} q_{ik_i} = \sum_{\mu=1}^{k_i} \hat{\gamma}_{ij}^{\mu} q_{i\mu} \tag{4.25}$$

with

$$\hat{\gamma}_{ij}^{\mu} = \frac{\gamma_{ij}^{\mu}}{a_i^{\mu+1} - a_i^{\mu}}, \ \mu = 1, 2, ..., k_i$$

It is straightforward from (4.25) that $V_i(l_i) = q_{i1} + q_{i2} + ... + q_{ik_i}$

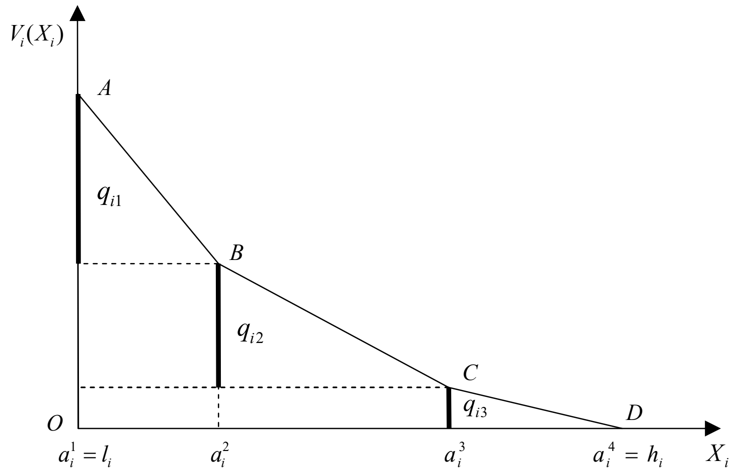


Figure 4.9: Value function for a non-linear input measure X_i

Figure 4.9 depicts a piece-wise linear value function for a non-linear input measure X_i decomposed in three segments. With the above transformations, the weight variables v_{i1}, v_{i2} and v_{i3}, which represent respectively the slopes of the line segments

AB, BC and CD, are replaced by the value variables q_{i1}, q_{i2} and q_{i3}, which represent the value decrements in the intervals $[a_i^1, a_i^2]$, $[a_i^2, a_i^3]$ and $[a_i^3, a_i^4]$ respectively.

Summing up (4.22) and (4.25), we get the value function (total virtual input) for the unit *j*, as follows:

$$V(X_j) = \sum_{i=1}^{t} \hat{x}_{ij} q_i + \sum_{i=t+1}^{m} \sum_{\mu=1}^{k_i} \hat{\gamma}_{ij}^{\mu} q_{i\mu} \tag{4.26}$$

In equation (4.26), the first summation refers to linear inputs, whereas the second summation refers to non-linear inputs.

Deriving the value based DEA model

Putting together the value functions for outputs and inputs given in (4.21) and (4.26) respectively, we get the overall value E_j of the unit *j* as a function of the value variables *p* and *q*:

$$E_j(p,q) = U(Y_j) + V(X_j) = \sum_{r=1}^{d} \hat{y}_{rj} p_r + \sum_{r=d+1}^{s} \sum_{\mu=1}^{k_r} \hat{\delta}_{rj}^{\mu} p_{r\mu} + \sum_{i=1}^{t} \hat{x}_{ij} q_i + \sum_{i=t+1}^{m} \sum_{\mu=1}^{k_i} \hat{\gamma}_{ij}^{\mu} q_{i\mu} \tag{4.27}$$

The non-linear value functions in (4.27) can be customized so as to acquire specific properties on the basis of individual preferences. This can be done by introducing restrictions on the variables $p_{r\mu}$, $\mu = 1,...,k_r$ and $q_{i\mu}$, $\mu = 1,...,k_i$. For example, homogeneous restrictions of the form

$$(b_r^{\mu+2} - b_r^{\mu+1}) p_{r\mu} - c_{r\mu}(b_r^{\mu+1} - b_r^{\mu}) p_{r,\mu+1} \geq 0 \;\; (c_{r\mu} \geq 1; \mu = 1,...,k_r - 2)$$

impose concavity on the value function of output *r,* with the parameters $c_{r\mu}$ adjusting the sharpness of the diminishing returns. Analogously, the restrictions

$$(a_i^{\mu+2} - a_i^{\mu+1}) q_{i\mu} - z_{i\mu}(a_i^{\mu+1} - a_i^{\mu}) q_{i,\mu+1} \geq 0 \;\; (z_{i\mu} \geq 1; \mu = 1,...,k_i - 2)$$

impose convexity on the value function of input *i*.

Completing the developments, the value based model to assess the efficiency of the evaluated unit j_0, with reference to the abstract model (4.17), is provided below:

$$\begin{aligned}
&\max E_{j_0}(p,q)=\sum_{r=1}^{d}\hat{y}_{rjo}p_r+\sum_{r=d+1}^{s}\sum_{\mu=1}^{k_r}\hat{\delta}_{rj_o}^{\mu}p_{r\mu}+\sum_{i=1}^{t}\hat{x}_{ijo}q_i+\sum_{i=t+1}^{m}\sum_{\mu=1}^{k_i}\hat{\gamma}_{ijo}^{\mu}q_{i\mu}\\
&s.t.\\
&\sum_{r=1}^{d}\hat{y}_{rj}p_r+\sum_{r=d+1}^{s}\sum_{\mu=1}^{k_r}\hat{\delta}_{rj}^{\mu}p_{r\mu}+\sum_{i=1}^{t}\hat{x}_{ij}q_i+\sum_{i=t+1}^{m}\sum_{\mu=1}^{k_i}\hat{\gamma}_{ij}^{\mu}q_{i\mu}\le 1 \;\; j=1,2,...,n\\
&p_r\ge 0 \quad (r=1,...,d)\\
&p_{r\mu}\ge 0 \;\; (r=d+1,...,s,\;\; \mu=1,...,k_r)\\
&q_i\ge 0 \quad (i=1,...,t)\\
&q_{i\mu}\ge 0 \;\; (i=t+1,...,m,\;\; \mu=1,...,k_i)\\
&p_{r\mu}\;,\; q_{i\mu}\in W
\end{aligned} \tag{4.28}$$

In the last constraint of (4.28), W denotes the region defined by user-specified restrictions on the variables that provide the non-linear value functions of outputs and inputs with properties reflecting the user’s preferences.

The formulations presented above actually transform the original input/output data set into an expanded data set by decomposing each one of the non-linear inputs and outputs in auxiliary linear inputs and linear outputs respectively. This transformation allows performing the efficiency assessments without drawing away from the grounds of the DEA methodology. As a practical guide to implement the data transformation, one may consider in the set of units two dummy DMUs, one comprised by the fixed minimum values for the inputs and the outputs, the other comprised by the fixed maximum values. Notice here that these dummy units are not taken into account for the efficiency assessments. Then, the transformation is carried out in two steps: In the first step the non-linear inputs and outputs are decomposed on the basis of the segments assumed for each one of them to derive the expanded data set. In a second step, the expanded data are normalized column-wise on the column ranges.

4.4.3 An application of the value based DEA model to the Efficiency assessment of a Portuguese retail chain in the pharmacy-cosmetics-hygiene sector

In this sub-section we revisit the case originally studied in Almeida and Dias (2012), which concerns the efficiency assessment of 19 stores of a Portuguese retail chain in the pharmacy-cosmetics-hygiene sector.

Table 4.3: Observed input/output data in original scales

DMU	*Inputs in original scales*					*Outputs in original scales*	
	X_{STK} *(L)*	X_{EMP} *(NL)*	X_{SAC} *(L)*	X_{RNT} *(L)*	X_{ARE} *(NL)*	Y_{SAL} *(L)*	Y_{F4} *(NL)*
1	360614	13.2	153071	275240	213	1994652	36.4
2	263736	9.5	111409	117916	213	1194289	44.6
3	628938	17.8	218122	492305	436	3841226	32.2
4	479582	16.5	189495	134824	262	2299879	23.8
5	600449	15.9	222567	411982	331	3905880	39.7
6	299876	12.3	159338	185368	208	1554821	37.2
7	171010	9.2	92436	124355	231	625315	22.0
8	354506	13.9	153228	231525	400	1570432	24.4
9	521819	13.1	155918	145527	222	2249522	28.0
10	357204	7.3	96041	179931	200	1505312	45.4
11	307347	11.3	135895	171760	313	1387585	28.1
12	701109	15.8	214814	300106	290	5425809	32.4
13	392894	15.2	170675	250726	216	2269410	40.5
14	604291	20.0	222424	387543	443	3410820	27.8
15	272851	12.0	148268	159532	197	1410839	33.9
16	327304	11.9	181352	168006	207	1263137	29.1
17	356157	11.5	130337	181693	286	1371183	26.5
18	152850	11.4	87223	147252	301	877671	32.6
19	295598	13.3	193606	160607	199	1634121	26.0

The five inputs considered, with their characterizations as linear (L) or non-linear (NL) and their scales of measurement, are: Average stock (STK – L - €), Number of employees (EMP – NL-full time equivalent), Salary costs (SAC – L - €), Rent (RNT – L- €) and Area (ARE – NL-m^2). Two outputs are considered: Global sales (SAL – L- €) and Family 4 sales/Global sales (F4 – NL- %). The input/output data are given in Table 4.3.

In a preliminary stage, the raw data were originally transformed in values. Fixed minimum and maximum levels for the inputs and outputs were set, beyond the observed minima and maxima, as shown in Table 4.4.

Table 4.4: Fixed minimum and maximum levels for inputs and outputs

Fixed min/max levels	X_{STK} (L)	X_{EMP} (NL)	X_{SAC} (L)	X_{RNT} (L)	X_{ARE} (NL)	Y_{SAL} (L)	Y_{F4} (NL)
l	100000	6	50000	100000	150	500000	20
h	1000000	24	250000	500000	450	6000000	50

Linear value functions were assumed for the three linear inputs X_{STK}, X_{SAC}, X_{RNT} and the linear output Y_{SAL}. The value functions of the non-linear inputs X_{ARE}, X_{EMP} and the non-linear output Y_{F4}, as shown in Figures 4.10-4.12, were obtained by interacting with the decision maker (Almeida and Dias, 2012).

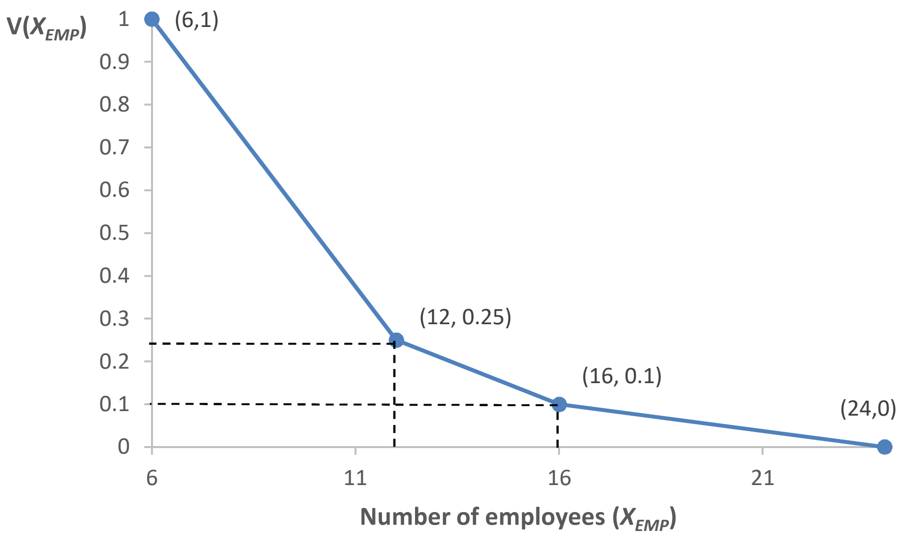


Figure 4.10 : Piece-wise linear value functions for the input X_{EMP}

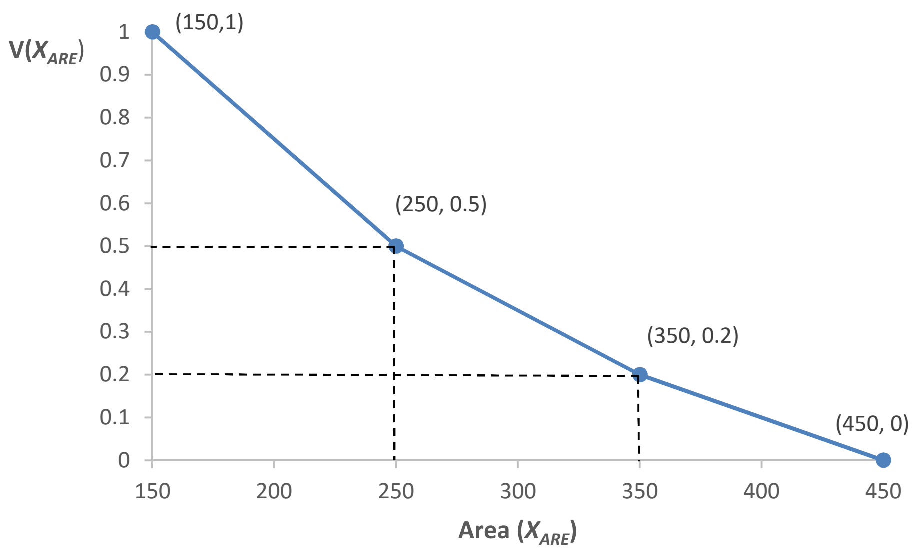


Figure 4.11: Piece-wise linear value functions for the input X_{ARE}

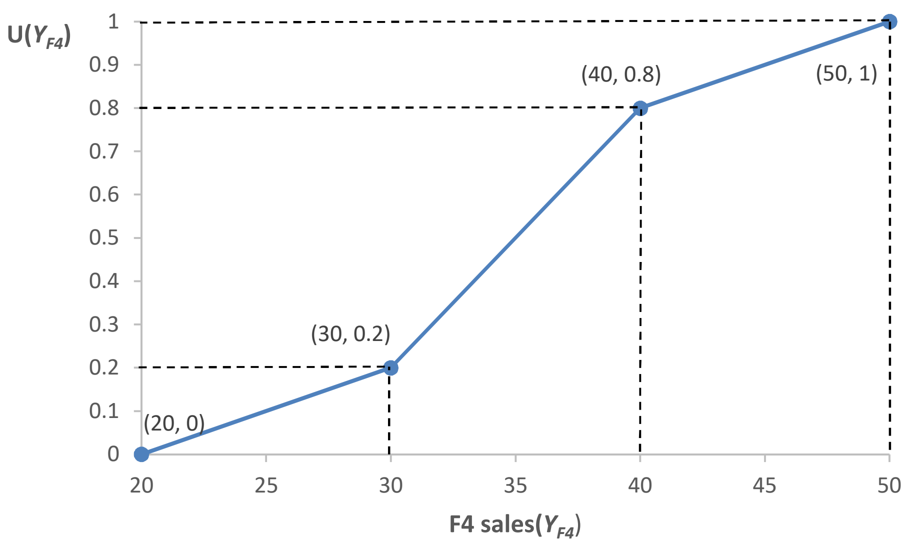


Figure 4.12: Piece-wise linear value functions for the output Y_{F4}

The efficiency estimates for the 19 DMUs, as given in Almeida and Dias (2012), are shown in the last column of Table 4.8 under the label z*. They were obtained by solving models (4.15) and (4.16), with the phase 2 model (4.16) augmented with the following intra-weight constraints:

$$W = \begin{cases} w_{SAL} \geq w_{STK} \geq w_{RNT} \geq w_{SAC} \geq w_{F4} \geq w_{EMP} \geq w_{ARE} \\ w_{SAL} \leq 11.1 w_{ARE} \end{cases}$$

The ordinal constraints derived by ranking the factors, whereas the last constraint provides a trade-off between the most and the least important factors. The latter constraint was introduced to avoid null weights (Almeida and Dias, 2012).

Table 4.5: Expanded data set in original scales

	Inputs										*Outputs*			
DMU	*X_{STK} (L)*	*X_{EMP} (NL)*			*X_{SAC}(L)*	*X_{RNT}(L)*	*X_{ARE} (NL)*			*Y_{SAL} (L)*	*Y_{F4} (NL)*		
		γ^1_{EMP}	γ^2_{EMP}	γ^3_{EMP}			γ^1_{ARE}	γ^2_{ARE}	γ^3_{ARE}		δ^1_{F4}	δ^2_{F4}	δ^3_{F4}
1	360614	0.00	2.80	8.00	153071	275240	37.00	100.00	100.00	1994652	10.00	6.40	0.00
2	263736	2.50	4.00	8.00	111409	117916	37.00	100.00	100.00	1194289	10.00	10.00	4.60
3	628938	0.00	0.00	6.20	218122	492305	0.00	0.00	14.00	3841226	10.00	2.20	0.00
4	479582	0.00	0.00	7.50	189495	134824	0.00	88.00	100.00	2299879	3.80	0.00	0.00
5	600449	0.00	0.10	8.00	222567	411982	0.00	19.00	100.00	3905880	10.00	9.70	0.00
6	299876	0.00	3.70	8.00	159338	185368	42.00	100.00	100.00	1554821	10.00	7.20	0.00
7	171010	2.80	4.00	8.00	92436	124355	19.00	100.00	100.00	625315	2.00	0.00	0.00
8	354506	0.00	2.10	8.00	153228	231525	0.00	0.00	50.00	1570432	4.40	0.00	0.00
9	521819	0.00	2.90	8.00	155918	145527	28.00	100.00	100.00	2249522	8.00	0.00	0.00
10	357204	4.70	4.00	8.00	96041	179931	50.00	100.00	100.00	1505312	10.00	10.00	5.40
11	307347	0.70	4.00	8.00	135895	171760	0.00	37.00	100.00	1387585	8.10	0.00	0.00
12	701109	0.00	0.20	8.00	214814	300106	0.00	60.00	100.00	5425809	10.00	2.40	0.00
13	392894	0.00	0.80	8.00	170675	250726	34.00	100.00	100.00	2269410	10.00	10.00	0.50
14	604291	0.00	0.00	4.00	222424	387543	0.00	0.00	7.00	3410820	7.80	0.00	0.00
15	272851	0.00	4.00	8.00	148268	159532	53.00	100.00	100.00	1410839	10.00	3.90	0.00
16	327304	0.10	4.00	8.00	181352	168006	43.00	100.00	100.00	1263137	9.10	0.00	0.00
17	356157	0.50	4.00	8.00	130337	181693	0.00	64.00	100.00	1371183	6.50	0.00	0.00
18	152850	0.60	4.00	8.00	87223	147252	0.00	49.00	100.00	877671	10.00	2.60	0.00
19	295598	0.00	2.70	8.00	193606	160607	51.00	100.00	100.00	1634121	6.00	0.00	0.00
Associated weight variables	v_{STK}	$v_{EMP,1}$	$v_{EMP,2}$	$v_{EMP,3}$	v_{SAC}	v_{RNT}	$v_{ARE,1}$	$v_{ARE,2}$	$v_{ARE,3}$	u_{SAL}	$u_{F4,1}$	$u_{F4,2}$	$u_{F4,3}$

Applying the data transformation-variable alteration technique developed in the previous sections, the expanded data set and its range-normalized counterpart are obtained, as shown in Table 4.5 and Table 4.6 respectively. For comparison purposes, the same fixed minimum and maximum values were assumed as shown in Table 4.4. The breakpoints for the non-linear factors are set to the values originally considered in Almeida and Dias (2012), as shown in Table 4.7.

Table 4.6: Transformed data set

DMU	*X_{STK}(L)*	*X_{EMP} (NL)*			*X_{SAC}(L)*	*X_{RNT}(L)*	*X_{ARE} (NL)*			*Y_{SAL} (L)*	*$Y_{\%F4}$ (NL)*		
	$\hat{x}_{STK}$	$\hat{\gamma}^1_{EMP}$	$\hat{\gamma}^2_{EMP}$	$\hat{\gamma}^3_{EMP}$	$\hat{x}_{SAC}$	$\hat{x}_{RNT}$	$\hat{\gamma}^1_{ARE}$	$\hat{\gamma}^2_{ARE}$	$\hat{\gamma}^3_{ARE}$	$\hat{y}_{SAL}$	$\hat{\delta}^1_{F4}$	$\hat{\delta}^2_{F4}$	$\hat{\delta}^3_{F4}$
1	0.71	0.00	0.70	1.00	0.48	0.56	0.37	1.00	1.00	0.27	1.00	0.64	0.00
2	0.82	0.42	1.00	1.00	0.69	0.96	0.37	1.00	1.00	0.13	1.00	1.00	0.46
3	0.41	0.00	0.00	0.78	0.16	0.02	0.00	0.00	0.14	0.61	1.00	0.22	0.00
4	0.58	0.00	0.00	0.94	0.30	0.91	0.00	0.88	1.00	0.33	0.38	0.00	0.00
5	0.44	0.00	0.02	1.00	0.14	0.22	0.00	0.19	1.00	0.62	1.00	0.97	0.00
6	0.78	0.00	0.93	1.00	0.45	0.79	0.42	1.00	1.00	0.19	1.00	0.72	0.00
7	0.92	0.47	1.00	1.00	0.79	0.94	0.19	1.00	1.00	0.02	0.20	0.00	0.00
8	0.72	0.00	0.53	1.00	0.48	0.67	0.00	0.00	0.50	0.19	0.44	0.00	0.00
9	0.53	0.00	0.73	1.00	0.47	0.89	0.28	1.00	1.00	0.32	0.80	0.00	0.00
10	0.71	0.78	1.00	1.00	0.77	0.80	0.50	1.00	1.00	0.18	1.00	1.00	0.54
11	0.77	0.12	1.00	1.00	0.57	0.82	0.00	0.37	1.00	0.16	0.81	0.00	0.00
12	0.33	0.00	0.05	1.00	0.18	0.50	0.00	0.60	1.00	0.90	1.00	0.24	0.00
13	0.67	0.00	0.20	1.00	0.40	0.62	0.34	1.00	1.00	0.32	1.00	1.00	0.05
14	0.44	0.00	0.00	0.50	0.14	0.28	0.00	0.00	0.07	0.53	0.78	0.00	0.00
15	0.81	0.00	1.00	1.00	0.51	0.85	0.53	1.00	1.00	0.17	1.00	0.39	0.00
16	0.75	0.02	1.00	1.00	0.34	0.83	0.43	1.00	1.00	0.14	0.91	0.00	0.00
17	0.72	0.08	1.00	1.00	0.60	0.80	0.00	0.64	1.00	0.16	0.65	0.00	0.00
18	0.94	0.10	1.00	1.00	0.81	0.88	0.00	0.49	1.00	0.07	1.00	0.26	0.00
19	0.78	0.00	0.68	1.00	0.28	0.85	0.51	1.00	1.00	0.21	0.60	0.00	0.00
Associated Value variables	q_{STK}	$q_{EMP,1}$	$q_{EMP,2}$	$q_{EMP,3}$	q_{SAC}	q_{RNT}	$q_{ARE,1}$	$q_{ARE,2}$	$q_{ARE,3}$	p_{SAL}	$p_{F4,1}$	$p_{F4,2}$	$p_{F4,3}$

Table 4.7: Breakpoints for the non-linear inputs and outputs in original scales

	Breakpoints for X_{EMP}				Breakpoints for X_{ARE}				Breakpoints for Y_{F4}			
	a_{EMP}^1	a_{EMP}^2	a_{EMP}^3	a_{EMP}^4	a_{ARE}^1	a_{ARE}^2	a_{ARE}^3	a_{ARE}^4	b_{F4}^1	b_{F4}^2	b_{F4}^3	b_{F4}^4
X_{EMP}	6	12	16	20								
X_{ARE}					150	250	350	450				
Y_{F4}									20	30	40	50

To build the model for the specific data set, without drawing away from the preferential information assumed in the original work, the following adjustments that imitate the same decision situation are made.

Value functions:

To maintain the preferential information assumed for the non-linear inputs and outputs in the current development, the following constraints in terms of the value variables are introduced:

$$\begin{aligned} &q_{EMP,1} - 5q_{EMP,2} = 0 \\ &2q_{EMP,2} - 3q_{EMP,3} = 0 \\ &3q_{ARE,1} - 5q_{ARE,2} = 0 \\ &2q_{ARE,2} - 3q_{ARE,3} = 0 \\ &3p_{F4,1} - p_{F4,2} = 0 \\ &p_{F4,2} - 3p_{F4,3} = 0 \end{aligned}$$

For example, as concerns the non-linear output *F4*, the slopes of the line segments of the value function are (see Figure 4.12):

$$u_{F4,1} = 0.02,\, u_{F4,2} = 0.06,\, u_{F4,3} = 0.02$$

As the ratio of these weights is of interest in the proposed model, the following variable transformations are applied to derive these ratios in terms of the corresponding value variables:

$$\frac{u_{F4,1}}{u_{F4,2}}=\frac{1}{3}\Leftrightarrow\frac{u_{F4,1}(b_{F4}^{2}-b_{F4}^{1})}{u_{F4,2}(b_{F4}^{3}-b_{F4}^{2})}=\frac{b_{F4}^{2}-b_{F4}^{1}}{3(b_{F4}^{3}-b_{F4}^{2})}\Leftrightarrow\frac{p_{F4,1}}{p_{F4,2}}=\frac{10}{3(40-30)}=\frac{1}{3}$$

$$\frac{u_{F4,2}}{u_{F4,3}}=3\Leftrightarrow\frac{u_{F4,2}(b_{F4}^{3}-b_{F4}^{2})}{u_{F4,3}(b_{F4}^{4}-b_{F4}^{3})}=\frac{3(b_{F4}^{3}-b_{F4}^{2})}{(b_{F4}^{4}-b_{F4}^{3})}\Leftrightarrow\frac{p_{F4,2}}{p_{F4,3}}=\frac{3(40-30)}{(50-40)}=3$$

Ranking and trade-off constraints:

Analogously, the ordinal and trade-off constraints *W* assumed in the original work (Almeida and Dias, 2012) are translated in terms of the proposed model as follows:

$$p_{SAL}\geq q_{STK}\geq q_{RNT}\geq q_{SAC}\geq p_{F4,1}+p_{F4,2}+p_{F4,3}\geq q_{EMP,1}+q_{EMP,2}+q_{EMP,3}\geq q_{ARE,1}+q_{ARE,2}+q_{ARE,3}$$
$$p_{SAL}\leq 11.1(q_{ARE,1}+q_{ARE,2}+q_{ARE,3})$$

On the basis of the above adjustments, the model (4.28) that assesses the relative efficiency of the evaluated unit j_0 takes the form of the model (4.29).

The efficiency scores and the optimal solutions (in terms of value variables) are presented in Table 4.8. Figures 4.13-4.19 exhibit the value functions assessed by the 19 DMUs for the inputs X_{STK}, X_{EMP}, X_{SAC}, X_{RNT}, X_{ARE} and the outputs Y_{SAL} and Y_{F4}.

$$\max E_{j_o} = \left[\begin{array}{l} \hat{y}_{SAL,j_0} p_{SAL} + \hat{\delta}^1_{F4,j_0} p_{F4,1} + \hat{\delta}^2_{F4,j_0} p_{F4,2} + \hat{\delta}^3_{F4,j_0} p_{F4,3} \\ + \hat{x}_{STK,j_0} q_{STK} + \hat{\gamma}^1_{EMP,j_0} q_{EMP,1} + \hat{\gamma}^2_{EMP,j_0} q_{EMP,2} + \hat{\gamma}^3_{EMP,j_0} q_{EMP,3} + \hat{x}_{SAC,j_0} q_{SAC} + \hat{x}_{RNT,j_0} q_{RNT} \\ + \hat{\gamma}^1_{ARE,j_0} q_{ARE,1} + \hat{\gamma}^2_{ARE,j_0} q_{ARE,2} + \hat{\gamma}^3_{ARE,j_0} q_{ARE,3} \end{array} \right.$$

$s.t.$

[section 1]

$$\begin{array}{l} \hat{y}_{SAL,j} p_{SAL} + \hat{\delta}^1_{F4,j} p_{F4,1} + \hat{\delta}^2_{F4,j} p_{F4,2} + \hat{\delta}^3_{F4,j} p_{F4,3} \\ + \hat{x}_{STK,j} q_{STK} + \hat{\gamma}^1_{EMP,j} q_{EMP,1} + \hat{\gamma}^2_{EMP,j} q_{EMP,2} + \hat{\gamma}^3_{EMP,j} q_{EMP,3} + \hat{x}_{SAC,j} q_{SAC} + \hat{x}_{RNT,j} q_{RNT} \\ + \hat{\gamma}^1_{ARE,j} q_{ARE,1} + \hat{\gamma}^2_{ARE,j} q_{ARE,2} + \hat{\gamma}^3_{ARE,j} q_{ARE,3} \le 1 \quad (j = 1,...n) \end{array} \tag{4.29}$$

[section 2]

$$q_{EMP,1} - 5q_{EMP,2} = 0$$
$$2q_{EMP,2} - 3q_{EMP,3} = 0$$
$$3q_{ARE,1} - 5q_{ARE,2} = 0$$
$$2q_{ARE,2} - 3q_{ARE,3} = 0$$
$$3p_{F4,1} - p_{F4,2} = 0$$
$$p_{F4,2} - 3p_{F4,3} = 0$$

[section 3]

$$p_{SAL} - q_{STK} \ge 0$$
$$q_{STK} - q_{RNT} \ge 0$$
$$q_{RNT} - q_{SAC} \ge 0$$
$$q_{SAC} - p_{F4,1} - p_{F4,2} - p_{F4,3} \ge 0$$
$$p_{F4,1} + p_{F4,2} + p_{F4,3} - q_{EMP,1} - q_{EMP,2} - q_{EMP,3} \ge 0$$
$$q_{EMP,1} + q_{EMP,2} + q_{EMP,3} - q_{ARE,1} - q_{ARE,2} - q_{ARE,3} \ge 0$$
$$p_{SAL} - 11.1q_{ARE,1} - 11.1q_{ARE,2} - 11.1q_{ARE,3} \le 0$$

$$p_{(.)}, q_{(.)} \ge 0$$

The [section 1] comprises the ordinary DEA constraints. The [section 2] constraints derive from the preferential information that drive the forms of the piece-wise linear value functions, whereas the [section 3] constraints are formed on the basis of the ranking and trade-off information assumed in the original study (Almeida and Dias, 2012).

Table 4.8: Efficiency scores and optimal solutions in terms of value variables

DMU	E_j	q_{STK}	$q_{EMP,1}$	$q_{EMP,2}$	$q_{EMP,3}$	q_{SAC}	q_{RNT}	$q_{ARE,1}$	$q_{ARE,2}$	$q_{ARE,3}$	p_{SAL}	$p_{F4,1}$	$p_{F4,2}$	$p_{F4,3}$	z^*
1	0.915	0.772	0.072	0.014	0.010	0.096	0.096	0.048	0.029	0.019	0.772	0.019	0.058	0.019	0.046
2	1.000	0.423	0.029	0.006	0.004	0.093	0.423	0.019	0.011	0.008	0.423	0.019	0.056	0.019	0.000
3	0.789	0.930	0.063	0.013	0.008	0.084	0.084	0.042	0.025	0.017	0.930	0.017	0.050	0.017	0.118
4	0.901	0.469	0.049	0.010	0.007	0.066	0.469	0.033	0.020	0.013	0.730	0.013	0.039	0.013	0.057
5	0.888	0.796	0.054	0.011	0.007	0.112	0.112	0.036	0.022	0.014	0.796	0.022	0.067	0.022	0.061
6	0.937	0.736	0.050	0.010	0.007	0.066	0.192	0.033	0.020	0.013	0.736	0.013	0.040	0.013	0.035
7	0.995	0.358	0.027	0.005	0.004	0.358	0.358	0.018	0.011	0.007	0.358	0.007	0.022	0.007	0.003
8	0.805	0.876	0.059	0.012	0.008	0.149	0.149	0.039	0.024	0.016	0.876	0.016	0.047	0.016	0.106
9	0.896	0.472	0.050	0.010	0.007	0.066	0.472	0.033	0.020	0.013	0.734	0.013	0.040	0.013	0.060
10	1.000	0.184	0.138	0.028	0.018	0.184	0.184	0.092	0.055	0.037	0.661	0.037	0.110	0.037	0.000
11	0.876	0.788	0.053	0.011	0.007	0.071	0.206	0.035	0.021	0.014	0.788	0.014	0.043	0.014	0.067
12	1.000	0.085	0.064	0.013	0.008	0.085	0.085	0.042	0.025	0.017	0.942	0.017	0.051	0.017	0.000
13	0.937	0.754	0.051	0.010	0.007	0.107	0.107	0.034	0.020	0.014	0.754	0.021	0.064	0.021	0.034
14	0.753	0.974	0.066	0.013	0.009	0.088	0.088	0.044	0.026	0.018	0.974	0.018	0.053	0.018	0.137
15	0.947	0.729	0.049	0.010	0.007	0.066	0.190	0.033	0.020	0.013	0.729	0.013	0.039	0.013	0.029
16	0.859	0.540	0.036	0.007	0.005	0.049	0.540	0.024	0.015	0.010	0.540	0.010	0.029	0.010	0.080
17	0.841	0.439	0.030	0.006	0.004	0.391	0.439	0.020	0.012	0.008	0.439	0.008	0.024	0.008	0.089
18	1.000	0.357	0.024	0.005	0.003	0.357	0.357	0.016	0.010	0.006	0.357	0.006	0.019	0.006	0.000
19	0.920	0.750	0.051	0.010	0.007	0.068	0.196	0.034	0.020	0.014	0.750	0.014	0.041	0.014	0.044

The results obtained by the proposed approach are straightly comparable with those given in Almeida and Dias (2012). Indeed, as shown in the second and the last columns of Table 4.8, exactly the same units (namely, the units 2, 10, 12 and 18) are estimated efficient with both approaches. From a computational burden aspect, although the proposed linear program (4.29) is a little larger than the phase 2 and phase 3 programs (4.15) and (4.16) due to the additional variables derived from the segmentation of the non-linear factors and the associated [section 2]-constraints, it is only solved once for each unit. Recall here that according to the Almeida and Dias (2012) procedure, a phase 2 program is solved for each unit and then a phase 3 program is solved for each inefficient unit. In particular, and in the context of their study, the proposed program (4.29) is solved 19 times, whereas 34 runs are needed (19 for phase 2 and 15 for phase 3) to complete the assessments with their approach. Moreover, any ordinary ready-made DEA software is sufficient to solve the proposed model, which is not the case for the three-phase procedure introduced by Almeida and Dias (2012).

