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Γεώργιος Ι. Μπάρτζης

Διπλωματική Εργασία

που υποβλήθηκε στο Τμήμα Στατιστικής και Ασφαλιστικής Επιστήμης του Πανεπιστημίου Πειραιώς ως μέρος των απαιτήσεων για την απόκτηση του Μεταπτυχιακού Διπλώματος Ειδίκευσης στην Εφαρμοσμένη Στατιστική

Πειραιάς
Σεπτέμβριος 2014

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Η παρούσα Διπλωματική Εργασία εγκρίθηκε ομόφωνα από την Τριμελή Εξεταστική Επιτροπή που ορίσθηκε από τη ΓΣΕΣ του Τμήματος Στατιστικής και Ασφαλιστικής Επιστήμης του Πανεπιστημίου Πειραιώς στην υπ' αριθμ. …….. συνεδρίασή του σύμφωνα με τον Εσωτερικό Κανονισμό Λειτουργίας του Προγράμματος Μεταπτυχιακών Σπουδών στην Εφαρμοσμένη Στατιστική

Τα μέλη της Επιτροπής ήταν:

- Λεκτορας Μπερσίμης Σωτήριος (Επιβλέπων)
- Αναπληρωτής Καθηγητής Αντζουλάκος Δημήτριος
- Αναπληρωτής Καθηγητής Ψαράκης Στυλιανός

Η έγκριση της Διπλωματική Εργασίας από το Τμήμα Στατιστικής και Ασφαλιστικής Επιστήμης του Πανεπιστημίου Πειραιώς δεν υποδηλώνει αποδοχή των γνωμών του συγγραφέα.

UNIVERSITY OF PIRAEUS

DEPARTMENT OF STATISTICS
AND INSURANCE SCIENCE

POSTGRADUATE PROGRAM IN
APPLIED STATISTICS

INTERPERETING THE OUT-OF-CONTROL SIGNAL OF A MULTIVARIATE DISPERSION CONTROL CHART

By

Georgios I. Bartzis

MSc Dissertation

submitted to the Department of Statistics and Insurance Science of the University of Piraeus in partial fulfilment of the requirements for the degree of Master of Science in Applied Statistics

Piraeus, Greece
September 2014

Στην οικογένειά μου

Ευχαριστίες

Αρχικά θα ήθελα να ευχαριστήσω θερμά τον επιβλέποντα καθηγητή μου, κ. Σ. Μπερσίμη για την πολύτιμη βοήθεια που μου έχει δώσει κατά τη διάρκεια των προπτυχιακών αλλά και των μεταπτυχιακών μου σπουδών. Μέσω της συνεργασίας του μου προσέφερε ευκαιρίες, επιλογές αλλά και συμβουλές που ήταν καθοριστικές για την πορεία μου όλα αυτά τα έτη. Με την καθοδήγησή του με ενέπνευσε και η γνωριμία μου μαζί του αποτελεί σημείο σταθμό για ότι έχω επιτύχει. Τον ευχαριστώ από καρδιάς.

Επιπλέον θα ήθελα να ευχαριστήσω τα μέλη της τριμελούς επιτροπής, τον Αναπληρωτή Καθηγητή κ. Δ. Αντζουλάκο και τον Αναπληρωτή Καθηγητή κ. Σ. Ψαράκη όχι μόνον για τις συμβουλές τους και τις επιστημονικές γνώσεις τις οποίες μου έχουν προσφέρει αλλά επίσης και για το χρόνο που αφιέρωσαν στη διόρθωση της διπλωματικής αυτής.

Στη συνέχεια θα ήθελα να ευχαριστήσω όλους τους εξαίρετους επιστήμονες που είχα ως καθηγητές κατά τη διάρκεια των προπτυχιακών και μεταπτυχιακών μου σπουδών καθώς οι γνώσεις που μου προσέφεραν ήταν περισσότερο από πολύτιμες.

Παράλληλα θα ήθελα να αναφερθώ και να ευχαριστήσω τους φίλους μου που στάθηκαν δίπλα μου και με στήριξαν κατά τη διάρκεια της συγγραφής της διπλωματικής αυτής εργασίας. Επίσης θέλω να ευχαριστήσω θερμά τις συμφοιτήτριές και φίλες μου Β. Βασιλειάδη και Ε. Σταματοπούλου για την πολύτιμη βοήθεια που μου έχουν προσφέρει αλλά και για τη στήριξή τους. Σε αυτό το σημείο θα ήθελα να ευχαριστήσω και την Ε. Βουρλιώτη καθώς η υπομονή και η υποστήριξή της ήταν καθοριστικές για την ολοκλήρωση αυτής της προσπάθειας.

Ολοκληρώνοντας, δε θα μπορούσα να παραβλέψω φυσικά την ίδια μου την οικογένεια καθώς στεκόταν πάντα δίπλα μου διακριτικά κατά τη διάρκεια αυτής της προσπάθειας και πίστεψε σε μένα από την αρχή.

Περίληψη

Ο στατιστικός έλεγχος διεργασιών είναι ένα ευρέως χρησιμοποιούμενο εργαλείο που στοχεύει στον έλεγχο και τη βελτίωση βιομηχανικών διαδικασιών. Τεχνικές με μία μεταβλητή μπορούν να χρησιμοποιηθούν για την παρακολούθηση ενός χαρακτηριστικού σε μια διαδικασία, αλλά σε πολλές περιπτώσεις περισσότερα χαρακτηριστικά πρέπει να παρακολουθούνται ταυτόχρονα. Η ανεξάρτητη παρακολούθηση των χαρακτηριστικών, θα οδηγήσει σε εσφαλμένα συμπεράσματα, διότι η σχέση μεταξύ των μεταβλητών δεν λαμβάνεται υπόψη. Έτσι, η χρήση του Πολυμεταβλητού Στατιστικού Ελέγχου Διεργασιών είναι αναγκαία για την αποφυγή τέτοιων καταστάσεων.

Ενώ οι περισσότερες τεχνικές που αναπτύχθηκαν στην πολυμεταβλητή περίπτωση αφορούν την παρακολούθηση του μέσου επιπέδου της διαδικασίας, η παρακολούθηση της διασποράς είναι εξίσου σημαντική λόγω του γεγονότος ότι με το να συμμορφωθεί η διασπορά, θα οδηγηθούμε σε μικρότερη κύμανση του μέσου επιπέδου και επομένως σε μια πιο σταθερή διεργασία.

Η παρούσα διατριβή αρχικά παρέχει μια εισαγωγή στο Στατιστικό Έλεγχο Διεργασιών. Στη συνέχεια, παρουσιάζεται η απλή περίπτωση της μονομεταβλητής παρακολούθησης της διασποράς. Επιπροσθέτως, μια εκτεταμένη ανασκόπηση της βιβλιογραφίας σχετικά με τα πολυμεταβλητά διαγράμματα ελέγχου για τη διασπορά έχει γίνει και τέλος, παρουσιάζουμε μια σύγκριση των διαφόρων πολυμεταβλητών διαγραμμάτων ελέγχου που εμφανίζονται στη σχετική βιβλιογραφία.

Abstract

Statistical process control is a widely used tool which aims in controlling and improving an industrial or manufacturing process. Univariate techniques can be used for monitoring one characteristic in a process but in many situations more characteristics must be monitored simultaneously. By monitoring the characteristics independently, will lead on false conclusions because the relation between the variables is not taken into account. So, the usage of Multivariate Statistical Process Control is a necessity for avoiding such situations.

While most techniques developed in the multivariate case deal with monitoring the mean vector of the process, monitoring the dispersion is equally important due to the fact that by controlling the dispersion, the mean will fluctuate less leading in a more stable process.

The present dissertation initially provides an introduction in Statistical Process Control. Next, the simple case of univariately monitoring the dispersion is presented. In addition an extensive literature review on multivariate control charts for the dispersion has been made and finally, we present a comparison of several multivariate control charts that appear in literature.

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Abbreviations

UCL	**U**pper **C**ontrol **L**imit
LCL	**L**ower **C**ontrol **L**imit
CL	**C**enter **L**ine
ARL_0	In-Control **A**verage **R**un **L**ength
ARL_1	Out-of-Control **A**verage **R**un **L**ength
SHEWHART	Shewhart Control Chart
CUSUM	**Cu**mulative **Su**m Control Chart
EWMA	**E**xponentially **W**eighted **M**oving **A**verage Control Chart

Symbols

α	Probability of "Type I Error"
β	Probability of "Type II Error"
$Z_{1-\alpha}$	The $1-\alpha$ percentile of Standard Normal Distribution
$t_{n,1-\alpha}$	The $1-\alpha$ percentile of Student t-Distribution with n degrees of freedom
$\chi^2_{p,1-\alpha}$	The $1-\alpha$ percentile of Chi-Square (χ^2) Distribution with p degrees of freedom
$F_{m,n,1-\alpha}$	The $1-\alpha$ percentile of F-Distribution with m,n degrees of freedom
$W_{p,1-\alpha}$	The $1-\alpha$ percentile of Wishart W-Distribution with p degrees of freedom
μ_0	The in-control mean of a univariate Normal Distribution
$\overline{X}$	The sample mean from a univariate Normal Distribution
σ_0^2	The real variance of a univariate Normal Distribution
s^2	The sample variance of a univariate Normal Distribution
p	The dimensions of a multivariate Normal Distribution
m	The number of rational subgroups
n	The sample size of each rational subgroup
$\boldsymbol{x}$	The vector of the multivariate observations
$\boldsymbol{\mu}_0$	The real mean vector of a multivariate Normal Distribution
$\overline{\boldsymbol{x}}$	The sample mean vector of a multivariate Normal Distribution
$\boldsymbol{\Sigma}_0$	The real variance-covariance matrix of a multivariate Normal Distribution
$\boldsymbol{P}$	The real correlation matrix of a multivariate Normal Distribution

$\boldsymbol{\Sigma}_0^{-1}$	The invert of the real variance-covariance matrix of a multivariate Normal Distribution
$\boldsymbol{S}$	The sample variance-covariance matrix of a multivariate Normal Distribution
$\boldsymbol{S}^{-1}$	The invert of the sample variance-covariance matrix of a multivariate Normal Distribution
$\boldsymbol{A}$	The sample sum of squares and cross products matrix
$\overline{\overline{\boldsymbol{x}}}$	The pooled (working with rational subgroups) sample mean vector of a multivariate Normal Distribution
$\overline{\boldsymbol{S}}$	The pooled (working with rational subgroups) sample variance-covariance matrix of a multivariate Normal Distribution
$\overline{\boldsymbol{x}}_m$	The sample mean vector of a multivariate Normal Distribution (working with individual observations)
$\boldsymbol{S}_m$	The sample variance-covariance matrix of a multivariate Normal Distribution (working with individual observations)
$\boldsymbol{B}^t$ *or* $\boldsymbol{B}'$	The transpose of matrix $\boldsymbol{B}$
$\lvert\boldsymbol{B}\rvert$	The determinant of matrix $\boldsymbol{B}$
$trace(\boldsymbol{B})$	The trace of matrix $\boldsymbol{B}$

Chapter 1

Introduction

1.1 Introduction

Statistical process control (SPC) is the oldest and well tested method for controlling and improving the products' quality in an industrial or manufacturing process. By using statistical methods, the researcher can discover non-conforming standards of the product and contribute to the maintenance of the desirable quality.

In practice a products' quality is not related to one but more qualitative characteristics. In other words, it is necessary to monitor more than one characteristics simultaneously to ensure the total quality of the product. Jackson (1991) in his paper commented that a multivariate procedure should provide 4 information:

- an answer on whether or not the process is in-control,
- an overall probability for the event "procedure diagnoses an out-of-control state erroneously" must be specified,
- the relation between the variables/attributes should be taken into account".
- an answer to the question "If the process is out-of-control, what is the problem?"

Therefore Multivariate Statistical Process Control (MSPC) is a necessity and the fact that more and more scientists throughout the world contribute to the expansion of this specific scientific area, makes it even more important. Harold Hotteling in 1947 first applied the idea of Multivariate Statistical Process Control in collected data regarding bomb sights in World War II. After that a lot of studies followed Hotteling's idea including Hicks (1955), Jackson (1956, 1959, 1985), Montgomery-Wadsworth (1972), Alt (1985), Crosier (1988), Hawkins (1991, 1993), Pignatiello-Runger (1990), Tracy-Yang-Mason (1992), Lowry-Montgomery (1995), Maravelakis-Bersimis-Panaretos-Psarakis(2002), Koutras-Berssimis-Antzoulakos (2006) and Maravelakis-Bersimis(2009).

1.2 Control Charts

Among all techniques used in SPC and MSPC, the most common is the Control Chart (*CC*). The CC, can be displayed when the quality of a product is characterized by values of a variable and is the visualization of a measured quality characteristic versus time. A CC is equipped with border lines which help the researcher determine whether the process is in-control (operates with natural variation) or out-of-control (operates with special cause of variation). The border lines are the following:

- The Upper Control Limit (UCL),
- The Center Line (CL) and
- The Lower Control Limit (LCL)

It is noted that the UCL and the LCL represent the maximum and the minimum allowed values that indicate if the process is considered to operate with its natural variability. On the other hand, the CL represents the expected value of the measured statistic function arising from functioning an in-control process. Other optional features that may be included to a CC are the Upper and Lower Warning

Limits (UWL and LWL) which represent zones that warn the researcher of the process, in the case that the plotted statistic exceeds or falls short of them.

1.3 Phases of Statistical Process Control

The usage of CCs can be generally separated into two phases with a different objective.

In Phase I a set of preliminary data is collected and is analysed retrospectively for constructing control limits in order to establish reliable control limits for monitoring future production. So in Phase I, m subgroups are collected and a set of control limits is computed for these points. It is fair to assume that in Phase I the data collected are in-control so the control limits can be calculated by using these m subgroups. Points that are outside the control limits are usually investigated by technicians and it is determined if special cause of variation has occurred. If any special cause of variation is identified for the points that are initially outside the control limits, they are excluded so a new set of control limits is constructed by using the rest samples. The next step of the researcher is to collect a new set of data and plot them on the control limits that have already been obtained. If any sample is outside the control limits, is again investigated and new control limits are constructed. This analysis continues until the process is stable, so control limits and a set of in-control Phase I data are obtained.

Phase II begins after a clean set of data has been gathered. In Phase II, the control limits that have been constructed from Phase I are used for on-line monitoring the process so the purpose of this phase is to monitor the process and not try to bring it in-control. A sample statistic is calculated for every new sample drawn from the process and is compared to the control limits and if the statistic is plotted outside the control limits, the process is claimed to be out-of-control. Otherwise, the monitoring continuous. In this phase sensitizing rules can be applied for detecting small shifts or for reacting more quickly to prevent the process from being out-of-control.

1.4 Control Chart Types

From literature it is clear that different CCs can be constructed for different scenarios that can exist in every process.

First of all, there are CCs that can be used for monitoring the mean of the process and their purpose is to control the target of the specifications of a product. On the other hand, the researcher may be interested on keeping the dispersion of a product in a specific level. For this purpose, CCs for the variability can be applied. Finally, the most reasonable scenario is for the researcher to maintain control over both the mean and the dispersion of a product and thus to use CCs for both the characteristics.

Another useful and interesting set of options for the researcher is whether or not the sample size that can be collected from the process is equal to one (1). So CCs have been proposed for either individual observations or not.

Usually, for monitoring financial data, there have been proposed several CCs based on the time-series approach. The difference that time-series data have from regular data is that they are time dependent so every new sample is correlated with previous ones.

Another big difference that exists in the nature of the data is that not all processes assume normality of the data. It is common for the data to be distribution-free or non-normal and CCs based on this assumption have been proposed. In most cases though the researcher makes the normality assumption and uses a large variety of CCs.

Finally, all CCs can be classified into two big distinct categories depending on one simple property. If the points plotted are based only in the information given by the most recent sample taken, regardless any previous information, then the CC is characterized as a CC without memory. On the other hand, the last few years CCs have been developed in which a point plotted is based on information obtained not only from the most recent sample but from previous as well. These CCs are called CCs with memory.

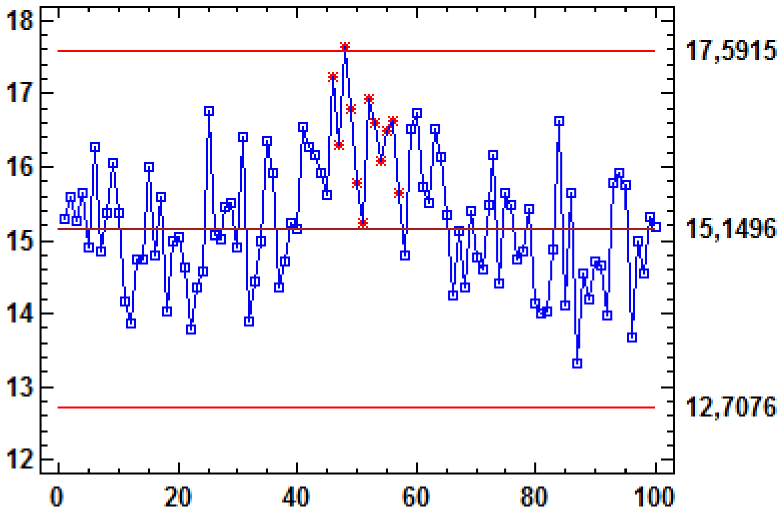
As mentioned before, CCs is the most common technique for maintaining the products quality. In practice three types of CCs are widely used:

- Shewhart Control Charts,
- Cumulative Sum Control Charts (CUSUM) and
- Exponentially Weighted Moving Average Control Charts (EWMA)

1.4.1 Control Charts Without Memory

In this category the most common type of CCs can be classified which are the Shewhart CCs [1.1] introduced by Walter A. Shewhart in 1920 while working for Bell Laboratories but the idea was published in 1931 in his book ”*Economic Control of Quality of Manufactured Product*”. Shewhart CCs can be used for monitoring either the mean or the dispersion of the process or both. These CCs can be used when the sample size is greater than two ($n > 2$) and the probability function of the plotted statistic is known or approximately known. A property in which the Shewhart CCs lack against the CCs with memory is that they cannot detect as easily small scale changes in the process as the latter. But in contrary they are usually preferred for detecting bigger scale shifts.

FIGURE 1.1: A Shewhart Control Chart.



1.4.2 Control Charts With Memory

From literature, it seems that two types of CCs can be classified into this category. These CCs are the CUSUM [1.2] and the EWMA [1.3] CCs introduced by Page (1954) and Roberts (1959) respectively. CUSUM and EWMA CCs are preferred when the shift that the researcher wants to detect is small. CUSUM CCs are mostly used when the probability function of the plotted statistic is known whereas EWMA CCs are more robust when the distribution is unknown.

The purpose of this thesis is to present multivariate CCs with the assumption of normality, with sample sizes bigger than one that are not time series. All these charts will also concern the dispersion of the process. For this reason, Normal distribution should also be presented.

FIGURE 1.2: A CUSUM Control Chart.