To conclude, there are three critical advantages of the proposed approach when compared to the approach of Almeida and Dias (2012) that motivated the current developments:

- It provides a measure of efficiency in the form of a ratio rather than in the form of a min-max loss of value
- It requires fewer linear programs to be solved.
- It provides a meaningful interpretation to the variables

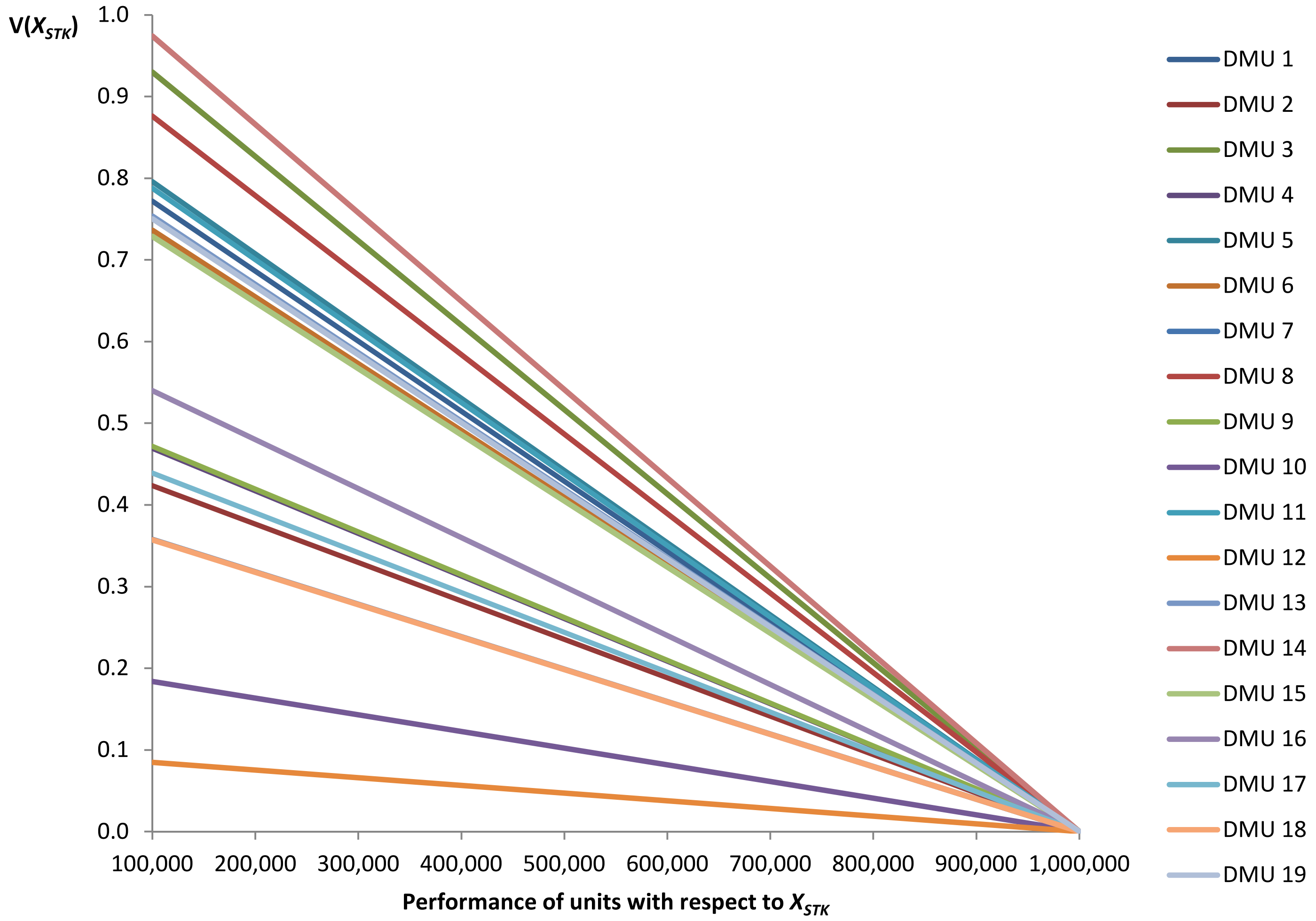


Figure 4.13: Value function assessed by the DMUS for X_{STK}

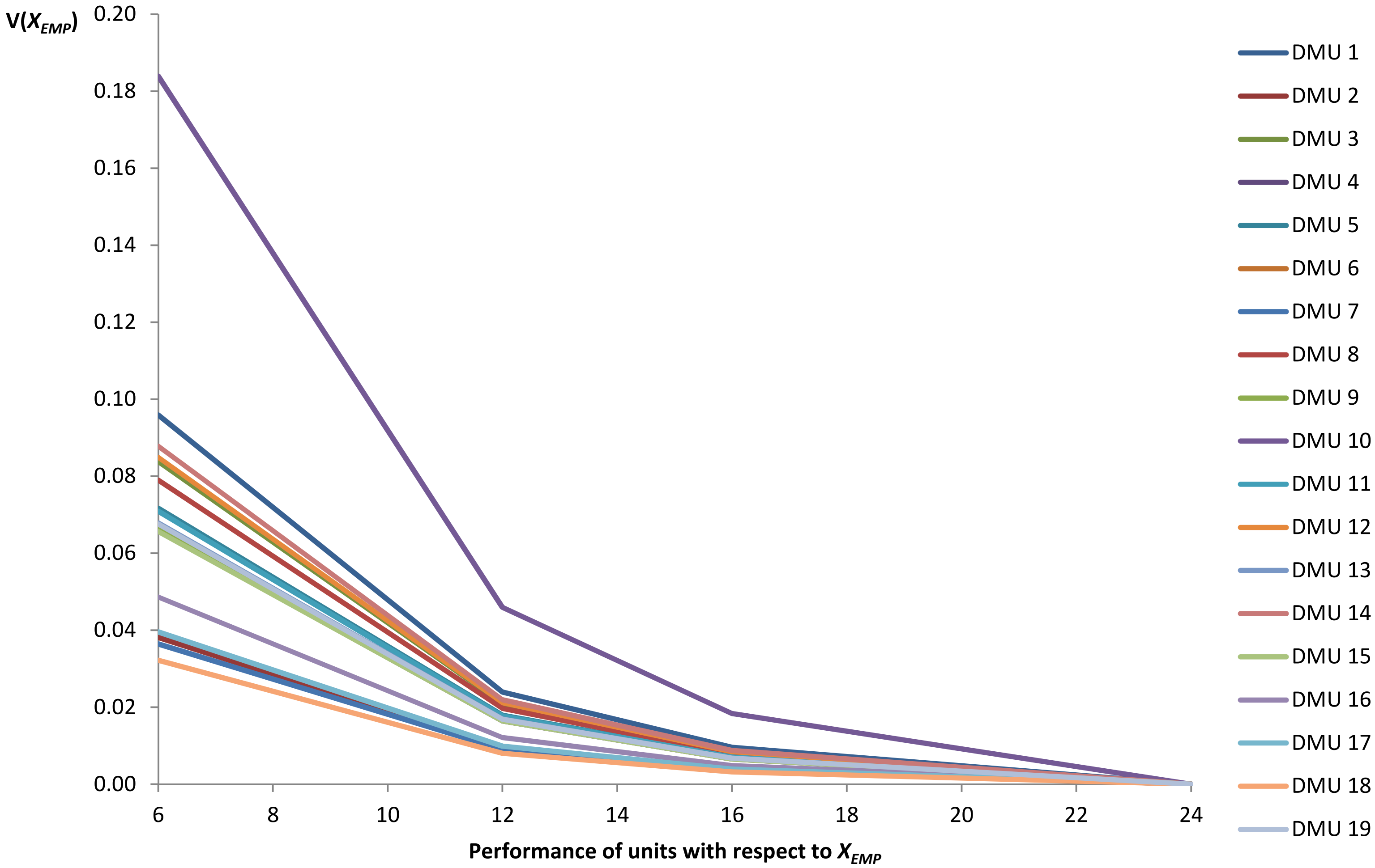


Figure 4.14: Value function assessed by the DMUS for X_{EMP}

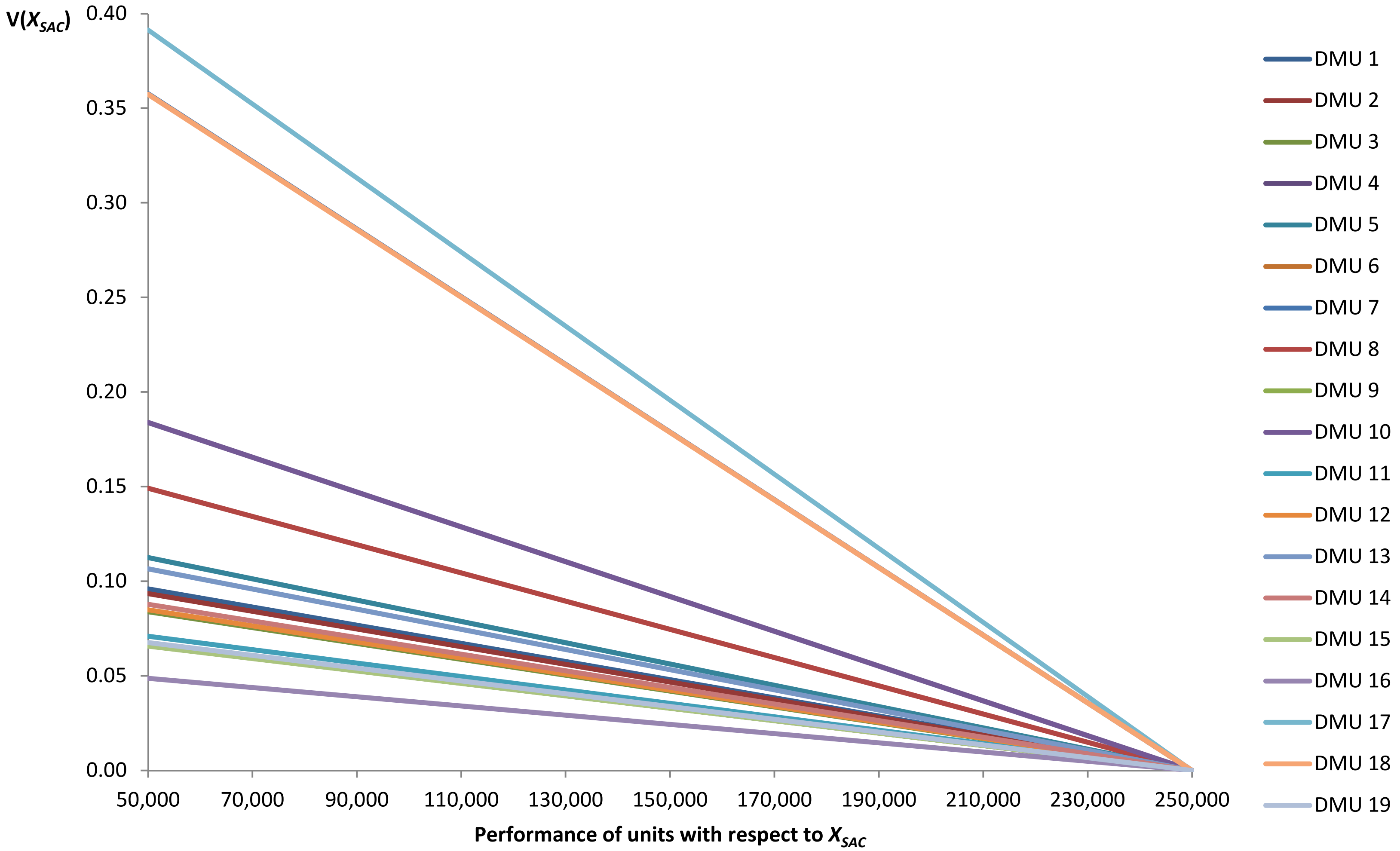


Figure 4.15: Value function assessed by the DMUS for X_{SAC}

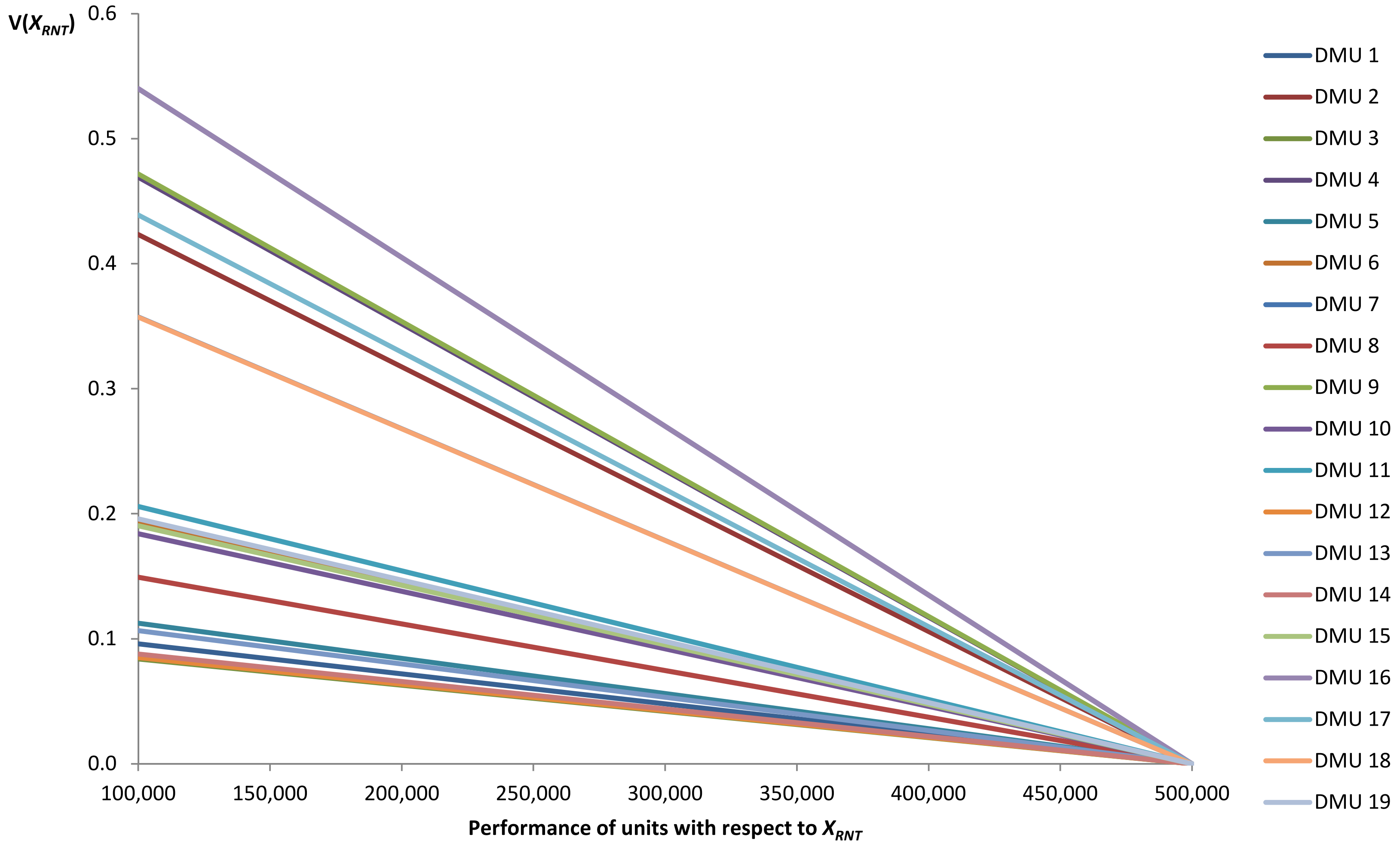


Figure 4.16: Value function assessed by the DMUS for X_{RNT}

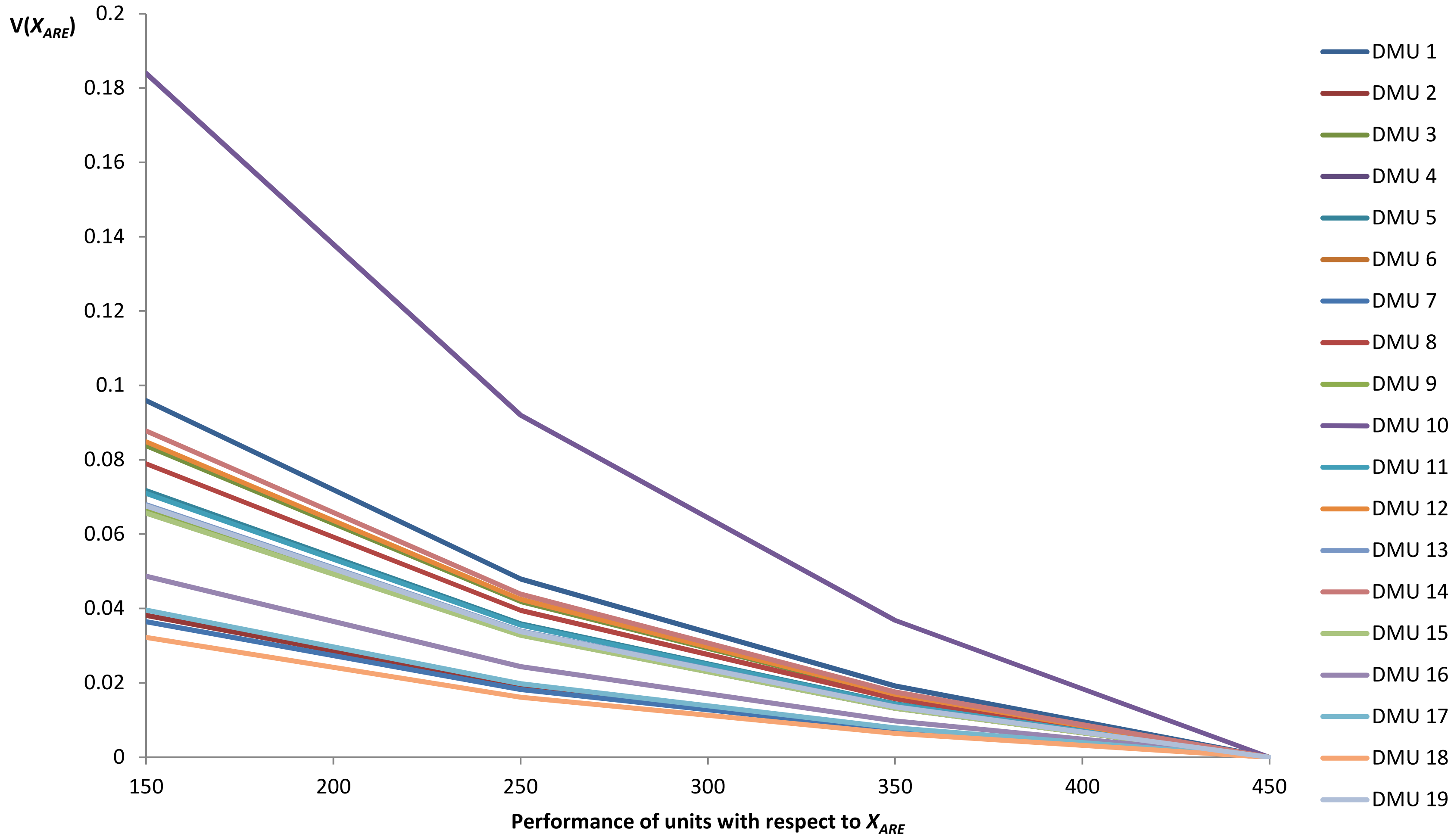


Figure 4.17: Value function assessed by the DMUS for X_{ARE}

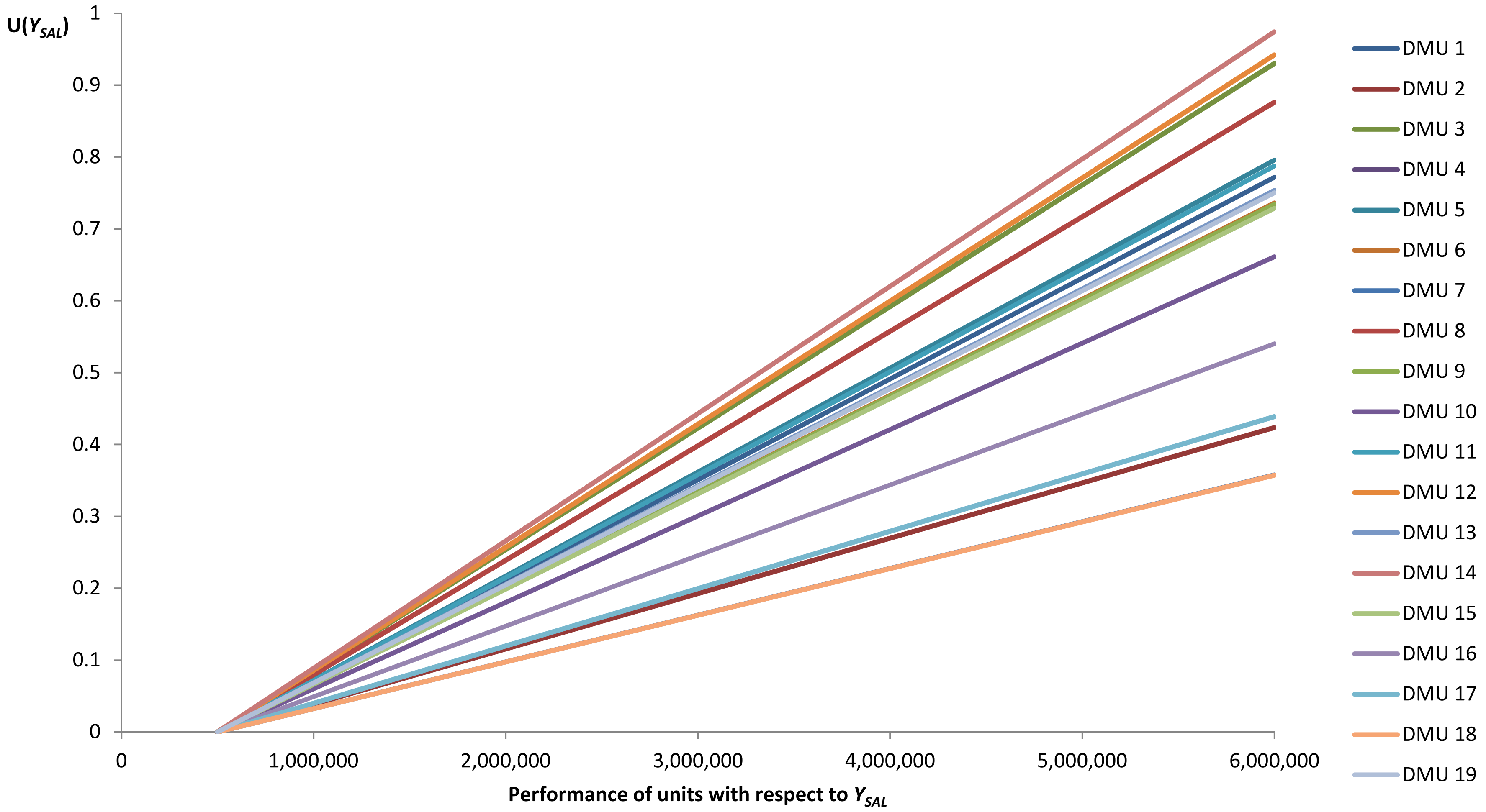


Figure 4.18: Value function assessed by the DMUS for Y_{SAL}

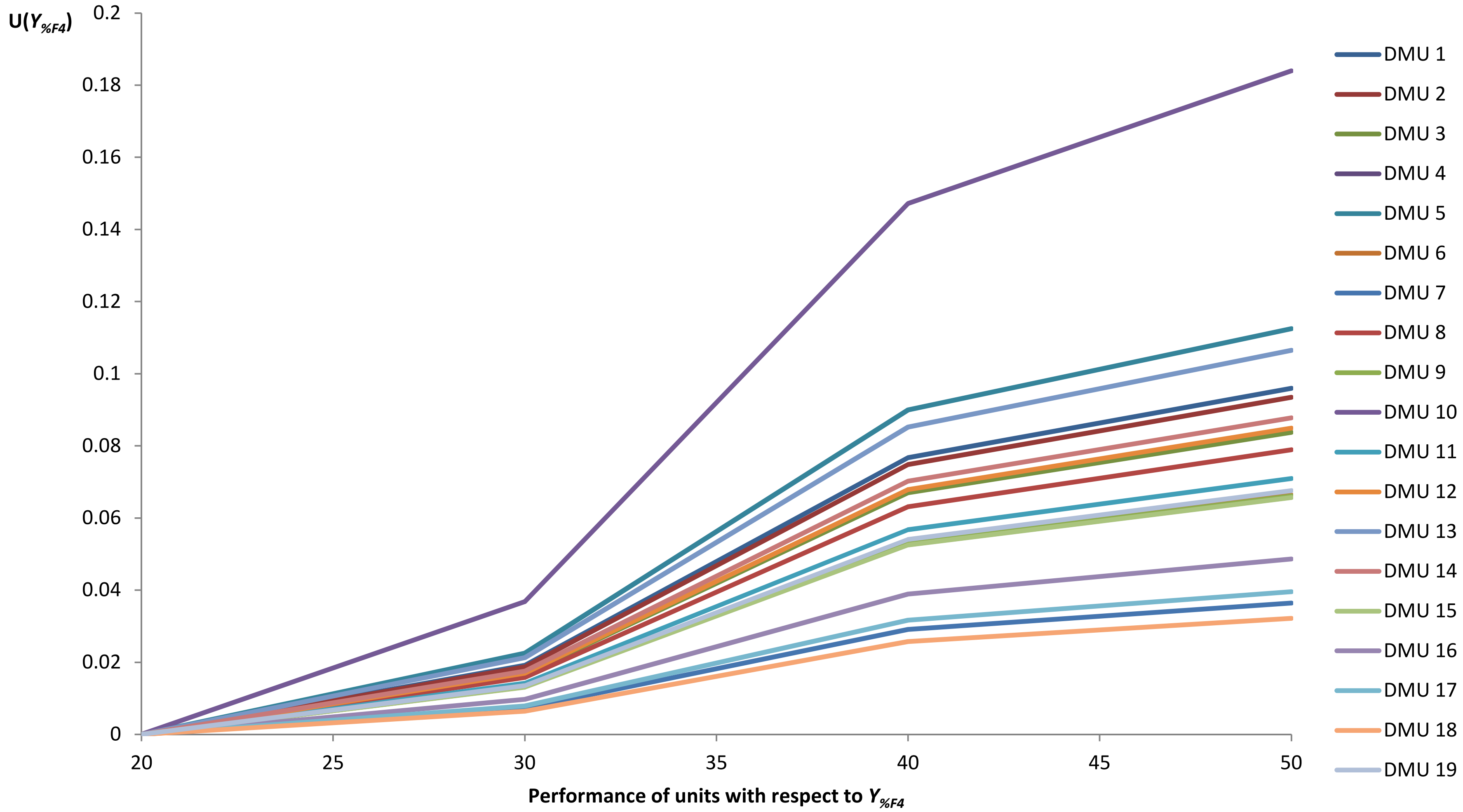


Figure 4.19: Value function assessed by the DMUS for $Y_{\%F4}$

4.5 Incorporating user preferences in value based DEA modeled by means of Ordinal Regression

In sub-section 4.4.3 we showed how one can incorporate the analyst's preferences using concepts of multi-attribute value theory. In particular, the value functions were estimated using direct preferential information for the desired levels of the inputs and the outputs. In this section, we develop an alternative indirect approach, based on ordinal regression analysis, to assess a prototype of the value functions. To this end, we utilize the preference elicitation protocol used in the ordinal regression method UTASTAR (Siskos and Yannacopoulos, 1985).

UTASTAR is an extension of UTA multi criteria method (Jacquet-Lagreze and Siskos, 1982), which is based on linear programming. It adopts the aggregation-disaggregation principle in order to assess value functions according to the analyst' preferential structure. Given a weak preference order on a subset of alternatives that the analyst is familiar with, the value functions of the criteria are adjusted so as to develop a preference model as consistent as possible with the analyst's individual preferences.

We develop a two-phase approach that bridges UTASTAR with DEA. Adjusting the UTASTAR formulation so as to be compatible with the developments presented in the previous section, we apply, in phase I, the UTASTAR method to assess the prototype preferential model of the analyst. Then, the assessed model is incorporated, in phase II, in the DEA efficiency assessments. A regular interpretation of DEA inputs and outputs to criteria in the MCDA terminology is that inputs are criteria to be minimized, whereas outputs are criteria to be maximized. With such a correspondence, the formulations in (4.21) and (4.26) developed in the previous section can be fully utilized in the UTASTAR context.

Given a subset AR of the n DMUs and a weak order on its items, that reflects the analyst's overall preference over AR , the LP model below assess piece-wise linear functions for the criteria (inputs and outputs) as consistent as possible with the analyst's stated preferences:

$$\begin{aligned}
&\min \ \mathrm{F} = \sum_{j \in A_R} \left(d_j^+ + d_j^- \right) \\
&\left(E_j(p,q) + d_j^- - d_j^+ \right) - \left(E_{j+1}(p,q) + d_{j+}^- - d_{j+1}^+ \right) > \delta \ if \ jPj+1 \\
&\left(E_j(p,q) + d_j^- - d_j^+ \right) - \left(E_{j+1}(p,q) + d_{j+}^- - d_{j+1}^+ \right) = 0 \ if \ jIj+1 \\
&\sum_{r=1}^{s} \sum_{\mu=1}^{k_r} p_{r\mu} + \sum_{i=1}^{m} \sum_{\mu=1}^{k_i} q_{i\mu} = 1 \\
&p_{r\mu} \geq 0, q_{i\mu} \geq 0, d_j^- \geq 0, d_j^+ \geq 0, j \in A_R
\end{aligned} \tag{4.30}$$

where $E_j(p,q)$ $j = 1,...,\mathrm{R}$ are given in the equation (4.27), d_j^+, d_j^- are overestimation and underestimation errors respectively, P denotes strict preference and I denotes indifference. As model (4.30) may have multiple optimal solutions, characteristic optimal solutions are investigated that maximize the value of one criterion at a time. This is accomplished in a post-optimality stage by solving a linear program each criterion (input and output) as showed below. The model (4.31) refers to outputs only and be adjusted for the inputs by replacing the objective function with $\varphi_i = \sum_{\mu=1}^{k_i} q_{i\mu}$, $i = 1,...,m$.

$$\begin{aligned}
&\max \ \varphi_r = \sum_{\mu=1}^{k_r} p_{r\mu} \quad \left(r = 1,...,s \right) \\
&s.t. \\
&\left(E_j(p,q) + d_j^- - d_j^+ \right) - \left(E_{j+1}(p,q) + d_{j+}^- - d_{j+1}^+ \right) > \delta \ if \ jPj+1 \\
&\left(E_j(p,q) + d_j^- - d_j^+ \right) - \left(E_{j+1}(p,q) + d_{j+1}^- - d_{j+1}^+ \right) = 0 \ if \ jIj+1 \\
&\sum_{r=1}^{s} \sum_{\mu=1}^{k_r} p_{r\mu} + \sum_{i=1}^{m} \sum_{\mu=1}^{k_i} q_{i\mu} = 1 \\
&\sum_{j \in A_R} \left(d_j^+ + d_j^- \right) \leq F^* + \varepsilon \\
&p_{r\mu} \geq 0, q_{i\mu} \geq 0, d_j^- \geq 0, d_j^+ \geq 0, j \in A_R
\end{aligned} \tag{4.31}$$

Totally, s+m LPs are solved (i.e. s LPs for the criteria associated with the outputs and m LPs for the criteria associated with the inputs). The last constraint in (4.31) is introduced in order to support the optimal value F^* of the objective function attained in model (4.30). Having obtained s+m alternative optimal solutions, the average, which is also optimal due to convexity, is used as representative of the analyst's preferences. This completes the phase I of our approach.

If $\left(\tilde{p}_{11},\tilde{p}_{12},...,\tilde{p}_{1k_1},....,\tilde{p}_{s1},\tilde{p}_{s2},...,\tilde{p}_{sk_s},\tilde{q}_{11},\tilde{q}_{12},...,\tilde{q}_{1k_1},....,\tilde{q}_{m1},\tilde{q}_{m2},...,\tilde{q}_{mk_m}\right)$ denote the average optimal solution, the assessed preferential model is incorporated in the DEA model (4.28), by appending the following constraint set *W*:

$$\begin{aligned}&\frac{p_{r,\mu+1}}{p_{r,\mu}}=\frac{\tilde{p}_{r,\mu+1}}{\tilde{p}_{r,\mu}},\ r=1,...,s;\mu=1,...,k_r\\&\frac{q_{i,\mu+1}}{q_{i,\mu}}=\frac{\tilde{q}_{i,\mu+1}}{\tilde{q}_{i,\mu}},\ i=1,...,m;\mu=1,...,k_i\end{aligned}\tag{4.32}$$

Solving model (4.28) with the additional constraints (4.32) for one DMU at a time we get the efficiency scores of the entire set of DMUs. This is the phase II which completes the approach.

4.5.1 Illustration

We provide in this sub-section a numerical illustration with 25 DMUs, one input $\left(X_1\right)$ and two outputs $\left(Y_1,Y_2\right)$, as depicted in Table 4.9.

For the sake of simplicity, three breakpoints are assumed for each factor as shown in Table 4.10 (i.e. the range of each factor is split in two segments).

Table 4.9: Observed input/output data in original scales

DMU	X_1	Y_1	Y_2
1	60	57	51
2	54	45	53
3	57	64	58
4	50	58	52
5	64	56	59
6	40	60	59
7	60	58	42
8	47	46	46
9	52	53	48
10	61	54	50
11	65	50	52
12	44	72	55
13	68	46	25
14	62	50	50
15	54	48	67
16	53	43	44
17	53	53	64
18	52	52	59
19	55	70	55
20	56	53	57
21	63	44	39
22	70	71	54
23	60	67	71
24	53	53	56
25	61	40	55

Table 4.10: Breakpoints for the non-linear inputs and outputs in original scales

Factors	b_1	b_2	b_3	α_1	α_2	α_3
X_1	-	-	-	40	60	75
Y_1	30	50	72	-	-	-
Y_2	10	40	71	-	-	-

Applying the transformations (4.18a) for the outputs and (4.23) for the inputs we get the expanded data set as shown in Table 4.11 and its range-normalized counterpart in Table 4.12.

Table 4.11: Expanded data set in original scales

DMU	γ_1^1	γ_1^2	δ_1^1	δ_1^2	δ_2^1	δ_2^2
1	0	15	20	7	30	11
2	6	15	15	0	30	13
3	3	15	20	14	30	18
4	10	15	20	8	30	12
5	0	11	20	6	30	19
6	20	15	20	10	30	19
7	0	15	20	8	30	2
8	13	15	16	0	30	6
9	8	15	20	3	30	8
10	0	14	20	4	30	10
11	0	10	20	0	30	12
12	16	15	20	22	30	15
13	0	7	16	0	15	0
14	0	13	20	0	30	10
15	6	15	18	0	30	27
16	7	15	13	0	30	4
17	7	15	20	3	30	24
18	8	15	20	2	30	19
19	5	15	20	20	30	15
20	4	15	20	3	30	17
21	0	12	14	0	29	0
22	0	5	20	21	30	14
23	0	15	20	17	30	31
24	7	15	20	3	30	16
25	0	14	10	0	30	15

Table 4.12: Range-normalized data set

DMU	$\hat{\gamma}_1^1$	$\hat{\gamma}_1^2$	$\hat{\delta}_1^1$	$\hat{\delta}_1^2$	$\hat{\delta}_2^1$	$\hat{\delta}_2^2$
1	0	1	1	0.32	1	0.35
2	0.30	1	0.75	0	1	0.42
3	0.15	1	1	0.64	1	0.58
4	0.50	1	1	0.36	1	0.39
5	0	0.73	1	0.27	1	0.61
6	1	1	1	0.45	1	0.61
7	0	1	1	0.36	1	0.06
8	0.65	1	0.80	0	1	0.19
9	0.40	1	1	0.13	1	0.26
10	0	0.93	1	0.18	1	0.32
11	0	0.68	1	0	1	0.39
12	0.80	1	1	1	1	0.48
13	0	0.47	0.80	0	0.50	0
14	0	0.87	1	0	1	0.32
15	0.30	1	0.90	0	1	0.87
16	0.35	1	0.65	0	1	0.13
17	0.35	1	1	0.14	1	0.77
18	0.40	1	1	0.09	1	0.61
19	0.25	1	1	0.91	1	0.48
20	0.20	1	1	0.14	1	0.55
21	0	0.80	0.70	0	0.97	0
22	0	0.33	1	0.95	1	0.45
23	0	1	1	0.77	1	1
24	0.35	1	1	0.14	1	0.52
25	0	0.93	0.50	0	1	0.48

Table 4.13 below depicts the selected subset A_R of DMUs with the preference ranking provided by a hypothetical analyst.

Table 4.13: A subset of DMUs and a preference ranking

DMU	$\hat{\gamma}_1^1$	$\hat{\gamma}_1^2$	$\hat{\delta}_1^1$	$\hat{\delta}_1^2$	$\hat{\delta}_2^1$	$\hat{\delta}_2^2$	Ranking
6	1	1	1	0.45	1	0.61	1
12	0.80	1	1	1	1	0.48	2
19	0.25	1	1	0.91	1	0.48	3
3	0.15	1	1	0.64	1	0.58	4
15	0.30	1	0.90	0	1	0.87	5
1	0	1	1	0.32	1	0.35	6
21	0	0.80	0.70	0	0.97	0	7

Applying model (4.30) and then performing the post-optimality analysis with model (4.31) on the data of Table 4.13, we get the following average optimal solution shown in Table 4.14 with F*=0.

Table 4.14: Average solution derived from the post-optimality model (4.31)

$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{p}_{11}$	$\tilde{p}_{12}$	$\tilde{p}_{21}$	$\tilde{p}_{22}$
0.009	0.333	0.255	0.016	0.333	0.054

As the optimal value of the objective function in model (4.30) is zero (F*=0), the assessed preference model is fully consistent with the ranking provided by the analyst. Figures 4.20 - 4.22 depict the value functions assessed for the input X1 and the outputs Y1 and Y2 on the basis of the optimal solution given in Table 4.14, which constitute a prototype of the analyst's value functions.

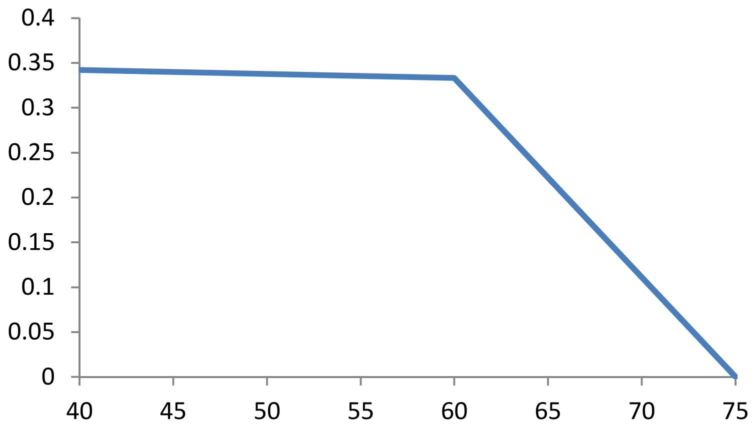


Figure 4.20: Value function for the input X_1

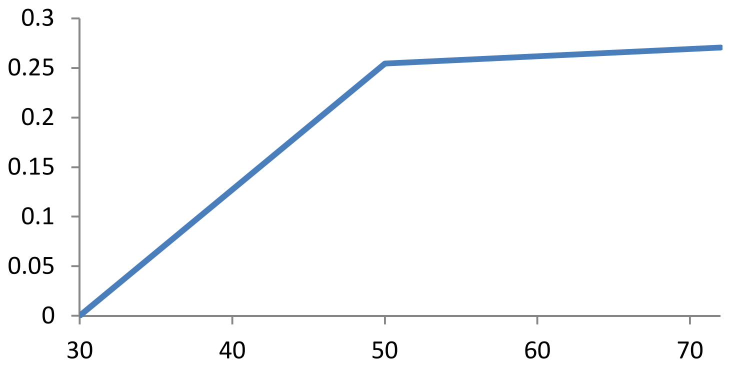


Figure 4.21: Value function for the output Y_1

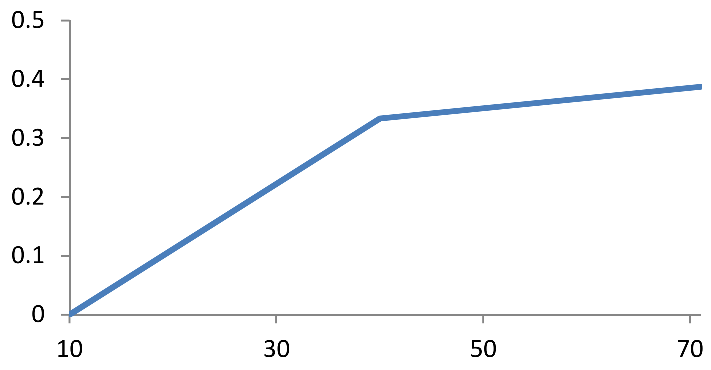


Figure 4.22: Value function for the output Y_2

The assessment of the value functions completes the phase I. The value based DEA efficiency assessments are made in phase II, by incorporating in the value based DEA model (4.28) the following set of constraints

$$W = \begin{cases} 0.009q_{12} - 0.333q_{11} = 0 \\ 0.255p_{12} - 0.016p_{11} = 0 \\ 0.333p_{22} - 0.054p_{21} = 0 \end{cases}$$

which translate the assessed value functions in terms of the variables *p* and *q*. The efficiency scores of the units, as shown in Table 4.15, are obtained by solving model (4.28) for one DMU at a time.

Table 4.15: Efficiency scores according to model (4.31) and the value restrictions

DMU	Efficiency	DMU	Efficiency
1	0.977	14	0.942
2	0.978	15	0.997
3	0.988	16	0.977
4	0.988	17	0.995
5	0.957	18	0.989
6	1.000	19	0.995
7	0.977	20	0.983
8	0.987	21	0.828
9	0.984	22	0.971
10	0.954	23	1.000
11	0.936	24	0.984
12	1.000	25	0.924
13	0.739		

Figures 4.23 - 4.25, exhibit the contribution of the input and the outputs to the efficiency index, as assessed by each evaluated DMU in order to maximize its efficiency score. This is the major characteristic DEA, which grants the flexibility to each DMU to assess its efficiency score by putting higher value to its advantageous features (inputs or outputs).

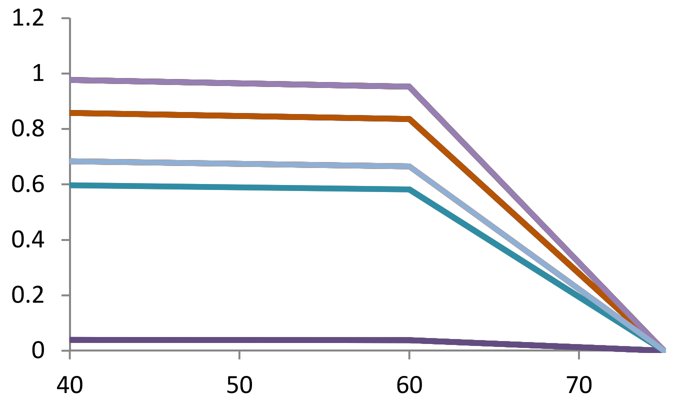


Figure 4.23: Value functions assessed by the DMUs for X_1

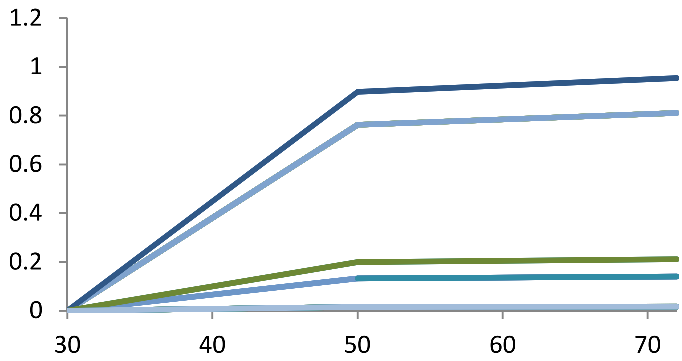


Figure 4.24: Value functions assessed by the DMUs for Y_1

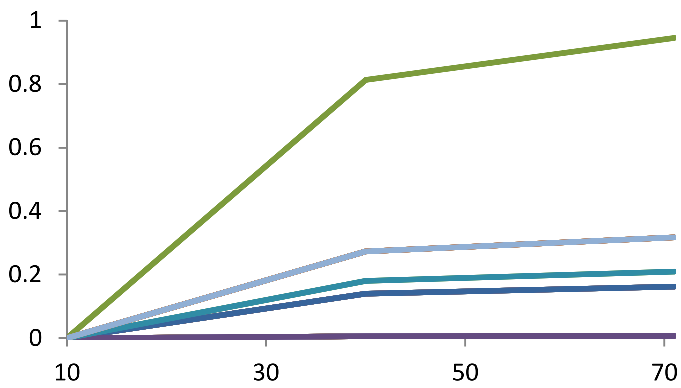


Figure 4.25: Value functions assessed by the DMUs for Y_2

As shown in Figures 4.23 - 4.25, the underlying value functions assumed by all the DMUs in the DEA efficiency assessments maintain the prototype preferential model assessed by UTASTAR on a sample of DMUs (Figures 4.20-4.22).

Chapter 5

Evaluation of the research activity of academic staff – A value based DEA approach

5.1 Introduction

Data Envelopment Analysis has been extensively used as a performance measurement framework in the different levels of the education sector (elementary, secondary and higher education). For example, Bessent and Bessent (1980), Bessent et al. (1982) and Chalos and Cherian (1995) utilized DEA to assess the efficiency of elementary schools. Arnold et al. (1996) evaluated the efficiency of 638 public secondary schools in Texas. As the results obtained by using Ordinary Least Squares (OLS) and Stochastic Frontier Analysis (SFA) were unsatisfactory, they used DEA in conjunction with regression analysis at a second phase to measure the impact of environmental factors and thus to improve the quality of the results. Bradley et al. (2001) assessed the technical efficiency of 2657 secondary schools in England by utilizing DEA to explain the inter-school variation in the observed efficiencies.

The literature on performance measurement in higher education is extensive and it is overviewed in the rest of this chapter. In this chapter, we develop an assessment framework to assess the research activity of academic staff in Higher Education (112 researchers, faculty members with Business and Economics of Greek Universities). The novelty of the proposed assessment approach is that it takes into account both the extent and the quality of the research work of the academics. To facilitate the incorporation of a quality aspect in the assessments, a value based piece-wise linear variant of the DEA model with intra- and inter – input/output value restrictions is employed. Assuming convex value functions for the publications in highly ranked journals and concave value functions for the publications in unranked journals, the quality research records are rewarded while the contribution of extensive publications in non-quality journals is diminished in the overall research performance. The effectiveness of the assessment framework in capturing the quality and the extent of the research work is further illustrated by comparing the results with those obtained by using standard DEA models.

Section 5.2 provides a literature review on performance measurement in Higher Education. Section 5.3 presents the motivation for developing the proposed assessment approach. Section 5.4 provides the factors (inputs and outputs) used and presents the descriptive statistics of the dataset. Section 5.5 presents the model that is employed and the parameters that were assumed in the assessment. The results and some concluding remarks are presented in sections 5.6 and 5.7 respectively.

5.2 Performance measurement in Higher Education - An overview

Traditionally, the assessment of Higher Education Institutes (HEIs) is based on teaching and research, which reflect, in a major extent, the quality of services that HEIs provide. Anderson and Walberg (1997) mentioned that, in education, it is difficult to use market mechanisms to determine the performance of an educational institution. Therefore, methods which encompass the essential characteristics of HEIs are needed. In earlier studies, the researchers attempted to synthesize the information produced by official agencies or public services in order to develop performance indicators. However, they attracted criticism due to controversial results; as Johnes and Taylor (1990) noticed, different indicators yield significant differing evaluations of the same HEI. Johnes (2006) explored the advantages and drawbacks of various methods for measuring the efficiency in the higher education sector and remarked the absence of input and output prices, the non-profit character of the institutions and the production of multiple outputs (e.g., research, teaching and community services) from multiple inputs. As Avkiran (2001) noticed, these inherent attributes render the higher education an attractive domain for DEA.

The application of DEA for the performance assessment in higher education has generally focused on the efficiencies of university programs or departments. A few years after the DEA was introduced by Charnes et al. (1978), the technique was straightforwardly applied to the higher education sector. For instance, Rhodes and

Southwick (1988) evaluated 96 public and 54 private universities in USA. Ahn et al. (1988) compared the efficiencies of U.S. universities obtained from DEA with findings derived by managerial accounting measures and econometric approaches. In the UK, Tomkins and Green (1988) studied twenty accounting departments of English universities using different DEA models. Beasley (1990) assessed chemistry and physics departments on the basis of research ratings. Johnes and Johnes (1993) used the data Research Assessment Exercise (RAE) of the year 1989 to assess the research performance of economics departments in the UK. The primary purpose of the periodic assessment RAE is to evaluate the quality of research undertaken by British HEIs and to support the distribution of the public funds for research. Similar assessment was conducted by Johnes (1995) on the basis of the data of RAE the year of 1992 by assessing the scale and technical efficiencies of economics departments in the UK with respect to their research output. Doyle et al. (1996) applied bootstrapping techniques and DEA on the basis of RAE - 1992 data to assess the research performance of business schools in the UK. They used cross-efficiencies to model peer appraisal and assurance regions to model various policy constraints. Athanassopoulos and Shale (1997) assessed the research performance of 45 universities in the UK using DEA by taking into account both the quantity and the quality of the research outputs as measured by research publications. They aggregated the publications of different categories in a single index obtained as a weighted average. Johnes (2006) assessed 100 English universities using data for the academic year 2000-2001. The input/output factors used were number of students, expenditures and grants provided by the Higher Education Funding Council for England (HEFCE).

Coelli (1996) studied the performance of the University of New England, Australia, relatively to 35 other Australian universities. He examined the performance of the academic and the administrative sections as well as the performance of the universities as a whole. He considered as outputs the number of students and the publication index (weighted by type) and various types of expenses and staff numbers as inputs. Avkiran (2001) applied also DEA in order to examine the relative efficiency of Australian universities. He developed three performance models, in particular one for the overall performance and two for the delivery of educational services and the fee-paying enrolments performance. In that study, he included the Research Quantum, which is the research component of Federal funds given to universities for their research activity, in order to reflect both the quality and quantity of research output (Department of Employment, Education, Training and Youth Affairs-DEETYA, 1997). Also, he noticed that the absence of market mechanisms to price educational outputs renders traditional production or cost functions inappropriate. Therefore, alternative efficiency analysis methods, such as DEA, should be used for the assessment of universities. Abbott and Doucouliagos (2003) utilized DEA to estimate technical and scale efficiency for the Australian public universities using data of 1995. They used a variety of output and input measures in order to illustrate the sensitivity of efficiency analysis. Concerning the measure of research output, they employed the Research Quantum Allocation that each university receives, as in the case of Avkiran (2001). They noted though, that it was the best measure of research output available for Australian universities. Stern et al. (1994) assessed 21 academic departments of the Ben-Gurion University in Israel. They used as inputs the operating costs and the salaries while as outputs they used the grants, the publications, the graduate students, and the contact hours. Korhonen et al. (2001) proposed the Value Efficiency Analysis approach as a means to incorporate the analyst's preferences in assessing the research performance of universities and research institutions. In particular, they used four composite indicators: quality of research, research activity, impact of research and doctoral student's activity, which were comprised of several simple indicators. For instance, the composite indicator quality of research comprised of the following simple indicators: number of articles published in international referred journals, scientific books and chapters in scientific books published by internationally well-known publishers and citations, which they were aggregated as a weighted average with the weights being obtained by experts. Kao and Hung (2008) assessed the relative efficiency of the academic departments at National Cheng Kung University in Taiwan. They used assurance-region constraints to incorporate a priori information provided by the top administrators of the university.

Ng and Li (2000) employed DEA for the assessment of the research performance of 84 key Chinese HEIs from 1993 to 1995. They used research staff and funding as inputs and publications data as outputs. Later, Johnes and Yu (2008) investigated the relative efficiency in the production of research of 109 Chinese regular universities in 2003 and 2004. They took into consideration the impact and the productivity of research as well as indicators regarding the staff, the students, the capital and the resources.

There is a limited number of studies conducted for assessing the performance of Greek academic institutions. Katharaki and Katharakis (2010) evaluated 20 Greek public universities by applying DEA and econometric models. Concerning the inputs, they included the number of academic staff with teaching and research activity, the number of non-academic staff, the number of active registered students and the operating expenses other than labour inputs such as expenditure of energy, non-salary expenses, administration services, buildings and grounds, libraries and student services. Concerning the outputs, they took into consideration the number of graduates including undergraduate, graduate and post-graduate degrees and the total economic resources of the university as a result of the research work, teaching and research staff. Also, Kounetas et al. (2011) assessed the research performance of the 18 academic departments of a single Greek University for the years 2001–2004. They considered six scenarios with various combinations of inputs and outputs. For instance, they considered the total expenditures, the number of the academic staff and the number of graduates as inputs, whilst they considered the number of publications, the number of conferences and the number of monographs as outputs. In addition, they applied a Tobit model in order to analyze the impact of the environmental effects on departmental efficiencies. They found that the infrastructure, the age and the schools' personnel have an important role. More recently, Halkos et al. (2012) estimated the performance of 16 departments of University of Thessaly by applying DEA and bootstrapping techniques. They used as inputs the number of academic staff, the number of auxiliary staff, the number of students and the total income, while as outputs teaching and research indicators.

5.3 Motivation and aim

Academic research is considered as one of the most important activities of academic staff in higher education. The extent and quality of academic research are determinants for the academics' appointment and advancement. However, the quality is a controversial topic because of the existence of a large volume of publications in journals of low quality. As the research activity in a university is strictly designated by the research activities of its staff members, the outcomes leverage its recognition and affect its position in international academic rankings (competitiveness). Moreover, there are countries where quality and performance issues play a crucial role in determining the funding that they receive from the government (e.g. in the UK and Australia). Therefore, the policy makers as well as the public draw significant attention to the results of the assessment of Higher Education Institutions (HEIs) and of their departments or faculties. Governments in many countries have already delivered policies with the aim to handle issues of accountability, cost control and enhancements of the quality of HEIs. In line with the above policies, in many countries periodical exercises are carried out by assessment bodies (committees). In the UK, for instance, the primary objective of Research Excellence Framework (REF) is the evaluation of the quality of research in publicly funded HEIs. It replaced the previous assessment system, last conducted in 2008, and named Research Assessment Exercise (RAE). In Australia, the Excellence in Research for Australia (ERA) initiative evaluates the quality of the research in Australian universities in order to provide advice to the Government on research matters and assist the National Competitive Grants Program (NCGP). Beyond the aforementioned initiatives, complementary policies, such as internal assessments are often adopted in many institutions. For instance, the research development group at Helsinki School of Economics established a two-person team in order to assess the research performance and assist the administration to the allocation of the resources (Korhonen et al., 2001).

In the sub-sequent sections we present an assessment framework to measure the performance of Greek universities academic staff. The aim is to encompass in the assessments both the volume as well as the quality of the research work. This is achieved by rewarding the researchers with qualitative research records (i.e.

publications in highly ranked journals with significant number of citations) and, contrary, by penalizing those that exhibit extensive publications in unranked journals with insignificant contribution.

5.4 Data

We study the research performance of 112 faculty members of Business and Economics Departments of Greek Universities. The factors (input and outputs) that are taken into account to measure the research performance of academics are summarized in Table 5.1 below.

Table 5.1: Input and Outputs included in the analysis

Input	-
I_{Yrs}	Time (years) since the first publication
Outputs	-
$O_{A+,A}$	Number of publications in highly ranked journals (rank A+ or A) according to the ERA 2010.
$O_{B,C}$	Number of publications in medium ranked journals (rank B or C) according to the ERA 2010.
O_D	Number of publications in unranked journals.
O_{CP}	Number of publications in proceedings of national and international conferences.
O_{RP}	Number of research projects that the researcher participated.
O_{Cit}	Number of citations (excluding self-citations) as per scopus.

A single input is used (I_{Yrs}) to measure the total time devoted in research by an academic since his/her first publication. Concerning the outputs, the publications are classified according to the quality of the journal they are published and they are treated as separate outputs. The journal rankings are drawn from the Excellence in Research for Australia (ERA) 2010 journal classification system, which classifies the journals in four quality classes (A+, A, B and C). A distinct class D is devoted to the journals that are not ranked in ERA. So, we considered three distinct outputs concerning journal publications ($O_{A+,A}$, $O_{B,C}$, and O_D) as shown in Table 5.1. The last three outputs (O_{CP}, O_{RP} and O_{Cit}) refer respectively to publications in conference proceedings, research projects that the individual has participated and the citations that the publications of the individual have received.

The data were drawn from Scopus, Google Scholar and the academic's personal Curriculum Vitaes (CVs). As the data may contain inaccuracies (for example outdated CVs) they are estimates of the research record rather than accurate performance metrics. However, the aim here is not to assess the true performance of the individuals or the institutions they belong to but only to provide the assessment framework with realistic data. Table 5.2 provides the descriptive statistics for the data that were estimated and used in the analysis.

Table 5.2: Descriptive statistics for the input and the outputs data

Variable	Mean	StDev	Minimum	Median	Maximum
I_{Yrs}	17.78	6.72	5.00	17.00	30.00
$O_{A+,A}$	6.14	4.67	0.00	5.000	19.00
$O_{B,C}$	11.74	7.51	0.00	10.00	29.00
O_D	14.88	10.93	0.00	13.00	47.00
O_{CP}	34.98	21.97	1.00	32.50	104.00
O_{RP}	6.35	4.58	1.00	5.000	15.00
O_{Cit}	56.23	78.21	0.00	23.00	350.00

As Avkiran (2001) mentioned, the performance indicators concerning the academic research are, among others, the number of publications. However, as there are major differences to the quality of the journals, it is necessary to classify the publications according to the quality of the journal they are published and then to aggregate them or to treat them as separate outputs. However, aggregation of the classes of the journals requires a priori knowledge about the relative importance among these classes and such information turns the assessment to be strict and inflexible (e.g. the DMUs cannot select the best weighting scheme among the classes of journals so as to achieve the maximum possible efficiency score).

In the current case study, publications are classified according to the quality of the journal they are published and treated as separated outputs. In addition, assurance region constraints (Type I) are introduced in the assessment framework so as to incorporate individual preferences over the relative importance of these classes. In contrast to the aggregation of the classes of journals to a single output, this modelling approach grants the flexibility to each evaluated scholar to select the most preferable weights over the classes remaining though consistent with the evaluator's preferences.

The journal rankings are drawn from the Excellence in Research for Australia (ERA) 2010 journal classification system. Journals which are not included in ERA are considered as unranked journals. The indicators used in ERA include a range of metrics such as citation profiles which are common to disciplines in the natural sciences, and peer review of a sample of research outputs which is more broadly common in the humanities and social sciences. ERA is a comprehensive collection. The data submitted by universities covers all eligible researchers and their research outputs. The precise set of indicators used has been developed in close consultation with the research community. This approach ensures that the indicators used are both appropriate and necessary, which minimizes the resourcing burden of ERA for Government and universities and ensures that ERA results are both robust and broadly accepted.

It is worthy to note here that the choice of the ERA journal classification system is an assumption in the current assessment framework. Other journal classification systems could be used instead. In this case though, because of the wide range of scope covered by the publications of the 112 academics of Business and Economics of Greek Universities under evaluation, a classification system that includes a wide list of journals was needed. ERA2010 is such a classification system as it comprises 20712 journals of a wide spectrum of scientific fields. For instance, the UK's Association of Business Schools (ABS) journal ranking includes a short list of journals relative to business and management science; as a result, it did not meet the needs of the current assessment.

5.5 Methodology

For the efficiency assessment of the academic staff we employ the output oriented VRS variant of the PL-DEA value based model (4.14) developed in chapter 4. The

VRS assumption is based on the fact that the corresponding outputs are not necessarily strictly analogous to the years since first publication. In addition, provided that an academic cannot reduce the years since his/her first publication an output orientation is chosen so as the targets to be based on the publications, conference proceedings, research projects and the citations that an academic should achieve given the total time devoted to research. The results of this set up acquire a meaningful interpretation for the academics and the policy makers; the efficiency scores denote the proportional expansion of all outputs so as inefficient academics to be rendered efficient and competitive. In order to facilitate the incorporation of the quality and the extent of the research output in the assessment, certain intra- and inter-variable constraints are introduced.

Intra-variable restrictions

To put emphasis on the quality of the research outcome, the outputs $O_{A+,A}$, $O_{B,C}$, O_D and O_{Cit} are considered as non-linear whereas the rest of the factors are assumed linear. Especially for the output $O_{A+,A}$, a convex value function is assumed so as to reward those academics showing high volume of quality publications. A single breakpoint is set to $b_{A+,A}^2 = 8$ that splits the range of values of $O_{A+,A}$ in two sub-intervals $\left[b_{A+,A}^1, b_{A+,A}^2\right]$ and $\left(b_{A+,A}^2, b_{A+,A}^3\right]$ where $b_{A+,A}^1 = l_{A+,A} = \min_j \{y_{(A+,A),j}\}$ and $b_{A+,A}^3 = h_{A+,A} = \max_j \{y_{(A+,A),j}\}$, while the convexity of the value function is driven by the condition $\frac{u_{(A+,A),1}}{u_{(A+,A),2}} \leq \frac{1}{2}$. Similar arrangements are made for the outputs $O_{B,C}$ and O_{Cit} for which the corresponding breakpoints are set to $b_{B,C}^2 = 18$ and $b_{Cit}^2 = 200$ respectively, and the convexity conditions are $\frac{u_{(B,C),1}}{u_{(B,C),2}} \leq 1$ and $\frac{u_{Cit,1}}{u_{Cit,2}} \leq \frac{1}{2}$. Contrarily, a concave value function is assumed for the output O_D so as to reduce the contribution of a large number of publications in non-quality journals in the efficiency. For this output the breakpoint is set to $b_D^2 = 20$ and the concavity of the value function is driven by the condition $\frac{u_{D,1}}{u_{D,2}} \geq 2$. Figures 5.1, 5.2, 5.3 and 5.4 present the non-linear value functions for the outputs $O_{A+,A}$, $O_{B,C}$, O_{Cit} and O_D respectively, based on the above parameters.

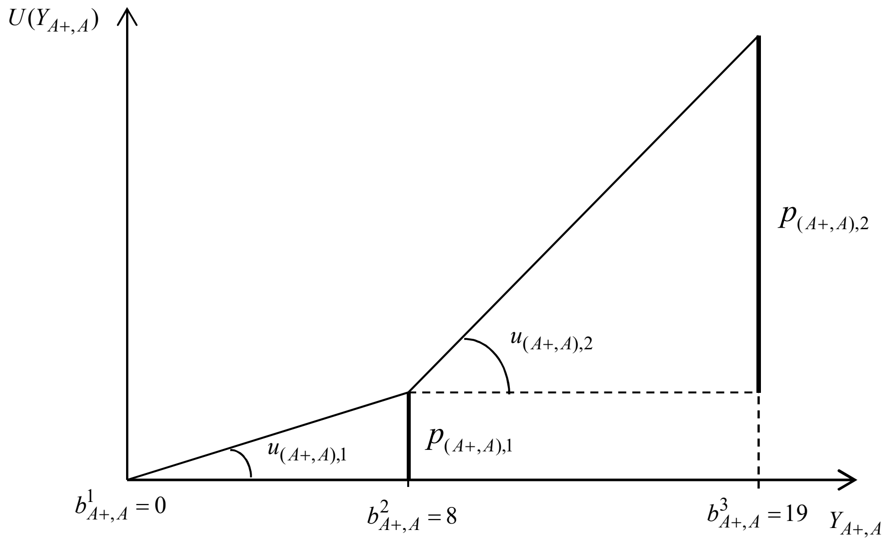


Figure 5.1: Convex value function for publications in A+, A journals

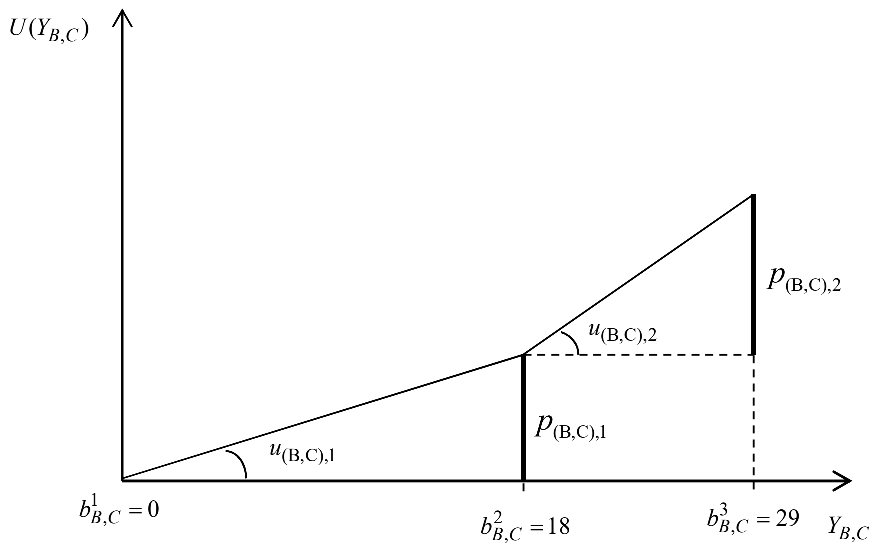


Figure 5.2: Convex value function for publications in B, C journals

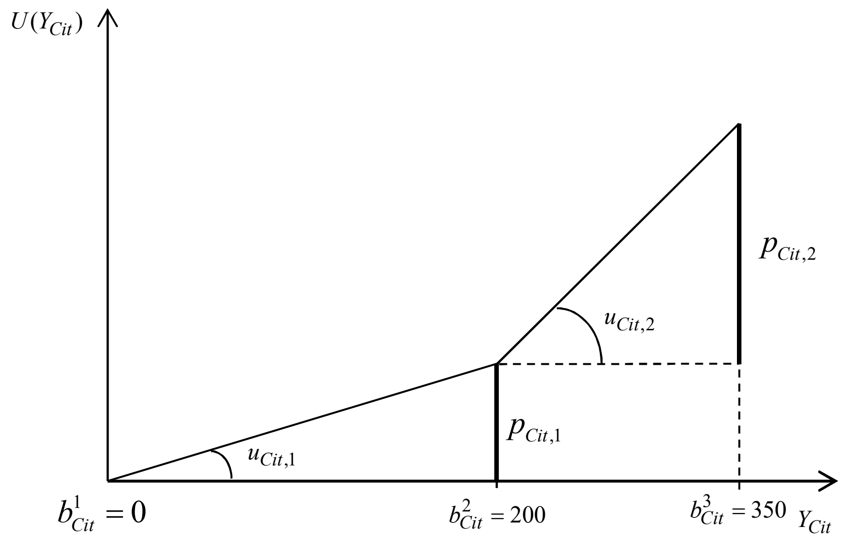


Figure 5.3: Convex value function for citations

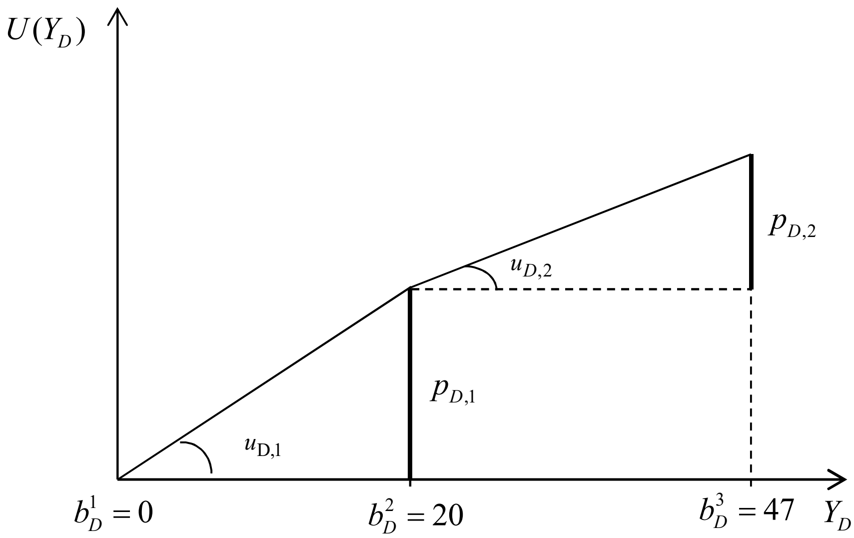
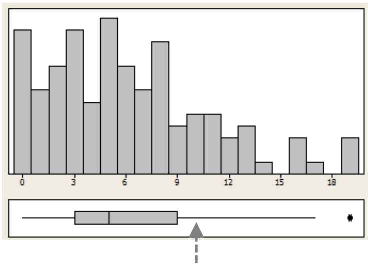


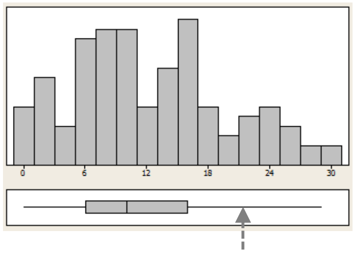
Figure 5.4: Concave value function for publications in unranked journals

The selection of the breakpoints is based on the distribution of the values of the corresponding outputs. Specifically, breakpoints were selected on points where the corresponding distributions were presenting a significant change. Additional information about the distribution of the values of the certain output factors is provided in Figures 5.5-5.8. The intra-variable restrictions are subjective estimates that can be considered as reflecting a hypothetical evaluator's point of view. These estimates play a crucial in the efficiency assessment and obviously different estimates may lead to different results.



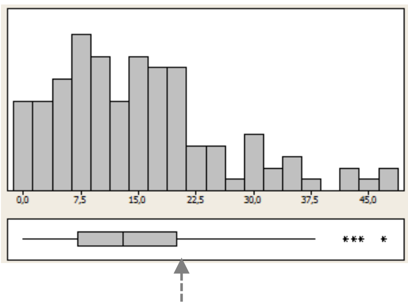
$b^1_{A+,A} = 0$ $b^2_{A+,A} = 8$ $b^3_{A+,A} = 19$

Figure 5.5: Publications in A+, A journals



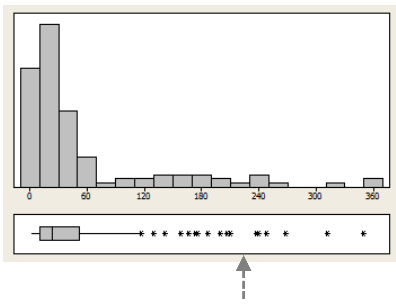
$b^1_{B,C} = 0$ $b^2_{B,C} = 18$ $b^3_{B,C} = 29$

Figure 5.6: Publications in B, C journals



$b^1_{D} = 0$ $b^2_{D} = 20$ $b^3_{D} = 47$

Figure 5.7: Publications in unranked journals



$b^1_{Cit} = 0$ $b^2_{Cit} = 200$ $b^3_{Cit} = 350$

Figure 5.8: Number of Citations

Inter-variable restrictions

In addition to the intra-variable restrictions that form the convex and concave value functions within the outputs, inter-variable restrictions are employed to define certain priorities across the outputs that describe the research outcome. Institutions and/or academics would normally have views as to the relative value of publications appearing in differently ranked journals and also the worth of citations versus publications. These views are subjective and possibly institution specific. Without claiming generality, and for illustrative purposes only, the inter-variable restrictions depicted in Table 5.3 have been incorporated in the assessment.

Table 5.3: Inter-variable value restrictions

*Value of Publications in A+, A journals ≥ 2*value of Publications in B, C journals*
Value of Publications in B, C journals ≥ 3 value of Publications in unranked journals*
Value of Publications in B, C journals ≥ 3 value of Conference proceedings*

Table 5.4 below summarizes the intra-variable and inter-variable constraints in terms of values incorporated in the value based PL-DEA model.