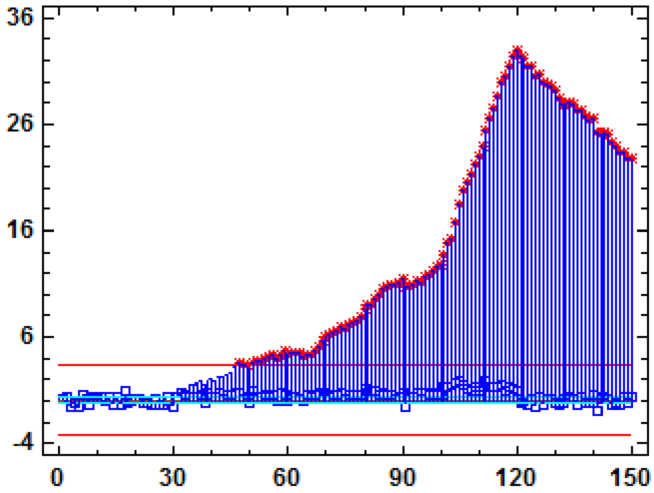
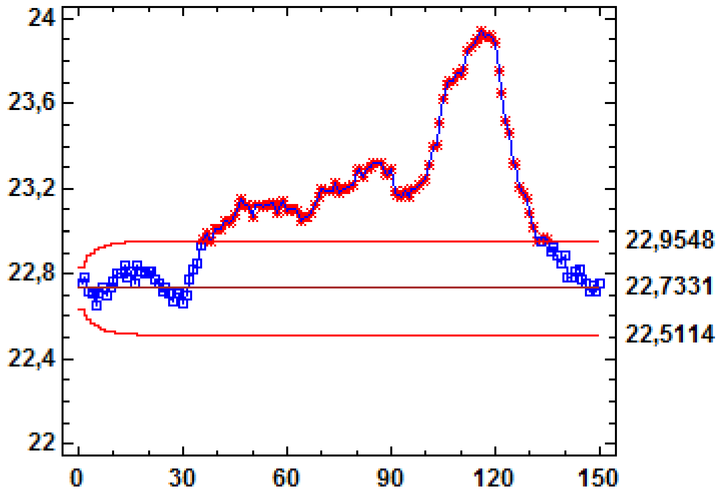


FIGURE 1.3: An EWMA Control Chart.



1.5 The Normal Distribution

Normal distribution is the most commonly used continuous distribution and the main reason that it is so popular is because it works or at least is good enough in many situations. The reason that the normal distribution works is because of the Central Limit Theorem which means that any variable that can be measured and is sufficiently large in terms of replicates will be approximately normal.

In the following chapters, the charts that have been recorded from the literature, have been proposed by assuming the normal distribution (multivariate or not) of the data. In order to develop the theory of the proposed methods, the reader must understand the assumed nature of the data.

Normal distribution, which is also known as Gaussian distribution was firstly proposed by Carl Friedrich Gauss in 1809 in a published monograph called " *Theoria motus corporum coelestium in sectionibus conicis solem ambientium*". In the same monograph it was also introduced the least squares method and the method of maximum likelihood.

1.5.1 The Univariate Normal Distribution

In Chapter 2, the CCs that will be presented assume that the data come from a univariate normal distribution so the simpler case must be presented first.

When a quality characteristic of interest must be monitored ($p = 1$), it is said that the vector $\boldsymbol{x} = (X)$ has a univariate normal distribution (symbolically $\boldsymbol{x} \sim N(\mu_0, \sigma_0^2)$) with in control mean μ_0 and in control variance σ_0^2. The density function has the following form:

$$f(\boldsymbol{x}) = f(X) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(X-\mu_0)^2}{2\sigma_0^2}\right]$$

1.5.2 The Bivariate Normal Distribution

All CCs in Chapter 3 have been proposed for monitoring two characteristics simultaneously and assume the multivariate normality or more specifically the bivariate normality.

In a two dimensional space ($p = 2$), a vector $\boldsymbol{x} = (X_1, X_2)$ has a bivariate normal distribution (symbolically $\boldsymbol{x} \sim N_2(\boldsymbol{\mu_0}, \boldsymbol{\Sigma}_0)$ where $\boldsymbol{\mu_0} = [\mu_1, \mu_2]$ is the in-control means and $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$ is the 2×2 variance covariance matrix with diagonal elements the in-control variances and off-diagonal elements the in-control covariance between the two characteristics). The probability density function has the following form:

$$f(\boldsymbol{x}) = f(X_1, X_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{z}{2(1-\rho^2)}\right]$$

where

$$z = \frac{(X_1 - \mu_1)^2}{\sigma_1^2} - \frac{2\rho(X_1 - \mu_1)(X_2 - \mu_2)}{\sigma_1\sigma_2} + \frac{(X_2 - \mu_2)^2}{\sigma_2^2}$$

and ρ is the correlation between the two variables.

1.5.3 The Multivariate Normal Distribution

Finally the multivariate normal distribution must be presented not only because it is the generalization of the univariate and bivariate normal distribution but also because the proposed CCs in Chapter 4 assume that the nature of the data are multivariate normal.

A multidimensional vector of variables

$$\boldsymbol{x}' = [X_1, X_2, \cdots, X_p]$$

with $p \geq 2$ has a multivariate normal distribution (symbolically $\boldsymbol{x} \sim N_p(\boldsymbol{\mu_0}, \boldsymbol{\Sigma}_0)$) if the probability density function has the following form:

$$f(\boldsymbol{x}) = f(X_1, X_2, \cdots, X_p) = (2\pi)^{-p/2}|\boldsymbol{\Sigma}_0|^{-1/2} \times \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_0)'\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_0)\right]$$

where

$$\boldsymbol{\mu}_0 = [\mu_1, \mu_2, \cdots, \mu_p]$$

is the in-control means for the p characteristics of interest and

$$\boldsymbol{\Sigma}_0 = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{bmatrix}$$

is the variance-covariance matrix with diagonal elements the in control variances for every characteristic and off-diagonal elements the in-control covariances between two characteristics.

A p-dimensional sample of size n can be illustrated as a data matrix (denoted as $\boldsymbol{X}_{p\times n}$) with the following form:

$$\boldsymbol{X}_{p\times n} = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ X_{p1} & X_{p2} & \cdots & X_{pn} \end{bmatrix}$$

1.6 The Case of the Dispersion

Controlling the dispersion of the process is as equally important as controlling the mean. By monitoring the mean in an industrial or manufacturing process, the researcher tries to achieve the specifications of the product. By assuming that a point in time is between the UCL and the LCL when monitoring the mean of the process, then the process is considered to be in-control on the condition that the dispersion of the quality characteristic has not changed through time. Also, it should be noted that when constructing a CC for the mean level of the process, the dispersion of the process is also taken into account indirectly through the control limits computed. In other words, the control limits of a process for the mean depend upon the dispersion. Therefore, the dispersion of the process should always be monitored. In the case that the dispersion is out-of-control, the mean level fluctuates more than it should leading not only the process out-of-control but also in some cases to wider control limits for the mean.

For the case of the dispersion there are several charts that have been proposed most of them based on different quantities.

1.6.1 Univariate Quantities for the Dispersion

There are several quantities in the univariate case that can be used for measuring the dispersion.

The first and most known quantity is the sample variance s^2. The sample variance is the second sample central moment and its mathematical expression for a sample of size n is the following:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2$$

Consecutively the sample standard deviation s can be derived as $s = \sqrt{s^2}$ and it measures how spread out the given data are. Of course, both quantities have positive values and small values indicate that the data tend to be really close, while high values indicate that the data are very spread out around the mean.

Also, if m subgroups of size n are available, then the pooled variance can be used as quantity for measuring the dispersion. Pooled variance has the following form:

$$s_p^2 = \frac{1}{m\left(n-1\right)}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij} - \overline{X}_i\right)^2$$

Another quantity that can easily be derived from the previous 2 is the $\overline{s}$ which is of the following form:

$$\overline{s} = \frac{\sum_{i=1}^{m} \sqrt{\frac{\sum_{j=1}^{n} \left(x_{ij} - \overline{\overline{x}}\right)^2}{n-1}}}{m}$$

where $\overline{\overline{x}} = \dfrac{\sum_{i=1}^{m} \overline{x}_i}{m}$.

The second univariate statistic that measures dispersion is the sample range (R) and it is defined as the maximum minus the minimum observed value of a sample. Its mathematical expression is:

$$R = X_{\max} - X_{\min}$$

Finally, another quantity for measuring the dispersion of the data is the sample Coefficient of Variation (CV). CV represents the ratio of the standard deviation to the mean and is useful for comparing the degree of variation from a data set to another even when the means are different. The mathematical expression is the following:

$$CV = \frac{s}{\overline{x}}$$

where $\overline{x}$ is the sample mean which is computed as $\overline{x} = \dfrac{1}{n} \sum_{i=1}^{n} X_i$ for a sample of size n.

1.6.2 Multivariate Quantities for the Dispersion

In this subsection, it is presented all the the multivariate analogues of the previous sample statistics.

For a variance covariance matrix $\mathbf{\Sigma}$ where:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{bmatrix}$$

we have the following.

The first multivariate quantity that can be used for measuring the variability is the generalized variance (GV) which is denoted as $|\mathbf{\Sigma}|$ and is the determinant of the variance covariance matrix. GV was introduced by Wilks (1932) and is a measure of the overall dispersion.

The GV can be computed as the generalized determinant of a $n \times n$ matrix. Let σ_{ij} be the entry on the i^{th} row and j^{th} column for $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, p$. Also let $\mathbf{\Sigma}_{ij}$ be the determinant of the square matrix of order $p-1$ obtained by $\mathbf{\Sigma}$ by removing the i^{th} row and the j^{th} column multiplied by $(-1)^{i+j}$. The GV is:

$$|\mathbf{\Sigma}| = \sum_{j=1}^{j=p} \sigma_{ij} \mathbf{\Sigma}_{ij}$$

for any given i. Although GV is widely used for measuring the multivariate variability, it seems to be a really simple approach of the multivariate structure. Lowery and Montgomery (1995) in a really famous example showed that three bivariate covariance matrices, have the same GV even though they have really different variances-covariances. These matrices are the following:

$$\mathbf{\Sigma}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

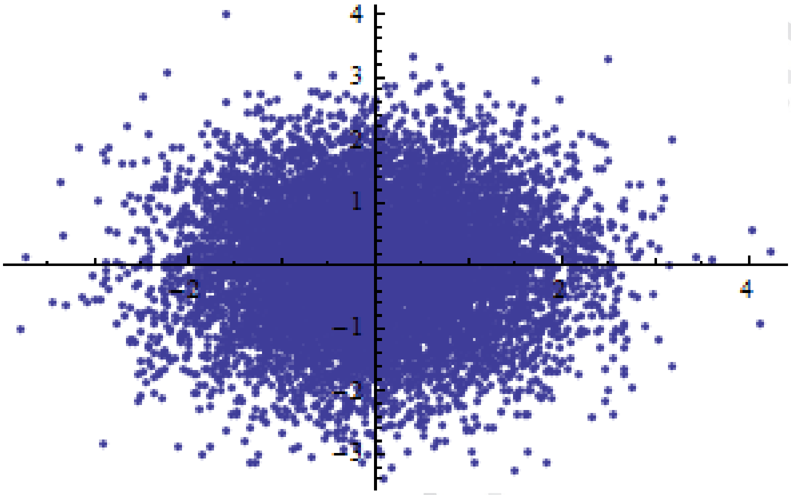
$$\mathbf{\Sigma}_2 = \begin{bmatrix} 2.32 & 0.40 \\ 0.40 & 0.50 \end{bmatrix}$$

$$\mathbf{\Sigma}_3 = \begin{bmatrix} 2.32 & -0.40 \\ -0.40 & 0.50 \end{bmatrix}$$

It can be easily seen that the GV in all three cases is equal to 1 but the correlation differ. For the matrices $\mathbf{\Sigma}_1$, $\mathbf{\Sigma}_2$ and $\mathbf{\Sigma}_3$ they are 0, 0.8 and -0.8 respectively.

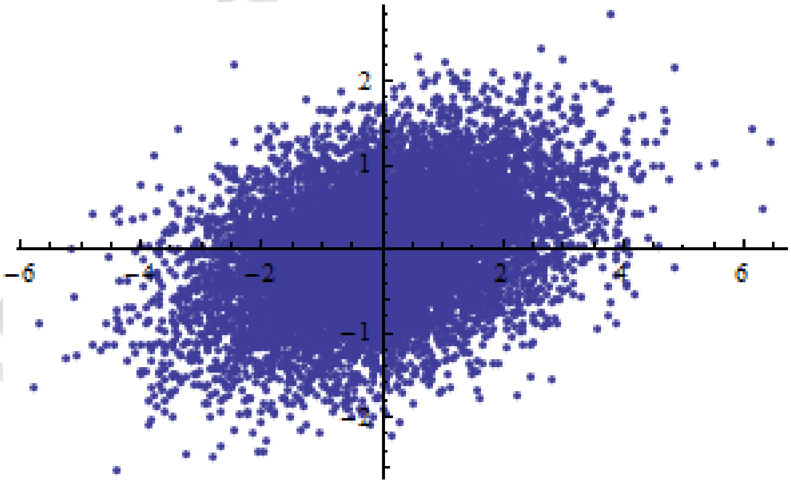
Following, a random sample of size $n = 10000$ in a process with mean vector $\boldsymbol{\mu} = [0, 0]$ and variance-covariance matrix $\mathbf{\Sigma}_1$ can be seen.

FIGURE 1.4: Correlation: 0.



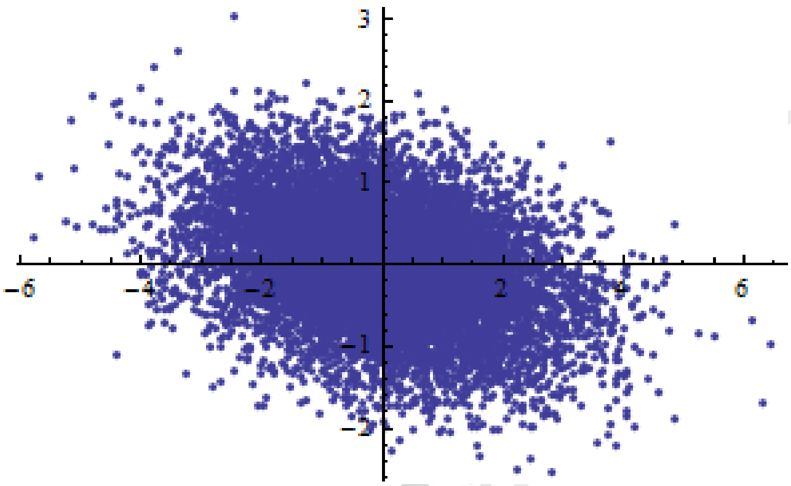
Furthermore, the process with mean vector $\boldsymbol{\mu} = [0, 0]$ and variance-covariance matrix $\mathbf{\Sigma}_2$ can be seen below.

FIGURE 1.5: Correlation: 0.8.



Finally, the process with mean vector $\boldsymbol{\mu} = [0, 0]$ and variance-covariance matrix $\boldsymbol{\Sigma}_3$ is the following:

FIGURE 1.6: Correlation: -0.8.



From the previous three graphs, it can be seen how different the nature of the data is. Even though the data are so different, the GV is the same so it can be justified GV's simplistic approach explained above.

Another quantity that can be used for measuring the variability is the total variance (TV) which is usually denoted as $tr\left(\boldsymbol{\Sigma}\right)$ and is the trace of the variance covariance matrix. The mathematical expression of the TV is the following:

$$tr\left(\boldsymbol{\Sigma}\right) = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_p^2 = \sum_{i=1}^{p} \sigma_i^2$$

Total Variance also has a defect. Although it is a good representation of the variance, it does not take into account the correlations between the variables.

A third quantity that measures variability in a multivariate space is the multivariate range. Gentle *et al.* (1975) in their paper present the bivariate range. For a sample $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ the bivariate range can be defined as:

$$R = \max_{i,j} \left\{ (X_i - X_j)^2 + (Y_i - Y_j)^2 \right\}^{1/2}$$

for $i, j = 1, 2, \cdots, n$.

A quantity which can measure the multivariate dispersion and yet has not been used for constructing CCs is the multivariate coefficient of variation (MCV).

Reyment (1960) was the first to propose a formal definition for the MCV. His proposal is denoted as CV_{RR} and is given as:

$$CV_{RR} = \left(\frac{|\boldsymbol{\Sigma}|^{1/p}}{\boldsymbol{\mu}'\boldsymbol{\mu}} \right)^{1/2}$$

Reyment suggested another MCV denoted as CV_{VV} with the following form:

$$CV_{VV} = \left(\frac{tr\left(\boldsymbol{\Sigma}\right)}{\boldsymbol{\mu}'\boldsymbol{\mu}} \right)^{1/2}$$

In another approach, Voinov and Nikulin (1996) proposed the following expression for the MCV denoted as CV_{VN}:

$$CV_{VN} \left(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)^{-1/2}$$

Finally, Albert and Zhang (2010) proposed the following expression:

$$CV_m = \left[\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}\boldsymbol{\mu}}{\left(\boldsymbol{\mu}'\boldsymbol{\mu}\right)^2} \right]^{1/2}$$

Concluding, another method used in multivariate data is the Principal Components Analysis (PCA). The purpose of the PCA is to replace a number of correlated variables $X_1, X_2, \cdots, X_p$ with fewer variables $C_1, C_2, \cdots$ which are a linear transformation of the initial variables and retain a significant amount of information.

For a p-variate vector of variables $\boldsymbol{X} = (X_1, X_2, \cdots, X_p)'$ with variance-covariance matrix $\boldsymbol{\Sigma}$, the first principal component can be specified as:

$$C_1 = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_p X_p = \boldsymbol{\alpha}'\boldsymbol{X}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_p)'$ is real vector with length equal to 1 meaning $||\boldsymbol{\alpha}||= \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_p^2} = 1$. Since $Var\left(C_1\right) = \boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}$, finding $\boldsymbol{\alpha}$ for the first component is the same as maximizing the quantity $\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}$. Therefore $\boldsymbol{\alpha}$ should be equal to the unit length eigenvector $\boldsymbol{u_1}$ which corresponds to the biggest eigenvalue (λ_1)

of the variance covariance matrix.

The second component can be defined as:

$$C_2 = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p = \boldsymbol{\alpha}' \boldsymbol{X}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_p)'$, $||\boldsymbol{\beta}|| = 1$, has the second largest possible variance and is uncorrelated to C_1. But

$$Cov\,(C_2, C_1) = Cov\,(\boldsymbol{\beta}'\boldsymbol{X}, \boldsymbol{u}_1\boldsymbol{X}) = \boldsymbol{\beta}'\boldsymbol{\Sigma}\boldsymbol{u}_1$$

and since $\boldsymbol{\Sigma}\boldsymbol{u}_1 = \lambda_1 \boldsymbol{u}_1$ it will be $Cov\,(C_2, C_1) = \lambda_1 \boldsymbol{\beta}'\boldsymbol{u}_1$. For having $Cov\,(C_2, C_1) = 0$ it must be $\boldsymbol{\beta}'\boldsymbol{u}_1 = 0$. Finally for finding $\boldsymbol{\beta}$ the following quantity should be maximized:

$$Var\,(C_2) = \boldsymbol{\beta}'\boldsymbol{\Sigma}\boldsymbol{\beta}$$

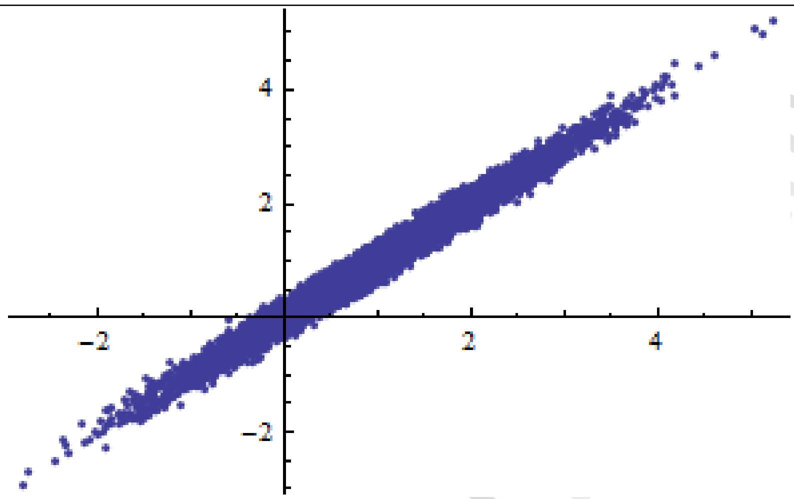
under the restriction that $\boldsymbol{\beta}$ is a unit vector and normal to $\boldsymbol{u}_1$.