Table 5.4: Restrictions translated in terms of values

Intra-variable restrictions	Inter-variable restrictions
$\frac{p_{(A+,A),1}}{p_{(A+,A),2}} \leq \frac{1}{2}\frac{\left(b_{A+,A}^{2}-b_{A+,A}^{1}\right)}{\left(b_{A+,A}^{3}-b_{A+,A}^{2}\right)}$	$p_{(A+,A),1}+p_{(A+,A),2} \geq 2(p_{(\mathrm{B,C}),1}+p_{(\mathrm{B,C}),2})$
$\frac{p_{(\mathrm{B,C}),1}}{p_{(\mathrm{B,C}),2}} \leq \frac{\left(b_{B,C}^{2}-b_{B,C}^{1}\right)}{\left(b_{B,C}^{3}-b_{B,C}^{2}\right)}$	$p_{(\mathrm{B,C}),1}+p_{(\mathrm{B,C}),2} \geq 3(p_{D,1}+p_{D,2})$
$\frac{p_{Cit,1}}{p_{Cit,2}} \leq \frac{1}{2}\frac{\left(b_{Cit}^{2}-b_{Cit}^{1}\right)}{\left(b_{Cit}^{3}-b_{Cit}^{2}\right)}$	$p_{(\mathrm{B,C}),1}+p_{(\mathrm{B,C}),2} \geq 3p_{CP}$
$\frac{p_{D,1}}{p_{D,2}} \geq 2\frac{\left(b_{D}^{2}-b_{D}^{1}\right)}{\left(b_{D}^{3}-b_{D}^{2}\right)}$	

The output oriented VRS value based PL-DEA model utilized for the assessment takes the following form:

$$
\begin{aligned}
&\min \mathrm{E}_{j_0} = \hat{x}_{Yrs,j_0} q_{Yrs} - w_0 \\
&s.t. \\
&[\text{section } 1] \\
&\hat{\delta}^1_{(A+,A),j_0} p_{(A+,A),1} + \hat{\delta}^2_{(A+,A),j_0} p_{(A+,A),2} + \hat{\delta}^1_{(B,C),j_0} p_{(B,C),1} + \hat{\delta}^2_{(B,C),j_0} p_{(B,C),2} \\
&+\hat{\delta}^1_{D,j_0} p_{D,1} + \hat{\delta}^2_{D,j_0} p_{D,2} + \hat{y}_{CP,j_0} p_{CP} + \hat{y}_{RP,j_0} p_{RP} \\
&+\hat{\delta}^1_{(Cit),j_0} p_{(Cit),1} + \hat{\delta}^2_{(Cit),j_0} p_{(Cit),2} = 1 \\
&\hat{\delta}^1_{(A+,A),j} p_{(A+,A),1} + \hat{\delta}^2_{(A+,A),j} p_{(A+,A),2} + \hat{\delta}^1_{(B,C),j} p_{(B,C),1} + \hat{\delta}^2_{(B,C),j} p_{(B,C),2} \\
&+\hat{\delta}^1_{D,j} p_{D,1} + \hat{\delta}^2_{D,j} p_{D,2} + \hat{y}_{CP,j} p_{CP} + \hat{y}_{RP,j} p_{RP} \\
&+\hat{\delta}^1_{(Cit),j} p_{(Cit),1} + \hat{\delta}^2_{(Cit),j} p_{(Cit),2} - \hat{x}_{Yrs,j} q_{Yrs} + w_0 \le 0, \ \ (j = 1,\ldots,112) \\
\\
&[\text{section } 2] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.1) \\
&2\left(b^3_{A+,A} - b^2_{A+,A}\right) p_{(A+,A),1} - \left(b^2_{A+,A} - b^1_{A+,A}\right) p_{(A+,A),2} \le 0 \\
&\left(b^3_{B,C} - b^2_{B,C}\right) p_{(B,C),1} - \left(b^2_{B,C} - b^1_{B,C}\right) p_{(B,C),2} \le 0 \\
&2\left(b^3_{Cit} - b^2_{Cit}\right) p_{Cit,1} - \left(b^2_{Cit} - b^1_{Cit}\right) p_{Cit,2} \le 0 \\
&2\left(b^2_{D} - b^1_{D}\right) p_{D,2} - \left(b^3_{D} - b^2_{D}\right) p_{D,1} \le 0 \\
\\
&[\text{section } 3] \\
&2 p_{(B,C),1} + 2 p_{(B,C),2} - p_{(A+,A),1} - p_{(A+,A),2} \le 0 \\
&3 p_{D,1} + 3 p_{D,2} - p_{(B,C),1} - p_{(B,C),2} \le 0 \\
&3 p_{CP} - p_{(B,C),1} - p_{(B,C),2} \le 0 \\
\\
&p_{(.)}, q_{(.)} \ge 0 \\
&w_0 \in \Re
\end{aligned}
$$

where the [section 1], [section 2] and [section 3] comprise the ordinary DEA constraints, the intra-variable restrictions and the inter-variable restrictions respectively.

5.6 Results

By applying the model (5.1) and the standard DEA model (2.11) for comparison purposes, a significant reduction of the efficient academics as well as of the average efficiency scores is observed (Table 5.5).

Table 5.5: Number of efficient academics and average efficiency scores

	Standard DEA model (2.11)	Value based PL-DEA model (5.1)
Number of efficient researchers	27	17
Average efficient score	0.641	0.425

To further illustrate the effectiveness of the value based PL-DEA approach in capturing the quality and the extent of the research output of the academics, two examples are analyzed.

(i) A subset of ten poor performing researchers satisfying the condition $I_{Yrs} \geq 20$, $O_{A+,A} \leq 4$ and $O_D \geq 17$ has been identified. The average values of efficiency score in cases of standard DEA model (2.11) and the Value Based PL-DEA model (5.1) are 0.563 and 0.199 respectively, indicating a significant reduction of their efficiency. None of them is detected as efficient by model (5.1) in contrast to model (2.11) which identified two of the ten academics as efficient.

(ii) Three academics #1, #2 and #3 are selected as typical cases representing a well performing academics with adequate years of research activity (case #1) and two young academics with significant and a poor activity (cases #2 and #3 respectively). Their performance and efficiency scores are presented in Table 5.6. The results show that the quality and extent of research activity in cases #1 and #2 has been rewarded (efficiency scores = 1) and the poor performance in case #3 has been further penalized by the Value based PL-DEA model (5.1).

Table 5.6: Research records and performance of three characteristic cases

Factor	Case #1	Case #2	Case #3
I_{Yrs}	28	9	6
$O_{A+,A}$	19	8	1
$O_{B,C}$	28	20	0
O_{D}	34	31	7
O_{CP}	91	15	14
O_{RP}	15	5	1
O_{Cit}	268	35	2
Efficiency Scores			
Value based PL-DEA model (5.1)	1	1	0.389
Standard DEA model (2.11)	1	1	0.823

5.7 Concluding remarks

In this chapter we developed a framework for assessing the research performance of the academic staff, which aims to encompass in the assessments both the volume as well as the quality of the research output. For the efficiency assessment we utilized the value based PL-DEA model that we developed in chapter 4. The effectiveness of this approach is justified by comparing the value based results with those obtained by the standard DEA model. The assessment exercise presented in this chapter is based on a number of assumptions, such as the selection of data sources, the classification of journals employed and the external preferential information adapted, that do affect the results. Nevertheless, these are parameters of the proposed framework, which can be adjusted by policy makers so as to reflect their value judgments.

Part B

NETWORK DATA ENVELOPMENT ANALYSIS: NEW MODELS AND APPLICATIONS

Chapter 6

Network DEA

6.1 Introduction

The conventional DEA models are based on the assumption that the internal structure of the Decision Making Unit is unknown. That is, the units are treated as black boxes with only the levels of the inputs that enter the system and the levels of the outputs that leave the system being known. Network DEA is an extension of conventional DEA, which takes into account the internal structure and the flow of intermediate measures of the DMUs. Network DEA conceives the production process that characterizes the DMUs as a network of sub-processes. Several models have been proposed in the network DEA literature. Castelli et al. (2010) provide a comprehensive categorized overview of models and methods developed for different multi-stage production configurations. Kao (2014a) provides a thorough classification of studies in network DEA, according to the type of the network structure and the model employed.

In this chapter, after a short overview of network DEA literature (section 6.2), we focus on and outline the four basic network DEA approaches established in the literature, namely, the *independent approach* (section 6.3), the *multiplicative decomposition approach* introduced by Kao and Hwang (2008) (section 6.4), the *additive decomposition approach* introduced by Chen, Cook, Li and Zhu (2009) (section 6.5) and the *SBM approach* introduced by Tone and Tsutsui (2009) (section 6.6). In section 6.7 we spot some limitations of the aforementioned basic approaches.

6.2 Network DEA: An overview

Lee and Billington (1992) where among the first who denoted the pitfalls in supply chain management and highlighted that although its overall performance depends on the joint performance of its sites (nodes of the supply chain), in most cases, the management of each site is performed by autonomous teams. Thus, measures for the overall performance of the supply chain should be adopted. In this line of thought, Fare and Whittaker (1995) and Fare and Grosskopf (1996), based on Fare (1991), utilized DEA to evaluate the efficiency in network structures, where the outputs from one process are used as inputs to another one (intermediate measures). They formulated the network activity as a DEA model and they presented the technology for each node (stage) of the network.

Wang et al. (1997) employed DEA in assessing the information technology impact on the performance of firms by assuming a two-stage series production process. They assessed the efficiency of each stage independently. Seiford and Zhu (1999) employed a two-stage structure in order to evaluate the overall performance of 55 U.S. commercial banks. In their setting, the first stage represented profitability and the second one marketability. They employed the independent approach and they calculated the efficiency score of each stage separately.

Cook et al. (2000) claimed that, in network structures, there are situations where particular inputs are shared among the stages. To this end, they developed a network DEA model, which determines the best resource split so as to maximize the overall efficiency of the network process. They considered the overall efficiency of the system as a convex combination of the individual stage efficiencies.

Zhu (2000) applied the independent assessment approach to determine a multi-dimensional financial performance model for the assessment of the Fortune 500 companies.

Cook and Hababou (2001) extended the additive DEA model and developed a dual-component measure to assess the performance of bank branches in both sales and service with shared inputs.

Chen and Zhu (2004) developed a linear model for two-stage series production processes, by taking into account the intermediate measures that link the two stages, in order to locate the efficient frontier of the production possibility set i.e. to provide information on how to project an inefficient DMU on the frontier so as to be rendered overall efficient.

Chen, Liang, Yang and Zhu (2006), proposed a non-linear model for assessing the stage and the overall efficiency series multistage production processes. The average of the stage efficiencies is maximized and the model is solved as a parametric linear program.

Chen, Liang and Yang (2006), approached the efficiency in supply chains as a DEA game model. They focused on a supply chain with two members and they showed that there are numerous Nash equilibrium efficiency plans for the two members.

Kao and Hwang (2008) introduced the multiplicative approach in two stage series processes. They assumed that the overall efficiency of the system is the product of the efficiencies of the two sub-processes. Thus, they proposed a linear program to estimate the overall efficiency of the system and then, they decomposed the overall efficiency to the stage efficiencies. They also introduced a technique to check the uniqueness of the stage efficiencies. This approach has drawn significant attention from the scientific community.

Tone and Tsutsui (2009) extended the SBM model in complex network structures. Within their setting, they provided input oriented, output oriented and non-oriented efficiency scores. Their model is applicable in both CRS and VRS situations and it provides projections for the inefficient units.

Chen, Cook, Li and Zhu (2009), introduced the additive efficiency decomposition in two-stage processes. They assumed that the overall efficiency of the system is a weighted average of the stage efficiencies. Assuming that the weights of the stage efficiencies should reflect their importance, they represented the "size" of each stage as the portion of total resources devoted to each stage. This representation allowed them to transform their model into a linear program and to estimate the overall efficiency of the system. Stage efficiencies are calculated a posteriori and similarly to the technique that Kao and Hwang (2008) introduced, they developed a technique to check the uniqueness of the stage efficiencies. An advantage of the additive efficiency decomposition approach over the multiplicative one is its straightforward extension to VRS situations.

Chen, Liang and Zhu (2009) studied the relationship between the models of Kao and Hwang (2008) and Chen and Zhu (2004) and they showed their equivalence.

Chen et al. (2010) provided an approach to derive the DEA frontier for two stage processes according to the multiplicative approach.

Kao (2009a, 2009b, 2012, 2013 and 2014b) extended the multiplicative approach in general network structures with parallel and series sub-systems.

Chen et al. (2013) discussed the pitfalls in network DEA concerning the estimation of the stage efficiencies, the efficient frontier and the projections of the inefficient units on the efficient frontier. They pointed out that the multiplier models and their duals, i.e. the envelopment models, use different concepts of efficiency and thus, their equivalence does not necessarily holds. They claimed that the projections of the inefficient units on the efficient frontier should be determined by the envelopment-based DEA models whereas the stages efficiencies should be estimated by the multiplier-based DEA models.

Lim and Zhu (2016) developed formulas to obtain frontier projections and divisional efficiency scores for two-stage processes, using the primal and dual solutions obtained by a multiplicative network DEA model.

Recently, Despotis et al. (2016) criticized the additive efficiency decomposition approach introduced by Chen, Cook, Li and Zhu (2009) and they proved that the efficiency scores obtained by the additive efficiency decomposition model are biased. They introduced the composition paradigm, where the efficiencies of the stages are estimated first and the overall efficiency of the system is obtained ex post. Their network DEA model provides unique and unbiased stage efficiency scores.

Applications of network DEA include: Avkiran (2009) and Fukuyama and Matousek (2011) where network DEA is employed to assess the efficiency of banks in the United Arab Emirates and Turkey respectively; Zhu (2011) and Adler et al. (2013) where the performance of airlines and airports is measured by presenting the production processes as network activities; Chen et al. (2012) where the performance of incineration plants in Taiwan is measured by using multi-activity network DEA.

6.3 The independent assessments approach

Wang et al. (1997) used DEA in assessing information technology (IT) impact on firm performance. In their case study they assumed a two-stage series production process where the first stage uses three inputs (IT budget, fixed assets, and employees) to produce one output (deposits), which is then used as the only input for the second stage, which in turn produces the final outputs of the whole production process (profits and %loans recovered). Figure 6.1 depicts the structure of the two-stage series production process.

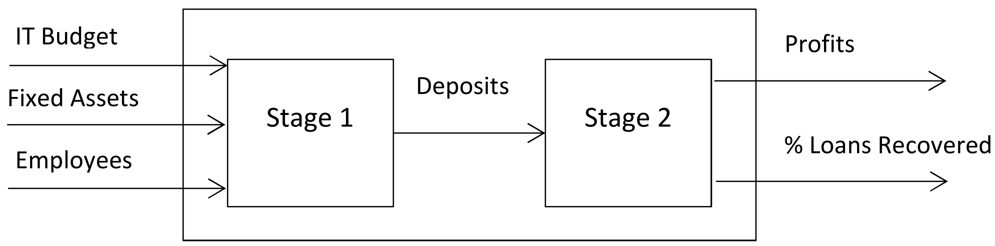


Figure 6.1: Example of a two-stage series production process

As presented in Figure 6.1, “deposits” is both output from the first stage and input to the second stage. Such factors are generally treated as intermediate measures, which link the sub-processes and play a key role in the efficiency assesment.

Wang et al. (1997) was the first to use the so called independent assessments approach. Specifically, treating the two stages independently they estimated the stage and the overall efficiency of the system by using standard DEA models. The overall efficiency of the system is defined as the ratio of the total virtual external outputs over the total virtual external inputs, ignoring the intermediate measures. The models (6.1-

6.3) are employed to calculate the efficiency of the first stage, the second stage and the overall efficiency of the system respectively.

$$
\begin{aligned}
& e_{j_0}^{1} = \max \ wZ_{j_0} \\
& s.t. \\
& vX_{j_0} = 1 \\
& wZ_j - vX_j \leq 0, \ j = 1,2,\ldots,n \\
& v, w \geq 0
\end{aligned}
\qquad (6.1)
$$

$$
\begin{aligned}
& e_{j_0}^{2} = \max \ uY_{j_0} \\
& s.t. \\
& wZ_{j_0} = 1 \\
& uY_{j_0} - wZ_{j_0} \leq 0, j = 1,2,\ldots,n \\
& w, u \geq 0
\end{aligned}
\qquad (6.2)
$$

$$
\begin{aligned}
& e_{j_0}^{0} = \max \ uY_{j_0} \\
& s.t. \\
& vX_{j_0} = 1 \\
& uY_j - vX_j \leq 0, j = 1,2,\ldots,n \\
& v, w, u \geq 0
\end{aligned}
\qquad (6.3)
$$

A rational convention is that a DMU to be characterized as overall efficient it must be efficient in all stages. Models that fail to incorporate this relation, can lead to misleading results. Actually, this shortcoming of the independent approach is attributed to the fact that the link among the sub processes is ignored in the evaluation process. The intermediate measures have a conflicting role in the efficiency assessments (e.g. the first stage aims to maximize their worth whereas the second stage aims to minimize it). Thus, this linkage affects the efficiency scores of the sub processes (stages) as well as the overall efficiency.

As the independent approach ignores the internal structure of a production process, it can be applied in order to estimate *realistic* upper bounds (ideal efficiency scores) for the efficiencies of the sub processes as well as for the overall one. Nevertheless, when a *cooperative* model that incorporates the linkage of the intermediate production processes is applied, the ideal efficiency scores of the independent approach are not always achievable.

6.4 The multiplicative efficiency decomposition approach

Kao and Hwang (2008) introduced a novel approach for the efficiency assessment in series two-stage processes where the first stage transforms external inputs to a number of intermediate measures, which then are used as inputs to the second stage that produces the final outputs as depicted in Figure 6.2.

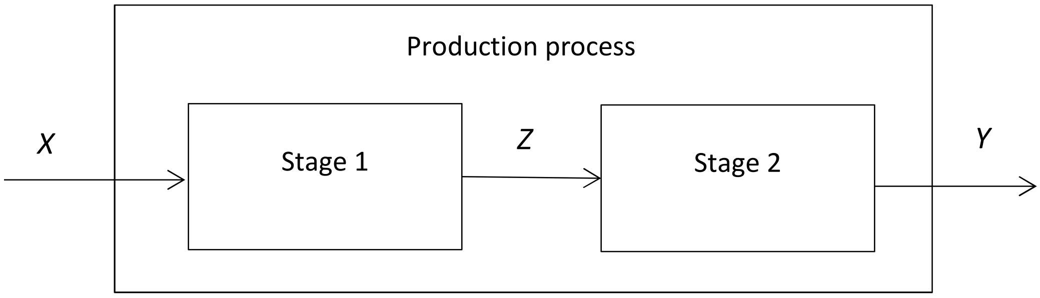


Figure 6.2: Representation of a two-stage series production process

Their approach is based on the reasonable assumption that the values of the intermediate measures (virtual intermediate measures) are the same, no matter if they are considered as outputs of the first stage or inputs to the second stage. For the evaluated DMU j_o, they define the overall efficiency of the system as the ratio of the total virtual external outputs over the total virtual external inputs $e^0_{j_0} = \frac{uY_{j_0}}{vX_{j_0}}$. The efficiencies of the first and the second stage are respectively $e^1_{j_0} = \frac{wZ_{j_0}}{vX_{j_0}}$ and $e^2_{j_0} = \frac{uY_{j_0}}{wZ_{j_0}}$. Thus, the overall efficiency of the system is the product of the efficiencies of the two sub-processes: $e^0_{j_0} = e^1_{j_0} \cdot e^2_{j_0}$.

Model (6.4) below is the fractional model to calculate the overall efficiency of the system and model (6.5) is the corresponding linear one which derives by applying the C-C transformation.

$$\begin{aligned} &\max \ \frac{uY_{j_0}}{vX_{j_0}} \\ &s.t. \\ &\frac{wZ_j}{vX_j} \le 1,\ j=1,2,...,n \\ &\frac{uY_j}{wZ_j} \le 1, j=1,2,...,n \\ &v,w,u \ge 0 \end{aligned} \qquad (6.4)$$

$$\begin{aligned} &\max \ uY_{j_0} \\ &s.t. \\ &vX_{j_0} = 1 \\ &wZ_j - vX_j \le 0,\ j=1,2,...,n \\ &uY_j - wZ_j \le 0, j=1,2,...,n \\ &v,w,u \ge 0 \end{aligned} \qquad (6.5)$$

Let $\left(v^*, w^*, u^*\right)$ be an optimal solution of model (6.5). Then, the overall efficiency of the system is $e_{j_0}^0 = \frac{u^* Y_{j_0}}{v^* X_{j_0}} = u^* Y_{j_0}$. The stage efficiencies can be then obtained from the optimal solution of model (6.5) as follows:

$$e_{j_0}^1 = \frac{w^* Z_{j_0}}{v^* X_{j_0}} = w^* Z_{j_0},\ e_{j_0}^2 = \frac{u^* Y_{j_0}}{w^* Z_{j_0}} = \frac{e_{j_0}^o}{e_{j_0}^1}$$

As the optimal solution of model (6.5) is not necessarily, the decomposition $e_{j_0}^0 = e_{j_0}^1 \cdot e_{j_0}^2$ may be not unique either. To deal with this the non-uniqueness issue, Kao and Hwang (2008) proposed a post-optimality phase where the efficiency of the first or the second stage (according to the priority given by the analyst) is maximized while maintaining the optimal overall efficiency of the system derived by model (6.5), as shown below.

$$\begin{aligned} &e_{j_0}^{1\max} = \max \ wZ_{j_0} \\ &s.t. \\ &vX_{j_0} = 1 \\ &wZ_j - vX_j \le 0,\ j=1,2,...,n \\ &uY_{j_0} - wZ_{j_0} \le 0, j=1,2,...,n \\ &uY_{j_0} = e_{j_0}^0 \\ &v,w,u \ge 0 \end{aligned} \qquad (6.6)$$

$$\begin{aligned} &e_{j_0}^{2\max} = \max \ uY_{j_0} \\ &s.t. \\ &wZ_{j_0} = 1 \\ &wZ_j - vX_j \le 0,\ j=1,2,...,n \\ &uY_{j_0} - wZ_{j_0} \le 0, j=1,2,...,n \\ &uY_{j_0} - e_{j_0}^0 vX_{j_0} = 0 \\ &v,w,u \ge 0 \end{aligned} \qquad (6.7)$$

where $e_{j_0}^{0}$ is the overall efficiency of the system as derived by model (6.5). Once $e_{j_0}^{1\max}$ is calculated by model (6.6), the efficiency of the second stage is $e_{j_0}^{2-} = \frac{e_{j_0}^{0}}{e_{j_0}^{1\max}}$. Analogously, if priority is given to the second stage, the efficiency of the first stage is $e_{j_0}^{1-} = \frac{e_{j_0}^{0}}{e_{j_0}^{2\max}}$. Notice that if $e_{j_0}^{1\max} = e_{j_0}^{1-}$ and $e_{j_0}^{2\max} = e_{j_0}^{2-}$ then, the efficiency decomposition is unique. However, the uniqueness of the stage efficiencies in the multiplicative approach does not necessarily holds.

Although the multiplicative model is not straightforwardly extended to fit to VRS situations, Kao and Hwang (2011) proposed a method to decompose technical and scale efficiencies.

6.4.1 Extensions to complex network structures

Kao (2014b) extended the multiplicative decomposition approach to general multi-stage processes arguing that any structure can be transformed into a series of parallel structures.

Multi-stage series processes

Assume a production process composed of q sub-processes (stages) in series, where the external inputs X enter the system through the first stage and external outputs Y are produced by the final stage q. Each one of the stages $1,\ldots,q-1$ produces intermediate measures $Z^{(1)},\ldots, Z^{(q-1)}$ that are used as inputs to the sub-sequent stage $2,\ldots,q$ respectively, as depicted in Figure 6.3.

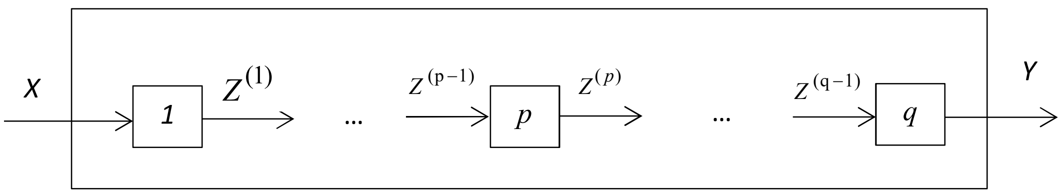


Figure 6.3: Representation of a multi-stage series production process

The model (6.8) provides the overall efficiency and the stage efficiencies of the system

$$
\begin{aligned}
&\max\ uY_{j_0} \\
&s.t. \\
&vX_{j_0} = 1 \\
&w^{(1)}Z_j^{(1)} - vX_j \le 0,\ j = 1,2,...,n \\
&w^{(p)}Z_j^{(p)} - w^{(p-1)}Z_j^{(p-1)} \le 0, j = 1,2,...,n; p = 2,...,q-1 \\
&uY_j - w^{(q-1)}Z_j^{(q-1)} \le 0, j = 1,2,...,n \\
&v, u \ge 0 \\
&w^{(p)} \ge 0, p = 1,...,q-1
\end{aligned}
\tag{6.8}
$$

where $Z^{(p)}$ $\left(p = 1,...,q-1\right)$ are the outputs of the p sub-process (intermediate measures) and $w^{(p)}$ the associated vector of weights. Let $\left(v^*, w^{*(1)},..., w^{*(q-1)}, u^*\right)$ be an optimal solution of model (6.8) when DMU j_0 is evaluated. Then, the overall efficiency of the system as well as the stage efficiencies for DMU j_0 are obtained as follows:

$$
\begin{aligned}
e_{j_0}^0 &= \frac{u^*Y_{j_0}}{v^*X_{j_0}} = u^*Y_{j_0} \\
e_{j_0}^1 &= \frac{w^{*(1)}Z_{j_0}^{(1)}}{v^*X_{j_0}} \\
e_{j_0}^p &= \frac{w^{*(p)}Z_{j_0}^{(p)}}{w^{*(p-1)}Z_{j_0}^{(p-1)}}, p = 2,...,q-1 \\
e_{j_0}^q &= \frac{u^*Y_{j_0}}{w^{*(q-1)}Z_{j_0}^{(q-1)}}
\end{aligned}
\tag{6.9}
$$

As noted previously, the stage efficiencies obtained by model (6.8) are not necessarily unique.

Parallel structures

The parallel structure is defined as a process, which is composed of independent sub-processes that can operate simultaneously. Assume a system with q parallel sub-processes that use $X^{(p)},\ p=1,...,q$ inputs to produce $Y^{(p)},\ p=1,...,q$ outputs as depicted in Figure 6.4. The summations $X=\sum_{p=1}^{q}X^{(p)}$, $Y=\sum_{p=1}^{q}Y^{(p)}$ denote the total system inputs and the total system outputs respectively.

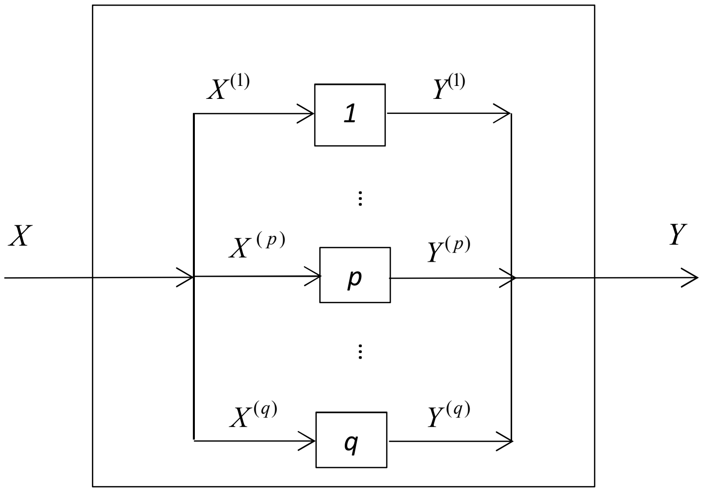


Figure 6.4: General representation of a parallel structure

To estimate the overall efficiency of the system as well as the efficiencies of each sub-process the following model (6.10) is employed

$$
\begin{aligned}
&\max\ uY_{j_0} \\
&s.t. \\
&vX_{j_0}=1 \\
&uY_j^{(p)}-vX_j^{(p)}\le 0, j=1,2,...,n; p=1,...,q \\
&v,u\ge 0
\end{aligned}
\qquad (6.10)
$$

Let $\left(v^*,u^*\right)$ be an optimal solution of model (6.10) when DMU j_0 is evaluated. Then, the overall efficiency of the system and the stage efficiencies are obtained as follows:

$$e_{j_0}^{0} = \frac{u^* Y_{j_0}}{v^* X_{j_0}} = u^* Y_{j_0}$$
$$e_{j_0}^{p} = \frac{u^* Y_{j_0}^{(\mathrm{p})}}{v^* X_{j_0}^{(\mathrm{p})}}, p = 1, \ldots, \mathrm{q} \tag{6.11}$$

The overall system efficiency is decomposed as a weighted average of the stage efficiencies where the weight for each stage is defined as the portion of the total virtual inputs that the stage consumes over the total virtual inputs consumed by the whole system, as follows:

$$\sum_{p=1}^{q} \omega^{(p)} e_{jo}^{p} = \sum_{p=1}^{q} \left[\frac{v^* X_{j_0}^{(p)}}{v^* X_{j_0}} * \frac{u^* Y_{j_0}^{(p)}}{v^* X_{j_0}^{(p)}} \right] = \frac{u^* Y_{j_0}}{v^* X_{j_0}} = e_{j_0}^{0} \tag{6.12}$$

Notice that the weight attached to each stage is a function of the variables of model (6.10) and they are calculated endogenously on the basis of the optimal solution of model (6.10). So, the weights assigned to the stages differ from one DMU to another and consequently, different priority is assumed to the stages for each DMU. Because model (6.10) may have multiple optimal solutions, the efficiency score of each stage as well as its associated weight in equation (6.12) may not be unique either.

Efficiency decomposition of general multistage processes

Kao (2014b) suggests that "*the key to decompose the system efficiency of a general multi-stage system is to find a transformation of series and parallel structures*". This can be achieved by introducing dummy processes so as to transfer the external inputs and the external outputs dedicated to particular sub-processes throughout the system. Figure 6.5 below exhibits a general multi-stage system and Figure 6.6 illustrates the transformed one with three series sub-systems, where each one of them has a parallel structure. Notice, that the circular nodes denote dummy processes.

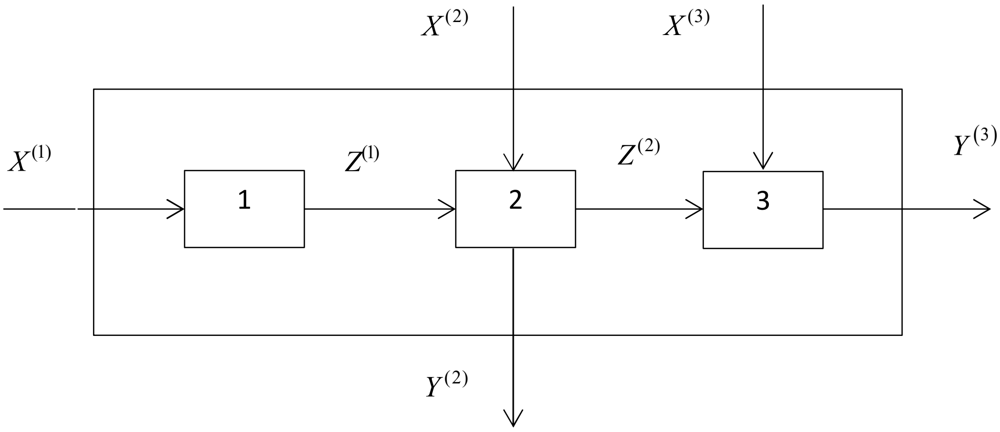


Figure 6.5: General multi-stage system

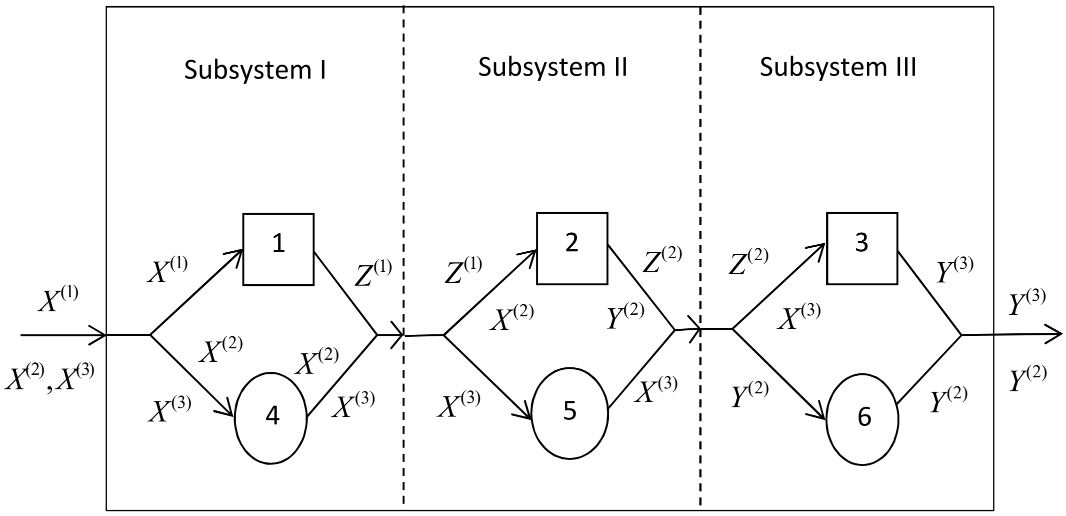


Figure 6.6: Transformed multi-stage system

The overall efficiency of the system depicted in Figure 6.5 is estimated by model (6.13).

$$
\begin{aligned}
& e^0_{j_o} = \max u^{(2)}Y^{(2)}_{j_o} + u^{(3)}Y^{(3)}_{j_o} \\
& s.t. \\
& v^{(1)}X^{(1)}_{j_o} + v^{(2)}X^{(2)}_{j_o} + v^{(3)}X^{(3)}_{j_o} = 1 \\
& w^{(1)}Z^{(1)}_{j} - v^{(1)}X^{(1)}_{j} \le 0, j = 1,...,n \\
& u^{(2)}Y^{(2)}_{j} + w^{(2)}Z^{(2)}_{j} - v^{(2)}X^{(2)}_{j} - w^{(1)}Z^{(1)}_{j} \le 0, j = 1,...,n \\
& u^{(3)}Y^{(3)}_{j} - v^{(3)}X^{(3)}_{j} - w^{(2)}Z^{(2)}_{j} \le 0, j = 1,...,n \\
& v^{(1)}, v^{(2)}, v^{(3)}, w^{(1)}, w^{(2)}, u^{(2)}, u^{(3)} \ge 0
\end{aligned}
\tag{6.13}
$$

According to the previous notations, the overall efficiency of the system, when DMU j_0 is evaluated, can be decomposed to the product of the efficiencies of the series sub-systems $e^0_{jo} = E^1_{jo} * E^2_{jo} * E^3_{jo}$. Each sub-system has a parallel structure and it is composed from one real and one dummy process whose efficiency is equal to one. Thus, the efficiency of each sub-system is calculated as follows:

$$
\begin{aligned}
E^1_{j_0} &= \omega^{(1)}e^1_{jo} + \omega^{(4)}e^4_{jo} = \omega^{(1)}e^1_{jo} + (1-\omega^{(1)}) \\
E^2_{j_0} &= \omega^{(2)}e^2_{jo} + \omega^{(5)}e^5_{jo} = \omega^{(2)}e^2_{jo} + (1-\omega^{(2)}) \\
E^3_{j_0} &= \omega^{(3)}e^3_{jo} + \omega^{(6)}e^6_{jo} = \omega^{(3)}e^3_{jo} + (1-\omega^{(3)})
\end{aligned}
$$

where

$$
\begin{aligned}
\omega^{(1)} &= v^{*(1)}X^{(1)}_{j_o} / (v^{*(1)}X^{(1)}_{j_o} + v^{*(2)}X^{(2)}_{j_o} + v^{*(3)}X^{(3)}_{j_o}) \\
\omega^{(2)} &= (w^{*(1)}Z^{(1)}_{j_o} + v^{*(2)}X^{(2)}_{j_o}) / (w^{*(1)}Z^{(1)}_{j_o} + v^{*(2)}X^{(2)}_{j_o} + v^{*(3)}X^{(3)}_{j_o}) \\
\omega^{(3)} &= (w^{*(2)}Z^{(2)}_{j_o} + v^{*(3)}X^{(3)}_{j_o}) / (w^{*(2)}Z^{(2)}_{j_o} + v^{*(3)}X^{(3)}_{j_o} + u^{*(2)}Y^{(2)}_{j_o})
\end{aligned}
$$

6.5 The additive efficiency decomposition approach

Chen, Cook, Li and Zhu (2009), introduced the additive efficiency decomposition approach for the simple two-stage process as depicted in Figure 6.2. They define the overall efficiency of the system as a weighted average of the stage efficiencies $e^0_{j_0} = t^1 e^1_{j_0} + t^2 e^2_{j_0}$ with $t^1 + t^2 = 1$. The stage efficiencies $\left(e^1_{j_0}, e^2_{j_0}\right)$ are defined as in the multiplicative approach $e^1_{j_0} = \frac{wZ_{j_0}}{vX_{j_0}}$, $e^2_{j_0} = \frac{uY_{j_0}}{wZ_{j_0}}$.