1.6.3 An Example on the Presented Quantities

In this subsection an example will be presented for comparing the various quantities presented that measure dispersion on a multivariate level. The scenarios are the following: On a bivariate space, 2 variables are considered with mean vector $\boldsymbol{\mu} = (1, 1)$ and variance covariance matrix with unit variances and covariances: 0.99, 0.50, 0, -0.50, -0.99. So all in all there are a total of 5 different scenarios. Each simulation was produced with 10000 repetitions.

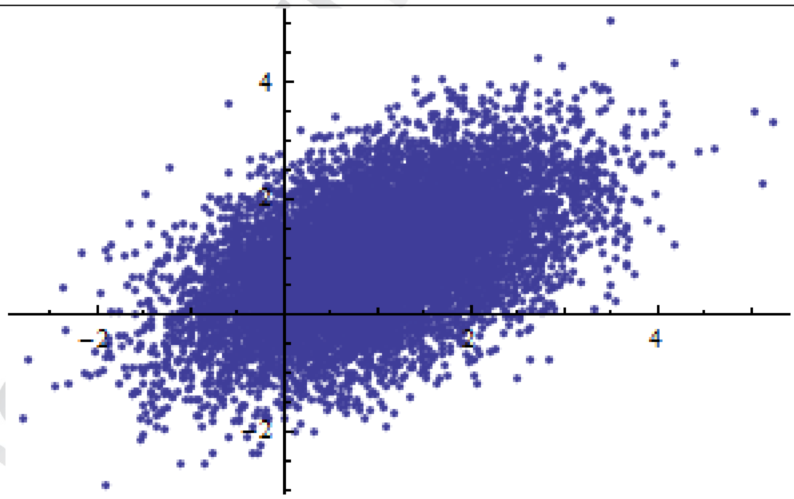
For $\boldsymbol{\mu} = (1, 1)$ and $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.99 \\ 0.99 & 1 \end{bmatrix}$ the produced points can be plotted as follows:

FIGURE 1.7: Correlation: 0.99.



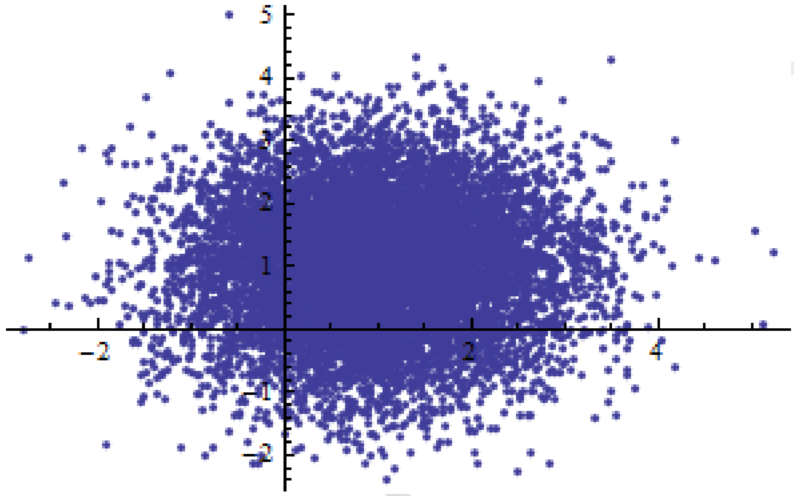
For $\boldsymbol{\mu} = (1, 1)$ and $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ the produced points can be plotted as follows:

FIGURE 1.8: Correlation: 0.50.



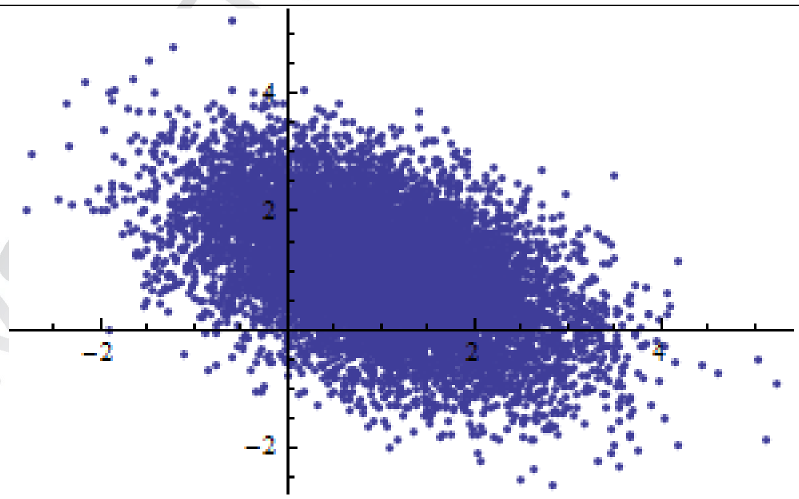
For $\boldsymbol{\mu} = (1,1)$ and $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ the produced points can be plotted as follows:

FIGURE 1.9: Correlation: 0.



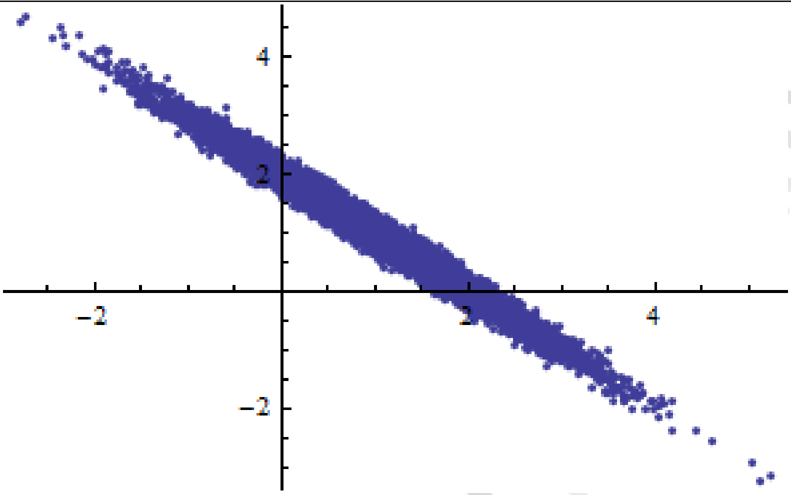
For $\boldsymbol{\mu} = (1,1)$ and $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$ the produced points can be plotted as follows:

FIGURE 1.10: Correlation: -0.50.



For $\boldsymbol{\mu} = (1,1)$ and $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & -0.99 \\ -0.99 & 1 \end{bmatrix}$ the produced points can be plotted as follows:

FIGURE 1.11: Correlation: -0.99.



In the following table, the summary of the quantities produced can be seen:

Correlation	*GV*	*TV*	*Range*	CV_m	*Var of* 1^{st} *PC*
0.99	0.019	1.966	11.447	0.997	1.956
0.50	0.717	1.963	9.633	0.866	1.479
0	0.956	1.955	8.149	0.707	0.986
-0.50	0.717	1.951	9.482	0.5	1.461
-0.99	0.019	1.962	11.212	0.071	1.952

It can be easily seen that the problem which was previously mentioned is now confirmed. GV has the same value for different scenarios. Also the problem that was mentioned about TV has been also confirmed.

1.7 Summary

In this section it was initially presented what is SPC and why is it important to generalize the idea to MSPC. Afterwards, the CCs were presented with their key features and the different types that have been proposed for every different scenario that can occur in a real process. Moreover, the two phases of a process were defined and after determining the purpose of the thesis, the normal distribution was defined. Finally, all quantities that measure dispersion were presented in both the univariate and multivariate case and some comparisons of these measures were made.

The outline of this thesis will be as follows. In Chapter 2, some univariate CCs for the dispersion will be presented. In Chapter 3, there are the bivariate CCs for the dispersion as for Chapter 4, the general case of the multivariate CCs will be presented. Chapter 5, compares both bivariate and multivariate charts in a two dimensional example. Finally, in Chapter 6 there is an overall summary of this thesis and proposal for further research is discussed.

Chapter 2

Univariate Control Charts for the Dispersion

2.1 Introduction

For achieving the predetermined target of a product the researcher should be able to monitor and control the process. As mentioned before, there are different types of CCs that can be used in practice for monitoring the process and most of them revolve around the mean of the process but are usually used in addition with the ones for the dispersion. In the univariate case, the most common and well-known univariate CCs for monitoring the process is by using the $\overline{X}$ and R charts. Phase II $\overline{X}$ CC has as plotted quantity the sample mean

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the following α probability control limits:

$$UCL = \mu + z_{\alpha/2}\sigma/\sqrt{n}$$

$$LCL = \mu - z_{\alpha/2}\sigma/\sqrt{n}$$

and with $CL = \mu$.

Phase II R CC has as plotted quantity the sample range:

$$X_{max} - X_{min}$$

and the following α probability control limits:

$$UCL = D_2\sigma$$

$$LCL = D_1\sigma$$

and with $CL = d_2\sigma$.

In this chapter, some univariate CCs for the dispersion will be presented. Section 2.2 deals with the Phase I CCs. In 2.2.1, the Shewhart CCs have been recorded while in subsection 2.2.2 and 2.2.3 are the Phase I CUSUM and EWMA CCs respectively. In section 2.3 the Phase II CCs for the dispersion can be found with 2.3.1 dealing with the Shewhart CCs, 2.3.2 discusses the CUSUM CCs and finally 2.3.3 is about the EWMA CCs.

2.2 Phase I Charts

2.2.1 Shewhart Charts

2.2.1.1 The s-Chart Control Limits

If the process variability σ is considered unknown, then by selecting m random subgroups of size n, an s-Chart can be used if as estimation of σ the quantity $\hat{\sigma} = \overline{s}/c_4$ is selected. The plotted statistic in this case is the sample standard deviation (s) and the 3σ control limits of the S-Chart are:

$$UCL = B_4 \times \overline{s} = \left[1 + 3\sqrt{1 - c_4^2}/c_4]\right] \times \overline{s}$$

$$LCL = B_3 \times \overline{s} = \left[1 - 3\sqrt{1 - c_4^2}/c_4]\right] \times \overline{s}$$

with Center Line $C.L. = \overline{s}$.

2.2.1.2 The s^2-Chart Control Limits

The unknown process variability σ^2 can be estimated by selecting as estimation of σ^2 the quantity $\overline{s^2}$. The s^2-Chart with probability α control limits has as plotted statistic the sample variance and the control limits are:

$$UCL = \frac{\overline{s^2}}{n-1} \times X^2_{n-1;\alpha/2}$$

$$LCL = \frac{\overline{s^2}}{n-1} \times X^2_{n-1;1-\alpha/2}$$

with Center Line $C.L. = \overline{s^2}$.

2.2.2 CUSUM Charts

2.2.2.1 $\log(s^2)$ CUSUM Chart

Chang and Gan (1995) proposed the use of the logarithmic transformation of the sample variance ($\log(s^2)$) in a CUSUM chart for monitoring the process dispersion. As an unbiased estimator of σ_0^2 from the m samples of size n the pooled variance can be used.

Chang and Gan (1995) introduced an upward CUSUM CC for detecting increases in the process variance using the scheme:

$$U_0 = u_0,$$

$$U_t = \max\left(U_{t-1} + \ln\left(s_t^2\right) - k_u, 0\right), t = 1, 2, \ldots,$$

where $0 \leq u_0 < UCL$ and k_u is a constant. This chart signals when $U_t \geq UCL$.

The corresponding downward CUSUM CC for detecting decreases in the process variance is given by the scheme:

$$D_0 = d_0,$$

$$D_t = min\left(D_{t-1} + \ln\left(s_t^2\right) + k_d, 0\right), t = 1, 2, \ldots,$$

where $LCL < d_0 \leq 0$ and k_d is constant. The downward CUSUM chart signals when $D_t \leq LCL$. The chart defined above cannot be used for individual observations. It should be noted that UCL and LCL can be chosen for a predetermined ARL.

2.3 Phase II Charts

2.3.1 Shewhart Control Charts

2.3.1.1 The S-Chart Control Limits

If a process with a Normally distributed quality characteristic X is considered, then the dispersion of the process can be monitored by using the sample standard deviation.

The CC for the dispersion of the quality characteristic X has the following 3σ control limits:

$$UCL = \mu_{s_t} + 3\sigma_{s_t} = \left(c_4 + 3\sqrt{1 - c_4^2}\right)\sigma$$

$$LCL = \mu_{s_t} - 3\sigma_{s_t} = \left(c_4 - 3\sqrt{1 - c_4^2}\right)\sigma$$

with $C.L. = \mu_{S_t} = c_4\sigma$.

and

$$\mu_{s_t} = c_4\sigma \ , \ \sigma_{s_t} = \sigma\sqrt{1 - c_4^2}$$

Moreover a CC with probability limits α can be easily developed by using the relation:

$$P\left(\chi_{n-1;1-\alpha/2} \leq \frac{s_t\sqrt{n-1}}{\sigma} \leq \chi_{n-1;\alpha/2}\right)$$

or equivalently:

$$P\left(\sigma\sqrt{\frac{\chi^2_{n-1;1-\alpha/2}}{n-1}} \leq s_t \leq \sigma\sqrt{\frac{\chi^2_{n-1;\alpha/2}}{n-1}}\right)$$

where χ_n is the distribution of the random variable $Y = \sqrt{X}$ (when $X \sim \chi^2_n$).

2.3.1.2 The s^2-Chart Control Limits

For monitoring the dispersion of the quality characteristic X the sample variance s_t^2 as plotted statistic can be used where

$$E\left(s_t^2\right) = \sigma^2, \frac{(n-1)\, s_t^2}{\sigma^2} \sim \chi^2_{n-1}$$

Since

$$P\left(\frac{\sigma^2}{n-1}\chi^2_{n-1;1-\alpha/2} \leq s_t^2 \leq \frac{\sigma^2}{n-1}\chi^2_{n-1;\alpha/2}\right)$$

the α probability control limits for the dispersion of the quality characteristic X where the sample variance s_t^2 is used as a dispersion measurement are:

$$UCL = \frac{\sigma^2}{n-1}\chi^2_{n-1;\alpha/2}$$

$$LCL = \frac{\sigma^2}{n-1}\chi^2_{n-1;1-\alpha/2}$$

with $C.L. = \sigma^2$.

2.3.2 CUSUM Control Charts

2.3.2.1 Scale CUSUM Chart

Hawkins (1981) proposed the following quantities to be used for the Scale CUSUM chart for the dispersion:

$$S_t^+ = \max\left[0, W_t - k + S_{t-1}^+\right], \text{where } S_0^+ = 0$$

$$S_t^- = \max\left[0, W_t - k + S_{t-1}^-\right], \text{where } S_0^- = 0$$

If either S_t^+ or S_t^- exceeds the quantity $= h\sigma_0$ (where h is usually equal to 5) then, the procedure is considered to be out of control. W_t derives from the following:

By considering individual observations X_t from $N\left(\mu_0, \sigma^2\right)$ and by putting:

$$Y_t = \frac{X_t - \mu_0}{\sigma}$$

the following statistic can be constructed:

$$W_t = \frac{\sqrt{|Y_t|} - E\left(|Y_t|\right)}{\sqrt{V\left(\sqrt{|Y_t|}\right)}}$$

where $E\left(|Y_t|\right) = 0.822$ and $\sqrt{V\left(\sqrt{|Y_t|}\right)} = 0.349$.

Hawkins observed that the distribution of the quantity W_t is **approximately** $N(0,1)$ and that W_t is sensitive in dispersion (σ^2) shifts.

$E\left(|Y_t|\right)$ and $V\left(\sqrt{|Y_t|}\right)$ can be computed as follows:

If $|Y_t| \sim \chi_1$ then,

$$E\left(\sqrt{|Y_t|}\right) = \frac{2^{1/4}\Gamma\left(3/4\right)}{\Gamma\left(1/2\right)}$$

$$E\left(|Y_t|\right) = \frac{2^{1/2}\Gamma\left(1\right)}{\Gamma\left(1/2\right)} = 0.79885$$

$$V\left(\sqrt{|Y_t|}\right) = E\left(|Y_t|\right) - \left[E\left(\sqrt{|Y_t|}\right)\right]^2 = 0.121906$$

2.3.2.2 P_σ and χ CUSUM Charts

Acosta-Mejia *et al.* (1998) proposed two new CCs for monitoring the dispersion of a normal process. These two new charts are two-sided and thus are able to detect both increases and decreases in process dispersion. P_σ, χ CUSUM charts are based on two different normalizing transformations of s^2. After transforming s^2 to an approximately normal variate Z, the standard two-sides CUSUM procedure can be applied which is the following:

$$S_t^+ = \max\left(0, Z_t - k_u + S_{t-1}^+\right),$$

$$S_t^- = \max\left(0, -Z_t - k_l + S_{t-1}^-\right),$$

where $S_0^+, S_0^- \geq 0$. The value S_t^+ is used to detect positive shifts while S_t^- is used to detect negative shifts. The constants k_u and k_t are called reference values and in most applications $k_l = k_u = k$. The statistics S_t^+ and S_t^- are compared to the decision limits h_u and h_l respectively where as stated by the authors, the decision limits are chosen by the user for achieving a desirable ARL. If either statistic exceeds its respective decision limit, the standard two-sided CUSUM procedure signals.

The statistics Z_t can be chosen to be one of the following quantities.

The P$_\sigma$ CUSUM CC for process dispersion

The first control chart for monitoring the process dispersion is based on the inverse normal transformation:

$$P_{\sigma_t} = \Phi^{-1}\left\{F_{\chi^2_{n-1}}\left(\frac{(n-1)\, s_t^2}{\sigma_0^2}\right)\right\}$$

P_{σ_t} has a standard normal distribution, where $F_{\chi^2_{n-1}}(y)$ is the cumulative distribution function for the χ^2 distribution with $n-1$ degrees of freedom and $\Phi(z)$ is the cumulative distribution function for the standard normal distribution. An increase (decrease) in σ will result in an increase (decrease) in the *mean* of P_{σ_t}. Thus the standard two-sided CUSUM scheme using P_{σ_t} as Z_t can be used to monitor the process variance. This procedure is called P_σ-CUSUM chart. For the P_σ-CUSUM chart the reference values k_u and k_l and the control limits h_u and h_l that give a desired ARL performance can be obtained through simulation.

The χ-CUSUM chart for process dispersion

The second approach for the CUSUM chart for the dispersion is based on a transformation given by Wilson and Hilferty (1931). Wilson and Hilferty showed that $\sqrt[3]{\chi_n^2/n}$ is approximately normally distributed with a mean $1 - 2/(9n)$ and variance $2/(9n)$. If the observations are *iid* $N(\mu, \sigma)$ then:

$$\chi_t = \left[\left(\frac{s_t^2}{\sigma_0^2}\right)^{1/3} - \left(1 - \frac{2}{9(n-1)}\right)\right] \Big/ \sqrt{\frac{2}{9(n-1)}}$$

will have an approximate standard normal distribution. This procedure is called χ-CUSUM chart and in this case, χ_t can be used as Z_t in the standard two-sided CUSUM scheme. For the χ-CUSUM the reference values k_u and k_l can be obtained as follows:

$$k_u = \frac{1}{2}\left[\left[\left(\sigma_{1^+}^2/\sigma_0^2\right)^{1/3} - 1\right]\left[1 - \frac{2}{(n-1)}\right] \Big/ \sqrt{\frac{2}{9(n-1)}}\right]$$

and

$$k_l = \frac{1}{2}\left[\left[1 - \left(\sigma_{1^-}^2/\sigma_0^2\right)^{1/3}\right]\left[1 - \frac{2}{(n-1)}\right] \Big/ \sqrt{\frac{2}{9(n-1)}}\right]$$

where $\sigma_1^{\pm} \neq \sigma_0$ is the process standard deviation that needs to be detected.

2.3.3 EWMA Control Charts

2.3.3.1 $\ln(\sigma^2)$ EWMA Control Chart

For monitoring the process dispersion the unknown in-control process variance σ_0^2 can be estimated as the pooled sample variance s_p^2 from a Phase I data set.

Crowder and Hamilton (1992) proposed an EWMA CC for monitoring the process' standard deviation using the scheme:

$$T_0 = \ln\left(\sigma_0^2\right),$$

$$T_t = \max\left(\lambda \ln\left(s_t^2\right) + (1-\lambda)\, T_{t-1}, \ln\left(s_p^2\right)\right), t = 1, 2, \cdots$$

where $0 < \lambda \leq 1$ is a smoother parameter. The UCL of this chart (for the T_t) in case of independent observations is given by:

$$UCL = K\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(\frac{2}{n-1} + \frac{2}{(n-1)^2} + \frac{4}{3\,(n-1)^3} - \frac{16}{15\,(n-1)^5}\right)}$$

where K is chosen together with λ for achieving a desired performance for the chart. The chart can be used only with subgroup data ($n > 1$) and the chart can be used to identify only upward shifts in the variability.

2.3.3.2 The CH EWMA Control Chart

Crowder and Hamilton (1992) EWMA CH chart is based on the following quantity:

$$Q_t = \max\left[(1-\lambda)\, Q_{t-1} + \lambda X_t, 0\right],$$

where $Q_0 = 0$. The chart detects an increase in the process variance if Q_t is greater than:

$$h = L\sqrt{\frac{\lambda}{2-\lambda}}\sigma_0$$

where L can be chosen to achieve the desired ARL.

Similarly, if one is interested in detecting a decrease in the process variance, the EWMA chart based on:

$$Q'_t = \min\left[(1-\lambda)\, Q'_{t-1} + \lambda X_t, 0\right],$$

can be used where $Q'_0 = 0$. The chart detects an decrease in the process variance if Q'_t is less than:

$$h' = -L'\sqrt{\frac{\lambda}{2-\lambda}}\sigma_0$$

where L' can be chosen to achieve the desired ARL.

2.3.3.3 The EWMA s^2 Control Chart

Castagliola (2005) following Crowder and Hamilton (1992) EWMA CC for monitoring the process standard deviation proposed the following three-parameter transformation to s^2:

$$T_t = \alpha + b\ln\left(s_t^2 + c\right)$$

with $c > 0$. The following EWMA can be derived:

$$Z_t = (1-\lambda)\, Z_{t-1} + \lambda T_t$$

If the value of $E\left(T_t\right)$ and $\sigma\left(T_t\right)$ of T_t which correspond to the parameters α, b and c are known, then the EWMA control limits for the transformed sample variance will be set at:

$$UCL = E\left(T_t\right) + K\left(\frac{\lambda}{2-\lambda}\right)^{1/2}\sigma\left(T_t\right)$$

$$LCL = E\left(T_t\right) - K\left(\frac{\lambda}{2-\lambda}\right)^{1/2}\sigma\left(T_t\right)$$

where K is a positive constant which is set for achieving a predetermined ARL_0.

By using the exact value for the standard deviation of the EWMA statistic (Montgomery (2001), Ryan (2000)) the control limits become:

$$UCL = E\left(T_t\right) + K\left(\frac{\lambda\left\{1-(1-\lambda)^{2t}\right\}}{2-\lambda}\right)^{1/2}\sigma\left(T_t\right)$$

$$LCL = E\left(T_t\right) - K\left(\frac{\lambda\left\{1-(1-\lambda)^{2t}\right\}}{2-\lambda}\right)^{1/2}\sigma\left(T_t\right)$$

In practice, the following simplified control limits for the s^2-EWMA CC can be used:

$$UCL = K\left(\frac{\lambda}{2-\lambda}\right)^{1/2}\sigma\left(T_t\right)$$

$$LCL = -K\left(\frac{\lambda}{2-\lambda}\right)^{1/2}\sigma\left(T_t\right)$$

The above control limits correspond to a two-sided EWMA CC but also the one-sided EWMA control limits can be considered.

The transformation proposed, belongs to a class of transformations originally proposed by Johnson (1949). This approach was chosen because α, b and c may result in approximate normality better than the approach of Crowder and Hamilton (1992).

Castagliola (2005) proves that α, b and c can be defined as:

$$b = B\left(n\right)$$

$$c = C\left(n\right)\sigma_0^2$$

$$\alpha = A\left(n\right) - 2B\left(n\right)\ln\left(\sigma_0\right)$$

where:

$$B\left(n\right) = \frac{1}{\sqrt{\ln\left(w^2+1\right)}}$$

$$A\left(n\right) = \frac{B\left(n\right)}{2}\ln\left(\frac{w^2\left(w^2+1\right)}{\mu_2\left(s_t^2\right)}\right)$$

$$C\left(n\right)\frac{\sqrt{\mu_2\left(s_t^2\right)}}{w} - E\left(s_t^2\right)$$

and

$$w = \left[\sqrt{\left(\gamma_1\left(S_t^2\right)/2\right)^2+1} + \left(\gamma_1\left(s_t^2\right)/2\right)\right]^{1/3} - \left[\sqrt{\left(\gamma_1\left(s_t^2\right)/2\right)^2+1} - \left(\gamma_1\left(s_t^2\right)/2\right)\right]^{1/3}$$

in which, $E\left(S_t^2\right)$, $\mu_2\left(s_t^2\right)$ and $\gamma_1\left(s_t^2\right)$ are the first three moments of s_t^2.

2.3.3.4 The SJ EWMA Control Chart

Shu and Jiang (2008) proposed an EWMA chart (*SJ* chart) based on:

$$W_t = \lambda\left(Z_t^+ - \frac{1}{\sqrt{2\pi}}\right) + (1-\lambda)\, W_{t-1},$$

where $W_0 = 0$. The chart declared to be out-of-control when W_t exceeds the UCL:

$$h = L\sqrt{\frac{\lambda}{2-\lambda}}\sigma^+_{Z_t^+}$$

where L_N can be chosen to achieve the desired ARL.

The following standardized quantity has been defined:

$$Z_t = \frac{X_t - \mu_{X|\sigma_t=\sigma_0}}{\sigma_X},$$

where $\mu_{X|\sigma_t=\sigma_0}$ is the approximate in-control mean of X_t. Also $Z_t^+ = \max(Z_t, 0)$. If Z_t has an exact standard normal distribution, Barr and Sherrill (1999) showed that $E\left(Z_t^+\right) = 1/\sqrt{2\pi}$ and $\sigma_{Z_t^2} = 1/2 - 1/(2\pi)$.

Similarly, if one is interested in detecting a decrease in the process variance, the following EWMA chart can be used:

$$W_t' = \lambda\left(Z_t^- - \frac{1}{\sqrt{2\pi}}\right) + (1-\lambda)\, W_{t-1}',$$

where $Z_t^- = \min(0, Z_t)$ and $W_0' = 0$. The LCL is given by:

$$h' = -L'\sqrt{\frac{\lambda}{2-\lambda}}\sigma^+_{Z_t^-},$$

where L' can be determined to achieve desired ARL. The chart is declared to be out-of-control when W_t' deceeds the LCL.

2.3.3.5 HHW1, HHW2 and HHW-C charts

Huwang *et al.* (2010) following Crowder and Hamilton (1992) and Shu and Jiang (2008) EWMA chart, proposed the following similar charts for detecting a decrease in the process variance.

The HHW1 EWMA chart

Huwang *et al.* (2010) define the following standardized statistic for monitoring either an increase or a decrease in the variance of the process in time t:

$$U_t = \frac{\ln\left[V_t - (1-\lambda)^t V_0\right] - \mu}{\sigma^2}$$

The one-sided UCL or LCL can be chosen to achieve a desired ARL_0.
First they obtain the EWMA statistic:

$$V_t = \sum_{i=1}^{t} \lambda (1-\lambda)^{t-i} \frac{s_i^2}{\sigma_0^2} + (1-\lambda)^t V_0$$

where $V_0 = 1$.

In their paper they use the fact that s_i^2/σ_0^2 follows a Gamma distribution. Due to independence, they use the following approximation by Box (1954)

$$V_t - (1-\lambda)^t V_0 = \sum_{i=1}^{t} \lambda (1-\lambda)^{t-i} \frac{S_i^2}{\sigma_0^2} \approx Gamma(\beta_1, \beta_2)$$

where:

$$\beta_1 = \frac{(n-1)(2-\lambda)\left[1-(1-\lambda)^t\right]^2}{2\lambda\left[1-(1-\lambda)^{2t}\right]}$$

and

$$\beta_2 = \frac{2\lambda\left[1-(1-\lambda)^{2t}\right]}{(n-1)(2-\lambda)\left[1-(1-\lambda)^t\right]}$$

The logarithm of $V_t - (1-\lambda)^t V_0$ follows approximately a Normal distribution with:

$$\mu = \ln(\beta_1\beta_2) - \frac{1}{2\beta_1} - \frac{1}{12\beta_t^2} + \frac{1}{120\beta_1^4}$$

and

$$\sigma^2 = \frac{1}{\beta_1} + \frac{1}{2\beta_1^2} + \frac{1}{6\beta_1^3} - \frac{1}{30\beta_1^5}$$

The HHW2 EWMA chart

Huwang *et al.* (2010) have proposed standardized statistic for monitoring the dispersion of a univariate process which is defined as follows:

$$D_t = \frac{H_t}{\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}}$$

The one-sided UCL and LCL can be chosen to achieve a desired ARL_0.
First they obtain the EWMA statistic:

$$H_t = \lambda M_t + (1-\lambda)\, H_{t-1}$$

where $H_0 = 0$. When the process is in control, H_t has a normal distribution with mean equal to 0 and variance $\lambda\left[1-(1-\lambda)^{2t}\right]/(2-\lambda)$.

The second proposition from Huwang *et al.* (2010) has come from the exact normal transformation of s_t^2/σ_0^2. It is known that when the process is in control, then the statistic $M_t = \Phi^{-1}\left\{F\left[(n-1)\, s_t^2/\sigma_0^2\right]\right\}$ follows a standard normal distribution, where $F(\cdot)$ is the distribution of a chi-squared random variable with *n-1* degrees of freedom.

In their paper, is used the standardized statistic because a possible change in the variance can result in both changes in the mean and variance of M_t.

The HHW – C EWMA chart

Finally, Huwang *et al.* (2010) conclude in their paper that the *HHW2* chart gives the best results for detecting an increase in the process variance. On the other hand, *HHW1* chart gives the best results for detecting a decrease in the process variance. By combining the lower-sided HHW1 CC with the upper-sided HHW2 chart, it results in a better performance for monitoring the process variance. The mixed CC is denoted as *HHW-C* chart.

Chapter 3

Bivariate Control Charts for the Dispersion

3.1 Introduction

In this section some bivariate control charts for the dispersion are presented. These control charts are specifically constructed for monitoring the dispersion of two characteristics of interest simultaneously. Section 3.2 discusses bivariate Phase *II* control charts for the dispersion since no charts that can be used only in Phase *I* could be found in literature. Subsection 3.2.1 presents Shewhart control charts while subsection 3.2.2 discusses two EWMA control charts.

3.2 Phase II Bivariate Control Charts

3.2.1 Shewhart Control Charts for the Dispersion

3.2.1.1 The CC2 Control Chart

By having **two quality characteristics** according to Alt (1985), the process variability can be monitored using as plotted statistic the GV of the variance-covariance matrix ($|\mathbf{S}|$) and the following chart can be constructed.

The (α probability) control limits are the following:

$$UCL = \frac{|\mathbf{\Sigma}_0|\left(\chi^2_{2n-4;1-\alpha/2}\right)^2}{\left(2\left(n-1\right)\right)^2}$$

$$LCL = \frac{|\mathbf{\Sigma}_0|\left(\chi^2_{2n-4;\alpha/2}\right)^2}{\left(2\left(n-1\right)\right)^2}$$

and center line $C.L. = |\mathbf{\Sigma}_0|$. Where

$$\frac{2(n-1)|\mathbf{S}|^{1/2}}{|\mathbf{\Sigma}_0|^{1/2}} \sim \chi^2_{2n-4}$$

3.2.1.2 The CC3 Control Chart

Alt and Smith (1988) proposed that by monitoring **two quality characteristics**, a CC with $|\mathbf{S}|^{1/2}$ as plotted statistic can be constructed with the following control limits:

$$UCL = \frac{|\mathbf{\Sigma}_0|^{1/2}\chi^2_{2n-4,1-\alpha/2}}{2\left(n-1\right)}$$

$$LCL = \frac{|\mathbf{\Sigma}_0|^{1/2}\chi^2_{2n-4,\alpha/2}}{2\left(n-1\right)}$$

and center line $C.L. = |\mathbf{\Sigma}_0|^{1/2}$.

3.2.1.3 The $|\boldsymbol{G}|$-Control Chart Based on the Gini Matrix

Riaz and Does (2008) proposed a bivariate CC for the process dispersion based on the sample Gini mean differences matrix. In their paper Riaz and Does use the quantity $|\boldsymbol{G}|^{1/2}= \left(G_y^2 G_x^2 - G_{yx} G_{xy}\right)^{1/2}$ for monitoring the quantity $|\boldsymbol{\Sigma}_0|^{1/2}$. The α probability limits for the proposed chart are:

$$UCL = |\boldsymbol{G}|_u^{1/2} \text{ with } F_n\left(|\boldsymbol{G}|^{1/2}= |\boldsymbol{G}|_u^{1/2}\right) \geq 1 - \alpha_u$$

$$LCL = |\boldsymbol{G}|_l^{1/2} \text{ with } F_n\left(|\boldsymbol{G}|^{1/2}= |\boldsymbol{G}|_l^{1/2}\right) \leq \alpha_l$$

After some simplifications, the previous limits can be modified to:

$$UCL = |\boldsymbol{G}|_u^{1/2}= B_u|\overline{\boldsymbol{G}}|^{1/2}/b_0 \text{ with } F_n\left(B = B_u\right) \geq 1 - \alpha_u$$

$$LCL = |\boldsymbol{G}|_l^{1/2}= B_l|\overline{\boldsymbol{G}}|^{1/2}/b_0 \text{ with } F_n\left(B = B_l\right) \leq \alpha_l$$

It is noted that the CL is $|\overline{\boldsymbol{G}}|^{1/2}$. For constructing the CC, the following matrix must be defined:

$$\boldsymbol{G} = \begin{bmatrix} G_y^2 & G_{yx} \\ G_{xy} & G_x^2 \end{bmatrix}$$

where

$$G_y = \left(\sqrt{\pi}/2\right) 4Cov\left(Y, F\left(Y\right)\right)$$

$$G_x = \left(\sqrt{\pi}/2\right) 4Cov\left(X, F\left(X\right)\right)$$

$$G_{yx} = \left(\sqrt{\pi}/2\right) 4Cov\left(Y, F\left(Y\right)\right) G_x$$

$$G_{xy} = \left(\sqrt{\pi}/2\right) 4Cov\left(X, F\left(X\right)\right) G_y$$

where F is cumulative distribution function. For developing the proposed chart, the authors define a new quantity for showing the relation between $|\boldsymbol{G}|^{1/2}$ and $|\boldsymbol{\Sigma}_0|^{1/2}$. The quantity B is defined as follows:

$$B = 2\left(n-1\right)|\boldsymbol{G}|^{1/2}/|\boldsymbol{\Sigma}_0|^{1/2}$$

From B three more quantities can be derived. These quantities are b_0, b_1 and B_α which represent the mean, the standard deviation and the α^{th} quantile point of the distribution of B. All three quantities can be obtained using a simulation approach.

3.2.1.4 VMIX Chart of a Bivariate Process

Quinino *et al.* (2012) in their paper, propose a new statistic for controlling the covariance matrix of a bivariate normal process with known means and variances. The CC is known as VMIX chart and the monitoring statistic VMIX is:

$$VMIX = \frac{\sum_{t=1}^{n} X_t^2 + \sum_{t=1}^{n} Y_t^2}{2n}$$

The chart signals when VMIX > CL, where CL is the control limit which is selected for a predefined ARL_0.

The CC is defined by considering X^* and Y^* as two quality characteristics of interest with means μ_{X^*} and μ_{Y^*} respectively. The variances are defined as σ^2_{X*} and $\sigma^2_{Y^*}$ and the covariance is defined as $\sigma_{X^*Y^*}$. If all the parameters are known, the new variables can be defined as:

$$X_t = \frac{(X_t^* - \mu_{X^*})}{\sigma_{x^*}}$$

and

$$Y_t = \frac{(Z_t^* - \rho X_t)}{\sqrt{1-\rho^2}}$$

where $Z_t^* = \dfrac{(Y_t^* - \mu_{Y^*})}{\sigma_{Y^*}}$. When the process is in control, X_t and Y_t follow the standardized normal distribution and become free of the correlation parameter ρ.

Therefore, after the assignable cause occurrence, at least one of the two variances σ_x^2 or σ_y^2, of the transformed variables X and Y, increases without changing the means $\mu_X = 0$ and $\mu_Y = 0$.

3.2.2 EWMA Control Charts for the Dispersion

3.2.2.1 EWMA Scheme Based on the VMAX Statistic

Machado and Costa (2008) proposed an EWMA scheme based on VMAX for detecting changes in the covariance matrix $\mathbf{\Sigma}$ of a bivariate process. The EWMA scheme is based on the statistic:

$$Z_t = \lambda Y_t + (1-\lambda) Z_{t-1}, t = 1, 2, \cdots$$

and a signal is given if $Z_t > CL$ where:

$$CL = \frac{\left(\chi^2_{2n-4,\alpha}\right)^2 |\mathbf{\Sigma}_0|}{4(n-1)^2}$$

It is denoted that $Y_t = \max\left\{S^2_{x_t}, S^2_{y_t}\right\}$. $S^2_{x_t}$ and $S^2_{y_t}$ are the sample variances of X and Y respectively. The starting value Z_0 is often taken to be the expected in-control value of Z as defined by Lucas and Saccucci (1990).

3.2.2.2 A Bivariate EWMA Control Chart Based on the Decomposition Method of Mason (1995)

Nezhad (2011) defined the following statistic for monitoring the dispersion of a bivariate process:

$$Q_t = \lambda S^2_t + (1-\lambda) Q_{t-1}$$

with $Q_0 = 2$. The time varying control limits for E_k can be defined as follows:

$$UCL_k = 2 + c\sqrt{\frac{4\lambda}{2-\lambda}}$$

The control limits for Q_t can also be obtained as:

$$UCL = \frac{2\chi^2_{\nu,1-\alpha/2}}{\nu} \text{ and}$$

$$LCL = \frac{2\chi^2_{\nu,\alpha/2}}{\nu}$$

where $\nu = \dfrac{2\,(2-\lambda)}{\lambda}$.