However, when the weights $\left(t^1, t^2\right)$ are treated as user-defined parameters, the authors ended to a non-linear model. For the sake of linearity, the weights $\left(t^1, t^2\right)$ are defined as functions of the decision variables in a manner that they reflect the size of each stage as viewed by the portion of the total resources devoted to each stage as follows:

$$t_j^1 = \frac{vX_j}{vX_j + wZ_j}, \quad t_j^2 = \frac{wZ_j}{vX_j + wZ_j}$$

Thus, the overall efficiency of the system is:

$$e_{j_0}^0 = t_{j_0}^1 e_{j_0}^1 + t_{j_0}^2 e_{j_0}^2 = \frac{vX_{j_0}}{vX_{j_0} + wZ_{j_0}} \cdot \frac{wZ_{j_0}}{vX_{j_0}} + \frac{wZ_{j_0}}{vX_{j_0} + wZ_{j_0}} \cdot \frac{uY_{j_0}}{wZ_{j_0}} = \frac{wZ_{j_0} + uY_{j_0}}{vX_{j_0} + wZ_{j_0}}$$

and it is obtained by the fractional model (6.14) or its linear equivalent (6.15), which is derived by applying the C-C transformation.

$$\begin{aligned}
&\max \ \frac{wZ_{j_0} + uY_{j_0}}{vX_{j_0} + wZ_{j_0}} \\
&s.t. \\
&\frac{wZ_j}{vX_j} \le 1, \ j = 1, 2, ..., n \\
&\frac{uY_j}{wZ_j} \le 1, j = 1, 2, ..., n \\
&v, w, u \ge 0
\end{aligned} \qquad (6.14)$$

$$\begin{aligned}
&\max \ wZ_{j_0} + uY_{j_0} \\
&s.t. \\
&vX_{j_0} + wZ_{j_0} = 1 \\
&wZ_j - vX_j \le 0, \ j = 1, 2, ..., n \\
&uY_{j_0} - wZ_{j_0} \le 0, j = 1, 2, ..., n \\
&v, w, u \ge 0
\end{aligned} \qquad (6.15)$$

Let $\left(v^*, w^*, u^*\right)$ be an optimal solution of model (6.15). Then, the overall efficiency of the system is $e_{j_0}^0 = \frac{w^* Z_{j_0} + u^* Y_{j_0}}{v^* X_{j_0} + w^* Z_{j_0}} = w^* Z_{j_0} + u^* Y_{j_0}$. The stage efficiencies can be obtained from the optimal solution of model (6.15) as follows:

$$e_{j_0}^1 = \frac{w^* Z_{j_0}}{v^* X_{j_0}}, \ e_{j_0}^2 = \frac{u^* Y_{j_0}}{w^* Z_{j_0}}$$

As in the case of the multiplicative approach, the decomposition of the overall efficiency to the stage efficiencies may not be unique. To deal with this issue, Chen, Cook, Li and Zhu (2009) followed the post-optimality check introduced by Kao and Hwang (2008). To this end, according to the priority given by the analyst, they maximize the efficiency of the first or the second stage by the models (6.16) and (6.17) below, respectively.

$$
\begin{aligned}
& e_{j_0}^{1\max} = \max \ wZ_{j_0} \\
& s.t. \\
& vX_{j_0} = 1 \\
& wZ_j - vX_j \leq 0, \ j = 1,2,...,n \\
& uY_j - wZ_j \leq 0, \ j = 1,2,...,n \\
& \left(1 - e_{j_0}^0\right) wZ_{j_0} + uY_{j_0} = e_{j_0}^0 \\
& v, w, u \geq 0
\end{aligned}
\tag{6.16}
$$

$$
\begin{aligned}
& e_{j_0}^{2\max} = \max \ uY_{j_0} \\
& s.t. \\
& wZ_{j_0} = 1 \\
& wZ_j - vX_j \leq 0, \ j = 1,2,...,n \\
& uY_j - wZ_j \leq 0, \ j = 1,2,...,n \\
& wZ_{j_0} + uY_{j_0} - e_{j_0}^0 vX_{j_0} = e_{j_0}^0 \\
& v, w, u \geq 0
\end{aligned}
\tag{6.17}
$$

If priority is given to the first stage, model (6.16) is applied to calculate the maximum efficiency of the first stage while maintaining the overall efficiency of the system $\left(e_{j_0}^0\right)$ as estimated by model (6.15). Once the maximum efficiency of the first stage is obtained, the efficiency of the second stage is calculated as $e_{j_0}^{2-} = \dfrac{e_{j_0}^0 - t_{j_0}^{*1} e_{j_0}^{1\max}}{t_{j_0}^{*2}}$ where $t_{j_0}^{*1}$ and $t_{j_0}^{*2}$ derive form the optimal solution of model (6.15). Analogously, if priority to the second stage is given, the maximum efficiency of the second stage is estimated by applying model (6.17). Once the optimal value of the objective function $\left(e_{j_0}^{2\max}\right)$ is calculated, the efficiency of the first stage is calculated as $e_{j_0}^{1-} = \frac{e_{j_0}^{0} - t_{j_0}^{*2} e_{j_0}^{2\max}}{t_{j_0}^{*1}}$. Notice that the efficiency decomposition is unique only if $e_{j_0}^{1\max} = e_{j_0}^{1-}$ and $e_{j_0}^{2\max} = e_{j_0}^{2-}$, which generally does not hold.

6.5.1 Extension to VRS

The additive efficiency decomposition approach is readily extended under the VRS assumption as given in the fractional model (6.18) and its linear equivalent (6.19) below.

$$\begin{aligned}
&\max \ \frac{wZ_{j_0} + \omega^1 + uY_{j_0} + \omega^2}{vX_{j_0} + wZ_{j_0}} \\
&s.t. \\
&\frac{wZ_j + \omega^1}{vX_j} \le 1, \ j = 1,2,\ldots,n \\
&\frac{uY_j + \omega^2}{wZ_j} \le 1, j = 1,2,\ldots,n \\
&v, w, u \ge 0 \\
&\omega^1, \omega^2 \in \Re
\end{aligned} \tag{6.18}$$

$$\begin{aligned}
&\max \ wZ_{j_0} + \omega^1 + uY_{j_0} + \omega^2 \\
&s.t. \\
&vX_{j_0} + wZ_{j_0} = 1 \\
&wZ_j + \omega^1 - vX_j \le 0, \ j = 1,2,\ldots,n \\
&uY_{j_0} + \omega^2 - wZ_{j_0} \le 0, j = 1,2,\ldots,n \\
&v, w, u \ge 0 \\
&\omega^1, \omega^2 \in \Re
\end{aligned} \tag{6.19}$$

Once the overall efficiency of the system is calculated, models (6.20) and (6.21) can be applied to estimate the maximum efficiency of the first and the second stage respectively. After calculating the maximum efficiency of one stage, the efficiency score of the other one can be estimated in an analogous manner as in the CRS case.

$$
\begin{aligned}
& e_{j_0}^{1\max} = \max \; wZ_{j_0} + \omega^1 \\
& s.t. \\
& vX_{j_0} = 1 \\
& wZ_j + \omega^1 - vX_j \le 0, \; j = 1,2,...,n \\
& uY_j + \omega^2 - wZ_j \le 0, \; j = 1,2,...,n \\
& \left(1 - e_{j_0}^0\right) * wZ_{j_0} + uY_{j_0} + \omega^1 + \omega^2 = e_{j_0}^0 \\
& v, w, u \ge 0 \\
& \omega^1, \omega^2 \in \Re
\end{aligned}
\tag{6.20}
$$

$$
\begin{aligned}
& e_{j_0}^{2\max} = \max \; uY_{j_0} + \omega^2 \\
& s.t. \\
& wZ_{j_0} = 1 \\
& wZ_j + \omega^1 - vX_j \le 0, \; j = 1,2,...,n \\
& uY_j + \omega^2 - wZ_j \le 0, \; j = 1,2,...,n \\
& wZ_{j_0} + uY_{j_0} - e_{j_0}^0 * vX_{j_0} + \omega^1 + \omega^2 = e_{j_0}^0 \\
& v, w, u \ge 0 \\
& \omega^1, \omega^2 \in \Re
\end{aligned}
\tag{6.21}
$$

6.6 The Slacks-Based Measure for network DEA

The multiplicative and the additive efficiency decomposition approaches utilize the radial measure of efficiency. Tone and Tsutsui (2009), developed an alternative network DEA approach (network SBM) which employs the slacks-based measure. The advantage of this approach is based on the estimation of the efficiency score when changes in inputs and outputs are not proportional. They used the weighted SBM (Tsutsui and Goto, 2009). Specifically, they set exogenous weights on the stages so as to incorporate the importance of the stages in the efficiency assessment.

Under the assumption of variable returns to scale, the evaluated DMU j_0 is expressed as follows:

$$
\begin{aligned}
& x_o^k = X^k \lambda^k + s^{k-} \\
& y_o^k = Y^k \lambda^k - s^{k+} \\
& e\lambda^k = 1 \\
& \lambda^k \geq 0, s^{k-} \geq 0, s^{k+} \geq 0 \ , k = 1,...,K
\end{aligned}
\tag{6.22}
$$

where K is the number of the stages, $X^k = \left(x_1^k,...,x_n^k\right)$, $Y^k = \left(y_1^k,...,y_n^k\right)$ and k denotes the k^{th} stage $k = 1,...,K$.

They considered two cases to describe the way two stages *k* and *h* ($h \neq k$) are linked by means of the intermediate measures. In the first case, it is assumed that the intermediate measures can be freely determined by the optimization (free link assumption), a situation that it represented by the following constraints:

$$
Z^{(k,h)} \lambda^h = Z^{(k,h)} \lambda^k \tag{6.23}
$$

In the second case, it is assumed that the intermediate measures are fixed (fix link assumption) with the corresponding constraints being as follows:

$$
\begin{aligned}
& z_o^{(k,h)} = Z^{(k,h)} \lambda^h \\
& z_o^{(k,h)} = Z^{(k,h)} \lambda^k
\end{aligned}
\tag{6.24}
$$

Depending on the orientation selected, three models have been proposed to assess the efficiency scores of the units.

Input oriented network SBM

Model (6.25) below is proposed to assess the input oriented efficiency score of the evaluated unit j_0:

$$
\begin{aligned}
&\theta_o^* = \min \ \sum_{k=1}^{K} w^k \left[1 - \frac{1}{m_k} \left(\sum_{i=1}^{m_k} \frac{s_i^{k-}}{x_{io}^k} \right) \right] \\
&s.t \\
&x_o^k = X^k \lambda^k + s^{k-} \\
&y_o^k = Y^k \lambda^k - s^{k+} \\
&e\lambda^k = 1 \\
&\lambda^k \geq 0, s^{k-} \geq 0, s^{k+} \geq 0 \ , \ k = 1,...,K \\
&\qquad and \\
&Z^{(k,h)} \lambda^h = Z^{(k,h)} \lambda^k \\
&\qquad or \\
&z_o^{(k,h)} = Z^{(k,h)} \lambda^h \\
&z_o^{(k,h)} = Z^{(k,h)} \lambda^k
\end{aligned}
\tag{6.25}
$$

where m_k is the number of inputs that the k stage consumes, w^k represents the relative importance of stage k with $\sum_{k=1}^{K} w^k = 1$. The constraints for the intermediate measures depend on the assumption made (free or fixed link). Let $\left(\theta_o^*, \lambda^{k*}, s^{k-*}, s^{k+*}\right)$ be an optimal solution of model (6.25). If $\theta_o^* = 1$, then the evaluated DMU is overall efficient. The stage efficiency scores as well as their relation with the system efficiency are given in the following equations:

$$
\theta_k = 1 - \frac{1}{m_k} \left(\sum_{i=1}^{m_k} \frac{s_i^{k-*}}{x_{io}^k} \right), \ \theta_o^* = \sum_{k=1}^{K} w^k \theta_\kappa \tag{6.26}
$$

Output oriented network SBM

To implement the output orientation, it is sufficient to replace the objective function in model (6.25) with the following:

$$
\frac{1}{\tau_o^*} = \max \sum_{k=1}^{K} w^k \left[1 + \frac{1}{r_k} \left(\sum_{r=1}^{r_k} \frac{s_r^{k+}}{y_{ro}^k} \right) \right] \tag{6.27}
$$

where r_k is the number of outputs that stage k produces. The stage efficiency scores as well as their relation with the overall efficiency are given in the following equations:

$$\tau_\kappa = \frac{1}{1+\frac{1}{r_k}\left(\sum_{r=1}^{r_k}\frac{s_r^{k+*}}{y_{ro}^k}\right)}, \quad \frac{1}{\tau_o^*} = \sum_{\kappa=1}^{\mathrm{K}}\frac{w_k}{\tau_\kappa} \tag{6.28}$$

Non-oriented network SBM

If no orientation is assumed, then the efficiency assessment can be performed by model (6.25), where the objective function is replaced by the following:

$$\rho_o^* = \min \frac{\sum_{k=1}^{K} w^k\left[1-\frac{1}{m_k}\left(\sum_{i=1}^{m_k}\frac{s_i^{k-}}{x_{io}^k}\right)\right]}{\sum_{k=1}^{K} w^k\left[1+\frac{1}{r_k}\left(\sum_{r=1}^{r_k}\frac{s_r^{k+}}{y_{ro}^k}\right)\right]} \tag{6.29}$$

Then, the stage efficiencies can be obtained analogously from the following equation:

$$\rho_\kappa = \frac{1-\frac{1}{m_k}\left(\sum_{r=1}^{m_k}\frac{s_i^{k-*}}{x_{i0}^k}\right)}{1+\frac{1}{r_k}\left(\sum_{r=1}^{r_k}\frac{s_r^{k+*}}{y_{r0}^k}\right)} \tag{6.30}$$

However, as Chen et al. (2013) pointed out, the network SBM approach cannot be applied in multi-stage processes where the stages have no additional external inputs and/or outputs and thus, the network SBM approach cannot be conceived as a general network DEA approach.

6.7 Drawbacks and limitations

The efficiency assessment in network structures is not straightforward. The stages (nodes of the network) are linked with intermediate measures, which are treated both as outputs from one stage and inputs to another stage. Thus, the intermediate measures have a conflicting role in the efficiency assessment. The standard DEA models (independent approach) do not take into account the linkage of the stages in the efficiency assessment and thus, they lead to misleading results (Wang et al., 1997). Several methods have been proposed in the network DEA literature to overcome this issue, which can generally unfold in two general approaches; the decomposition approach and the composition approach.

The decomposition approach is based on the assumption that the stages should cooperate so as the system to achieve its maximum efficiency. Thus, in this approach, the overall efficiency of the system is estimated first and then, the stage efficiencies are obtained ex post from the optimal solution. It is noteworthy that the proposed methods that are based on the decomposition approach differ only in the definition of the overall efficiency. For example the multiplicative method of Kao and Hwang (2008) assumes that the overall efficiency of the system is the squared geometric mean of the stages efficiencies whereas the additive model introduced by Chen, Cook, Li and Zhu (2009) assumes that the overall efficiency of the system is a weighted average of the stage efficiencies. However, the stage efficiencies obtained by the methods materializing the efficiency decomposition approach are not unique. This is the main drawback of all the efficiency decomposition methods. Moreover, the additive efficiency decomposition method provides biased stage efficiency scores, which is an additional drawback of this method. The multiplicative efficiency decomposition method, apart from that it is limited only to cases where constant returns to scale are assumed, when maximizing the overall efficiency of a unit, may implicitly, yet unreasonably, assume different DMU-specific priorities for the stages. Thus, the decomposition of the overall efficiency to the stage efficiencies may bias the efficiency assessments in favor of one stage over the other and it does not provide the analyst with the necessary information to communicate the results, as concerns the priorities of the stages.

Unlike the efficiency decomposition approach, in the composition approach, introduced by Despotis et al. (2016), the efficiencies of the two stages are estimated first and the overall efficiency of the DMU is obtained ex post. A major advantage of the assessment method presented in Despotis et al. (2016), over the additive and the multiplicative decomposition methods is that the former provides unique and unbiased efficiency scores for two-stage processes. Its disadvantage, however, is that it cannot be readily extended in series processes with more than two stages. This is an effect of the different orientations selected for the first and the second stage, which in fact was made to simplify the models and keep them within the field of linear programming (simplicity at the expense of generality).

In the next chapter, we provide a novel approach that extends the composition paradigm in general multi-stage processes and eliminates the drawbacks and the limitations mentioned above.

Chapter 7

A novel network DEA approach for general series multi-stage processes

7.1 Introduction

In this chapter, we introduce the composition paradigm in general series multi-stage processes, by proposing a multi-objective programming approach. Without harming simplicity, our approach overcomes the lack of generality in Despotis et al. (2016), as long as our model and the solution method proposed can handle any type of series multi-stage process. Our developments make the direct comparison of the new approach with the multiplicative method (Kao and Hwang, 2008) possible and fruitful, in a manner that enables us to point out some critical issues that one should take into account when using the multiplicative decomposition method. Unlike the additive (Chen, Cook, Li and Zhu, 2009) and the multiplicative efficiency decomposition (Kao and Hwang, 2008) methods, our new general approach secures the uniqueness of the efficiency scores. Moreover, the efficiency assessments are neutral, in the sense that no implicit priority is assumed for some stages over the others.

The chapter is organized as follows. Section 7.2 is devoted to two-stage processes. We identify four distinct types of processes that cover all possible configurations. In sub-section 7.2.1 we introduce our modeling approach in detail with respect to the elementary two-stage process, which assumes that nothing but the external inputs to the first stage enters the system and nothing but the outputs of the second stage leaves the system. A thorough comparison of our method with the multiplicative approach (Kao and Hwang, 2008) highlights the advantages of the former and points out some critical shortcomings of the latter. In sub-sections 7.2.2 to 7.2.4, we generalize our approach to more complicated two-stage configurations.

When case data are available in the literature, we compare the results obtained by our method with those from other methods. Otherwise, we provide synthetic data and the corresponding results for testing and validation. In section 7.3 we extend our formulations in general multi-stage processes. Conclusions are drawn in section 7.4.

7.2 Two-stage processes

In this section we develop our novel network DEA approach for the case of two-stage series processes. We follow the composition paradigm introduced in Despotis et al. (2016). In the composition paradigm, as opposed to the decomposition approach (Kao and Hwang, 2008, Chen, Cook, Li and Zhu, 2009), the stage efficiencies are estimated without any a priori definition of the overall efficiency of the system. In Despotis et al. (2016), once the stage efficiencies are estimated, the overall efficiency is computed a posteriori by aggregating the stage efficiencies additively or multiplicatively. In this chapter we define the overall efficiency of the system as the ratio of the weighted external outputs over the weighted external inputs of the system.

We consider four types of processes that cover all possible two-stage series configurations, as depicted in Fig. 7.1.

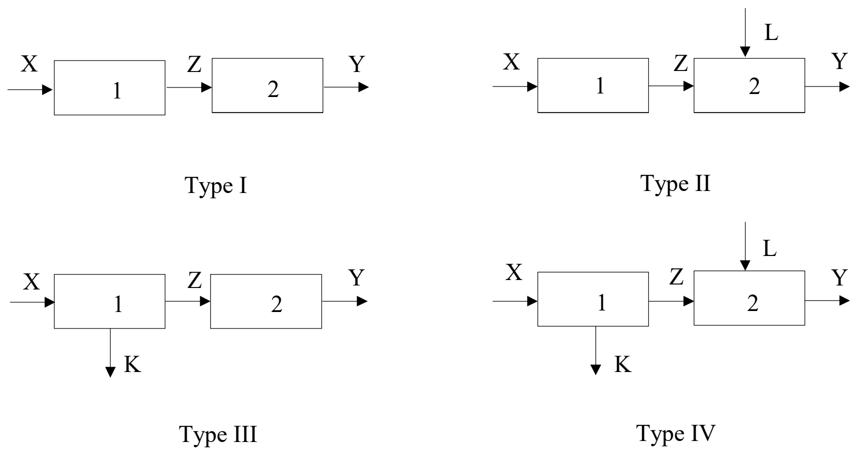


Figure 7.1: The four types of series two-stage processes

Let us introduce the following basic notation:

$j \in J = \{1,\ldots,n\}$: The index set of the *n* DMUs.

$j_0 \in J$: Denotes the evaluated DMU.

$X_j = \left(x_{ij}, i = 1,\ldots,m\right)$: The vector of stage-1 external inputs used by DMU_j (all types).

$Z_j = \left(z_{pj}, p = 1,\ldots,q\right)$: The vector of intermediate measures for DMU_j (all types).

$Y_j = \left(y_{rj}, r = 1,\ldots,s\right)$: The vector of stage-2 final outputs produced by DMU_j (all types).

$L_j = \left(l_{dj}, d = 1,\ldots,a\right)$: The vector of stage-2 external inputs (types I and IV).

$K_j = \left(k_{cj}, c = 1,\ldots,b\right)$: The vector of stage-1 final outputs (types III and IV).

$\eta = \left(\eta_1,\ldots,\eta_m\right)$: The vector of weights for the stage-1 external inputs in the fractional model.

$v = \left(v_1,\ldots,v_m\right)$: The vector of weights for the stage-1 external inputs in the linear model.

$\varphi = \left(\varphi_1,\ldots,\varphi_q\right)$: The vector of weights for the intermediate measures in the fractional model.

$w = \left(w_1,\ldots,w_q\right)$: The vector of weights for the intermediate measures in the linear model.

$\omega = \left(\omega_1,\ldots,\omega_s\right)$: The vector of weights for the stage-2 outputs in the fractional model.

$u = \left(u_1,\ldots,u_s\right)$: The vector of weights for the stage-2 outputs in the linear model.

$\gamma = \left(\gamma_1,\ldots,\gamma_a\right)$: The vector of weights for the stage-2 external inputs.

$\mu = \left(\mu_1,\ldots,\mu_b\right)$: The vector of weights for the stage-1 final outputs.

e_j^o : The overall efficiency of DMU_j.

e_j^1: The efficiency of the first stage for DMU$_j$.

e_j^2: The efficiency of the second stage for DMU$_j$.

E_j^1: The independent efficiency score of the first stage for DMU$_j$.

E_j^2: The independent efficiency score of the first stage for DMU$_j$.

7.2.1 Type I structure

Consider the elementary case (Type I) where each DMU transforms some external inputs X to final outputs Y via the intermediate measures Z with a two-stage process, as depicted in Fig. 7.1. In this basic setting, nothing but the external inputs to the first stage enters the system and nothing but the outputs of the second stage leaves the system. Typically, the efficiency of the first and the second stage of a DMU j are defined as follows:

$$e_j^1 = \frac{\varphi Z_j}{\eta X_j},\ e_j^2 = \frac{\omega Y_j}{\varphi Z_j}$$

The overall efficiency of DMU j is defined as the ratio of the total virtual exogenous output to the total virtual exogenous input:

$$e_j^o = \frac{\omega Y_j}{\eta X_j}$$

Consider the basic input oriented CRS-DEA models that estimate the stage-1 and the stage-2 efficiency for the evaluated unit j_0 independently:

$$\begin{aligned} & \mathrm{E}_{j_0}^1 = \max \frac{\varphi Z_{j_0}}{\eta X_{j_0}} \\ & s.t. \\ & \varphi Z_j - \eta X_j \le 0,\quad j = 1,\ldots,n \\ & \eta \ge \varepsilon, \varphi \ge \varepsilon \end{aligned} \tag{7.1}$$

$$\begin{aligned}
& \mathrm{E}_{j_0}^2 = \max \frac{\omega Y_{j_0}}{\varphi Z_{j_0}} \\
& s.t. \\
& \omega Y_j - \varphi Z_j \leq 0, \quad j = 1, \ldots, n \\
& \varphi \geq \varepsilon, \omega \geq \varepsilon
\end{aligned} \tag{7.2}$$

In order to link the efficiency assessments of the two stages, it is universally accepted that the weights associated with the intermediate measures are the same, no matter if these measures are considered as outputs of the first stage or inputs to the second stage. Appending the constraints of model (7.1) to model (7.2) and vice versa we get the following augmented models (7.3) and (7.4) for the first and the second stage respectively:

$$\begin{aligned}
& \mathrm{E}_{j_0}^1 = \max \frac{\varphi Z_{j_0}}{\eta X_{j_0}} \\
& s.t. \\
& \varphi Z_j - \eta X_j \leq 0, \quad j = 1, \ldots, n \\
& \omega Y_j - \varphi Z_j \leq 0, \quad j = 1, \ldots, n \\
& \eta \geq \varepsilon, \varphi \geq \varepsilon, \omega \geq \varepsilon
\end{aligned} \tag{7.3}$$

$$\begin{aligned}
& \mathrm{E}_{j_0}^2 = \max \frac{\omega Y_{j_0}}{\varphi Z_{j_0}} \\
& s.t. \\
& \varphi Z_j - \eta X_j \leq 0, \quad j = 1, \ldots, n \\
& \omega Y_j - \varphi Z_j \leq 0, \quad j = 1, \ldots, n \\
& \eta \geq \varepsilon, \varphi \geq \varepsilon, \omega \geq \varepsilon
\end{aligned} \tag{7.4}$$

As noticed in Despotis et al. (2016), the optimal solutions of (7.1) and (7.2) are also optimal in (7.3) and (7.4) respectively. Models (7.3) and (7.4) have common constraints and, thus, they form the following bi-objective program:

$$\begin{aligned}
&\max \frac{\varphi Z_{j_0}}{\eta X_{j_0}} \\
&\max \frac{\omega Y_{j_0}}{\varphi Z_{j_0}} \\
&s.t. \\
&\varphi Z_j - \eta X_j \leq 0, \quad j = 1, \ldots, n \\
&\omega Y_j - \varphi Z_j \leq 0, \quad j = 1, \ldots, n \\
&\eta \geq \varepsilon, \varphi \geq \varepsilon, \omega \geq \varepsilon
\end{aligned} \tag{7.5}$$

Applying the C-C transformation with respect to the first objective function, i.e. multiplying all the terms of the fractional objective functions and the constraints by $t > 0$, such that $t\eta X_{j_0} = 1$ and setting $t\eta = v, t\omega = u, t\varphi = w$ we get the following equivalent bi-objective program, whose second objective function is still fractional.

$$\begin{aligned}
&\max wZ_{j_0} \\
&\max \frac{uY_{j_0}}{wZ_{j_0}} \\
&s.t. \\
&vX_{j_0} = 1 \\
&wZ_j - vX_j \leq 0, \quad j = 1, \ldots, n \\
&uY_j - wZ_j \leq 0, \quad j = 1, \ldots, n \\
&v \geq \varepsilon, w \geq \varepsilon, u \geq \varepsilon
\end{aligned} \tag{7.6}$$

Solving the linear equivalents of models (7.3) and (7.4) one gets the independent efficiency scores $\mathrm{E}^1_{j_0}$ and $\mathrm{E}^2_{j_0}$ of the two stages respectively. In terms of multi-objective programming (MOP), the vector $\left(\mathrm{E}^1_{j_0}, \mathrm{E}^2_{j_0}\right)$ constitutes the ideal point of the bi-objective program (7.6) in the objective functions space. The efficiencies of the two stages can be obtained by solving the bi-objective program (7.6). However, as the ideal point is not generally attainable, solving a MOP means finding efficient (Pareto optimal) solutions in the variable space that are mapped on the Pareto front in the objective functions space, i.e. solutions that they cannot be altered to increase the value of one objective function without decreasing the value of at least one other objective function. The model (7.7) below employs the weighted Tchebycheff norm

(L_∞ norm) to locate a point on the Pareto front, by minimizing the maximum of the weighted deviations $t_1\left(\mathrm{E}_{j_0}^1 - e_{j_0}^1\right)$ and $t_2\left(\mathrm{E}_{j_0}^2 - e_{j_0}^2\right)$ of $\left(e_{j_0}^1 = wZ_{j_0}, e_{j_0}^2 = uY_{j_0} / wZ_{j_0}\right)$ from the ideal point $\left(\mathrm{E}_{j_0}^1, \mathrm{E}_{j_0}^2\right)$, with weights $t_1 > 0$ and $t_2 > 0$.

$$
\begin{aligned}
&\min \delta \\
&s.t. \\
&t_1(\mathrm{E}_{j_0}^1 - wZ_{j_0}) \leq \delta \\
&t_2\left(\mathrm{E}_{j_0}^2 - \frac{uY_{j_0}}{wZ_{j_0}}\right) \leq \delta \\
&vX_{j_0} = 1 \\
&wZ_j - vX_j \leq 0, \quad j = 1,\ldots,n \\
&uY_j - wZ_j \leq 0, \quad j = 1,\ldots,n \\
&v \geq \varepsilon, w \geq \varepsilon, u \geq \varepsilon, \delta \geq 0
\end{aligned}
\tag{7.7}
$$

Every optimal solution of (7.7) is *weakly* efficient (*weakly* Pareto optimal) solution for (7.6) (Ehrgott, 2000). At optimality, at least one the first two constraints in (7.7) will be binding. Assuming that there is no stated preference information that gives priority to one of the two stages, we employ in our assessments the unweighted Tchebycheff norm, i.e. we assume $t_1 = t_2 = 1$, and we get the following:

$$
\begin{aligned}
&\min \delta \\
&s.t. \\
&\mathrm{E}_{j_0}^1 - wZ_{j_0} \leq \delta \\
&\left(\mathrm{E}_{j_0}^2 - \delta\right) wZ_{j_0} - uY_{j_0} \leq 0 \\
&vX_{j_0} = 1 \\
&wZ_j - vX_j \leq 0, \quad j = 1,\ldots,n \\
&uY_j - wZ_j \geq 0, \quad j = 1,\ldots,n \\
&v \geq \varepsilon, w \geq \varepsilon, u \geq \varepsilon, \delta \geq 0
\end{aligned}
\tag{7.8}
$$

Although model (7.8) is non-linear, it can be easily solved by bisection search (c.f. Despotis, 1996). Clearly, $0 \leq \delta \leq 1$. Hence bisection search can be performed in the bounded interval [0,1] as follows. Let $\underline{\delta}$ be a lower bound of δ for which the constraints of (7.8) are not consistent (initially $\underline{\delta} = 0$) and $\overline{\delta}$ an upper bound of δ

for which the constraints are consistent (initially $\overline{\delta}=1$). Then, the consistency of the constraints is tested for $\delta'=\left(\underline{\delta}+\overline{\delta}\right)/2$. If they are consistent, δ' will replace $\overline{\delta}$; if they are not it will replace $\underline{\delta}$. The bisection continues until both bounds come sufficiently close to each other. Let $\left(\delta^*,v^*,w^*,u^*\right)$ be an optimal solution of (7.8) and

$$e_{j_0}^{1*}=\frac{w^*Z_{j_0}}{v^*X_{j_0}}=w^*Z_{j_0},\; e_{j_0}^{2*}=\frac{u^*Y_{j_0}}{w^*Z_{j_0}}$$

The model (7.9) below provides a Pareto optimal solution to (7.6). The model (7.9) is equivalent to employing lexicographically (in a second phase) the L_1 norm on the set of optimal solutions of (7.8) (see, e.g. Steuer and Choo, 1983).

$$\begin{aligned}
&\max s_1+s_2\\
&s.t.\\
&\mathrm{E}_{j_0}^1-wZ_{j_0}+s_1=\delta^*\\
&\left(\mathrm{E}_{j_0}^2-\delta^*\right)wZ_{j_0}-uY_{j_0}+s_2w^*Z_{j_0}=0\\
&vX_{j_0}=1\\
&wZ_j-vX_j\le 0,\;\; j=1,\ldots,n\\
&uY_j-wZ_j\le 0,\;\; j=1,\ldots,n\\
&v\ge\varepsilon, w\ge\varepsilon, u\ge\varepsilon\\
&\delta^*\ge s_1\ge 0, \delta^*\ge s_2\ge 0
\end{aligned}\qquad (7.9)$$

In (7.9), δ^* is the optimal value of the objective function of (7.8) and $w^*Z_{j_0}$ the optimal virtual intermediate measure derived by model (7.8). Notice here that the term $w^*Z_{j_0}$ is used as an effective substitute of wZ_{j_0} to secure the linearity of the model. In case that $s_2>0$ in the optimal solution of (7.9), the program is solved iteratively by replacing in each iteration the weights w in the coefficient of s_2 with the optimal weights w obtained in the preceding iteration, until the stage efficiencies in two successive iterations remain unchanged (c.f. Despotis, 1996) for a similar treatment). The same holds for the second phase programs in types II-IV as well as in the general case presented in the next sub-sections. The optimal solution $\left(\hat{v},\hat{w},\hat{u}\right)$ of (7.9) is a

Pareto optimal solution of (7.6) and the efficiency scores for unit j_0 in the first and the second stage as well as the overall efficiency of the system are respectively:

$$\hat{e}^1_{j_0} = \frac{\hat{w}Z_{j_0}}{\hat{v}X_{j_0}} = \hat{w}Z_{j_0},\ \hat{e}^2_{j_0} = \frac{\hat{u}Y_{j_0}}{\hat{w}Z_{j_0}},\ \hat{e}^o_{j_0} = \frac{\hat{u}Y_{j_0}}{\hat{v}X_{j_0}} = \hat{u}Y_{j_0}$$

with $\hat{e}^o_{j_0} = \hat{e}^1_{j_0} \cdot \hat{e}^2_{j_0}$. Since the optimal solution of (7.8) is weakly Pareto optimal, in (7.9), at most one of the two optimal values of the variables $\hat{s}_1$ and $\hat{s}_2$ will be strictly positive. If $\hat{s}_1 = 0$ and $\hat{s}_2 = 0$, then the optimal solution of (7.8) is Pareto optimal.

Illustration

For comparison purposes, we apply models (7.8) and (7.9) to the data originally presented in (Kao and Hwang, 2008) and used in many other studies. The case concerns the performance measurement of 24 Taiwanese non-life insurance companies. The authors considered a two-stage production process with two inputs (Operation expenses-X1 and Insurance expenses-X2), two intermediate measures (Direct written premiums-Z1 and Reinsurance premiums-Z2) and two final outputs (Underwriting profit-Y1 and Investment profit-Y2). For the complete data set the reader is referred to the original article (Kao and Hwang, 2008). Table 7.1 summarizes the results obtained by applying the *additive decomposition* method (Chen, Cook, Li and Zhu, 2009) (columns 2-4) and the *multiplicative decomposition* method (Kao and Hwang, 2008) (columns 5-7).

Table 7.2 exhibits the results obtained by applying the proposed approach. Specifically, columns 2 and 3 present the independent (ideal) efficiency scores for stage-1 and stage-2 respectively, columns 4 and 5 present the stage-1 and stage-2 efficiency scores, whereas the last column presents the overall efficiency scores. Notice here that in all cases (DMUs), the model (7.8) provided Pareto optimal solutions, i.e. model (7.9) did not alter the efficiency scores obtained from (7.8).

Table 7.1: Results obtained from the additive and the multiplicative decomposition methods

	Chen, Cook, Li and Zhu (2009)			Kao and Hwang (2008)		
DMU	e^1	e^2	e^o	e^1	e^2	e^o
1	0.9926	0.7045	0.8491	0.9926	0.7045	0.6992
2	0.9985	0.6257	0.8122	0.9985	0.6257	0.6248
3	0.6900	1	0.8166	0.6900	1	0.6900
4	0.7243	0.4200	0.5965	0.7243	0.4200	0.3042
5	0.8307	0.9233	0.8727	0.8307	0.9233	0.7670
6	0.9606	0.4057	0.6887	0.9606	0.4057	0.3897
7	0.7521	0.3522	0.5804	0.6706	0.4124	0.2766
8	0.7256	0.3780	0.5795	0.6630	0.4150	0.2752
9	1	0.2233	0.6116	1	0.2233	0.2233
10	0.8615	0.5408	0.7131	0.8615	0.5408	0.4660
11	0.7291	0.2068	0.5088	0.6468	0.2534	0.1639
12	1	0.7596	0.8798	1	0.7596	0.7596
13	0.8107	0.2431	0.5565	0.6720	0.3093	0.2078
14	0.7246	0.3740	0.5773	0.6699	0.4309	0.2886
15	1	0.6138	0.8069	1	0.6138	0.6138
16	0.8856	0.3615	0.6395	0.8856	0.3615	0.3202
17	0.7232	0.4597	0.6126	0.6276	0.5736	0.3600
18	0.7935	0.3262	0.5868	0.7935	0.3262	0.2588
19	1	0.4112	0.7056	1	0.4112	0.4112
20	0.9332	0.5857	0.7654	0.9332	0.5857	0.5465
21	0.7505	0.2623	0.5412	0.7321	0.2743	0.2008
22	0.5895	1	0.7418	0.5895	1	0.5895
23	0.8426	0.4989	0.6854	0.8426	0.4989	0.4203
24	1	0.0870	0.5435	0.4287	0.3145	0.1348

Table 7.2: Results obtained from model (7.9) (same as from model (7.8)).