Nezhads (2011) proposal relied on the decomposition method of Mason *et al.*(1995) for two quality characteristics. According to Masons decomposition method, the following two statistics can be defined:

$$T_{t1} = \left(\frac{x_{t1} - E\,(x_{t1})}{\sigma_1}\right)$$

$$T_{t2.1} = \left(\frac{x_{t2} - E\,(x_{i2}|x_{t1})}{\sigma_{2.1}}\right)$$

where $\sigma_{2.1}$ is the conditional standard deviation of the second characteristic given the first. Finally, the S_t^2 statistic defined as follows:

$$S_t^2 = T_{t1}^2 + T_{t2.1}^2$$

When the process is in-control, the statistics S_t^2 follows a χ_2^2 distribution.

Chapter 4

Multivariate Control Charts for the Dispersion

4.1 Introduction

This chapter discusses all multivariate control charts for monitoring the dispersion of the process. In the multivariate case, the most well-known control chart for monitoring the process' mean in the case of normality is the D^2 control chart. The control chart is based on plotting the following statistic against time: $D_t^2 = n\left(\overline{\boldsymbol{x}}_t - \boldsymbol{\mu}_0\right)' \boldsymbol{\Sigma}_0^{-1}\left(\overline{\boldsymbol{x}}_t - \boldsymbol{\mu}_0\right)$. It is assumed that the in-control mean vector and the variance-covariance matrix is known or are estimated from Phase *I*. The D_t^2 statistic represents the distance between any point and $\boldsymbol{\mu}_0$. The UCL for the D^2 is $\chi^2_{p,1-\alpha}$. For the case of the dispersion there are several charts that have been proposed and will be discussed in this chapter.

In section 4.2 all Phase I control charts will be presented. 4.2.1 discusses Shewhart control charts while 4.2.2 EWMA control charts. No Phase *I* CUSUM control charts where found in literature. Section 4.3 discusses Phase *II* control charts where 4.3.1 presents the Shewhart control charts and 4.3.2 and 4.3.3 present the CUSUM and EWMA control charts respectively.

4.2 Phase I Control Charts

4.2.1 Shewhart Control Charts for the Dispersion

4.2.1.1 The Phase *I* CC1 Control Chart

By considering unknown the variance-covariance matrix $|\mathbf{\Sigma_0}|$ the unbiased estimator $|\overline{\mathbf{S}}|/b_1$ defined by Alt(1985) can be used. The control limits of the Phase I CC for the dispersion with monitoring statistic the GV of the sample variance covariance matrix ($|\boldsymbol{S}_t|$) are:

$$UCL = \left(|\overline{\mathbf{S}_t}|/b_1\right)\left(b_1 + 3b_2^{1/2}\right)$$

$$LCL = \left(|\overline{\mathbf{S}_t}|/b_1\right)\left(b_1 - 3b_2^{1/2}\right)$$

with $C.L. = |\overline{\mathbf{S}_t}|$. b_1 can be defined as $\dfrac{\prod_{t=1}^{p} n-t}{(n-1)^p}$ and also, b_2 can be defined as $\dfrac{\prod_{t=1}^{p} n-t}{(n-1)^{2p}}$.

4.2.2 EWMA Control Charts for the Dispersion

4.2.2.1 The EWMA V Chart

For a p-variate normal process with a mean vector $\boldsymbol{\mu}_0$ and a variance-covariance matrix $\mathbf{\Sigma}_0$ ($N_p(\boldsymbol{\mu}_0, \mathbf{\Sigma}_0)$) in which $\boldsymbol{\mu}_0$ and $\mathbf{\Sigma}_0$ can be estimated from the Phase *I* data samples as the total mean and the pooled variance respectively the EWMA V CC can be constructed.

Yeh *et al.* (2003) propose the next multivariate EWMA CC based on the function:

$$S_\nu(t) = \lambda \times (\nu_t - 0.5) + (1-\lambda) \times S_\nu(t-1), t \geq 1$$

where $S_\nu(0) = 0$.

The control limits for the multivariate V EWMA CC are:

$$UCL = L \times \sqrt{\frac{1}{12}\left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2t}\right)}$$

$$CL = 0$$

$$LCL = -L \times \sqrt{\frac{1}{12}\left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2t}\right)}$$

where $t = 1, 2, \ldots$ and L are chosen for a predetermined ARL_0. The authors propose $m \geq 50, n \geq 10$ and $3 \leq p \leq 8$.

Yeh *et al.* (2003) defined the following probability for $t \geq 1$:

$$\nu_t = P\left(\prod_{j=1}^{p} F_{n-j,N-i+1-j} \leq \left(\prod_{j=1}^{p} \frac{N-m+1-j}{n-j}\right) \times \frac{|n\mathbf{S}_t|}{|N\overline{\mathbf{S}}|}\right)$$

where for any given λ and $t \geq 1$:

$$E\left(S_\nu(t)\right) = 0 \text{ and } V\left(S_\nu(t)\right) = \frac{1}{12}\left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2t}\right)$$

4.2.2.2 EWMA Chart Based on Generalized Variance

It is known that if the process is in control ($\boldsymbol{X}_i \sim N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$) then, the distribution of:

$$Y_t = \sqrt{\frac{n-1}{2p}} \ln \frac{|\boldsymbol{S}_t|}{|\boldsymbol{\Sigma}_0|}$$

follows asymptotically the standardized normal distribution. If the process is out-of-control and more specifically if $\boldsymbol{\Sigma}_0$ changes to $\boldsymbol{\Sigma}_1$, then Y_t is asymptotically distributed as $N\left(\ln|\boldsymbol{\Sigma}_1|/|\boldsymbol{\Sigma}_0|, 1\right)$. So, a change in the generalized variance is characterized in a change of the mean of Y_t. Therefore a univariate EWMA chart can be used for detecting a mean shifts in Y_t. If $\boldsymbol{\Sigma}_0$ is known, Yeh *et al.* (2006) defined the following:

$$G_i = \lambda Y_t + (1-\lambda) Y_{t-1}$$

where $G_0 = 0$ and λ is a smoothing constant. The control limits for the EWMA charts are:

$$UCL = L \times \sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}$$

and

$$LCL = -L \times \sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}$$

If $\mathbf{\Sigma}_0$ is not known, it can be estimated by $\overline{\boldsymbol{S}}$ and the statistic Y_t is modified to:

$$Y_t^* = \sqrt{\frac{k\,(n-1)}{2p\,(k+1)}}\ln\frac{|\boldsymbol{S}_t|}{|\overline{\boldsymbol{S}}|}$$

and follows asymptotically the standardized normal distribution. If the process is out-of-control and more specifically if $\mathbf{\Sigma}_0$ changes to $\mathbf{\Sigma}_1$, then Y_t^* is asymptotically distributed as $N\left(\sqrt{k/k+1}\,\ln|\mathbf{\Sigma}_1|/|\mathbf{\Sigma}_0|, 1\right)$ and the EWMA statistic is given by:

$$G_t^* = \lambda Y_t^* + (1-\lambda)\,Y_{t-1}^*$$

4.3 Phase II Control Charts

4.3.1 Shewhart Control Charts for the Dispersion

4.3.1.1 W-Statistic Based Chart

For monitoring the dispersion of a multivariate process Alt (1985) proposes the following statistic:

$$W_t = -pn + pn\ln n - n\ln\left(\frac{|\mathbf{A}_t|}{|\mathbf{\Sigma}_0|}\right) + trace\left(\mathbf{\Sigma}_0^{-1}\mathbf{A}_t\right)$$

The α probability limits of the chart are:

$$UCL = X^2_{p(p+1)/2;1-\alpha}$$

$$LCL = 0$$

and it is noted that $\mathbf{A}_t$ is the sum of squares and cross products matrix. Also, $\mathbf{A}_t = (n-1)\mathbf{S}_t$. If W statistic plots over UCL then the process is considered out-of-control.

4.3.1.2 The Phase *II* CC1 Control Chart

Alt(1985) proposes the development of a $|\mathbf{S}|$-CC by using the first two moments of $|\mathbf{S}|$. The 3σ control limits are:

$$UCL = |\mathbf{\Sigma}_0|\left(b_1 + 3\sqrt{b_2}\right)$$

$$C.L. = |\mathbf{\Sigma}_0| b_1$$

$$LCL = |\mathbf{\Sigma}_0|\left(b_1 - 3\sqrt{b_2}\right)$$

where

$$b_1 = \frac{\prod_{i=1}^{p}(n-i)}{(n-1)^p}$$

and

$$b_2 = \frac{\prod_{i=1}^{p}(n-i)}{(n-1)^{2p}} \times \left[\prod_{j=1}^{p}(n-j+2) - \prod_{j=1}^{p}(n-j)\right]$$

If the LCL is computed as a negative number ($LCL < 0$), then it must be replaced by zero (0).

4.3.1.3 Shewhart Chart Based on Conditional Entropy

Guerrero-Cusumano (1995) states that by measuring the difference between sample and theoretical entropy for the independent case, the following statistic (E) for monitoring the variance-covariance matrix can be obtained:

$$E_t = \sqrt{\frac{n-1}{2p}} \sum_{i=1}^{p} \ln\left(\frac{s_i^2}{\sigma_{i0}^2}\right)$$

The mentioned statistic E follows a univariate standard normal distribution. The UCL and LCL are calculated using simulation as follows:

$$UCL = gp\left[G'\left(\frac{n-1}{2}\right) - \ln\left(\frac{n-1}{2}\right)\right] + z_{a/2}k\sqrt{pG''\left(\frac{n-1}{2}\right)}$$

$$LCL = gp\left[G'\left(\frac{n-1}{2}\right) - \ln\left(\frac{n-1}{2}\right)\right] - z_{a/2}k\sqrt{pG''\left(\frac{n-1}{2}\right)}$$

where $g = (2\,(n-1)\,/p)^{1/2}$, $G'(\cdot)$ and $G''(\cdot)$ are the first and second derivative of the natural logarithm of the gamma function.

The result derived from the following suggestion of expressing entropy ($H(x)$):

$$H(x) = \frac{1}{2}p\ln(2\pi e) + \frac{1}{2}2\ln|\mathbf{\Sigma}_{d_0}^2| + \frac{1}{2}\ln|\boldsymbol{P}_0| =$$

$$= \frac{1}{2}p\ln(2\pi e) + \frac{1}{2}\mathbf{\Sigma}_{i=1}^{p}\ln\left(\sigma_{i0}^2\right) - T(X)$$

where $\boldsymbol{P}_0 = \mathbf{\Sigma}_{d_0}^{-1}\mathbf{\Sigma}_0\mathbf{\Sigma}_{d_0}^{-1}$ is the correlation matrix, $\mathbf{\Sigma}_{d_0} = diag\,(\sigma_{i0})$ with σ_{i0}, being the in-control standard deviation for the ith component of X. The function $T(X)$ is called the mutual information of the random variable X. By estimating σ_{i0}^2 with the sample variance of the ith component s_i^2, $\widehat{\boldsymbol{H}}(x)$ is obtained.

For the dependent case another statistic can be used:

$$E_{2t} = k\sum_{i-1}^{p}\left[\ln\left(\chi_{\alpha}^{n-1}\right) - \ln(n-1)\right]$$

with control limits that can be calculated from the following:

$$UCL = gp\left[G'\left(\frac{n-1}{2}\right) - \ln\left(\frac{n-1}{2}\right)\right] + z_{a/2}k\sqrt{pG''\left(\frac{n-1}{2}\right) + \frac{2}{n-1}tr\left(\boldsymbol{P}_0 - \boldsymbol{I}\right)^2}$$

$$LCL = gp\left[G'\left(\frac{n-1}{2}\right) - \ln\left(\frac{n-1}{2}\right)\right] - z_{a/2}k\sqrt{pG''\left(\frac{n-1}{2}\right) + \frac{2}{n-1}tr\left(\boldsymbol{P}_0 - \boldsymbol{I}\right)^2}$$

4.3.1.4 Shewhart Chart Based on the Decomposition of $\boldsymbol{S}_t$

Tang and Barnett (1996) proposed a multivariate Shewhart chart based on the decomposition of $\boldsymbol{S}_t$ into a sum of independent χ^2 statistics.Their chart is based on plotting the following statistic for each sample of n observation:

$$T_t = \sum_{j=1}^{2p-1} Z_j^2$$

When the process is in control, Z_j's are independently and identically distributed as $N(0,1)$ and therefore T is distributed as χ^2_{2p-1}. An out-of-control signal is detected as soon as T exceeds UCL which is determined from χ^2_{2p-1}. As mentioned,

$$T_t = \sum_{j=1}^{2p-1} Z_j^2$$

where:

$$Z_1 = \Phi^{-1}\left\{\chi^2_{n-1}\left[\frac{(n-1)\,s_1^2}{\sigma_1^2}\right]\right\}$$

$$Z_j = \Phi^{-1}\left\{\chi^2_{n-j}\left[\frac{(n-1)\,s^2_{j\cdot 1,2,\cdots,j-1}}{\sigma^2_{j\cdot 1,2,\cdots,j-1}}\right]\right\} \text{ for } j = 2,3,\cdots,p$$

$$Z_{p+1} = \Phi^{-1}\left\{\chi^2_{p-1}\left[(n-1)\,s_1^2\,(d_2-\theta_2)'\,\boldsymbol{\Sigma}^{-1}_{2,3,\cdots,p\cdot 1}\,(d_2-\theta_2)\right]\right\}$$

and

$$Z_{p+j-1} = \Phi^{-1}\left\{\chi^2_{p-j+1}\left[(n-1)\,s^2_{j-1\cdot 1,2,\cdots,j-2}\,(d_j-\theta_j)'\,\boldsymbol{\Sigma}^{-1}_{j,j+1,\cdots,p\cdot 1,2,\cdots,j-1}\,(d_j-\theta_j)\right]\right\}$$

for $j = 3,4,\cdots,p$. Also the following statistics must be defined. $\boldsymbol{S}'_{(j-1)\times(p-j+1)} = (\boldsymbol{S}_{j,j-1}, \boldsymbol{S}_{j+1,j-1}, \cdots, \boldsymbol{S}_{p,j-1})$ and $\boldsymbol{S}_{k,j}$ represent the row vector of sample covariances between the kth variable and each of the first j variables. The same goes for $\boldsymbol{\Sigma}_0$ by replacing the sample statistics with the corresponding population parameters. The conditional sample variance of the jth variable given the first *j-1* variables is defined as follows:

$$s^2_{j\cdot 1,2,\cdots,j-1} = s_j^2 - \boldsymbol{S}'_{j,j-1}\boldsymbol{S}^{-1}_{j,j-1}\boldsymbol{S}_{j,j-1}$$

The conditional sample covariance matrix of the last $p-j+1$ variables given the first $j-1$ can be expressed as:

$$\boldsymbol{S}_{j,j+1,\cdots,p\cdot 1,2,\cdots,j-1} = \boldsymbol{S}_{*p-j+1} - \boldsymbol{S}'_{(j-1)\times(p-j+1)}\boldsymbol{S}^{-1}_{(j-1)\times(p-j+1)}$$

where $\boldsymbol{S}_j$ and $\boldsymbol{S}_{*k}$ are the sample covariance matrix of the first j variables and of the last k variables respectively.

d_j (θ_j) for $j = 2, 3, \cdots, p$ denote the vector of sample (population) regression coefficients when each of the last $p-j+1$ variables is regressed on the *(j-1)*th variable while the first *j-2* variables are held fixed.

$$d_j = \frac{\left[S_{(j-1)\times(p-j+1)} - S'_{j-1,j-2}S^{-1}_{j-2}\left(S'_{j,j-2}S'_{j+1,j-2}S'_{p,j-2}\right)'\right]'}{s^2_{j-1} - S'_{j-1,j-2}S^{-1}_{j-2}S'_{j-1,j-2}}$$

Likewise, θ_j is similarly expressed by replacing the sample with the population statistics.

4.3.1.5 |S|-Control Chart

In their paper, Aparisi *et al.* (1999) studied the distribution of the |**S**|-CC and presented two |**S**|-Charts that are suitable for more than two quality characteristics. The first procedure, consists only by an UCL and the second with both an UCL and a LCL. In both cases the plotted statistic is the GV.

In the case were only an UCL is needed, the control limit is:

$$UCL = \frac{J^{1-a}_{n,p}|\boldsymbol{\Sigma}_0|}{(n-1)^p}$$

If both UCL and $L.C.L$ are needed, the (α probability) control limits are the following:

$$UCL = \frac{J^{1-a/2}_{n,p}|\boldsymbol{\Sigma}_0|}{(n-1)^p}$$

$$LCL = \frac{J^{a/2}_{n,p}|\boldsymbol{\Sigma}_0|}{(n-1)^p}$$

with center line $C.L. = |\mathbf{S}|$ in both cases. $J_{n,p}^{1-a/2}$ corresponds to the 1-α percentile of the distribution of the transformed variable:

$$J_{n,p} = \frac{(n-1)^p |\mathbf{S}|}{|\mathbf{\Sigma}|}$$

In the paper, tables for the values of $J_{n,p}^{1-a/2}$ and $J_{n,p}^{a/2}$ for various number of quality characteristics p and sample sizes n, are also presented.

4.3.1.6 Shewhart Control Chart Based on $H_0 : \mathbf{\Sigma} = \mathbf{\Sigma}_0$

Levinson *et al.* (2002) proposed the following statistic for $i \geq 1$ for treating the problem as testing $H_0 : \mathbf{\Sigma} = \mathbf{\Sigma}_0$ v.s. $H_1 : \mathbf{\Sigma} \neq \mathbf{\Sigma}_0$:

$$mM_i = m\left[(k+1)(n-1)\ln|\boldsymbol{S}_p| - k(n-1)\ln|\overline{\boldsymbol{S}}| - (n-1)\ln|\boldsymbol{S}_i|\right]$$

When the process is in control, mM_i follows $\chi^2_{p(p+1)/2}$ and thus, the UCL and LCL can be determined. Also,

$$m = 1 - \left[\frac{1}{k(n-1)} + \frac{1}{n-1} - \frac{1}{(k+1)(n-1)}\right] \times \left[\frac{2p^2+3p-1}{6(p+1)}\right]$$

and

$$\boldsymbol{S}_p = \frac{k(n-1)\overline{\boldsymbol{S}} + (n-1)\boldsymbol{S}_i}{(k+1)(n-1)}$$

where $\overline{\boldsymbol{S}}$ comes from a Phase I data set.

4.3.1.7 Shewhart Control Chart Based on Probability Integral Transformation

Yeh *et al.* (2002) proposed using the probability integral transformation to transform different statistics into the same random variable. The part dealing with the covariance matrix can be written as:

$$\nu_t = P\left[\prod_{i=1}^{p} F_{n-1-i,k(n-1)-k+1-i} \leq \left(\prod_{i=1}^{p} \frac{k\,(n-1)-k+1-i}{n-1-i}\right) \times \frac{|(n-1)\,\boldsymbol{S}_t|}{|k\,(n-1)\,\overline{\boldsymbol{S}}|}\right]$$

where $\overline{\boldsymbol{S}}$ comes from a Phase I data set.

When the process is in-control, ν_t are a sequence of independently and identically distributed Uniform ($U\,(0,1)$) random variables, therefore the control limits can be set up based on $U\,(0,1)$.

4.3.1.8 Double Sampling $|\boldsymbol{S}|$ Chart

The Multivariate Double Sampling (MDS) $|\boldsymbol{S}|$ chart from Grigoryan and He (2007) consists of five steps. For these steps, three quantities must be defined. V_1 and V_2 which are the control limits at the first stage of the process and V_3 which is the control limit for the second stage.

For the **first step** of the process, a set of size n_1 can be taken and the following statistic can be computed:

$$Y = \left(|\boldsymbol{S}_1| - b_1|\boldsymbol{\Sigma}_0|\right)/b_2^{1/2}|\boldsymbol{\Sigma}_0|$$

where $\boldsymbol{S}_1$ is the variance-covariance matrix of a sample of size n_1. **Step 2**: If Y falls in the interval $[-V_1, V_2]$ then the process is in control.

Step 3: If Y falls in the interval $(V_2, +\infty)$ or $(-\infty, -V_2)$ then the process is out of control.

Step 4: If Y falls in the interval $[-V_2, -V_1]$ or $[V_1, V_2]$, then a second sample of size n_2 can be taken and the following statistic is computed based on the combined sample of size $n_1 + n_2$:

$$Y_1 = \left(|\boldsymbol{S}_{12}| - b_{11}|\boldsymbol{\Sigma}_0|\right)/b_{22}^{1/2}|\boldsymbol{\Sigma}_0|$$

where $\boldsymbol{S}_{12}$ is the variance-covariance matrix of a sample of size $n_1 + n_2$. **Step 5**: If Y_1 falls in the interval $[-V_3, V_3]$ the process is in control. Otherwise is considered out-of-control.

The previous five steps take into account the quantities: b_1, b_2, b_{11} and b_{22} which are determined below:

$$b_1 = [1/(n_1-1)^p]\prod_{i=1}^{p}(n_1-i)$$

$$b_2 = \left[1/(n_1-1)^{2p}\right]\prod_{i=1}^{p}(n-i)\left[\prod_{j=1}^{p}(n_1-j+2)-\prod_{j=1}^{p}(n_1-j)\right]$$

$$b_{11} = [1/(n_1+n_2-1)^p]\prod_{i=1}^{p}(n_1+n_2-i)$$

$$b_{22} = \left[1/(n_1+n_2-1)^{2p}\right]\prod_{i=1}^{p}(n-1)\left[\prod_{j=1}^{p}(n_1+n_2-j+2)-\prod_{j=1}^{p}(n_1+n_2-j)\right]$$

Finally, for constructing the MDS $|\boldsymbol{S}|$ chart, one has to determine the parameters n_1, n_2, V_1, V_2 and V_3.

4.3.1.9 Monitoring Variation Using the Wilk's Statistic

Assuming that $\boldsymbol{S}$ is the sample covariance estimator of $\boldsymbol{\Sigma}$ based on a historical data set of size n and $\boldsymbol{S}_A$ is the sample covariance estimator obtained from the HDS and the m new samples, the following statistic can be used according to Mason *et al.* (2009) to compare the variation in the above two samples using Wilk's statistic (1962):

$$W_t = \left(\frac{n-1}{n+m-1}\right)^p \frac{|\boldsymbol{S}_t|}{|\boldsymbol{S}_A|}$$

Wilk's statistic has values between 0 and 1. Values near 1 correspond that the estimated covariance matrix $\boldsymbol{S}_A$ is similar to the estimated covariance matrix $\boldsymbol{S}$, while values near 0 indicate otherwise. For a given level of significance α, the LCL of W is determined by:

$$P(W < w_\alpha) = \alpha$$

where w_α is the α^{th}quantile of the distribution of W.

Regarding the quantiles of the distribution of W, various approximations have been developed. The most well-known approximation developed by Bartlett (1938) and is based on the χ^2 distribution. The approximation is as follows:

$$-f \ln (W) \approx \chi^2_{mp}$$

where $f = (2n + m - p - 3)/2$. It is stated that the approximation can be used when $(p^2 + m^2) \leq f/3$. There is also another approximation developed by Rao (1951) based on the F distribution. The approximation is given by:

$$K\left(\frac{1 - W^{1/t}}{W^{1/t}}\right) \approx F_{pm,ft-g}$$

where $K = \dfrac{ft - g}{pm}$, $t = \left(\dfrac{p^2 m^2 - 4}{p^2 + m^2 - 5}\right)^{1/2}$ and $g = \dfrac{pm - 2}{2}$. Finally it is noted that the approximation is valid for $ft > g$.

For the χ^2 approximation, the α^{th} quantile of W is approximated by:

$$w_\alpha \simeq \exp\left\{-\frac{1}{f}\chi^2_{mp}(1-\alpha)\right\}$$

By using the F-distribution approximation, the α^{th} quantile of W is approximated by:

$$w_\alpha \simeq \left\{\frac{K}{K + F_{pm,ft-g}(1-\alpha)}\right\}^t$$

4.3.1.10 Monitoring Variation Using Scatter Ratios Decomposition

Mason *et al.* (2010) continued the idea of using Wilks' statistic and decomposing Wilks' ratio statistic by noting that the sample generalized variance of the covariance estimator $\boldsymbol{S}$ can be written as:

$$|\boldsymbol{S}| = s_{11}\left[s_{22}\left(1 - r^2_{2.1}\right)\right]\left[s_{33}\left(1 - r^2_{3.12}\right)\right]\cdots\left[s_{pp}\left(1 - r^2_{p.12\cdots p-1}\right)\right]$$

where $r^2_{k.12\cdots k-1}$ represents the squared multiple correlation coefficient for the regression of x_k on $x_1, x_2, \cdots, x_{k-1}$. Applying the previous result, Wilks' statistic can be written as:

$$W_t = \left(\frac{n-1}{n+m-1}\right)^p \left(\frac{s_{11}}{s'_{11}}\right) \frac{s_{22}\left(1-r^2_{2.1}\right)}{s'_{22}\left(1-r'^2_{2.1}\right)} \cdots \frac{s_{pp}\left(1-r^2_{p.12\cdots p-1}\right)}{s'_{pp}\left(1-r'^2_{p.12\cdots p-1}\right)}$$

where s'_{jj} and r'^2 refer to the sample variances and correlation coefficients for the variables in the combined data set. The decomposition of the W statistic can also be written as:

$$W = W_1 \times W_{2.1} \times W_{3.1,2} \times \cdots \times W_{p.1,2,\cdots,p-1}$$

where $W_1 = \dfrac{(n-1)\, s_1^2}{(n+m-1)\, s_1'^2}$, $W_{j.1,\cdots,j-1} = \dfrac{(n-1)\, s^2_{j.1\cdots j-1}}{(n+m-1)\, s'^2_{j.1\cdots j-1}}$ and $s^2_{j.1\cdots j-1}$ represents the conditional variance of x_j on $x_1, x_2, \cdots, x_{j-1}$. With this methodology, q distinct factors can be monitored and thus, Bonferroni limits can be used. The chart signals as soon as a factor plots below its corresponding LCL (for a pre determined LCL). Mason *et al.* in their paper also gave a way for monitoring the dispersion with individual observations.

4.3.1.11 Shewhart Chart Using the Eigenvalues

Mohd Noor A. and Djauhari M.A. (2011) proposed measuring the performance of multiple eigenvalue CCs for monitoring the multivariate process variability. The j^{th} CC, has as plotted statistic the j^{th} eigenvalue of the i^{th} future variance-covariance matrix denoted as $\lambda_{i\boldsymbol{S}}$.

The associated control limits for an individual eigenvalue chart are given by:

$$UCL = \lambda_{j\boldsymbol{\Sigma}_0} + L\left(\sqrt{\frac{2}{n-1}\left(\lambda_{j\boldsymbol{\Sigma}_0}\right)^2}\right)$$

$$LCL = \lambda_{j\boldsymbol{\Sigma}_0} - L\left(\sqrt{\frac{2}{n-1}\left(\lambda_{j\boldsymbol{\Sigma}_0}\right)^2}\right)$$

$L = \Phi\left(Z_\nu\right)$ where $\nu = \sqrt[p]{\alpha}/2$ and $\Phi\left(\cdot\right)$ is the cumulative distribution function of the standard normal distribution. The process is said to be out-of-control when at least one CC gives an out of control signal.

4.3.1.12 One-Sided LRT-Based Control Chart

Yen *et al.*(2011) in their study focus on monitoring the dispersion of a multivariate process if the dispersion is decreasing. In a previous work, Yen and Shiau (2010) derived the LRT statistic for monitoring increases of the dispersion. The proposed one-sided CC of Yen and Shiau (2010) had the following statistic:

$$T_I = n \sum_{i=1}^{p_I^*} \left[(d_i - 1) - \log d_i \right], \text{ for } p_I^* > 0$$

and $T_I = 0$ for $p_I^* = 0$. It is noted that $d_1 \geq \cdots \geq d_p > 0$ are the roots of $|\boldsymbol{S}_t - d\boldsymbol{\Sigma}_0| = 0$ and p_I^* is the number of $d_i > 1$.

In a similar way, Yen *et al.* (2010) in their paper propose following statistic for monitoring decreases in dispersion:

$$T_D = n \sum_{i=1}^{p_D^*} \left[(d_i - 1) - \log d_i \right], \text{ for } p_D^* > 0$$

and $T_D = 0$ for $p_D^* = 0$. In this case it is noted that p_D^* is the number of $0 < d_i < 1$. The chart signal whenever T_D exceeds $T_D(\alpha)$ for the case of decrease and whenever T_I exceeds $T_I(\alpha)$ in the case of increase.

It is stated by the authors that the distribution of T_D is difficult to be obtained analytically so in their paper they used a Monte Carlo simulation to estimate the critical value of T_D.

Finally, Yen *et al.* (2012) propose the usage of a combined chart which signals an out-of-control alarm if:

$$T_I > T_I(\alpha_I) \text{ or } T_D > T_D(\alpha_D)$$

4.3.1.13 Test of Covariance Changes Without Large Data

Hung and Chen (2012) in their paper have proposed two statistics for monitoring the variance-covariance matrix (T1 and T2). Their form is:

$$T1 = \{t_{ii}^2 \text{ for } 1 \leq i \leq p, T_{off-diag}\} \text{ and}$$

$$T2 = \{T_{diag}, T_{off-diag}\}$$

$T1$ statistic signals if:

$$\{t_{11}^2 < \chi^2_{n-1,(a_1/2)} \text{ or } t_{11}^2 > \chi^2_{n-1,1-(a_1/2)}\},$$

$$\text{or}$$

$$\vdots$$

$$\text{or}$$

$$\{t_{pp}^2 < \chi^2_{n-1,(a_p/2)} \text{ or } t_{pp}^2 > \chi^2_{n-1,1-(a_p/2)}\}$$

$$\text{or}$$

$$\{T_{off-diag} > \chi^2_{(p/2)(p-1),1-\alpha_{off-diag}}\}$$

$T2$ signals if:

$$\{T_{diag} < \chi^2_{(p/2)(2n-p-1),\left(\alpha_{diag}/2\right)}\}$$

$$\text{or}$$

$$\{T_{diag} > \chi^2_{(p/2)(2n-p-1),1-\left(\alpha_{diag}/2\right)}\}$$

$$\text{or}$$

$$\{T_{off-diag} > \chi^2_{(p/2)(p-1),1-\alpha_{off-diag}}\}$$

Their proposal is based on the assumption that $(n-1)\,\boldsymbol{S}$ follows a Wishart distribution with parameters $(n-1)$ and $\boldsymbol{\Sigma}_0$ ($\boldsymbol{S} \sim W_p\,(n-1, \boldsymbol{\Sigma}_0)$). Since $\boldsymbol{\Sigma}_0$ is positive definite, there is a matrix $\boldsymbol{A}$ satisfying $\boldsymbol{A}\boldsymbol{\Sigma}_0\boldsymbol{A}' = \boldsymbol{I}_p$ which leads to $(n-1)\,\boldsymbol{A}\boldsymbol{\Sigma}_0\boldsymbol{A}' \sim W_p\,(n-1, \boldsymbol{I_p})$. Using the Cholesky's decomposition theorem, $\boldsymbol{\Sigma}_0$ can be decomposed into $\boldsymbol{M}\boldsymbol{M}'$ where $\boldsymbol{M}$ is the unique lower triangular matrix with positive diagonal elements. $\boldsymbol{A}$ can be chosen to be $\boldsymbol{M}^{-1}$. Applying the Cholesky's decomposition theorem once more to the $(n-1)\,\boldsymbol{A}\boldsymbol{S}\boldsymbol{A}'$ another lower triangular matrix $\boldsymbol{T}$ can be obtained. Hung and Chen (2012) proved that t_{ij} are mutually independent distributed as:

$$t_{ii}^2 \sim \chi_{n-1}^2, \text{ for } 1 \leq i \leq p$$

$$t_{ii} N(0,1), \text{ for } 1 \leq j \leq i \leq p$$

The following hypothesis test is considered: $H_0 : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$ versus $H_1 : \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0$. Any departure from the null hypothesis will make certain t_{ij} behave abnormally. Two test statistics are constructed (T_{diag} and $T_{off-diag}$) with exact null distributions.

Finally T_{diag} and $T_{off-diag}$ are defined as:

$$T_{diag} = \sum_{1 \leq i \leq p} t_{ij}^2 \sim \chi_{(p/2)(2n-p-1)}^2 \text{ and}$$

$$T_{off-diag} = \sum_{1 \leq j \leq i \leq p} t_{ij}^2 \sim \chi_{(p/2)(p-1)}^2$$

4.3.1.14 Penalized Likelihood Ratio (PLR) Chart

In their paper, Li *et al.* (2012) assume that $\boldsymbol{X}$ follows a p-dimensional normal distribution with known (or estimated from Phase I data) $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$. Without loss of generality, they state, that they assume that $\boldsymbol{X}$ follows $N_p(\mathbf{0}, \boldsymbol{I}_p)$ when the process is in control. The chart that has been constructed supposes that the out of control matrix $\boldsymbol{\Sigma}_1$ remains sparse, meaning that only a few diagonal elements are not equal to one and only a few off-diagonal elements are not equal to zero. The first step for the construction of the chart was to estimate $\boldsymbol{\Omega} = \boldsymbol{\Sigma}_0^{-1}$ using a penalized likelihood function. In the second step, the charting statistic was calculated based on the negative log-likelihood ratio of testing $H_0 : \boldsymbol{\Sigma}_0 = \boldsymbol{I}_p$ versus $H_1 : \boldsymbol{\Sigma}_0 \neq \boldsymbol{I}_p$. For the estimation of $\boldsymbol{\Omega}$ Li *et al.* (2012) found that by penalizing all elements of $\boldsymbol{\Omega}$ produces a more effective CC. The penalized negative likelihood function can be written as:

$$l(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n; \boldsymbol{\Omega}) = tr(\Omega \boldsymbol{S}) - \ln|\boldsymbol{\Omega}| + \lambda ||\boldsymbol{\Omega}||_1$$

where $||\boldsymbol{\Omega}||_1 = \sum_{j=1}^{p}\sum_{i=1}^{p}|\omega|$ and λ is a parameter that can be tuned to achieve different levels of sparsity of the $\boldsymbol{\Omega}$ estimate. $\boldsymbol{\Omega}_\lambda$ is the solution to the previous function for a given λ:

$$\boldsymbol{\Omega}_\lambda = \arg\min_{\boldsymbol{\Omega}>0}\left\{(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n; \boldsymbol{\Omega})\right\}.$$

After obtaining $\boldsymbol{\Omega}_\lambda$, the CC calculates for each given sample:

$$\Lambda_\lambda = tr\left(\boldsymbol{S}\right) - tr\left(\boldsymbol{\Omega}_\lambda \boldsymbol{S}\right) + \ln|\boldsymbol{\Omega}_\lambda|$$

The PLR chart, signals when $\Lambda_\lambda > UCL_\lambda$ where UCL_λ is chosen for a given λ to achieve a predetermined ARL.

4.3.1.15 Covariance Matrix Monitor with Fewer Observations than Variables

Mahaboudou-Tchao E. and Agboto V. (2013) propose a Shewhart-type CC based on the statistic:

$$c_t = tr\left(\boldsymbol{S}_t\right) - \ln|\boldsymbol{S}_t| - p.$$

The plot signals if $c_t > h$ where h is chosen to achieve a specified in-control ARL.

The CC is based on a sample of size n less than p. For cases with fewer observations than dimensions the data are unable to compute a non-singular sample covariance matrix. They use a matrix $\boldsymbol{A}$ with the property of $\boldsymbol{A}\boldsymbol{\Sigma}_0\boldsymbol{A}' = \boldsymbol{I}_p$ and transform $\boldsymbol{x}$ to $\boldsymbol{u} = \boldsymbol{A}\left(\boldsymbol{x} - \boldsymbol{\mu}_0\right)$ where $\boldsymbol{u}$ follows $N_p\left(\boldsymbol{0}, \boldsymbol{I}_p\right)$ when the process is in control. In their research they propose a two step mechanism. In the first step, for each sample i, $\boldsymbol{V}_t = \boldsymbol{U}_t'\boldsymbol{U}_t$ is computed. Using $\boldsymbol{V}_t$, an estimate of the inverse covariance matrix can be found using:

$$Q\left(\boldsymbol{\Omega}\right) = -\ln|\boldsymbol{\Omega}| + tr\left(\boldsymbol{\Omega}\widehat{\boldsymbol{\Sigma}}\right) + \rho||\boldsymbol{\Omega}||_1,$$

where $||\boldsymbol{\Omega}||_1 = \sum_{j=1}^{p}\sum_{i=1}^{p}|\omega_{ij}|$, $\boldsymbol{\Omega} = \boldsymbol{\Sigma}_0^{-1}$ and ρ is a data dependent tuning parameter.

Next, they obtain an estimate of the covariance matrix $\boldsymbol{S}_t$ by inverting $\widehat{\boldsymbol{\Omega}}_t$. Finally, the matrix $\boldsymbol{S}_t$ is compared with the identity matrix using c_t.

4.3.2 CUSUM Control Charts

4.3.2.1 MCUSUM for the Dispersion

Healy (1987), by considering a shift on the variance-covariance matrix from $\boldsymbol{\Sigma_0}$ to $\boldsymbol{\Sigma_1} = C\boldsymbol{\Sigma_0}$ proposed a multivariate *CUSUM* CC for the dispersion of the process given that the vector $\boldsymbol{\mu}$ is constant throughout the whole process. The *CUSUM* CC is based on the following function:

$$MC_k k = \max\left[MC_{k-1} + Y_k - K, 0\right], k = 1, 2, \ldots, m$$

where,

$$Y_k = (\mathbf{x}_k - \boldsymbol{\mu})' \boldsymbol{\Sigma^{-1}} (\mathbf{x}_k - \boldsymbol{\mu})$$

and

$$K = p \ln(C) \frac{C}{C-1}$$

The procedure is considered to be out of control if $MC_k \geq H$.

4.3.2.2 CUSUM Chart Based on Projection Pursuit

Chan and Zhang (2001) use the following statistics for monitoring the dispersion of a multivariate process:

$$Q_t^+ = \max\left\{0, Q_{t1}^+, Q_{t2}^+, \cdots, Q_{tt}^+\right\}$$

and

$$Q_t^- = \min\left\{0, Q_{t1}^-, Q_{t2}^-, \cdots, Q_{tt}^-\right\}$$

where $Q_0^+ = Q_0^- = 0$. The projection method signals as soon as either $Q_t^+ > h_+$ or $Q_t^- < h_-$ where h_+ and h_- are decision values.