DMU	E^1	E^2	$\hat{e}^1$	$\hat{e}^2$	$\hat{e}^o$
1	0.9926	0.7134	0.9847	0.7055	0.6946
2	0.9985	0.6275	0.9971	0.6260	0.6242
3	0.6900	1	0.6900	1	0.6900
4	0.7243	0.4323	0.7125	0.4205	0.2996
5	0.8375	1	0.7912	0.9537	0.7545
6	0.9637	0.4057	0.9618	0.4038	0.3884
7	0.7521	0.5378	0.6385	0.4243	0.2709
8	0.7256	0.5113	0.6375	0.4232	0.2698
9	1	0.2920	0.9408	0.2328	0.2190
10	0.8615	0.6736	0.7557	0.5678	0.4290
11	0.7405	0.3267	0.6594	0.2455	0.1619
12	1	0.7596	1	0.7596	0.7596
13	0.8107	0.5435	0.6075	0.3404	0.2068
14	0.7246	0.5178	0.6463	0.4395	0.2840
15	1	0.7047	0.9341	0.6389	0.5968
16	0.9072	0.3847	0.8843	0.3618	0.3199
17	0.7233	1	0.4419	0.7186	0.3175
18	0.7935	0.3737	0.7572	0.3373	0.2554
19	1	0.4158	0.9962	0.4120	0.4104
20	0.9332	0.9014	0.7289	0.6970	0.5081
21	0.7505	0.2795	0.7400	0.2690	0.1991
22	0.5895	1	0.5895	1	0.5895
23	0.8501	0.5599	0.8020	0.5119	0.4106
24	1	0.3351	0.7978	0.1328	0.1060

Comparison of the new approach with the multiplicative decomposition approach

In the following, we will show the relation of our approach with the multiplicative decomposition method of Kao and Hwang (2008). Recall here that the multiplicative decomposition model assumes that the overall efficiency is the product of the stage efficiencies:

$$e_j^1 = \frac{wZ_j}{vX_j},\ e_j^2 = \frac{uY_j}{wZ_j},\ e_j^o = e_j^1 \cdot e_j^2 = \frac{uY_j}{vX_j}$$

The model below estimates the stage efficiencies by optimizing the overall efficiency:

$$\begin{aligned}
& e_{j_0}^o = \max uY_{j_0} \\
& s.t. \\
& vX_{j_0} = 1 \\
& wZ_j - vX_j \leq 0,\ \ j = 1,\ldots,n \\
& uY_j - wZ_j \leq 0,\ \ j = 1,\ldots,n \\
& v \geq \varepsilon, w \geq \varepsilon, u \geq \varepsilon
\end{aligned} \tag{7.10}$$

Once an optimal solution $\left(v^*, w^*, u^*\right)$ of model (7.10) is obtained, the overall efficiency and the stage efficiencies are calculated as follows:

$$e_{j_0}^o = u^* Y_{j_0},\ e_{j_0}^1 = \frac{w^* Z_{j_0}}{v^* X_{j_0}} = w^* Z_{j_0},\ e_{j_0}^2 = \frac{u^* Y_{j_0}}{w^* Z_{j_0}} = \frac{e_{j_0}^o}{e_{j_0}^1}$$

The difference between our method and the multiplicative decomposition method is conceptual rather than structural. In fact, our method follows the composition paradigm introduced in (Despotis et al., 2016). Structurally, models (7.6) and (7.10) have exactly the same constraints and differ only in the objective functions. That is both models have the same feasible region. Model (7.6) is a bi-objective program (vector-maximization model) with the objectives representing the stage-1 and stage-2 efficiencies. The overall efficiency of the system is obtained by the Pareto optimal solution of (7.6) that locates the stage efficiencies as close as possible to their ideal values in the minmax sense. In model (7.10), on the other hand, the overall efficiency of the system is maximized and the stage efficiencies are obtained as offspring, by decomposing the overall efficiency. The structural similarity of models (7.6) and

(7.10) enables plotting their objective functions space jointly. Fig. 7.2 below is a general representation of the objective functions space of models (7.6) and (7.10) for an evaluated unit (X_0, Z_0, Y_0). Actually, it is the plane in the three-dimensional space (vX_0, wZ_0, uY_0) that is vertical to the axis vX at $vX_0 = 1$. The horizontal axis represents for both models the stage-1 efficiency. The vertical axis represents the overall efficiency as per model (7.10), i.e. the product of stage-1 and stage-2 efficiencies for both models.

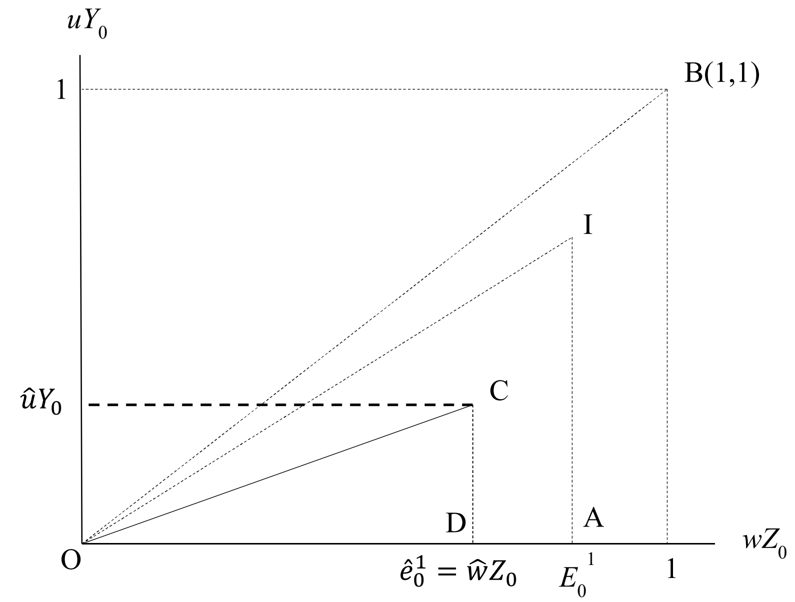


Figure 7.2: General representation of the objective functions space of models (7.6) and (7.10)

The point B(1,1) represents the boundaries of the objective functions values and corresponds to an overall efficient unit with $e_{j_0}^1 = w^* Z_{j_0} = 1$ and $e_{j_0}^o = u^* Y_{j_0} = 1$. Then, the efficiency of stage-2 is $e_{j_0}^2 = u^* Y_{j_0} / w^* Z_{j_0} = 1$ and is represented by the slope of the bisecting line OB. The point I corresponds to the stage-1 and stage-2 ideal (independent) efficiency scores of the evaluated unit and is formed as the intersection of the vertical line to the horizontal axis at E_0^1 and a line form the origin with slope E_0^2, i.e. $E_0^2 = tan\widehat{IOA}$. The point C is located by the model (7.9) on the Pareto front of model (7.6) and is formed as the intersection of the vertical line to the horizontal axis at $\hat{e}_0^1 = \hat{w}Z_0$ and the line from the origin with slope $\hat{e}_0^2 = \hat{u}Y_0 / \hat{w}Z_0$. The abscissa of C is the stage-1 efficiency, whereas its ordinate is the overall efficiency of the evaluated unit as defined in the multiplicative model. Thus, C represents the Pareto front point derived by the multiplicative model (7.10) if and only if its ordinate is maximal.

Consider now the parametric version of model (7.8) that is solved for different values of the parameters $t_1 > 0$ and $t_2 > 0$, such that $t_1 + t_2 = 1$.

$$
\begin{aligned}
& \min \delta \\
& s.t. \\
& t_1 wZ_{j_0} + \delta \geq t_1 \mathrm{E}_{j_0}^1 \\
& t_2 uY_{j_0} - \left(t_2 \mathrm{E}_{j_0}^2 - \delta\right) wZ_{j_0} \geq 0 \\
& vX_{j_0} = 1 \\
& wZ_j - vX_j \leq 0, \quad j = 1, \ldots, n \\
& uY_j - wZ_j \leq 0, \quad j = 1, \ldots, n \\
& v \geq \varepsilon, w \geq \varepsilon, u \geq \varepsilon, \delta \geq 0
\end{aligned}
\tag{7.11}
$$

For every t_1 and t_2, model (7.11) locates a point on the Pareto front of model (7.6). That is, model (7.8) can be used as an instrument to generate the Pareto front of model (7.6). The greatest is the value of t_1 than t_2 the highest is the priority given to stage-1 over stage-2 and vice versa. The crooked line ABCD in Fig. 7.3 represents the Pareto front of model (7.6) for DMU 17 (c.f. Tables 7.1 and 7.2). Point I depicts the ideal (independent) stage efficiencies of this unit. Particularly, its abscissa is $E_{17}^1 = 0.7233$ and the slope of the line OI is $E_{17}^2 = 1$. Point C is the point on the Pareto front that corresponds to the solution obtained by the multiplicative model of Kao and Hwang (2008). Its ordinate is $e_{17}^o = 0.36$, which is maximal, its abscissa is $e_{17}^1 = 0.6276$ whereas the stage-2 efficiency is $e_{17}^2 = e_{17}^o / e_{17}^1 = 0.5736$ and is represented by the slope of the line OC. Point B depicts the point on the Pareto front obtained by our model (7.9). The abscissa of point B is the stage-1 efficiency $\hat{e}_{17}^1 = 0.4419$, the slope of the line OB is the stage-2 efficiency $\hat{e}_{17}^2 = 0.7186$, whereas the ordinate of point B is the overall efficiency $\hat{e}_{17}^{o} = 0.3175$. It is clear that generally holds $\hat{e}_{j}^{o} \le e_{j}^{o}$ because e_{j}^{o} is maximal.

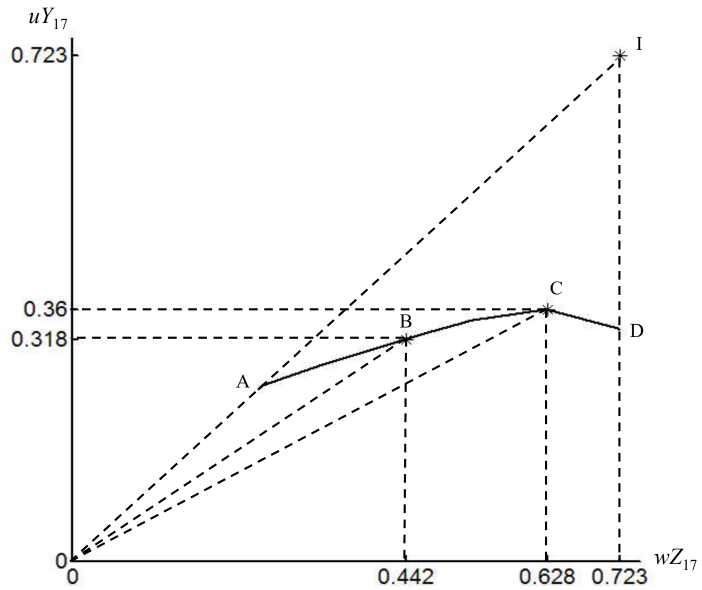


Figure 7.3: The Pareto front of DMU 17

Fig. 7.4 exhibits the conventional Pareto front for DMU 17 in the objective functions space of (7.6) with the horizontal and the vertical axes representing respectively the stage-1 and the stage-2 efficiency scores. Point I is formed by the ideal efficiency scores $\left(E_{17}^{1} = 0.7233, E_{17}^{2} = 1\right)$. The curve ABCD is the Pareto front for unit 17, the point $B(0.4419,\ 0.7186)$ is the Pareto optimal solution obtained by model (7.8) and is uniquely formed by the intersection of the Pareto front with a ray from the ideal point I with direction $(-1,-1)$. Point $C(0.6276,\ 0.5736)$ represents the solution obtained by the multiplicative model (7.10).

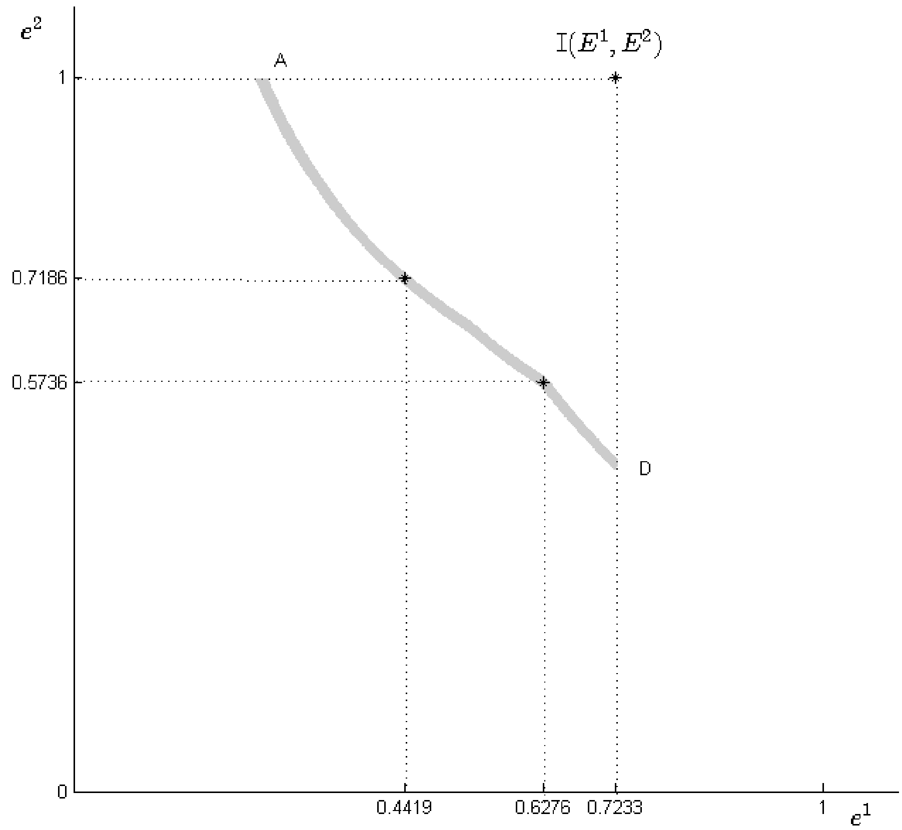


Figure 7.4: The conventional Pareto front for DMU 17 in the (e1, e2) space

The model (7.9) locates a unique point on the Pareto front, i.e. it estimates unique efficiency scores for the two stages. Given that the unweighted Tchebycheff norm is employed in (7.8), no priority is assumed for one stage over the other. If, however, one is to assign different priorities to the two stages, the efficiency assessment can be performed via the weighted variant (7.11), with specific values for the parameters t_1 and t_2 that reflect the analyst's preference. Each distinct pair (t_1, t_2) locates a point on the Pareto front. Since the model (7.11) can locate any point on the Pareto front, it can locate point C in Fig.7.3 (point C in Fig. 7.4) as well. Indeed, solving model (7.11) for $t_1 = 0.81668,\ t_2 = 0.18332$ we get the same stage and overall efficiencies as those obtained by the multiplicative method. Notice however, that in this case the stage-1 is over-weighted significantly at the expense of the stage-2. This is an indication that the multiplicative decomposition method, when maximizing the overall efficiency of a unit, may implicitly, yet unreasonably, assume different and, interestingly, DMU-specific priorities for the two stages. Thus, the decomposition of the overall efficiency to the stage efficiencies may bias the efficiency assessments in favor of one stage over the other and it does not provide the analyst with the necessary information to communicate the results, as concerns the priorities of the stages.

Kao and Hwang (2008) proposed a pair of post-optimality models to check the uniqueness of the efficiency decomposition. As shown in Fig. 7.3, the efficiency decomposition for DMU 17 is unique at point C. Although this holds for all the 24 units in Table 7.1, it is not a general property of the multiplicative decomposition in model (7.10). Table 7.3 presents a synthetic case of 30 DMUs with two inputs (X1, X2), two intermediate measures (Z1, Z2) and two outputs (Y1,Y2) drawn form a uniform distribution in the interval [10,100]. Columns 8-10 present the overall and stage efficiency scores obtained by the multiplicative decomposition model (7.10). Columns 11-14 present alternative efficiency decompositions that maintain the optimal overall efficiency score e^o. They are calculated by applying the post-optimality check proposed in (Kao and Hwang, 2008). Specifically, columns 11-12 provide the maximal and the minimal efficiencies for stage-1 that maintain the overall efficiency score. Respectively, the maximal and the minimal efficiencies for stage-2 are given in columns 13-14. These results show that the efficiency decomposition for the units 8, 13, 18, 19, 23 and 30 is not unique.

The crooked line ABD in Fig. 7.5 depicts the Pareto front generated by model (7.11) for unit 18. Notice again that applying model (7.8) to the data of Table 7.3 generates Pareto optimal solutions for all the units, i.e. the second-phase model (7.9) does not alter the efficiency scores obtained by the former. The point I depicts the ideal solution of (7.6) $\left(E_{18}^1 = 0.5046, E_{18}^2 = 1\right)$. Actually, the independent (ideal) efficiency score of stage-2 is represented by the slope of the line OI. The segment BD of the Pareto front is parallel to the horizontal axis and all the points on it correspond to equivalent efficiency decompositions that maintain the same overall efficiency $e_{18}^0 = 0.2477$. Points B and D depict the two extreme decompositions $(e_-^1 = e^0 / e_{max}^2 = 0.3021,\ e_{max}^2 = 0.8201)$ and $(e_{max}^1 = 0.5046,\ e_-^2 = e^0 / e_{max}^1 = 0.4910)$ respectively. The slopes of the lines OB and OD represent the stage-2 efficiency scores e^2_{max} and e^2_{-} respectively. Point C represents the unique Pareto optimal point obtained by our model (7.8) with $\hat{e}^1 = 0.3082$, $\hat{e}^2 = 0.8037$ and $\hat{e}^o = 0.2477$. Fig. 7.6 exhibits the conventional form of the Pareto front for unit 18. The counterpart in Fig. 7.6 of the segment BD of the Pareto front in Fig. 7.5 is the curve BD, which, in fact, consists of an infinite number of alternative efficiency decompositions of the overall efficiency $e^o = 0.2477$. Contrariwise, model (7.8) generates the unique pair of Pareto optimal efficiency scores depicted on point C. Summarizing, unlike the Kao and Hwang's (2008) multiplicative efficiency decomposition method, our approach generates unique and unbiased efficiency scores.

Table 7.3: Synthetic data and results obtained by model (7.10) and post-optimality analysis

DMU	X1	X2	Z1	Z2	Y1	Y2	e^1	e^2	e^o	e^1_{max}	e^1_{-}	e^2_{max}	e^2_{-}	DMU
1	69.5	68.6	56.6	84.4	48.7	62.8	0.2316	0.5070	0.1174	0.2316	0.2316	0.5070	0.5070	1
2	40.2	66.2	88	47.2	85.8	28.3	0.3265	0.7508	0.2451	0.3265	0.3265	0.7508	0.7508	2
3	81.3	89.8	44.4	18.4	38.3	20.7	0.0866	0.6844	0.0593	0.0866	0.0866	0.6844	0.6844	3
4	55	97.9	28.7	41.6	38.2	10.3	0.1344	0.5830	0.0784	0.1344	0.1344	0.5830	0.5830	4
5	56.2	59.1	26.5	52.7	44.2	17.4	0.1688	0.5823	0.0983	0.1688	0.1688	0.5823	0.5823	5
6	64.8	64.4	14.7	70.5	86.6	22.9	0.1685	1	0.1685	0.1685	0.1685	1	1	6
7	79.2	68.1	63.5	39.3	47.6	35	0.1644	0.5613	0.0923	0.1644	0.1644	0.5613	0.5613	7
8	36	74.3	66.6	57.4	40.3	94.8	0.4297	0.6953	0.2987	0.4297	0.4145	0.7208	0.6953	8
9	10.8	10.3	46.5	47.9	57.5	95.2	1	1	1	1	1	1	1	9
10	17.7	93.6	35.9	58.7	45.9	12	0.5235	0.5149	0.2696	0.5235	0.5235	0.5149	0.5149	10
11	38.8	97.5	55.2	41.7	60.5	82.7	0.3113	0.8189	0.2550	0.3113	0.3113	0.8189	0.8189	11
12	60.9	96.4	86	28.9	93.1	72.3	0.2006	1	0.2006	0.2006	0.2006	1	1	12
13	70.3	45.8	65.3	35.3	34.3	98.8	0.3158	0.7390	0.2334	0.3158	0.2334	1	0.7390	13
14	20.5	75.6	13.1	60	53.3	18.3	0.3759	0.7190	0.2703	0.3759	0.3759	0.7190	0.7190	14
15	17.9	74.8	54.2	66.7	52.1	15.8	0.6443	0.4696	0.3026	0.6443	0.6443	0.4696	0.4696	15
16	51.8	19.8	52.3	74.2	73.6	84.7	0.6980	0.8163	0.5698	0.6980	0.6980	0.8163	0.8163	16
17	11.3	27.3	42.7	72.3	68.9	37.4	1	0.6694	0.6694	1	1	0.6694	0.6694	17
18	58.7	42.1	95.9	26.6	51.6	96.4	0.5046	0.4910	0.2477	0.5046	0.3021	0.8201	0.4910	18
19	41.4	51.6	83	75.4	20.5	72	0.4656	0.4237	0.1973	0.4656	0.4537	0.4348	0.4237	19
20	99.7	87.1	87.5	96.9	58.6	39	0.2311	0.3739	0.0864	0.2311	0.2311	0.3739	0.3739	20
21	25.6	14.6	52	19.1	44.3	51.3	0.5293	0.8807	0.4662	0.5293	0.5293	0.8807	0.8807	21
22	65.1	97.3	79.4	68	53.8	55.5	0.2438	0.4927	0.1201	0.2438	0.2438	0.4927	0.4927	22
23	40.4	33	74.5	21.7	13.9	55.7	0.5001	0.3652	0.1826	0.5001	0.3031	0.6025	0.3652	23
24	19.4	20.1	77.5	74.1	60.9	71	0.8855	0.5673	0.5023	0.8855	0.8855	0.5673	0.5673	24
25	54.2	99.3	20.8	69.9	47.8	12.2	0.1867	0.5291	0.0988	0.1867	0.1867	0.5291	0.5291	25
26	80.1	27.5	51.3	95.8	21.7	12.6	0.5850	0.1674	0.0979	0.5850	0.5850	0.1674	0.1674	26
27	82.9	38.1	43.3	75.3	16.8	26.6	0.3403	0.2273	0.0774	0.3403	0.3403	0.2273	0.2273	27
28	98.6	81.8	93.8	15.9	40.3	35.8	0.1455	0.4735	0.0689	0.1455	0.1455	0.4735	0.4735	28
29	77.3	40.3	95.6	52.5	96.1	44.2	0.3766	0.7660	0.2885	0.3766	0.3766	0.7660	0.7660	29
30	38.6	58.3	37.8	66.1	16	69.9	0.2274	0.9032	0.2054	0.2618	0.2274	0.9032	0.7847	30

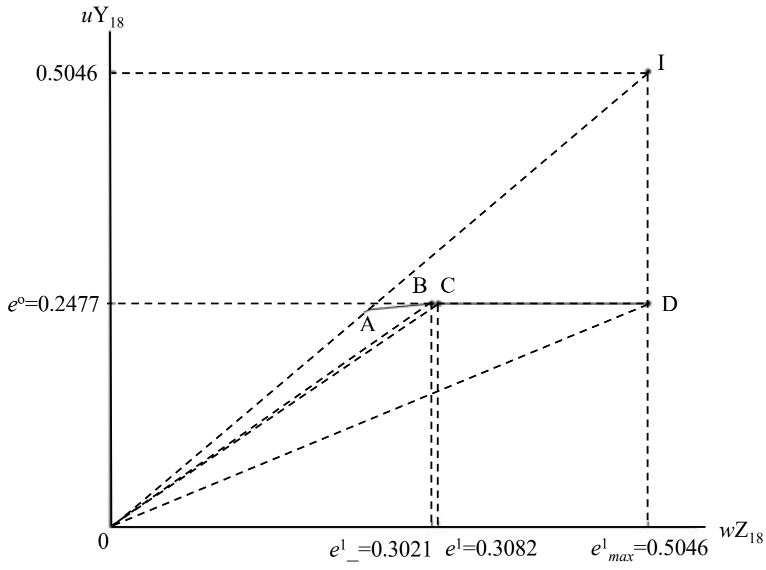


Figure 7.5: Non-unique efficiency decomposition of unit 18

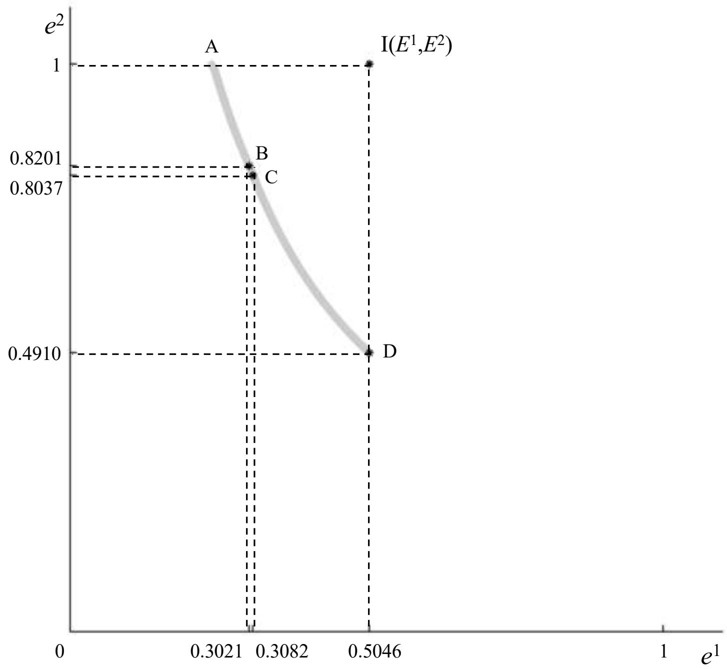


Figure 7.6: The conventional Pareto front for unit 18 in the (e1,e2) space

7.2.2 Type II structure

In the structure of type II, the second stage uses some extra external inputs L beyond the intermediate measures as depicted in Fig. 7.1. In this case, the efficiency of the first and the second stage of DMU j are defined as follows:

$$e_j^1 = \frac{wZ_j}{vX_j},\ e_j^2 = \frac{uY_j}{wZ_j + \gamma L_j}$$

The overall efficiency of DMU j is defined as the ratio of the total virtual exogenous output to the total virtual exogenous input:

$$e_j^o = \frac{uY_j}{vX_j + \gamma L_j}$$

Similarly to Type I, the bi-objective program for estimating the efficiencies of the two stages is as follows:

$$\begin{aligned}
&\max \frac{wZ_{j_0}}{vX_{j_0}} \\
&\max \frac{uY_{j_0}}{wZ_{j_0} + \gamma L_{j_0}} \\
&s.t. \\
&wZ_j - vX_j \le 0, \quad j = 1, \ldots, n \\
&uY_j - wZ_j - \gamma L_j \le 0, \quad j = 1, \ldots, n \\
&v \ge \varepsilon, w \ge \varepsilon, u \ge \varepsilon, \gamma \ge \varepsilon
\end{aligned} \tag{7.12}$$

Applying the C-C transformation to (7.12) on the basis of the denominator of the first objective function, we get the following:

$$\begin{aligned}
&\max wZ_{j_0} \\
&\max \frac{uY_{j_0}}{wZ_{j_0}+\gamma L_{j_0}} \\
&s.t. \\
&vX_{j_0}=1 \\
&wZ_j - vX_j \le 0, \quad j=1,\ldots,n \\
&uY_j - wZ_j - \gamma L_j \le 0, \quad j=1,\ldots,n \\
&v \ge \varepsilon, w \ge \varepsilon, u \ge \varepsilon, \gamma \ge \varepsilon
\end{aligned} \tag{7.13}$$

Notice here that there is a variable transformation from (7.12) to (7.13) (see previous section) but we use the same variable names for the economy of notation. The same simplification is adopted in the next sections.

The minmax model that calculates the stage-1 and stage-2 efficiency scores at a minimum distance (unweighted L_∞ norm) from their ideal counterparts is as follows:

$$\begin{aligned}
&\min \delta \\
&s.t. \\
&E_{j_0}^1 - wZ_{j_0} \le \delta \\
&(E_{j_0}^2 - \delta)\left(wZ_{j_0}+\gamma L_{j_0}\right) - uY_{j_0} \le 0 \\
&vX_{j_0}=1 \\
&wZ_j - vX_j \le 0, \quad j=1,\ldots,n \\
&uY_j - wZ_j - \gamma L_j \le 0, \quad j=1,\ldots,n \\
&v \ge \varepsilon, w \ge \varepsilon, u \ge \varepsilon, \gamma \ge \varepsilon, \delta \ge 0
\end{aligned} \tag{7.14}$$

The ideal efficiency scores are obtained by considering (7.12) with one objective function at a time and solving its linear equivalent derived by the C-C transformation. The optimal solution of (7.14) is weakly Pareto optimal solution of (7.13). As explained in the previous section, model (7.14) can be solved by bisection search. Let $(\delta^*, v^*, w^*, u^*, \gamma^*)$ be an optimal solution of (7.14) and

$$e_{j_0}^{1*} = \frac{w^* Z_{j_0}}{v^* X_{j_0}} = w^* Z_{j_0}, \; e_{j_0}^{2*} = \frac{u^* Y_{j_0}}{w^* Z_{j_0} + \gamma^* L_{j_0}}$$

The second phase program (7.15) below provides a Pareto optimal solution to (7.13):

$$
\begin{aligned}
&\max s_1 + s_2 \\
&s.t. \\
&\mathrm{E}^1_{j_0} - wZ_{j_0} + s_1 = \delta^* \\
&(E^2_{j_0} - \delta^*)\left(wZ_{j_0} + \gamma L_{j_0}\right) - uY_{j_0} + s_2\left(w^* Z_{j_0} + \gamma^* L_{j_0}\right) = 0 \\
&vX_{j_0} = 1 \\
&wZ_j - vX_j \le 0, \quad j = 1,\ldots,n \\
&uY_j - wZ_j - \gamma L_j \le 0, \quad j = 1,\ldots,n \\
&v \ge \varepsilon, w \ge \varepsilon, u \ge \varepsilon, \gamma \ge \varepsilon \\
&\delta^* \ge s_1 \ge 0, \delta^* \ge s_2 \ge 0
\end{aligned}
\tag{7.15}
$$

Given the optimal solution $\left(\hat{s}_1, \hat{s}_2, \hat{v}, \hat{w}, \hat{u}, \hat{\gamma}\right)$ of (7.15), the efficiency scores for unit j_0 in the first and the second stage as well as the overall efficiency of the system are respectively:

$$
\hat{e}^1_{j_0} = \frac{\hat{w}Z_{j_0}}{\hat{v}X_{j_0}} = \hat{w}Z_{j_0}, \; \hat{e}^2_{j_0} = \frac{\hat{u}Y_{j_0}}{\hat{w}Z_{j_0} + \hat{\gamma}L_{j_0}}, \; \hat{e}^o_{j_0} = \frac{\hat{u}Y_{j_0}}{\hat{v}X_{j_0} + \hat{\gamma}L_{j_0}}
$$

If $\hat{s}_1 = \hat{s}_2 = 0$, then the optimal solution of (7.14) is already Pareto optimal, and model (7.15) does not alter the efficiency scores obtained by (7.14).

Illustration

We illustrate models (7.14) and (7.15) on a two-stage process of type II drawn from Li et al. (2012). The case concerns the assessment of regional R&D process of 30 Provincial Level Regions in China. The stage-1 represents the technology development whereas the stage-2 represents the economic application. The stage-1 inputs are: R&D personnel (X1), R&D expenditure (X2) and the proportion of regional science and technology funds in regional total financial expenditure (X3). The outputs (intermediate measures) of stage-1, which are inputs to stage-2 are: number of patents (Z1) and number of papers (Z2). The extra input to stage-2 is contract value in technology market (L). The final outputs are GDP (Y1), total exports (Y2), urban per capita annual income (Y3) and gross output of high-tech industry (Y4). The reader is referred to Li et al. (2012) for the complete data set.

For comparison, we present in Table 7.4 the results given in Li et al. (2012) and those obtained by model (7.15). Li et al. (2012) calculate the stage-1 and stage-2 efficiency scores parametrically and then they give the overall efficiency as the product of the stage efficiencies, although in their case, the overall efficiency is not readily decomposed to the stage efficiencies, as in the case of the simple structure of Type I (Kao and Hwang, 2008, Liang et al., 2008). However, to be in line with their results, we present the product of the stage efficiencies obtained by our approach as well.

Notice that, for all DMUs, the model (7.14) provided Pareto optimal solutions. This is validated by the fact that in the second phase program (7.15), the optimal values of the slacks were $\hat{s}_1 = \hat{s}_2 = 0$. Fourteen out of the thirty units show identical individual efficiency scores. Notice also that $\hat{e}^1 \cdot \hat{e}^2 \leq e^0$. This is a natural effect of the fact that in Li et al. (2012), among the parametrically generated pairs of stage efficiency scores, the one that shows the maximal squared geometric average is selected. However, as it is explained in section 7.2.1, such an approach often assumes implicitly different priorities for the two stages, with one stage arbitrarily favored over the other. Indeed, the stage efficiency scores given in Li et al. (2012) for the units 3, 17, 18, 19, 22 and 26, for example, can be obtained by the weighted variant of model (7.14) with the couples of weights $(t_1 = 0.256745,\ t_2 = 0.74325)$, $(t_1 = 0.966292,\ t_2 = 0.033708)$, $(t_1 = 0.975312,\ t_2 = 0.024688)$, $(t_1 = 0.199731,$ $t_2 = 0.800269)$, $(t_1 = 0.779891, t_2 = 0.220109)$ and $(t_1 = 0.169132,\ t_2 = 0.830868)$ respectively. The advantage of our approach is that it provides unique and unbiased efficiency scores. However, if it is to assign explicitly different priorities to the two stages, the weighted variant of (7.14) could be used.