It must be denoted that $Q_{tj}^{+} = \lambda_{tj}^{\max} - (t-j+1)\, r_{+}$ and $Q_{tj}^{-} = \lambda_{tj}^{\min} - (t-j+1)\, r_{-}$ where r_{+} and r_{-} are two reference values. Also when observation are collected, under the null hypothesis, $\lambda_{tj}^{\max}$ and $\lambda_{tj}^{\min}$ denote respectively the largest and smallest eigenvalue of the subgroup sample matrix.

4.3.2.3 Multiple CUSUM Charts Based on Regression Adjusted Variables

Yeh *et al.* (2004), Yeh *et al.* (2005) and Huwang *et al.* (2005) following Hawkin's (1991,1992) proposal expanded the idea of a multivariate CC for monitoring the process mean based on regression adjusted variables.
For a given process, one calculates:

$$S_{ti}^{+} = \max\left(0, S_{(t-1)i}^{+} + W_{ti} - r\right)$$

and

$$S_{ti}^{-} = \min\left(0, S_{(t-1)i}^{-} + W_{ti} - r\right)$$

where $S_{ti}^{-} = S_{ti}^{+} = 0$ and r is a reference value. An out-of-control signal is detected on the multiple CUSUM chart as soon as

$$\max_{1 \leq i \leq p} \left\{\max\left(S_{ti}^{+}, -S_{ti}^{-}\right)\right\} > h$$

where h is the decision value.

For a data set, the following statistic can be computed:

$$\boldsymbol{Z}_t = \left[diag\left(\boldsymbol{\Sigma}_0^{-1}\right)\right]^{-1/2} \boldsymbol{\Sigma}_0^{-1}\left(\boldsymbol{X}_t - \boldsymbol{\mu}_0\right)$$

where $\boldsymbol{Z}_t = \left(\boldsymbol{Z}_{t1}, \boldsymbol{Z}_{t2}, \cdots, \boldsymbol{Z}_{tp},\right)'$. When the process is in-control, $\boldsymbol{Z}_t$ is distributed as $N_p\left(\boldsymbol{0}, \boldsymbol{I}_p\right)$.

For detecting changes in the variance of the i^{th} component the following statistic is defined:

$$W_{ti} = \frac{|Z_{ti}|^{1/2} - 0.822}{0.349}$$

When the process is in control, W_{ti} follows the standardized normal distribution. If the distribution of Z_{ti} changes to $N\left(0, \sigma^2\right)$ then the distribution of W_{ti} changes approximately to $N\left(2.355\left(\sigma^{1/2} - 1\right), \sigma\right)$. Therefore, the usual univariate CUSUM chart is constructed to monitor mean shifts in W_{ti} (thus the variance of Z_{ti}).

4.3.3 EWMA Control Charts for the Dispersion

4.3.3.1 Multivariate EWMA Chart Based on Regression Adjusted Variables

From Hawkins' (1991,1992) proposal, the multiple CUSUM chart based on regression adjusted variables can be transformed to multiple EWMA charts. For $t \geq 1$ and $i = 1, 2, \cdots, p$ one calculates:

$$E_{ti} = \lambda W_{ti} + (1 - \lambda) E_{(t-1)i}$$

where $E_{0i} = 0$ and an out-of-control signal is given when:

$$\max_{1 \leq i \leq p} \{|E_{ti}|\} > L \times \sqrt{\frac{\lambda}{2 - \lambda}}$$

where L is a predefined value selected for a predetermined ARL_0. It must be noted that $W_{ti} = \frac{|Z_{ti}|^{1/2} - 0.822}{0.349}$.

4.3.3.2 The EWMLR control chart

Yeh *et al.* (2004) in their paper state that when the monitoring begins, independent samples of size n are taken from the process. The *EWMLR* CC for $t \geq 1$ is:

$$R_t = \lambda r_t + (1 - \lambda) R_{t-1}$$

where $R_0 = r_1$. The process is considered to be out-of-control if R_t is greater than the UCL ($UCL = UCL\,(m, n, w)$) and is chosen to achieve a predetermined ARL and is given from tables in Yeh *et al.* (2004) paper. It must be noted that r_t is defined as follows:

$$r_t = (mn + n - 2)\ln|\mathbf{A} + \mathbf{B}_t| - (mn - 1)\ln|\mathbf{A}| - (n - 1)\ln|\mathbf{B}_t|$$

where

$$\mathbf{B}_t = \sum_{j=1}^{n} \left(\mathbf{X_{tj}} - \overline{\mathbf{X}}_\mathbf{t}\right)\left(\mathbf{X_{tj}} - \overline{\mathbf{X}}_\mathbf{t}\right)'$$

and

$$\boldsymbol{A} = \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\boldsymbol{X}_{ij} - \overline{\overline{\boldsymbol{X}}}_t\right)\left(\boldsymbol{X}_{ij} - \overline{\overline{\boldsymbol{X}}}_t\right)'$$

4.3.3.3 The Maximum Multivariate Exponentially Weighted Moving Variability (MaxMEWMV) Control Chart

Yeh *et al.* (2004) propose plotting the following statistic:

$$MaxD_t = \max\left[\frac{D_{t1} - E\,(D_{t1})}{\sqrt{Var\,(D_{t1})}}, \frac{D_{t2} - E\,(D_{t2})}{\sqrt{Var\,(D_{t2})}}\right]$$

The chart signals as soon as the value of $MaxD$ plots a predetermined UCL. For constructing the chart they define the following statistic after they assume that $\boldsymbol{\Sigma}_0 = \boldsymbol{I}_{p\times p}$:

$$\boldsymbol{S}_t = \lambda \boldsymbol{X}_t \boldsymbol{X}_t' + (1 - \lambda)\,\boldsymbol{S}_{t-1}$$

where $\boldsymbol{S}_0 = \boldsymbol{X}_1\boldsymbol{X}_1'$. The proposed CC for monitoring the variance-covariance matrix has been derived from $\boldsymbol{S}_t$. In their approach, Yeh *et al.* (2004) have examined the variance and covariance components of $\boldsymbol{S}_t$ separately. So the following two quantities have been defined:

$$\boldsymbol{S}_{t_v} = \left(S_{t(11)}, S_{t(22)}, \cdots, S_{t(pp)}\right)'$$

and

$$\boldsymbol{S}_{t_c} = \left(S_{t(12)}, S_{t(13)}, \cdots, S_{t(ij)}, \cdots, S_{t((p-1)p)}\right)'$$

where $\boldsymbol{S}_{t_v}$ is a $p \times 1$ vector with the p diagonal elements of $\boldsymbol{S}_t$ and $\boldsymbol{S}_{t_c}$ is a $p(p-1)/2 \times 1$ vector of the upper triangular off-diagonal elements of $\boldsymbol{S}_t$. With this approach the deviation of $\boldsymbol{S}_{t_v}$ and $\boldsymbol{S}_{t_c}$ is measured from:

$$\boldsymbol{I}_{p\times 1} \text{ and } \mathbf{0}_{p(p-1)/2\times 1}$$

Yeh *et al.* (2004) have defined the following approach for measuring the distance between the two vectors:

$$D_{t1} = ||\boldsymbol{S}_{t_v} - \boldsymbol{I}_{p\times 1}||^2 = \sum_{j=1}^{p} \left(\sum_{k=1}^{t} a_k \boldsymbol{X}_{kj}^2 - 1 \right)^2$$

and

$$D_{t2} = ||\boldsymbol{S}_{t_c} - \mathbf{0}_{p(p-1)/2\times 1}||^2 = \sum_{i<j}^{p} \left(\sum_{k=1}^{t} a_k \boldsymbol{X}_{ki} \boldsymbol{X}_{kj} \right)^2$$

where $a_k = \lambda (1-\lambda)^{t-k}$, $a_1 = (1-\lambda)^{t-1}$ and $k = 2, 3, \cdots, t$. Also, $i, j = 1, 2, \cdots, p$.

Finally it should be noted that:

$$\mu_{D_{t1}} = 2p \frac{\lambda}{2-\lambda} \text{ and } \sigma_{D_{t1}} = p \left[\frac{48\lambda^4}{1-(1-\lambda)^4} + \frac{8\lambda^2}{(2-\lambda)^2} \right]$$

and also

$$\mu_{D_{t2}} = \frac{p(p-1)}{2} \frac{\lambda}{2-\lambda} \text{ and } \sigma_{D_{t2}} = p(p-1)(2p-1) \left[\frac{\lambda^4}{1-(1-\lambda)^4} \right] + p(p-1) \left(\frac{\lambda}{2-\lambda} \right)^2$$

4.3.3.4 The MEWMA V Control Chart

For monitoring the dispersion of the process, the following statistic proposed by Yeh *et al.* (2010) and is defined as follows:

$$\nu_t = P \left(\prod_{m=1}^{p} F_{n-i, N-k+1-i} \leq \left(\prod_{i=1}^{p} \frac{N-m+1-i}{n-i} \right) \times \frac{|n\boldsymbol{S}_t|}{|N\overline{\boldsymbol{S}}|} \right)$$

where m is the number of samples taken for estimating the Σ_0 in Phase I. Also, $|n\boldsymbol{S}_t|$ and $|N\overline{\boldsymbol{S}}|$ denote the determinant of the matrix $n\boldsymbol{S}_t$ and $N\overline{\boldsymbol{S}}$ $(N = n \times m)$. When the process is in-control, ν_t is distributed as $U\,(0,1)$. The EWMA chart is given below:

$$S_\nu\,(t) = \lambda \times (\nu_t - 0.5) + (1-\lambda) \times S_\nu\,(t-1)$$

where $S_\nu\,(0) = 0$. The authors state that $S_\nu\,(t)$ is symmetric at 0 so the two control limits can be the following:

$$UCL = L \times \sqrt{\frac{1}{12}\left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2t}\right)}$$

and

$$LCL = -L \times \sqrt{\frac{1}{12}\left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2t}\right)}$$

with Center Line 0 (*CL=0*)

4.3.3.5 The ELR Control Chart

The following CC has been constructed for simultaneously monitoring the mean and also the dispersion of the process. Zhang *et al.* (2010) consider the following hypothesis test:

$$H_0 : \boldsymbol{\mu} = \mathbf{0} \text{ and } \boldsymbol{\Sigma} = \boldsymbol{I}_p \text{ versus } H_1 : \boldsymbol{\mu} \neq \mathbf{0} \text{ or } \boldsymbol{\Sigma} \neq \boldsymbol{I}_p$$

The generalized likelihood ratio statistic for this test can be obtained and it is the following:

$$LR_t = np\,(a - \log g - 1) + n||\overline{\boldsymbol{X}}_t||^2$$

where $a = \dfrac{1}{p}tr\,(\boldsymbol{S}_t)$, $g = (|\boldsymbol{S}_t)^{1/p}$ and $||\cdot||$ represents the Euclidean distance of a vector. In the paper, it is mentioned that the terms $||\overline{\boldsymbol{X}}_t||^2$ and $a - \log g$ contribute to the changes of the process mean and variance respectively. Finally, the charting statistic has the following form:

$$ELR_t = np\,(a' \log g' - 1) + n||\boldsymbol{u}_t||^2$$

where $a' = \frac{1}{p} tr\left(\boldsymbol{v}_t\right)$ and $g' = \left(|\boldsymbol{v}_t|\right)^{1/p}$. Also,

$$\boldsymbol{u}_t = \lambda \overline{\boldsymbol{X}}_t + (1-\lambda)\, \boldsymbol{u}_{t-1}$$

$$\boldsymbol{v}_t = \lambda \overline{\boldsymbol{S}}_t^* + (1-\lambda)\, \boldsymbol{v}_{t-1}$$

with $\boldsymbol{S}_t^* = \sum_{j=1}^{n} \left(\boldsymbol{X}_{ij} - \boldsymbol{u}_t\right)' \left(\boldsymbol{X}_{ij} - \boldsymbol{u}_t\right)/n$ and $\boldsymbol{u_0} = \boldsymbol{0}$, $\boldsymbol{v_0} = \boldsymbol{I}_p$. The control limits for this particular CC are mostly available from the authors upon request. In general, the *ELR* statistic follows an asymptotic χ^2 distribution.

4.3.3.6 The Max Norm Control Chart

Shen *et al.*(2013) proposed an EWMA CC for monitoring the variance covariance matrix denoted as *Max Norm*. The proposed statistic to plot has the following form:

$$T_{t_{MaxNorm}} = \max\left[\frac{T_{t_1} - E\left(T_1\right)}{\sqrt{Var\left(T_1\right)}}, \frac{T_{t_2} - E\left(T_2\right)}{\sqrt{Var\left(T_2\right)}}\right]$$

and it signals as soon as $T_{t_{MaxNorm}}$ exceeds a pre-determined UCL. The CC derived from trying to determine if the covariance matrix of:

$$\boldsymbol{\Sigma}_t = (1-\lambda)\, \boldsymbol{\Sigma}_{t-1} + \lambda \boldsymbol{S}_t$$

is significantly different from the identity matrix. If

$$\boldsymbol{C}_t = \boldsymbol{\Sigma}_t - \boldsymbol{I}_{p\times p}$$

then, the deviation of variance-covariance matrix can be examined by the deviation of $\boldsymbol{C}_i$ from $\boldsymbol{0}$. In their study, the authors adopt a certain way for measuring the distance between the two vectors. The two measures used are defined as:

$$T_{t_1} = ||\boldsymbol{d}_t||_2 = \sum_{i=1}^{p} \sum_{j=i}^{p} c_{t(ij)}^2$$

and

$$T_{t_2} = ||\boldsymbol{d}_t||_\infty = max\left(|c_{t(11)}|, \cdots, |c_{t(pp)}|\right)$$

where $c_{t(ij)}$ is an element (in the i^{th} row and j^{th} column) in the covariance matrix $\boldsymbol{C}_t$ for $i \leq j$. When $\boldsymbol{\Sigma}$ deviates from $\boldsymbol{I}_{p\times p}$, then T_{t_1} and T_{t_2} tend to have larger values. Shen *et al.* (2013) in their study estimate through Monte Carlo simulation the asymptotic limits of $E\left(T_1\right)$, $E\left(T_2\right)$, $Var\left(T_1\right)$ and $Var\left(T_2\right)$.

Chapter 5

Comparisons

5.1 Introduction

Every case that can be considered in real life is special and it differs with any other. A practitioner may deal with various problems that can occur in his line of production and of course he wants to be ahead of them. So, for every scenario that can be dealt, it must be known to him the best way to catch up to it for less time and money to be consumed by the process. In other words, the practitioner should know which is the fastest CC to signal if the process is out-of-control depending on the possible shift that may occur. This Chapter, deals with this problem and various scenarios based on the available sample, the possible shift have been considered. In subsection 5.2 the competing CCs and Scenarios are presented. In subsection 5.2.1 the charts for the bivariate case will be presented and in subsection 5.2.2 the comparison will take place to determine the best available choice for every scenario. Finally in subsection 5.2.3 is the summary of the chapter.

5.2 Competing Control Charts and Scenarios

In this section, it will presented comparisons between some of the multivariate CCs presented in chapter 3 and 4. The first multivariate CC that will be used for comparisons is the CC1 introduced by Frank Alt (1985)which was presented in subsection 4.3.1.2

The second multivariate CC is the W-Chart which was described in section 4.3.1.1 and was also introduced by Frank Alt (1985).

The third chart is a bivariate CC and it is the CC2 Control Chart which was presented in section 3.2.1.1 and was proposed by Alt (1985).

Another bivariate CC (CC3) was proposed by Alt and Smith (1988) and will be presented in this chapter. The chart corresponds to section 3.3.1.2.

Two more CCs have been proposed in the same paper from Hung and Chen (2012) and are based on the Cholesky decomposition theorem. Namely, T1 and T2 and were presented in section 4.3.1.13.

On an approach for a bivariate case, Quinino et al. (2012) in their paper propose a new statistic for controlling the covariance matrix a normal process with known means and variances. The *VMIX* statistic was presented in section 3.2.2.1.

It is known that in most cases the performance of CCs is measured by the Average Run Length (ARL) which is the expected waiting time until the first occurrence of an event creating an out-of-control signal. In literature there are two distinct cases for the ARL. The in-control ARL and the out-of-control ARL. The in-control ARL is the average number of plotted samples until an out-of-control signal even though the process is in-control. The out-of-control ARL is the average number of plotted samples until an out-of-control signal when the process is considered to be out-of-control.

Regarding the comparison of the various CCs, the control limits of the charts were computed for achieving an in-control ARL equal to 200. Also one scenario has been taken into account for the number of variables ($p = 2$) because for more scenarios computational difficulties were encountered. Furthermore, scenarios for different sample sizes have been considered with $n = 5, 10, 20$. In addition, the scenarios were made for simulating a process with mean vector $\boldsymbol{\mu} = (0, 0)$ and variance-covariance matrix $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \sigma_{11}\rho\sigma_{22} \\ \sigma_{11}\rho\sigma_{22} & 1 \end{bmatrix}$ with $\rho = -0.75, -0.3, 0, 0.5$ and 0.75. Finally, the out-of-control ARL are compared for a shift in one or two variances and the shifts had the form $k\sigma^2$ with $k = 1, 1.1, 1.2, \ldots, 2$.

In the diagrams it is plotted the volume of the shift and the $\ln(ARL)$ for a better presentation. The number of simulations were set to 10000.

5.2.1 The Bivariate Case ($p = 2$)

In this section the various figures for the different scenarios will be presented. As a reminder, the graphs were made with the X-axis representing the volume of the shift in the variance for a CC with fixed control limits for achieving $ARL_0 = 200$ and the Y-axis representing the $\ln(ARL)$.

5.2.1.1 Scenario with $\rho = -0.75$

The first scenario assumes that the correlation between the variables is -0.75 meaning that the variables have a strong negative correlation.