Table 7.4: Results from Li et al. (2012) and from model (7.15) (same as from model (7.14))

	Li et al. (2012)			Model (7.15)						
DMU	e^1	e^2	$e^o = e^1 \cdot e^2$	E^1	E^2	$\hat{e}^1$	$\hat{e}^2$	$\hat{e}^1.\hat{e}^2$	$\hat{e}^o$	DMU
1	1	0.1598	0.1598	1	0.1598	1	0.1598	0.1598	0.1598	1
2	1	0.2489	0.2489	1	0.2489	1	0.2489	0.2489	0.2489	2
3	0.8950	0.5365	0.4802	1	0.5728	0.9314	0.5042	0.4696	0.4696	3
4	0.6774	0.5704	0.3864	0.7426	0.5704	0.7021	0.5300	0.3721	0.3721	4
5	0.6697	0.3895	0.2608	0.6697	0.3895	0.6697	0.3895	0.2608	0.3310	5
6	0.5668	1	0.5668	0.5668	1	0.5668	1	0.5668	0.6137	6
7	1	0.2207	0.2207	1	0.3121	0.9177	0.2298	0.2109	0.2113	7
8	1	1	1	1	1	1	1	1	1	8
9	0.9398	1	0.9398	0.9398	1	0.9398	1	0.9398	0.9534	9
10	1	1	1	1	1	1	1	1	1	10
11	0.8885	0.8351	0.7420	0.8885	0.8351	0.8885	0.8351	0.7420	0.7756	11
12	0.9328	0.2648	0.2470	0.9328	0.2703	0.9278	0.2653	0.2462	0.2566	12
13	0.8493	0.7373	0.6262	0.8504	0.7373	0.8495	0.7364	0.6256	0.6707	13
14	0.9060	0.2816	0.2551	0.9060	0.3360	0.8545	0.2845	0.2431	0.2431	14
15	1	0.3685	0.3685	1	0.3780	0.9921	0.3702	0.3673	0.3689	15
16	0.9225	1	0.9225	0.9225	1	0.9225	1	0.9225	0.9225	16
17	0.5644	0.9914	0.5595	0.5647	1	0.5572	0.9925	0.5531	0.6958	17
18	0.7152	0.4947	0.3538	0.7158	0.5184	0.6986	0.5012	0.3501	0.4158	18
19	0.6671	0.3668	0.2447	0.6969	0.3742	0.6810	0.3583	0.2440	0.2440	19
20	0.4573	1	0.4573	0.4573	1	0.4573	1	0.4573	0.4629	20
21	0.7101	0.8176	0.5806	0.7101	0.8498	0.6854	0.8251	0.5656	0.4573	21
22	0.5708	0.5156	0.2943	0.5864	0.5709	0.5495	0.5340	0.2935	0.3976	22
23	1	0.1941	0.1941	1	0.2509	0.9441	0.1951	0.1842	0.1905	23
24	1	0.4566	0.4566	1	0.4817	0.9758	0.4574	0.4463	0.4517	24
25	1	0.5846	0.5846	1	0.6159	0.9756	0.5915	0.5770	0.5839	25
26	0.7293	0.9171	0.6688	0.9111	0.9541	0.7869	0.8299	0.6530	0.7304	26
27	1	1	1	1	1	1	1	1	1	27
28	0.3599	1	0.3599	0.3599	1	0.3599	1	0.3599	0.3599	28
29	0.4300	1	0.4300	0.4300	1	0.4300	1	0.4300	0.4308	29
30	1	1	1	1	1	1	1	1	1	30

7.2.3 Type III structure

In the structure of type III, the first stage produces some final outputs K that exit the system, beyond the intermediate measures as depicted in Fig. 7.1. In this case, the efficiency of the first and the second stage of DMU j are typically defined as follows:

$$e_j^1 = \frac{wZ_j + \mu K_j}{vX_j}, \quad e_j^2 = \frac{uY_j}{wZ_j}$$

The overall efficiency of the system is $e_j^o = \dfrac{uY_j + \mu K_j}{vX_j}$

On the basis of the above definitions, the bi-objective program for estimating the efficiencies of the two stages is as follows:

$$\begin{aligned}
&\max \frac{wZ_{j_0} + \mu K_{j_0}}{vX_{j_0}} \\
&max \frac{uY_{j_0}}{wZ_{j_0}} \\
&s.t. \\
&wZ_j + \mu K_j - vX_j \le 0, \quad j = 1, \ldots, n \\
&uY_j - wZ_j \le 0, \quad j = 1, \ldots, n \\
&v \ge \varepsilon, w \ge \varepsilon, u \ge \varepsilon, \mu \ge \varepsilon
\end{aligned} \tag{7.16}$$

Applying the C-C transformation to (7.16) with respect to the first objective function we get the model (7.17) below:

$$\begin{aligned}
&\max wZ_{j_0} + \mu K_{j_0} \\
&\max \frac{uY_{j_0}}{wZ_{j_0}} \\
&s.t. \\
&vX_{j_0} = 1 \\
&wZ_j + \mu K_j - vX_j \le 0, \quad j = 1, \ldots, n \\
&uY_j - wZ_j \le 0, \quad j = 1, \ldots, n \\
&v \ge \varepsilon, w \ge \varepsilon u \ge \varepsilon, \mu \ge \varepsilon
\end{aligned} \tag{7.17}$$

The following model calculates the stage-1 and stage-2 efficiency scores at a minimum deviation (unweighted L_∞ norm) from their ideal efficiency values:

$$\begin{aligned}
&\min \delta \\
&s.t. \\
&E^1_{j_0} - wZ_{j_0} - \mu K_{j_0} \le \delta \\
&(E^2_{j_0} - \delta) wZ_{j_0} - uY_{j_0} \le 0 \\
&vX_{j_0} = 1 \\
&wZ_j + \mu K_j - vX_j \le 0, \quad j = 1, \ldots, n \\
&uY_j - wZ_j \le 0, \quad j = 1, \ldots, n \\
&v \ge \varepsilon, w \ge \varepsilon, u \ge \varepsilon, \mu \ge \varepsilon, \delta \ge 0
\end{aligned} \tag{7.18}$$

The ideal efficiency scores are obtained by solving (7.16) with one objective function at a time, after transforming it to its linear equivalent. Solving the non-linear model

(7.18) by bisection search for $\delta \in [0,1]$ we get an optimal solution $(\delta^*, v^*, w^*, u^*, \mu^*)$, which is weakly Pareto optimal for the MOP (7.17) and

$$e_{j_0}^{1*} = \frac{w^* Z_{j_0} + \mu^* K_{j_0}}{v^* X_{j_0}} = w^* Z_{j_0} + \mu^* K_{j_0},\ e_{j_0}^{2*} = \frac{u^* Y_{j_0}}{w^* Z_{j_0}}$$

The second phase program (7.19) below provides a Pareto optimal solution to (7.17):

$$\begin{aligned}
&\max s_1 + s_2 \\
&s.t. \\
&E_{j_0}^1 - wZ_{j_0} - \mu K_{j_0} + s_1 = \delta^* \\
&(E_{j_0}^2 - \delta^*) wZ_{j_0} - uY_{j_0} + w^* Z_{j_0} s_2 = 0 \\
&vX_{j_0} = 1 \\
&wZ_j + \mu K_j - vX_j \le 0,\ \ j = 1, \dots, n \\
&uY_j - wZ_j \le 0,\ \ j = 1, \dots, n \\
&v \ge \varepsilon, w \ge \varepsilon, u \ge \varepsilon, \mu \ge \varepsilon \\
&\delta^* \ge s_1 \ge 0, \delta^* \ge s_2 \ge 0
\end{aligned} \tag{7.19}$$

Given the optimal solution $(\hat{s}_1, \hat{s}_2, \hat{v}, \hat{w}, \hat{u}, \hat{\mu})$ of (7.19), the efficiency scores for unit j_0 in the first and the second stage as well as the overall efficiency of the system are respectively:

$$\hat{e}_{j_0}^1 = \frac{\hat{w} Z_{j_0} + \hat{\mu} K_{j_0}}{\hat{v} X_{j_0}} = \hat{w} Z_{j_0} + \hat{\mu} K_{j_0},\ \hat{e}_{j_0}^2 = \frac{\hat{u} Y_{j_0}}{\hat{w} Z_{j_0}},\ \hat{e}_{j_0}^o = \frac{\hat{u} Y_{j_0} + \hat{\mu} K_{j_0}}{\hat{v} X_{j_0}} = \hat{u} Y_{j_0} + \hat{\mu} K_{j_0}$$

If $\hat{s}_1 = \hat{s}_2 = 0$, then the optimal solution of (7.18) is already Pareto optimal, and model (7.19) does not alter the efficiency scores obtained by the former.

Illustration

For validation of our computations, we give the data and the results of a synthetic numerical example with 30 DMUs, two inputs (X1, X2), two intermediate measures (Z1, Z2), two final outputs from stage-1 (K1, K2) and two final outputs from stage-2 (Y1, Y2). The data shown in Table 7.5 are random and drawn column-wise from a uniform distribution in the intervals given in the last row of the Table 7.5.

Table 7.5: Synthetic data for type III structure and results obtained from models (7.18) and (7.19)

DMU	X1	X2	X3	Z1	Z2	K1	K2	Y1	Y2	E^1	E^2	e^{1*}	e^{2*}	$\hat{e}^1$	$\hat{e}^2$	$\hat{e}^o$	DMU
1	71.2	6.3	46.6	61.4	109	54.2	64.3	8.4	36.1	0.9577	0.8675	0.8261	0.7358	0.8261	0.7358	0.6680	1
2	21.3	12.6	59.4	93.5	94.4	31.7	60	5.4	22.5	0.7471	0.4769	0.6897	0.4195	0.6897	0.4195	0.2894	2
3	65.8	16.6	30.1	134.4	57.3	23.1	42.8	12.3	12.8	0.7216	0.8008	0.4995	0.5787	0.4995	0.5787	0.3247	3
4	71.9	12.5	54.2	113	94.8	63.6	35	6.7	36	0.5012	0.7008	0.4988	0.6984	0.4988	0.6984	0.4488	4
5	78.4	5.9	19.5	132	140.8	52.8	19.5	9.1	44	1	0.6409	1	0.6409	1	0.6409	0.6409	5
6	30.4	15	43.3	57.3	100.1	60.4	17.8	7.8	14.1	0.7040	0.4007	0.6297	0.3264	0.6297	0.3264	0.2055	6
7	30.2	11.1	34.5	148	66.6	51.6	43.8	6.9	23.7	0.8818	0.5956	0.7346	0.4484	0.7346	0.4484	0.3294	7
8	11.4	5	33.5	122	69.2	60.6	74.8	2.1	18.5	1	0.4063	1	0.4063	1	0.4063	0.6461	8
9	62.7	15.1	29.2	146.7	102.8	73.6	25.8	7.6	13.6	0.8210	0.2802	0.7551	0.2143	0.7551	0.2143	0.2115	9
10	75.9	11.9	17.4	84.8	141.4	18.9	16.3	13.9	20.1	0.9664	0.4963	0.9657	0.4956	0.9657	0.4956	0.4785	10
11	87.8	19.4	50.9	109.8	53.6	63.2	45.5	11.1	18.3	0.4268	0.7725	0.3723	0.7180	**0.3723**	**0.7420**	0.3723	11
12	17.1	10.8	15.5	98.3	66.4	16.2	31.1	17.8	34.6	1	1	0.9862	0.9862	0.9862	0.9862	0.9785	12
13	48.9	8.9	50.8	133.1	65.6	57.8	64.2	3.4	45.7	0.6282	1	0.5831	0.9549	0.5831	0.9549	0.5714	13
14	82.4	8.1	51	57.1	98.8	73.6	10.5	19.4	38.7	0.7835	1	0.7541	0.9858	**0.7541**	**1**	0.7541	14
15	55.9	14.5	23.4	96.4	88.1	20.8	27.6	4.4	44.6	0.6494	0.9708	0.6382	0.9597	0.6382	0.9597	0.6125	15
16	46.4	10.3	14.8	70.8	127.1	75.7	56.6	12.7	43.5	1	0.9065	1	0.9065	1	0.9065	0.9558	16
17	51.9	7	37.8	119.9	64.1	75.9	23.7	8.8	10.8	0.9267	0.5121	0.9119	0.4973	**0.9119**	**0.5047**	0.9119	17
18	43.2	8.3	51.6	123.2	117	40.6	79.6	9.2	23.2	0.8530	0.4097	0.8192	0.3758	0.8192	0.3758	0.3079	18
19	60.6	14.1	43.8	58	107.5	27.2	62.9	16.2	12.3	0.5695	0.8221	0.5250	0.7776	**0.5250**	**0.7999**	0.5250	19
20	59.7	11.7	22.8	89.9	131.9	15	33	11.7	36.9	0.8400	0.6671	0.8064	0.6335	0.8064	0.6335	0.5108	20
21	26.4	12.2	23.7	142.3	110.7	69.8	47.9	10.2	19.5	1	0.3648	1	0.3648	1	0.3648	0.3648	21
22	70.6	18.3	23.8	93.6	123.1	51.6	43.5	2.3	29	0.7008	0.5350	0.6993	0.5334	0.6993	0.5334	0.3730	22
23	67.8	5.2	46.3	73.7	124.4	50.3	25.2	7.7	12.2	1	0.3134	0.9824	0.2958	0.9824	0.2958	0.6289	23
24	83.5	13.7	47.5	72.9	111.9	34.6	65.5	6.9	43.6	0.5683	0.9473	0.5257	0.9047	0.5257	0.9047	0.5036	24
25	33.8	6	27.6	77.1	74.8	70.5	54.7	9.2	34	1	0.8944	1	0.8944	1	0.8944	0.9735	25
26	81.9	9.4	57.7	131	64.5	14.7	63.5	6.1	21.2	0.5791	0.5489	0.5125	0.4823	0.5125	0.4823	0.3998	26
27	38	12.3	57.6	94.4	98.6	24.1	61.8	14.3	48.6	0.6075	1	0.5587	0.9512	0.5587	0.9512	0.5314	27
28	74.4	7.6	40	141	88.5	73.6	18.9	8.6	13.5	0.8706	0.3625	0.8493	0.3412	0.8493	0.3412	0.7296	28
29	92.2	12.4	45.8	94.5	54.8	23.5	57.9	5.5	11.9	0.5081	0.4043	0.4888	0.3849	**0.4888**	**0.3964**	0.4888	29
30	73.3	14.3	54.8	91.4	53.3	19.2	62.3	4.2	34.7	0.4444	1	0.4444	1	0.4444	1	0.4444	30
	[10,100]	[5,20]	[10,60]	[50,150]	[50,150]	[10,80]	[10,80]	[2,20]	[10,50]								

For five out of the thirty units (namely, units 11, 14, 17, 19 and 29), the second phase program (7.19) corrected the efficiency scores derived by the minmax model (7.18), providing Pareto optimal solutions. For the rest of the units, Pareto optimal solutions were obtained early by model (7.18).

7.2.4 Type IV structure

The Type IV structure is the most general two-stage series process. Multiples of this structure in series composes the general multi-stage series process, which will be studied in the next section in the light of our proposed approach. In this case, the efficiency of the first and the second stage of DMU *j* are defined as follows

$$e_j^1 = \frac{wZ_j + \mu K_j}{vX_j},\ e_j^2 = \frac{uY_j}{wZ_j + \gamma L_j}:$$

The overall system efficiency is $e_j^o = \dfrac{uY_j + \mu K_j}{vX_j + \gamma L_j}$.

The bi-objective program for estimating the efficiencies of the two stages is as follows:

$$\begin{aligned}
&\max \frac{wZ_{j_0} + \mu K_{j_0}}{vX_{j_0}} \\
&\max \frac{uY_{j_0}}{wZ_{j_0} + \gamma L_{j_0}} \\
&s.t. \\
&wZ_j + \mu K_j - vX_j \leq 0, \quad j = 1,\ldots,n \\
&uY_j - wZ_j - \gamma L_j \leq 0, \quad j = 1,\ldots,n \\
&v \geq \varepsilon, w \geq \varepsilon, u \geq \varepsilon, \gamma \geq \varepsilon, \mu \geq \varepsilon
\end{aligned} \tag{7.20}$$

Applying the C-C transformation to (7.20) on the basis of the denominator of the first objective function, we get the following:

$$\begin{aligned}
&\max wZ_{j_0} + \mu K_{j_0} \\
&\max \frac{uY_{j_0}}{wZ_{j_0} + \gamma L_{j_0}} \\
&s.t. \\
&vX_{j_0} = 1 \\
&wZ_j + \mu K_j - vX_j \leq 0, \quad j = 1,\ldots,n \\
&uY_j - wZ_j - \gamma L_j \leq 0, \quad j = 1,\ldots,n \\
&v \geq \varepsilon, w \geq \varepsilon, u \geq \varepsilon, \gamma \geq \varepsilon, \mu \geq \varepsilon
\end{aligned} \tag{7.21}$$

Similarly to the previous structures, the minmax model that calculates the stage-1 and stage-2 efficiency scores at a minimum distance (unweighted L_∞ norm) from their independent counterparts is as follows:

$$
\begin{aligned}
& \min \delta \\
& s.t. \\
& E_{j_0}^1 - wZ_{j_0} - \mu K_{j_0} \le \delta \\
& (E_{j_0}^2 - \delta)\left(wZ_{j_0} + \gamma L_{j_0}\right) - uY_{j_0} \le 0 \\
& vX_{j_0} = 1 \\
& wZ_j + \mu K_j - vX_j \le 0, \quad j = 1, \ldots, n \\
& uY_j - wZ_j - \gamma L_j \le 0, \quad j = 1, \ldots, n \\
& v \ge \varepsilon, w \ge \varepsilon, u \ge \varepsilon, \gamma \ge \varepsilon, \mu \ge \varepsilon, \delta \ge 0
\end{aligned}
\tag{7.22}
$$

Solving the model (7.22) by bisection we get a weakly Pareto optimal solution of the MOP (7.21) $(\delta^*, v^*, w^*, u^*, \gamma^*, \mu^*)$ and

$$
e_{j_0}^{1*} = \frac{w^* Z_{j_0} + \mu^* \mathrm{K}_{j_0}}{v^* X_{j_0}} = w^* Z_{j_0} + \mu^* \mathrm{K}_{j_0}, \; e_{j_0}^{2*} = \frac{u^* Y_{j_0}}{w^* Z_{j_0} + \gamma^* L_{j_0}}
$$

The second phase program (7.23) below provides a Pareto optimal solution to the MOP (7.21):

$$
\begin{aligned}
& \max s_1 + s_2 \\
& s.t. \\
& E_{j_0}^1 - wZ_{j_0} - \mu K_{j_0} + s_1 = \delta^* \\
& (E_{j_0}^2 - \delta^*)\left(wZ_{j_0} + \gamma L_{j_0}\right) - uY_{j_0} + \left(w^* Z_{j_0} + \gamma^* L_{j_0}\right) s_2 = 0 \\
& vX_{j_0} = 1 \\
& wZ_j + \mu K_j - vX_j \le 0, \quad j = 1, \ldots, n \\
& uY_j - wZ_j - \gamma L_j \le 0, \quad j = 1, \ldots, n \\
& v \ge \varepsilon, w \ge \varepsilon, u \ge \varepsilon, \gamma \ge \varepsilon, \mu \ge \varepsilon \\
& \delta^* \ge s_1 \ge 0, \delta^* \ge s_2 \ge 0
\end{aligned}
\tag{7.23}
$$

Once an optimal solution $\left(\hat{s}_1, \hat{s}_2, \hat{v}, \hat{w}, \hat{u}, \hat{\gamma}, \hat{\mu}\right)$ of (7.23) is obtained, the efficiency scores for unit j_0 in the first and the second stage as well as the overall efficiency of the system are respectively:

$$
\hat{e}_{j_0}^1 = \frac{\hat{w} Z_{j_0} + \hat{\mu} K_{j_0}}{\hat{v} X_{j_0}} = \hat{w} Z_{j_0} + \hat{\mu} K_{j_0}, \; \hat{e}_{j_0}^2 = \frac{\hat{u} Y_{j_0} + \hat{\gamma} L_{j_0}}{\hat{w} Z_{j_0}}, \; \hat{e}_{j_0}^o = \frac{\hat{u} Y_{j_0} + \hat{\mu} K_{j_0}}{\hat{v} X_{j_0} + \hat{\gamma} L_{j_0}}
$$

If $\hat{s}_1 = \hat{s}_2 = 0$, then the optimal solution of (7.22) is already Pareto optimal, and model (7.23) does not alter the efficiency scores obtained by (7.22).

Illustration

For testing and validation purposes, we provide the reader with the data (Table 7.6) and the results (Table 7.7) of a synthetic numerical example with 30 DMUs, three inputs to stage-1 (X1, X2, X3), two intermediate measures (Z1, Z2), to final outputs from stage-1 (K1, K2), two extra inputs to stage-2 (L1, L2) and two final outputs from stage-2 (Y1, Y2). The data exhibited in Table 7.6 are random and drawn column-wise from a uniform distribution in the intervals given in the last row of the Table 7.6.

Table 7.6: Synthetic data for type IV structure

DMU	X1	X2	X3	Z1	Z2	K1	K2	L1	L2	Y1	Y2
1	22.3	13.2	54.6	110.1	66.1	21.8	44.6	18	31	13.3	12.5
2	68.3	8.3	15.8	75.4	116.4	19.8	12	19.6	25.8	2.4	18.2
3	52	19.2	31.2	94.3	59.9	47.3	47.4	11.5	22.5	2.3	36
4	31.8	12	40.3	66.4	127.2	10.5	35.8	16.8	37.1	3	19.5
5	95.3	12	29	108.9	52.3	15	22.5	14	27.2	15.8	16.7
6	52.8	6.1	22.6	102.4	78.8	69.6	27	14.8	44.9	12.6	20.4
7	50.5	9.3	48.7	124.6	120.6	52.2	49.8	5.9	38.5	9.5	20.5
8	80.1	17.4	58.4	64.5	131.2	37.7	14.6	10.3	65.6	16.7	39.9
9	53.9	14	36.9	129.8	122.1	60.9	24.1	11.9	49.5	16.8	15
10	20.9	9.5	48.8	66.4	132.5	12.2	68.7	10.1	54.5	10	28.3
11	82.5	7.1	16.8	71.9	138.9	47.7	60.7	5.6	19.1	19.7	33.6
12	27	10.6	25.6	51.9	84.4	47.3	63.3	11	39.6	12.2	43.7
13	49.6	10.7	20.6	125.5	97.3	15.3	32.6	17.7	38.9	18.9	44.7
14	55.7	19.4	46.6	91.5	117.3	79	60.3	11.8	26.4	7.5	38.7
15	55.1	18.2	52.5	90.1	61	12.2	24.9	17	33.5	17.2	43.9
16	66.3	8	34.9	131.1	63.7	57	30.7	10.7	52.5	11.2	15.5
17	93.3	6.3	43.5	53.5	133.9	38.6	32.1	13.4	45	19.7	15.4
18	10.8	11.9	31.5	118.7	89.4	34.9	23.6	11	67.3	8	20.5
19	98.5	6.8	21.3	75.4	133	28	28.9	16.1	26.7	9.5	20.3
20	27.8	17.1	24.9	81	52.2	30.6	14.3	16.9	35.3	17.4	15.4
21	42	7.2	59.7	98.4	147.5	29.2	39.4	14.8	42.3	10.7	44.5
22	98.7	8.5	51	132.8	60.6	27.3	69.3	19.8	61.9	19.9	33.3
23	53.5	15.6	25.7	93.5	121.6	31.3	34.6	19.7	56.5	13	47.6
24	25.1	16.7	56.8	81.6	145.6	62.1	74.8	11.7	17.4	7.6	29.9
25	96.3	15.3	45.1	120.5	133.6	25.7	56.8	19.7	16.9	14.9	38.5
26	97.9	6.8	53.1	103.8	89.8	45.7	49.6	17.7	56.3	4.9	12.5
27	37.4	15	15.5	63.1	128.2	53.1	22	5.5	57.7	5.8	11.8
28	70	12.8	21.5	126.1	97.2	28.3	44.3	11.4	56.7	4.9	47.2
29	24	5.8	33.8	91.2	82.6	73.7	76.2	19.4	42.4	7.8	10.3
30	48.6	18.6	55.9	126	73.8	15.4	57.6	17.1	76.2	13.2	25.6
	[10,100]	[5,20]	[10,60]	[50,150]	[50,150]	[10,80]	[10,80]	[5,20]	[10,80]	[2,20]	[10,50]

Table 7.7: Results obtained from models (7.22) and (7.23) applied to the data of Table 7.6

DMU	E^1	E^2	e^{1*}	e^{2*}	$\hat{e}^1$	$\hat{e}^2$	$\hat{e}^o$
1	0.7338	0.7327	0.6751	0.6739	0.6751	0.6739	0.5573
2	1	0.4751	0.9868	0.4619	0.9868	0.4619	0.4643
3	0.7608	1	0.7287	0.9679	0.7287	0.9679	0.7805
4	0.8938	0.4268	0.8280	0.3610	0.8280	0.3610	0.3361
5	0.6914	1	0.5488	0.8574	0.5488	0.8574	0.5360
6	1	0.6708	1	0.6708	1	0.6708	0.8084
7	0.8812	0.6316	0.8529	0.6033	0.8529	0.6033	0.5606
8	0.5496	0.9796	0.4727	0.9028	0.4727	0.9028	0.4842
9	0.8656	0.7555	0.8298	0.7198	0.8298	0.7198	0.6305
10	1	0.6316	1	0.6316	1	0.6316	0.8970
11	1	1	1	1	1	1	1
12	1	1	1	1	1	1	1
13	1	0.8690	0.9715	0.8405	0.9715	0.8405	0.8321
14	0.6687	0.9337	0.6686	0.9336	0.6686	0.9336	0.6687
15	0.4215	1	0.4119	0.9904	0.4119	0.9904	0.6253
16	0.9813	0.7660	0.7815	0.5662	0.7815	0.5662	0.4829
17	1	1	0.9344	0.9344	0.9344	0.9344	0.8910
18	1	0.5189	1	0.5189	1	0.5189	0.5189
19	1	0.5096	0.9858	0.4954	0.9858	0.4954	0.4922
20	0.7676	1	0.7067	0.9390	0.7067	0.9390	0.7086
21	1	0.7871	0.9975	0.7846	0.9975	0.7846	0.7838
22	0.9591	1	0.7993	0.8403	0.7993	0.8403	0.6978
23	0.8255	0.7260	0.8178	0.7184	0.8178	0.7184	0.6697
24	1	0.9035	0.9623	0.8658	**1**	**0.8658**	0.8658
25	0.6613	1	0.6613	1	0.6613	1	0.7661
26	0.9447	0.2427	0.9343	0.2323	0.9343	0.2323	0.2395
27	1	0.3791	1	0.3791	1	0.3791	0.5963
28	1	1	0.9030	0.9030	0.9030	0.9030	0.8742
29	1	0.3762	1	0.3762	1	0.3762	0.3762
30	0.6246	0.6813	0.5512	0.6079	0.5512	0.6079	0.4320

As shown in Table 7.7, in all units except one (namely the unit 24) the second phase program (7.23) did not alter the efficiency scores obtained by model (7.22). For unit 24, the second phase program increased the stage-1 efficiency score from 0.9623 to 1 without decreasing the efficiency score of stage-2 (0.8658).

7.3 Multi-stage processes

A multi-stage series process is actually a multiple of type I-IV structures in series, where links exist only between successive stages. Thus, our developments for two-stage processes can be straightforwardly generalized in multi-stage configurations depicted in Fig. 7.7.

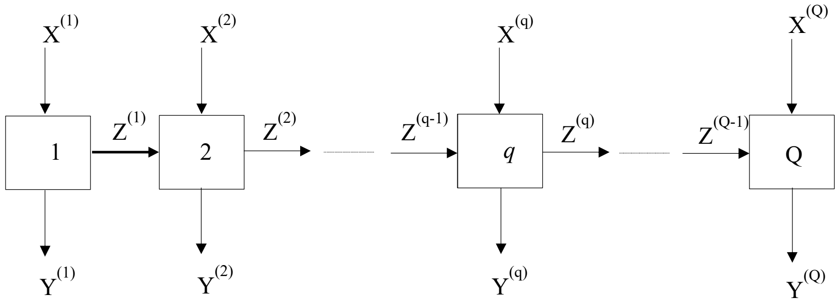


Figure 7.7: General multi-stage series process

We adjust the notation as follows:

$j \in J = \{1,\dots,n\}$: The index set of the *n* DMUs.

$j_0 \in J$: Denotes the evaluated DMU.

$q = 1,\dots,Q$: The index of one of the *Q* stages.

$X_j^{(q)}, q = 1,\dots,Q$: The vector of stage-*q* external inputs used by DMU_j.

$Z_j^{(q)}, q = 1,\dots,Q-1$: The vector of intermediate measures passed from stage-*q* to the next one, for DMU_j.

$Y_j^{(q)}, q = 1,\dots,Q$: The vector of stage-*q* final outputs produced by DMU_j.

$v^{(q)}, q = 1,\dots,Q$: The vector of weights for the stage-*q* external inputs.

$w^{(q)}, q = 1,\dots,Q-1$: The vector of weights for the stage-*q* intermediate measures.

$u^{(q)}, q = 1,\dots,Q$: The vector of weights for the stage-*q* outputs.

e_j^o : The overall efficiency of DMU_j.

$e_j^{(q)}, q = 1,\dots,Q$: The efficiency of stage-*q* for DMU_j.

$E_j^{(q)}, q = 1,\dots,Q$: The independent efficiency score of stage-*q* for DMU_j.

In this general case, the efficiency $e_j^{(q)}, q = 1,\ldots,Q$ of each stage is defined as follows:

$$
\begin{aligned}
e_j^{(1)} &= \frac{u^{(1)}Y_j^{(1)} + w^{(1)}Z_j^{(1)}}{v^{(1)}X_j^{(1)}} \\
e_j^{(q)} &= \frac{u^{(q)}Y_j^{(q)} + w^{(q)}Z_j^{(q)}}{v^{(q)}X_j^{(q)} + w^{(q-1)}Z_j^{(q-1)}}, \quad q = 2,\ldots,Q-1 \\
e_j^{(Q)} &= \frac{u^{(Q)}Y_j^{(Q)}}{v^{(Q)}X_j^{(Q)} + w^{(Q-1)}Z_j^{(Q-1)}}
\end{aligned}
$$

Model (7.24) below is a multi-objective program with Q objective functions, each representing the efficiency of stage-q, q=1,..., Q.

$$
\begin{aligned}
&\max \frac{u^{(1)}Y_{j_0}^{(1)} + w^{(1)}Z_{j_0}^{(1)}}{v^{(1)}X_{j_0}^{(1)}} \\
&\max \frac{u^{(q)}Y_{j_0}^{(q)} + w^{(q)}Z_{j_0}^{(q)}}{v^{(q)}X_{j_0}^{(q)} + w^{(q-1)}Z_{j_0}^{(q-1)}}, \; q = 2,\ldots,Q-1 \\
&\max \frac{u^{(Q)}Y_{j_0}^{(Q)}}{v^{(Q)}X_{j_0}^{(Q)} + w^{(Q-1)}Z_{j_0}^{(Q-1)}} \\
&s.t. \\
&u^{(1)}Y_j^{(1)} + w^{(1)}Z_j^{(1)} - v^{(1)}X_j^{(1)} \le 0, j = 1,\ldots,n \\
&u^{(q)}Y_j^{(q)} + w^{(q)}Z_j^{(q)} - v^{(q)}X_j^{(q)} - w^{(q-1)}Z_j^{(q-1)} \le 0, j = 1,\ldots,n, q = 2,\ldots,Q-1 \\
&u^{(Q)}Y_j^{(Q)} - v^{(Q)}X_j^{(Q)} - w^{(Q-1)}Z_j^{(Q-1)} \le 0, j = 1,\ldots,n \\
&v^{(q)} \ge \varepsilon, u^{(q)} \ge \varepsilon, \; q = 1,\ldots,Q \\
&w^{(q)} \ge \varepsilon, \; q = 1,\ldots,Q-1
\end{aligned}
\tag{7.24}
$$

The ideal values (independent efficiency scores) $E_{j_0}^{(q)}, q = 1,\ldots,Q$ of the Q stages are obtained by considering each objective function separately and solving the linear equivalent of model (7.24) derived by the C-C transformation. Given the ideal efficiency scores that each stage attains when considered independently from the others, the program (7.25) below provides a weakly Pareto optimal solution to the MOP (7.24) and estimates efficiency scores for the Q stages as close as possible to their ideal counterparts with respect to the unweighted L_∞ norm. Model (7.25) below is derived by applying the C-C transformation to (7.24) on the basis of the denominator of the first objective function and is solved by bisection search.

$$
\begin{aligned}
&\min \delta \\
&s.t. \\
&\mathrm{E}_{j_0}^{(1)} - u^{(1)}Y_{j_0}^{(1)} - w^{(1)}Z_{j_0}^{(1)} \le \delta \\
&(\mathrm{E}_{j_0}^{(q)} - \delta)\left(v^{(q)}X_{j_0}^{(q)} + w^{(q-1)}Z_{j_0}^{(q-1)}\right) - u^{(q)}Y_{j_0}^{(q)} - w^{(q)}Z_{j_0}^{(q)} \le 0, q = 2,\ldots,Q-1 \\
&(\mathrm{E}_{j_0}^{(Q)} - \delta)\left(v^{(Q)}X_{j_0}^{(Q)} + w^{(Q-1)}Z_{j_0}^{(Q-1)}\right) - u^{(Q)}Y_{j_0}^{(Q)} \le 0 \\
&v^{(1)}X_{j_0}^{(1)} = 1 \\
&u^{(1)}Y_j^{(1)} + w^{(1)}Z_j^{(1)} - v^{(1)}X_j^{(1)} \le 0, j = 1,\ldots,n \\
&u^{(q)}Y_j^{(q)} + w^{(q)}Z_j^{(q)} - v^{(q)}X_j^{(q)} - w^{(q-1)}Z_j^{(q-1)} \le 0, j = 1,\ldots,n, q = 2,\ldots,Q-1 \\
&u^{(Q)}Y_j^{(Q)} - v^{(Q)}X_j^{(Q)} - w^{(Q-1)}Z_j^{(Q-1)} \le 0, j = 1,\ldots,n \\
&v^{(q)} \ge \varepsilon, u^{(q)} \ge \varepsilon,\ q = 1,\ldots,Q \\
&w^{(q)} \ge \varepsilon,\ q = 1,\ldots,Q-1 \\
&\delta \ge 0
\end{aligned}
\tag{7.25}
$$

Let $\left(\delta^*, v^{*(q)}, q = 1,\ldots,Q,\ w^{*(q)}, q = 1,\ldots,Q-1,\ u^{*(q)}, q = 1,\ldots,Q\right)$ be an optimal solution of model (7.25), which is weakly Pareto optimal for (7.24) and

$$
\begin{aligned}
e_{j_0}^{*(1)} &= \frac{u^{*(1)}Y_{j_0}^{(1)} + w^{*(1)}Z_{j_0}^{(1)}}{v^{*(1)}X_{j_0}^{(1)}} = u^{*(1)}Y_{j_0}^{(1)} + w^{*(1)}Z_{j_0}^{(1)} \\
e_{j_0}^{*(q)} &= \frac{u^{*(q)}Y_{j_0}^{(q)} + w^{*(q)}Z_{j_0}^{(q)}}{v^{*(q)}X_{j_0}^{(q)} + w^{*(q-1)}Z_{j_0}^{(q-1)}},\ \ q = 2,\ldots,Q-1 \\
e_{j_0}^{*(Q)} &= \frac{u^{*(Q)}Y_{j_0}^{(Q)}}{v^{*(Q)}X_{j_0}^{(Q)} + w^{*(Q-1)}Z_{j_0}^{(Q-1)}}
\end{aligned}
$$

The second phase program that provides a Pareto optimal solution for the MOP (7.24) is as follows:

$$
\begin{aligned}
&\max \sum_{q=1}^{Q} s^{(q)} \\
&s.t. \\
&\mathrm{E}_{j_0}^{(1)} - u^{(1)} Y_{j_0}^{(1)} - w^{(1)} Z_{j_0}^{(1)} + s^{(1)} = \delta^* \\
&\left(\mathrm{E}_{j_0}^{(q)} - \delta^*\right)\left(v^{(q)} X_{j_0}^{(q)} + w^{(q-1)} Z_{j_0}^{(q-1)}\right) - u^{(q)} Y_{j_0}^{(q)} - w^{(q)} Z_{j_0}^{(q)} \\
&+\left(v^{*(q)} X_{j_0}^{(q)} + w^{*(q-1)} Z_{j_0}^{(q-1)}\right) s^{(q)} = 0, q = 2,\ldots,Q-1 \\
&\left(\mathrm{E}_{j_0}^{(Q)} - \delta^*\right)\left(v^{(Q)} X_{j_0}^{(Q)} + w^{(Q-1)} Z_{j_0}^{(Q-1)}\right) - u^{(Q)} Y_{j_0}^{(Q)} + \left(v^{*(Q)} X_{j_0}^{(Q)} + w^{*(Q-1)} Z_{j_0}^{(Q-1)}\right) s^{(Q)} = 0 \\
&v^{(1)} X_{j_0}^{(1)} = 1 \qquad (7.26) \\
&u^{(1)} Y_j^{(1)} + w^{(1)} Z_j^{(1)} - v^{(1)} X_j^{(1)} \le 0, j = 1,\ldots,n \\
&u^{(q)} Y_j^{(q)} + w^{(q)} Z_j^{(q)} - v^{(q)} X_j^{(q)} - w^{(q-1)} Z_j^{(q-1)} \le 0, j = 1,\ldots,n, q = 2,\ldots,Q-1 \\
&u^{(Q)} Y_j^{(Q)} - v^{(Q)} X_j^{(Q)} - w^{(Q-1)} Z_j^{(Q-1)} \le 0, j = 1,\ldots,n \\
&v^{(q)} \ge \varepsilon, u^{(q)} \ge \varepsilon, q = 1,\ldots,Q \\
&w^{(q)} \ge \varepsilon, q = 1,\ldots,Q-1 \\
&\delta^* \ge s^{(q)} \ge 0,\ q = 1,\ldots,Q
\end{aligned}
$$

Given an optimal solution $\left(\hat{v}^{(q)}, \hat{u}^{(q)}, \hat{s}^{(q)}, q = 1,\ldots,Q,\ \hat{w}^{(q)}, q = 1,\ldots,Q-1\right)$ of (7.26) the stage efficiency scores for the evaluated unit j_0 and the overall system efficiency are:

$$
\begin{aligned}
\hat{e}_{j_0}^{(1)} &= \frac{\hat{u}^{(1)} Y_{j_0}^{(1)} + \hat{w}^{(1)} Z_{j_0}^{(1)}}{\hat{v}^{(1)} X_{j_0}^{(1)}} = \hat{u}^{(1)} Y_{j_0}^{(1)} + \hat{w}^{(1)} Z_{j_0}^{(1)} \\
\hat{e}_{j_0}^{(q)} &= \frac{\hat{u}^{(q)} Y_{j_0}^{(q)} + \hat{w}^{(q)} Z_{j_0}^{(q)}}{\hat{v}^{(q)} X_{j_0}^{(q)} + \hat{w}^{(q-1)} Z_{j_0}^{(q-1)}},\ \ q = 2,\ldots,Q-1 \\
\hat{e}_{j_0}^{(Q)} &= \frac{\hat{u}^{(Q)} Y_{j_0}^{(Q)}}{\hat{v}^{(Q)} X_{j_0}^{(Q)} + \hat{w}^{(Q-1)} Z_{j_0}^{(Q-1)}} \\
\hat{e}_{j_0}^{o} &= \frac{\sum_{q=1}^{Q} \hat{u}^{(q)} Y_{j_0}^{(q)}}{\sum_{q=1}^{Q} \hat{v}^{(q)} X_{j_0}^{(q)}}
\end{aligned}
$$

Illustration

For testing and validation to be made possible, we provide a synthetic example with 30 DMUs operating as three-stage processes, as depicted in Fig. 7.8. The randomly generated data and the results obtained by solving model (7.26) for each DMU are given in Tables 7.8 and 7.9 respectively.

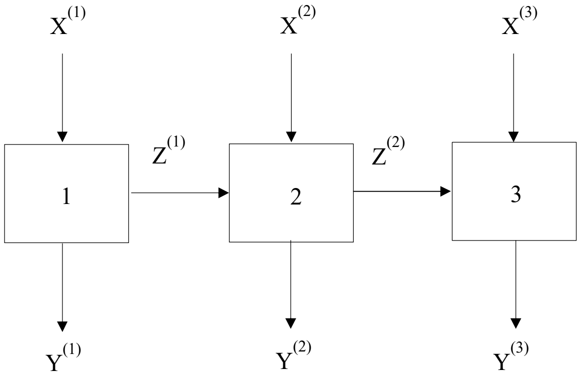


Figure 7.8: A three-stage process

Table 7.8: Synthetic data for the multi-stage process of Fig. 7.8

DMU	$X_1^{(1)}$	$X_2^{(1)}$	$Y_1^{(1)}$	$Y_2^{(1)}$	$Z_1^{(1)}$	$Z_2^{(1)}$	$X_1^{(2)}$	$X_2^{(2)}$	$Y_1^{(2)}$	$Y_2^{(2)}$	$Z_1^{(2)}$	$Z_2^{(2)}$	$X_1^{(3)}$	$X_2^{(3)}$	$Y_1^{(3)}$	$Y_2^{(3)}$
1	56.7	8.5	10.9	37.5	39	47.3	54.7	47	36.8	23.1	21.4	25.7	58.2	49.7	36.4	37.2
2	50.9	7.7	3.3	42.8	73	36.7	16.9	57.7	34	11	36.9	35.1	12.1	37.2	43.4	58.5
3	16.6	17.4	18	32.1	72.4	18.3	29.5	11.6	12.2	37.8	36.6	35.9	58.6	44.3	26.6	50.5
4	67.6	16.5	3.2	48.5	63.5	36.2	56.4	27.8	12.3	25.7	14.9	26.8	19.5	54.7	47.2	20.3
5	89.7	19	9.9	44.3	20.7	12.9	55.9	43.1	16.1	20.4	26.3	26.9	43.4	12.7	58.9	47.2
6	38.4	6.6	16.9	29.4	11.7	26.3	45.7	24.1	10.8	38	21	33.3	39.3	25.2	33.4	48.2
7	87.7	7.7	9.1	33.2	54.8	36.7	40.9	21.5	27.4	28	38.7	41.2	43.8	12.3	63.2	45.8
8	37.7	6.5	13	30.5	34.1	47.3	27.2	45.6	43.3	20.8	21.3	57.3	28.1	19.8	24.1	42.1
9	80.9	12.3	16.7	46	85.2	42.4	56.8	41.2	34.7	34.4	45.8	44.7	41	46	41.7	28.7
10	46.4	7.9	18	44.4	88.5	29.4	16.2	39.5	30.8	23.6	43.1	27	50.6	46.1	20.2	50.9
11	43.1	18.4	18.8	32.2	32.9	40.3	46.5	43	44.6	24.4	25.6	50.8	11	53.9	32.1	29.1
12	27.4	6.5	5.4	42.8	74.1	26.7	42.3	12.4	13.9	28.8	29.9	12.4	14.2	39.1	20.1	34.8
13	20.1	5.7	6.7	52.8	81.7	48.9	51.7	27.4	46.3	17.9	37.8	8.3	58.7	13.5	30.4	55.6
14	34.7	13.4	18.2	46.4	57.8	49.5	29.9	32.6	14.3	13.4	43.4	9.6	42.6	56.1	27.8	54.3
15	68.1	16.6	12.7	43.2	80.7	44.6	47.5	22	30.7	25.3	34.4	14	21.6	50	34.7	36.1
16	72.6	9.7	11.1	19.4	85.5	25.6	51.8	45.8	15.7	35.3	33	22.8	30.2	24.3	29.6	32.7
17	65.5	7.7	13	30.6	53.9	28.2	26.1	52.8	32.4	19.2	23	21.6	16.1	37.2	27.6	44.3
18	10.8	10.1	16.7	38.6	68.3	19.9	37.6	24.1	10.2	8.9	28.3	5.6	23.4	59.2	52.9	56.4
19	77.5	8.2	11.6	33.4	56.1	41.4	59	46.6	40.7	20.5	38.6	34.7	22.9	45.8	69.6	56.4
20	83.8	12.7	5.6	17	12.1	45.3	37.5	16.9	43.9	15.5	45.4	10.2	26.6	51.9	71.7	43.7
21	71.7	18.6	10.2	24.1	45.7	46.5	26.5	51.8	46.7	19	38.8	13.1	17.6	31.7	32.2	33.3
22	13.4	14.4	9.7	48.4	61.7	32.3	41	16.9	49.5	34.2	10.7	39.7	27.4	33.5	46.5	54.1
23	40.3	6.5	19.4	15.6	51.7	34	28	39.4	30.2	19.1	37	52.3	16.1	38	40.7	37.7
24	66.3	10.9	13.2	49.5	39.8	16	47.8	28.3	20.9	18.7	27.5	58.6	54.2	23.5	48.8	56.2
25	68.4	5.8	14.5	18.1	85	46	30.7	50.3	14	17.6	27.5	36.4	14.7	47.5	34.6	21.3
26	27.9	12.5	15	41.8	76.4	28	34.6	35.2	30.3	9.9	14.7	59.8	56.5	35.2	23.8	41.3
27	31.5	11.5	8.2	35	77.9	18.2	44.7	34.5	33.4	14.1	42.6	35.4	30	42.3	44	48.7
28	63.8	20	11.3	23.7	39.8	46	58.6	53.9	40.5	8	23	33.4	12.4	25.4	29.6	27.2
29	48.2	17.2	12	37.9	57.5	40.5	26.4	27.7	13.3	20	19.8	23.2	27.1	16.9	21.4	33.5
30	59.9	12.3	4.8	19.9	79.8	45.3	51.9	32.5	36.5	14	23.7	28.7	46.8	33.8	72.5	27.5
	[10,90]	[5,20]	[2,20]	[15,55]	[10,90]	[10,50]	[10,60]	[10,60]	[10,50]	[5,40]	[10,50]	[5,60]	[10,60]	[10,60]	[20,75]	[20,60]

Table 7.9: Results obtained from models (7.25) and (7.26) applied to the data of Table 7.8

DMU	E^1	E^2	E^3	e^{1*}	e^{2*}	e^{3*}	$\hat{e}^1$	$\hat{e}^2$	$\hat{e}^3$	$\hat{e}^o$	DMU
1	0.6979	0.7085	0.6474	0.6692	0.6798	0.6188	0.6692	0.6798	0.6188	0.6356	1
2	0.6594	1	1	0.6594	1	1	0.6594	1	1	0.8133	2
3	0.7012	1	0.6006	0.7012	1	0.6006	0.7012	1	0.6006	0.6891	3
4	0.3173	0.4927	1	0.2963	0.4717	0.9790	**0.3068**	**0.4717**	**0.9985**	0.3003	4
5	0.3083	1	1	0.3083	1	1	0.3083	1	1	0.3083	5
6	1	1	0.8725	1	1	0.8724	1	1	0.8724	0.9933	6
7	0.6428	0.9838	1	0.6275	0.9685	0.9847	**0.6275**	**0.9685**	**0.9983**	0.8213	7
8	0.9935	1	0.9554	0.9527	0.9593	0.9147	**0.9935**	**0.9593**	**0.9350**	0.9593	8
9	0.6355	0.8040	0.4696	0.6340	0.8025	0.4681	**0.6347**	**0.8025**	**0.4681**	0.6054	9
10	1	1	0.6376	1	1	0.6376	1	1	0.6376	0.7888	10
11	0.5796	1	0.8851	0.5322	0.9526	0.8378	0.5322	0.9526	0.8378	0.5421	11
12	0.7939	0.8150	0.8203	0.7534	0.7746	0.7798	**0.7939**	**0.7746**	**0.8249**	0.7746	12
13	1	0.8043	1	1	0.8043	1	1	0.8043	1	0.9403	13
14	0.8064	0.8718	0.7640	0.6911	0.7566	0.6488	**0.7178**	**0.7566**	**0.6488**	0.6566	14
15	0.4241	0.6843	0.6668	0.3929	0.6531	0.6356	0.3929	0.6531	0.6356	0.3929	15
16	0.6169	0.7314	0.6621	0.5491	0.6635	0.5942	**0.5491**	**0.6635**	**0.6082**	0.5454	16
17	0.7395	0.9225	0.9456	0.7380	0.9210	0.9440	**0.7380**	**0.9210**	**0.9441**	0.7391	17
18	1	0.7168	1	0.9779	0.6868	0.9700	**1**	**0.6868**	**0.9700**	0.8158	18
19	0.6961	0.8030	1	0.6887	0.7956	0.9926	**0.6887**	**0.7956**	**0.9999**	0.6784	19
20	0.4158	1	1	0.4158	1	1	0.4158	1	1	0.6516	20
21	0.3437	1	0.8453	0.3437	1	0.8453	0.3437	1	0.8453	0.9219	21
22	1	1	1	1	1	1	1	1	1	1	22
23	1	1	0.7844	1	1	0.7844	1	1	0.7844	0.9360	23
24	0.6497	1	0.9073	0.6449	0.9951	0.9024	0.6449	0.9951	0.9024	0.6454	24
25	1	0.6358	0.7559	0.9700	0.6058	0.7259	**1**	**0.6058**	**0.7259**	0.7349	25
26	0.7068	1	0.6771	0.5676	0.8608	0.5379	**0.6752**	**0.8608**	**0.5379**	0.5908	26
27	0.5689	1	0.6982	0.4697	0.9008	0.5990	**0.4697**	**0.9008**	**0.6615**	0.4713	27
28	0.3655	0.7667	0.7944	0.3359	0.7372	0.7649	**0.3359**	**0.7372**	**0.7824**	0.3095	28
29	0.4378	0.6626	0.8439	0.4050	0.6297	0.8110	**0.4050**	**0.6298**	**0.8132**	0.4048	29
30	0.4510	0.6185	1	0.4373	0.6048	0.9863	**0.4373**	**0.6048**	**0.9999**	0.3540	30

The bold figures in columns 8-10 of Table 7.9 indicate the units, whose final Pareto optimal efficiency scores were obtained in the second phase.

7.4 Conclusion

We introduced in this chapter a novel network DEA approach to efficiency assessment in series multi-stage processes. Actually, it is a multi-objective programming approach that employs the L_∞ norm as distance measure to locate the stage efficiency scores as close as possible to their ideal values. Our approach is general, in the sense that it can handle series multi-stage processes of any type. It is exact, as it provides unique efficiency scores and it is neutral, as it treats the different stages equivalently. Also, it responds accurately to any different weighting scheme for the stages, by driving the efficiency assessments accordingly.

Chapter 8

The assessment of the academic research activity – A network DEA approach

8.1 Introduction

Data envelopment analysis has been commonly used as an instrument to measure the performance of academic units (Universities, faculties, departments or individuals) in various aspects of academic activities such as research, teaching and administration (Beasley, 1995, Athanassopoulos and Shale, 1997, Korhonen et al., 2001, Avkiran, 2001, Katharaki and Katharakis, 2010, Kounetas et al., 2011). Recently, network DEA has been also applied to assess the performance of entities in education in various aspects Monfared and Safi (2013) assessed the academic performance of colleges in Alzahra, Iran. They used a two-stage network structure with shared inputs to represent teaching and research as two separate activities. Johnes (2013) evaluated the efficiency of higher education institutions in England. He employed a two-stage network structure with the first stage representing the teaching activity and the second one, employability of the graduates. Lee and Worthington (2016) employed network DEA to evaluate the research performance of Australian universities. They utilized a two-stage network structure to represent the university research production as a two-stage process. The first stage represented research and the second one grant applications. Full time equivalent Academics and the number of PhD students were considered as the two external inputs to the first stage, whose output was a publication indicator (considered also as input to the second stage) measured as a weighted average of the number of publications of different categories. The output of the second stage was the value of grants (total research income).

In this chapter, we present a framework to assess the academic research activity in higher education. The aim of this assessment framework is to encompass both the extent and the quality of the research work as well as its impact. Thus, a two-stage network structure is used in which the first stage represents productivity and the second one the recognition of the research outputs (publications). To illustrate the proposed approach, an anonymous dataset of 40 academics with estimated realistic data is used. For the efficiency assessment, we utilize the network DEA approach developed in the previous chapter. The results of the analysis have a meaningful interpretation and the current application also highlights the applicability and the effectiveness of the network DEA approach developed in the previous chapter.

8.2 Assessing the research productivity and impact of academics

The scope of the proposed approach is to estimate the relative efficiency of academics with respect to their research activity and the impact of the research. The quality of the research output and the recognition it receives in the international scientific community affects the recognition of the researcher himself as well as the reputation of the hosting institution. In this context, the research activity of an individual staff member is viewed as a two-stage process as depicted in Figure 8.1.

The first stage represents the *productivity* of the individual: The inputs in this stage are *time in post* ($X^{(1)}$) and *total salary* since appointment ($X^{(2)}$). The output of the first stage is *publications* (Z). In order to make the assessment strict, the following assumptions were made regarding the publications:

- Only publications in journals indexed in Scopus were taken into account.
- In case of multiple authors, each individual author is credited with a fraction of the publication, actually with *1/n*, where *n* is the number of authors. So, the total number of publications of an individual is given as single-author equivalent (SAE). In this manner, each publication is counted at most once at the faculty or at the institution level, as there might be co-authors from other institutions.

- The journals, and thus the publications in that journals, are classified in four quality classes (A+, A, B, C) according to the ERA2010 journal classification system (ERA: Excellence in Research in Australia). A fifth class D is made for journals that are not indexed in ERA2010.

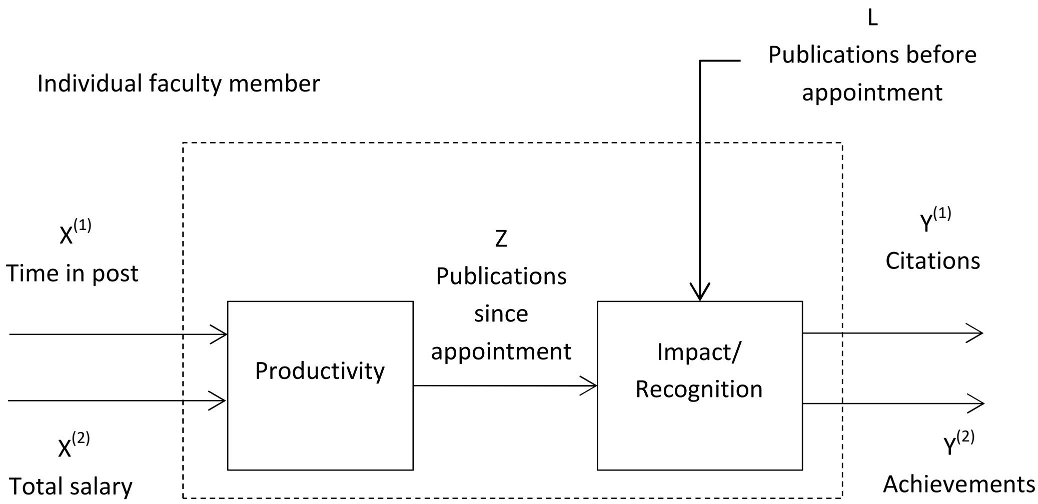


Figure 8.1: The academic research activity as a two stage-process

The second stage represents the impact that the research work of the individual has in academia and the recognition, which the researcher has gained as a result of his work. Once released, a publication becomes an independent entity, which, depending on its quality and dissemination, generates *citations* and recognition. The latter is measured through the *academic achievements* of the individual, such as being chief editor of scientific journals, associate editor or member of editorial boards, being invited as keynote speaker in conferences, participating in scientific or advisory committees of conferences. The number of occurrences of each one of the above are weighted and aggregated to derive a measure of *academic achievements*. The publications made by an individual before his appointment are extra inputs to the second stage, as in conjunction with those made in post, they contribute in the academic profile of the individual.

Two cases were examined: In case I, the total number of single-author equivalent (SAE) publications was considered with no distinction among the journals. In case II, the publications were broken down in the quality classes mentioned above, with the SAE publications in each class constituting a distinct measure. The descriptive statistics for the data considered in case I and case II are exhibited in Table 8.1 and Table 8.2 respectively.

Table 8.1: Descriptive statistics of the data for case I: Total number of publications (SAE)

	Factors	Min	Max	Average	St. Dev.
$X^{(1)}$	Years in post	3	28.50	11.73	6.90
$X^{(2)}$	Total Salary (tens of thousands)	6.42	105.68	35.55	25.81
Z	Total SAE	0.25	20.90	5.49	5.08
L	Total SAE	0.00	14.56	4.04	4.04
$Y^{(1)}$	Citations (tens of)	1.60	95.10	24.76	22.38
$Y^{(2)}$	Achievements	0.50	25.50	6.34	6.07

The breakdown of the publications in categories is made to introduce the quality dimension in the assessments. This is made by introducing assurance region constraints in the assessment models (7.14) and (7.15), which were presented in the previous chapter. In the current assessment, we assumed that the publications in category A^+ should be weighted at least 1.5 and at most twice as much as the publications in A (i.e. $1.5 \le w\left(A^+\right)/w\left(A\right) \le 2$). For the other categories, we assumed the following weight constraints: $2 \le w\left(A\right)/w\left(B\right) \le 2.5$, $1.5 \le w\left(B\right)/w\left(C\right) \le 2$ and $2 \le w\left(C\right)/w\left(D\right) \le 3$.

Although the aforementioned assurance region constraints reflect a global knowledge e.g. publication in journal ranked as A^+ are more important than publications in journals ranked as A, the intensity of preference is subjective. For instance, a policy maker could consider that publications in category A^+ should be weighted at least twice and at most thrice as much as the publications in A. Such changes in the parameters of the constraints can lead to different results. However, the aim of this application is not to assess the individuals or the institutions they belong to. Rather it is to illustrate the proposed framework with realistic data and then to present its effectiveness.

Table 8.2: Descriptive statistics of the data for case II: Publications broken down in quality classes

	Factors	Min	Max	Average	St. Dev.
$X^{(1)}$	Years in post	3.00	28.50	11.73	6.90
$X^{(2)}$	Total Salary (tens of thousands)	6.42	105.68	35.55	25.81
$Z^{(1)}$	A+	0.00	2.00	0.19	0.43
$Z^{(2)}$	A	0.00	6.42	1.11	1.72
$Z^{(3)}$	B	0.00	7.42	1.63	1.67
$Z^{(4)}$	C	0.00	9.2	1.78	2.03
$Z^{(5)}$	D	0.00	4.26	0.79	1.06
$L^{(1)}$	A+	0.00	4.45	0.39	0.84
$L^{(2)}$	A	0.00	8.03	1.09	1.55
$L^{(3)}$	B	0.00	6.33	1.16	1.52
$L^{(4)}$	C	0.00	8.53	1.1	1.66
$L^{(5)}$	D	0.00	1.58	0.29	0.48
$Y^{(1)}$	Citations (tens of)	1.60	95.10	24.76	22.38
$Y^{(2)}$	Achievements	0.50	25.50	6.34	6.07

8.3 Results

The results obtained by applying the model (7.14) and the second phase program (7.15), to the data summarized in Tables 8.1 and 8.2 are given in Table 8.3. Comparing the distributions in Figures 8.2 and 8.3, one can observe a decrease in the productivity scores (stage-1), on average, when the quality of the publications is taken into account in case II. This is exhibited in Table 8.3 as well, where the data for two faculty members are presented. Both records are almost identical and their difference is revealed only when their publications are broken down in categories of quality. They are both inefficient in case I, with the individual #10 outperforming a bit the individual #18 in terms of productivity. However, when the quality dimension of the publications is taken into account in case II, the individual #10 is rendered efficient whereas the #18 loses much of his productivity score. Comparing the distributions in Figures 8.4 and 8.5, it is observed that there is an increase of the average efficiency score. Concerning the impact of the research and the achievements, the faculty member #10 outperforms #18 in case I, whereas #10 is outperformed by #18 in case II. This reversal can be justified by the fact that, although both individuals have almost the same level of citations and academic achievements, the #18 achieves this level of outputs with publications of low quality. In other words, the assessment disfavors the faculty member #10 for whom one would expect higher achievements from his high level publications.

In terms of productivity (case II, stage-1), only one faculty member is rendered efficient. In terms of impact of the research, five faculty members are efficient. However, as shown in Figures 8.6 and 8.7, none of the faculty members is overall efficient, but this is not rare in network DEA.

Table 8.3: Data and results for two indicative individuals

Individual	#10	#18
Years in post	7	7
Total income in post (tens of thousands)	15.0	15.9
Publications after appointment (SAE total)	8.1	8.1
A^+	2.0	0.0
A	2.0	1.5
B	2.1	2.5
C	2.0	1.8
D	0.0	2.3
Publications before appointment (SAE total)	4.4	4.5
A^+	2.8	0.0
A	1.6	0.0
B	0.0	2.7
C	0.0	1.8
D	0.0	0.0
Citations (tens of)	39.0	40.4
Achievements	5.0	4.0
Case I - Productivity (Stage-1)	0.976	0.918
Case I - Impact (Stage-2)	0.242	0.208
Case I - Overall	0.236	0.191
Case II - Productivity (Stage-1)	1.000	0.554
Case II - Impact (Stage-2)	0.128	0.319
Case II - Overall	0.128	0.177

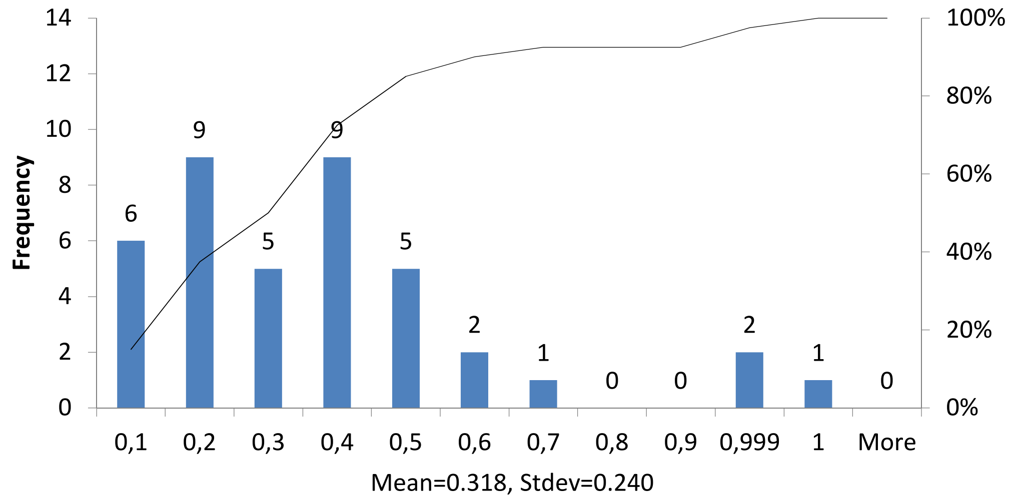


Figure 8.2: Stage-1 efficiency distributions in case I

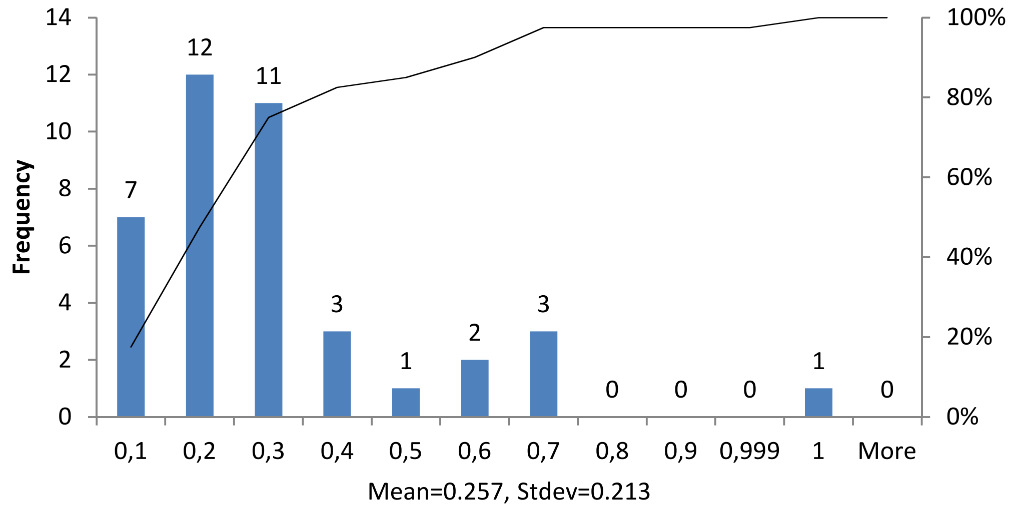


Figure 8.3: Stage-1 efficiency distributions in case II

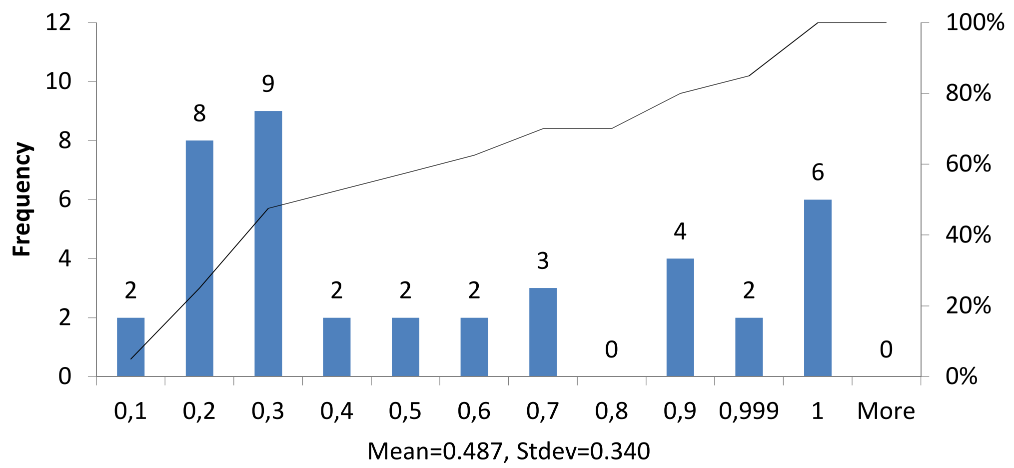


Figure 8.4: Stage-2 efficiency distributions in case I

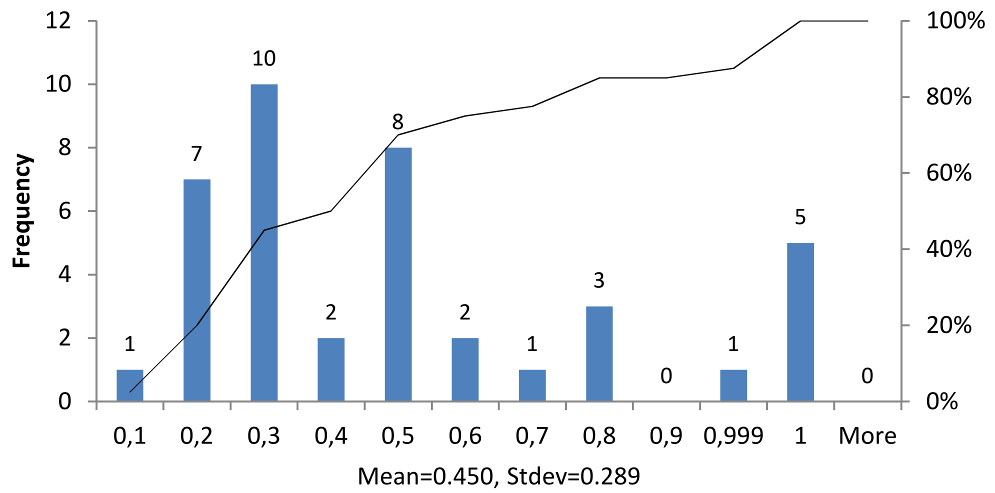


Figure 8.5: Stage-2 efficiency distributions in case I

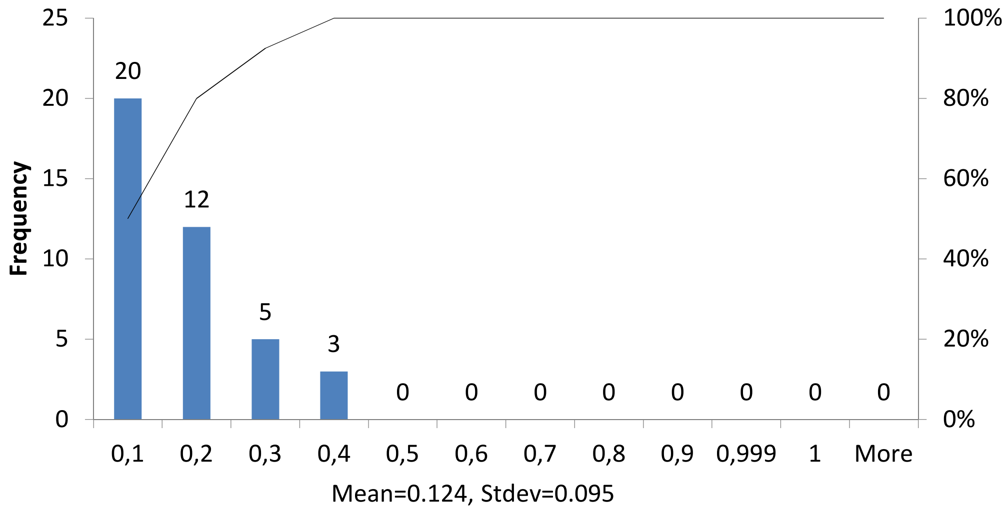


Figure 8.6: Overall efficiency distribution in case I

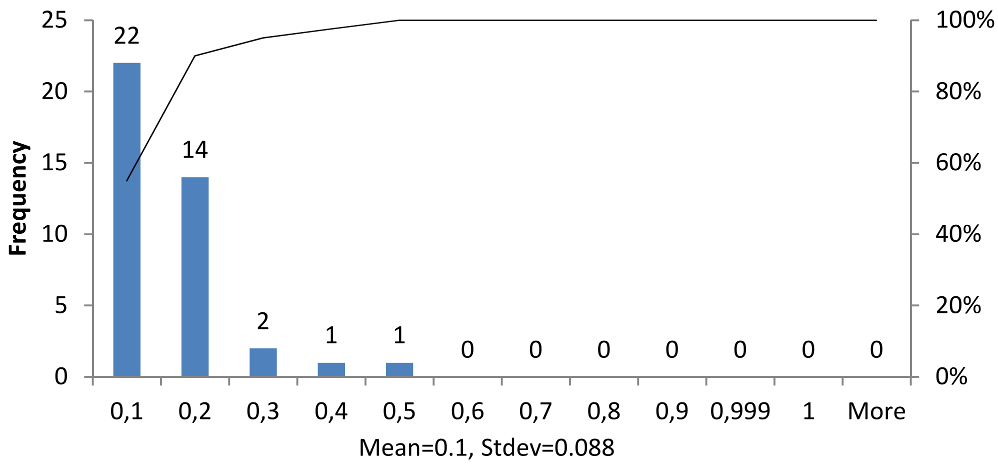


Figure 8.7: Overall efficiency distribution in case II

8.4 Conclusion

We developed a framework for the assessment of the research performance at a faculty member level. The research activity of each faculty member is viewed as a two-stage process. The first stage represents the research productivity of the individual while in post, whereas the second stage represents the impact of his research work. Disentangling productivity from impact is justified by the fact that a research paper, once published, becomes an independent entity. Utilizing the models (7.14) and (7.15), which were developed in the previous chapter, allows us to treat both stages equally and thus to obtain neutral and unbiased results. However, the ERA 2010 classification system that was selected as well as the subjective judgments for the priorities given to the journal quality classes are assumptions that do affect the results. Nevertheless, any other choice could be made.

Chapter 9

Summary and conclusion

In this dissertation, we focused on two extensions of the conventional DEA, namely, value based DEA and network DEA. We provided critical reviews on the value based and network DEA models proposed in the literature and we developed new models which overcome their limitations.

In the first part of this dissertation we dealt with value based DEA. Specifically, we introduced a data transformation – variable alteration technique as a means to transform the original input/output weights into values. We showed that this transformation enhances the conventional DEA models with additional properties such as units invariance, dimensionality and a meaningful interpretation of the variables in the DEA models. We provided a critical review on DEA with non-linear virtual inputs/outputs which spots the discontinuity issue of the value functions and then, we extended the data transformation – variable alteration technique to DEA models with non-linear virtual inputs and outputs, by employing piecewise linear value functions, that effectively treats the aforementioned discontinuity issue and provides a clearer representation of the value functions in such cases. These findings allowed us to develop a novel value based DEA model, which unlikely the value based DEA models proposed in the literature, provides a measure of efficiency for the evaluated units. To illustrate the effectiveness of our new developments, we revisited a case study drawn from the literature. By assimilating the preferential information given in the original work, the assessment results showed that our approach successfully locates the efficient DMUs and unlike the assessment method used in the original work that discriminates only between efficient and inefficient units, it provides a measure of efficiency. Moreover, we developed a two-phase approach to incorporate individual preferences in a DEA assessment framework by means of Ordinal Regression. The advantage of this new approach is that instead of using direct preferential information for the desired levels of the inputs and the outputs to estimate the value functions, it allows us to assess a prototype of the value functions based on Ordinal Regression. Finally, we further illustrated the effectiveness and the applicability of the novel value based DEA model by presenting an application concerning the assessment of the research performance of academics which takes into account both the quantity as well as the quality of the research output.

In the second part of this dissertation we dealt with network DEA by developing a novel network DEA approach for general series multi-stage processes. Particularly, we introduced a multi-objective programming approach, which employs the L_∞ norm as a distance measure to locate the stage efficiency scores as close as possible to their ideal values that are obtained independently through standard DEA models. Our new approach overcomes the defects of the network DEA models proposed in the literature as it provides unique and unbiased stage efficiency scores. When data were available in the literature, the advantages of our approach were illustrated by comparing the results obtained by our method with those obtained by other methods presented in the literature. When data were not available in the literature, synthetic data were used for testing and validation. The effectiveness and the applicability of our approach, was further illustrated by providing an application for the assessment of the academic research activity in higher education viewed as a two-stage network process. However, utilizing the proposed network DEA approach to cases where particular resources are shared among the stages of the system will lead to a highly non-linear model. This issue can be viewed as a limitation of this approach.

The extension of the current developed value based DEA model to network structures as well as the extension of the current developed network DEA approach in parallel network structures are subjects for further research. Moreover, it is worthy to mention that in network DEA the efficiency scores under the assumption of variable returns to scale are not necessarily higher than the efficiency scores obtained under the assumption of constant returns to scale, as it happens in the conventional DEA models. Moreover, in network DEA, although some DMUs may be identified as efficient in particular sub-processes (stages) of the network activity, none of them may be identified as overall efficient in the production process. This violates the assumption that the efficiency scores are estimated on a relative basis. These issues are considered as irregularities in network DEA and require further investigation.

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