FIGURE 5.1: For n=5 and Shift in One Variance

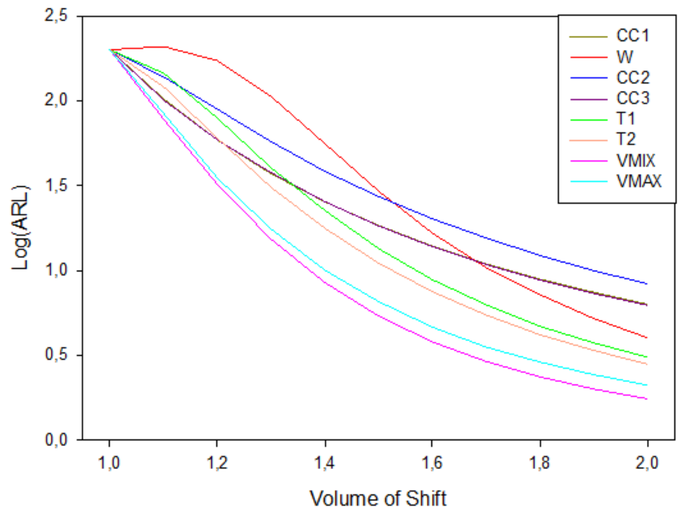
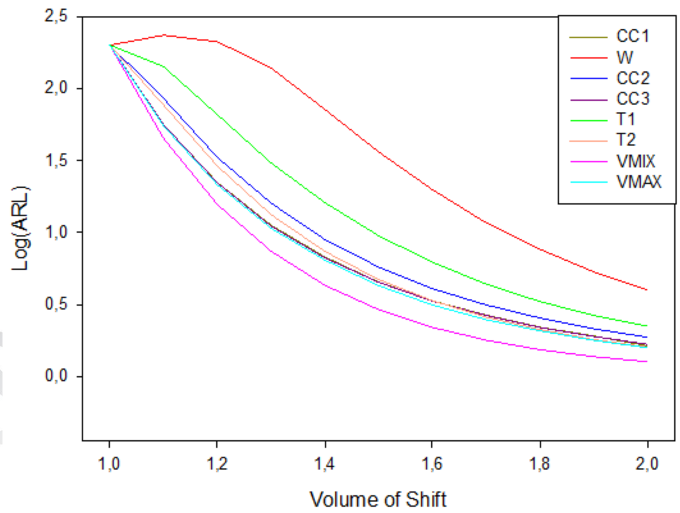


FIGURE 5.2: For n=5 and Shift in Two Variances



For a sample size of 10, the following can be derived:

FIGURE 5.3: For n=10 and Shift in One Variance

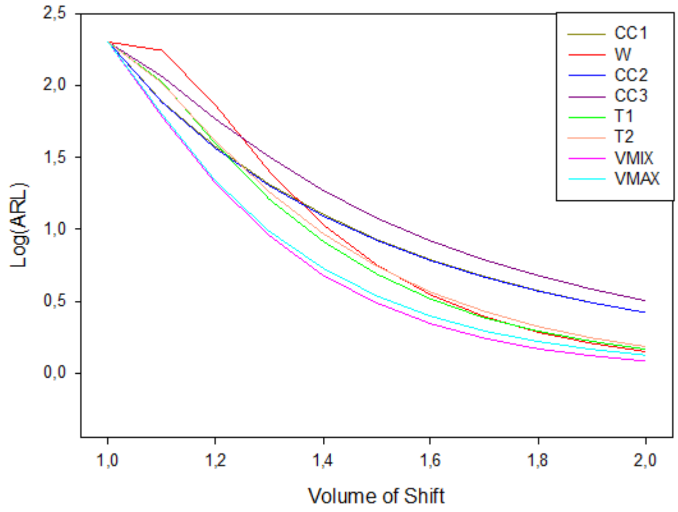
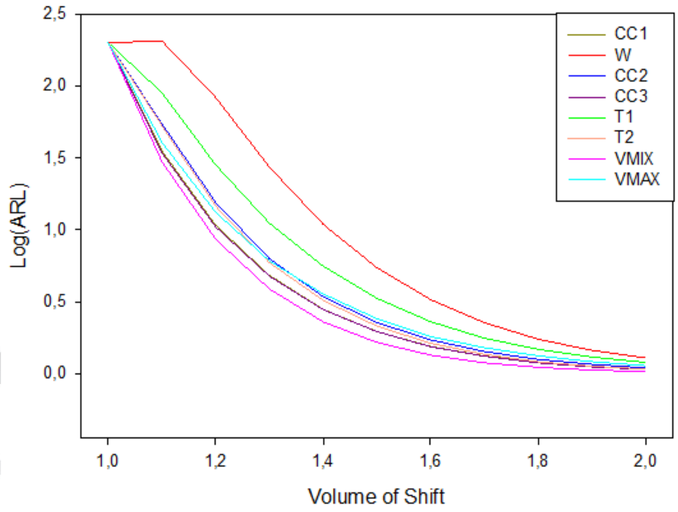


FIGURE 5.4: For n=10 and Shift in Two Variances



By having a sample size of 20 the following charts have been constructed:

FIGURE 5.5: For n=20 and Shift in One Variance

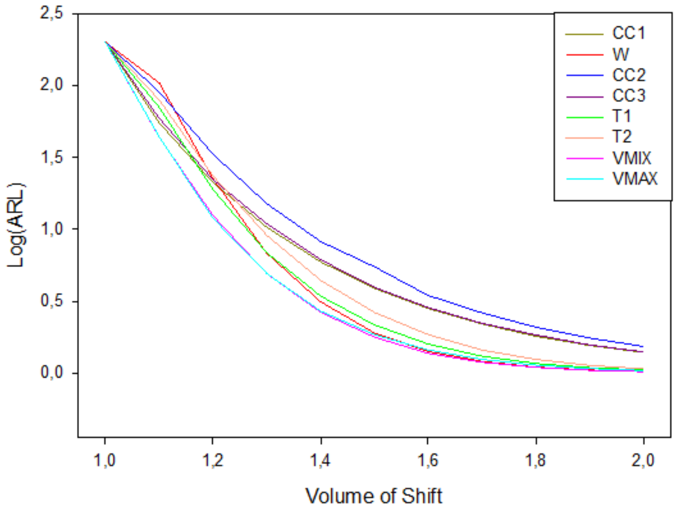
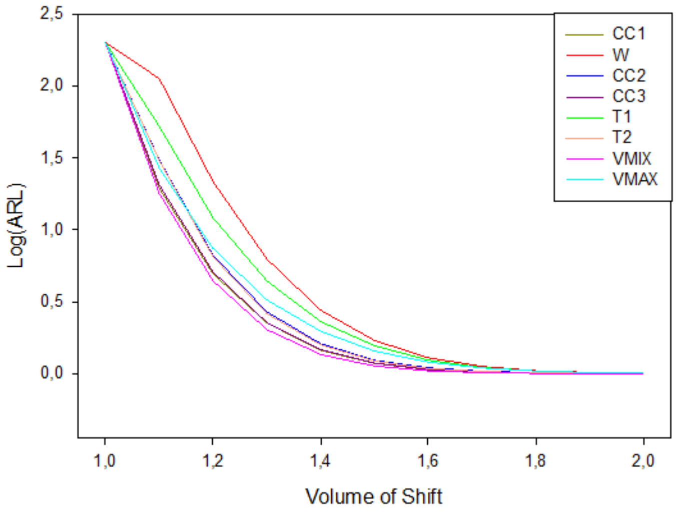


FIGURE 5.6: For n=20 and Shift in Two Variances



5.2.1.2 Scenario with $\rho = -0.30$

The second scenario assumes that the correlation between the variables is -0.30 meaning that the variables have a moderate negative correlation.

Figure 5.7: For n=5 and Shift in One Variance

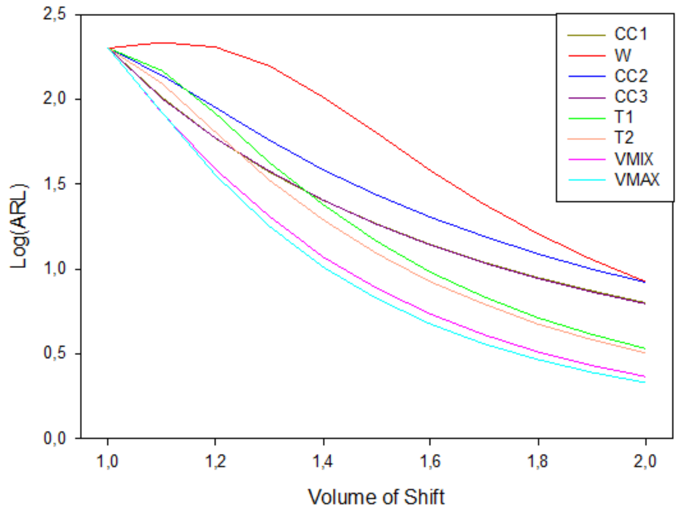
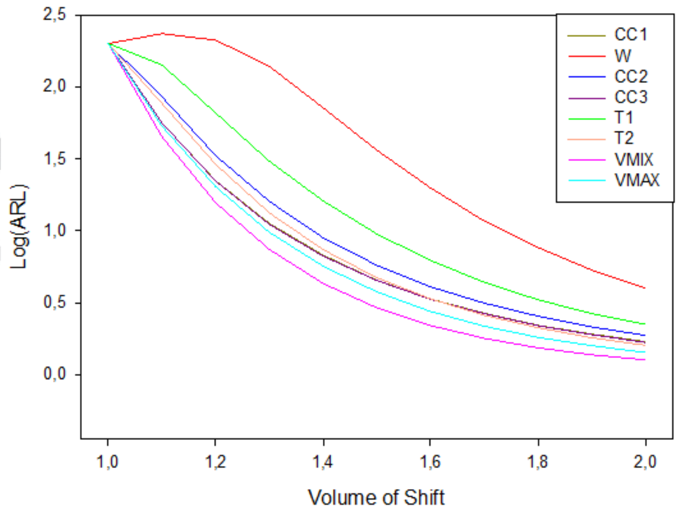


Figure 5.8: For n=5 and Shift in Two Variances



For a sample size of 10, the following can be derived:

FIGURE 5.9: For n=10 and Shift in One Variance

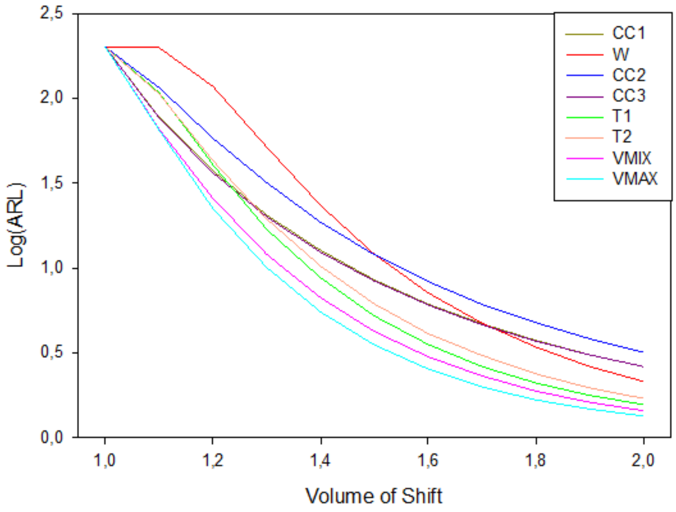
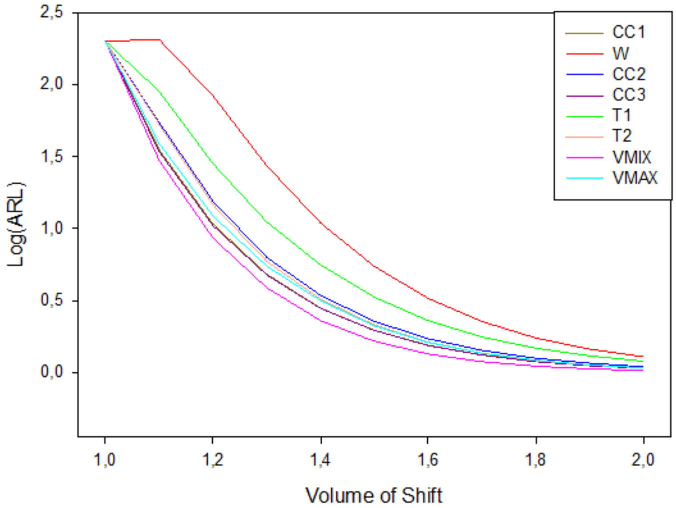


FIGURE 5.10: For n=10 and Shift in Two Variances



By having a sample size of 20 the following charts have been constructed:

FIGURE 5.11: For n=20 and Shift in One Variance

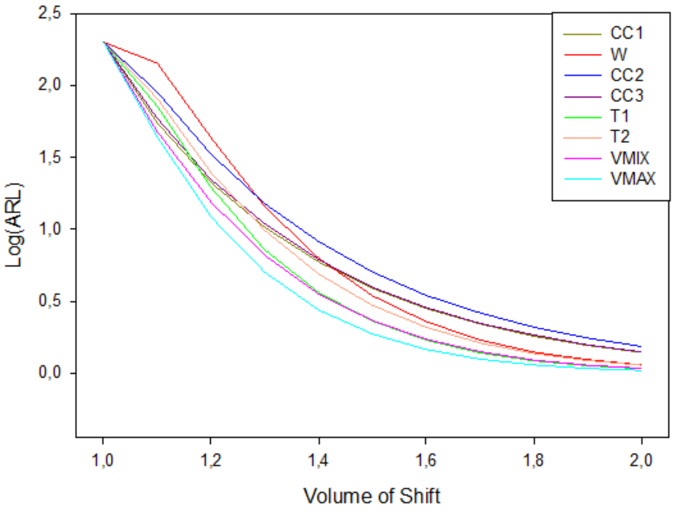
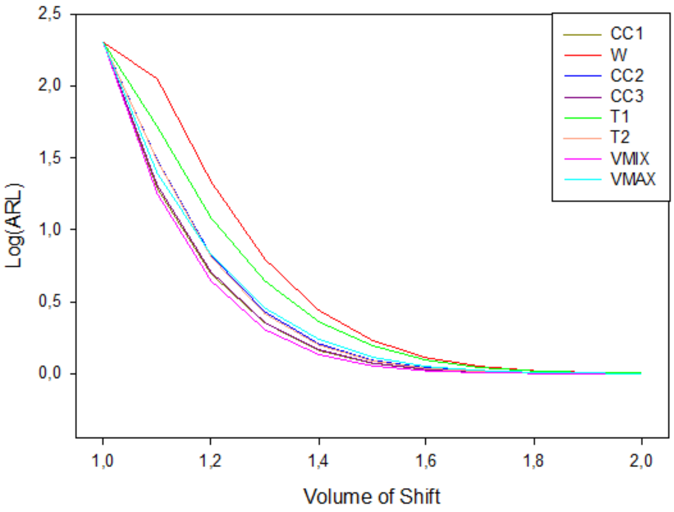


FIGURE 5.12: For n=20 and Shift in Two Variances



5.2.1.3 Scenario with $\rho = 0$

The third scenario assumes that the correlation between the variables is 0 meaning that the variables are uncorrelated.

FIGURE 5.13: For n=5 and Shift in One Variance

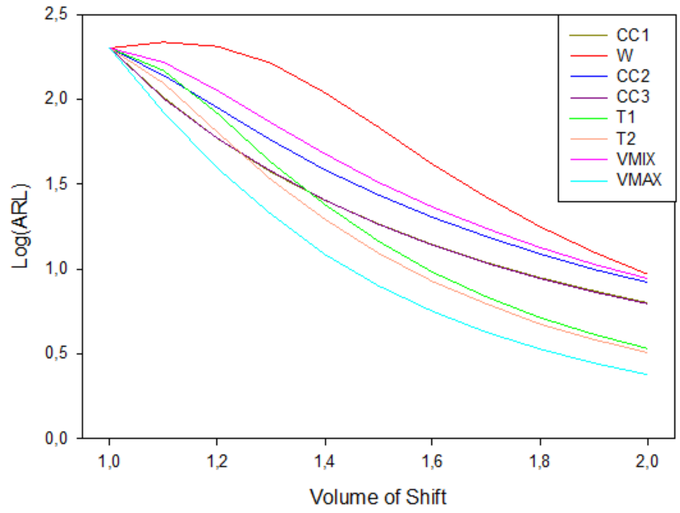
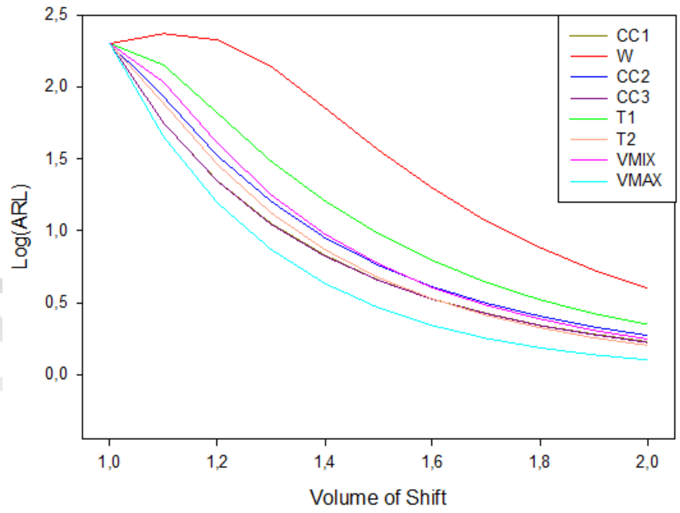


FIGURE 5.14: For n=5 and Shift in Two Variances



For a sample size of 10, the following can be derived:

Figure 5.15: For n=10 and Shift in One Variance

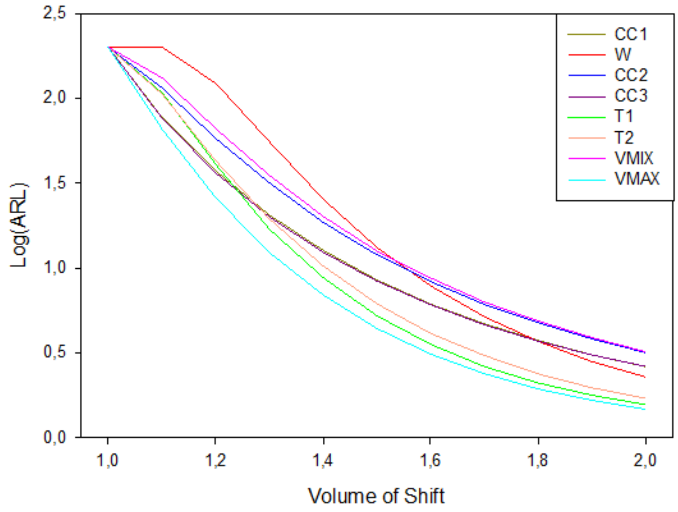
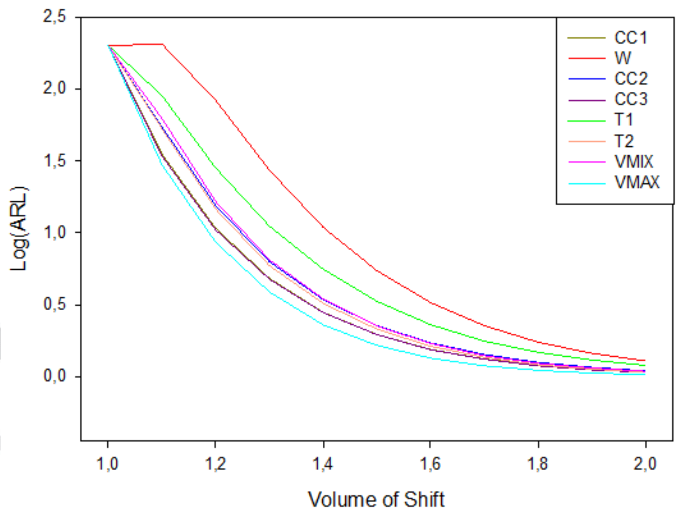


Figure 5.16: For n=10 and Shift in Two Variances



By having a sample size of 20 the following charts have been constructed:

FIGURE 5.17: For n=20 and Shift in One Variance

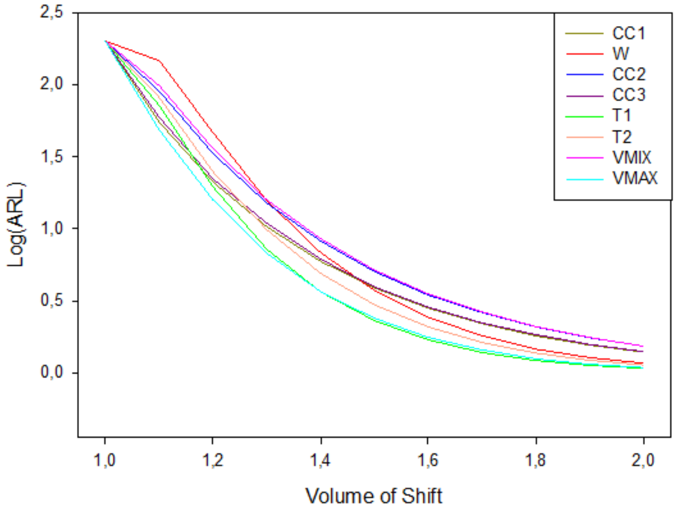
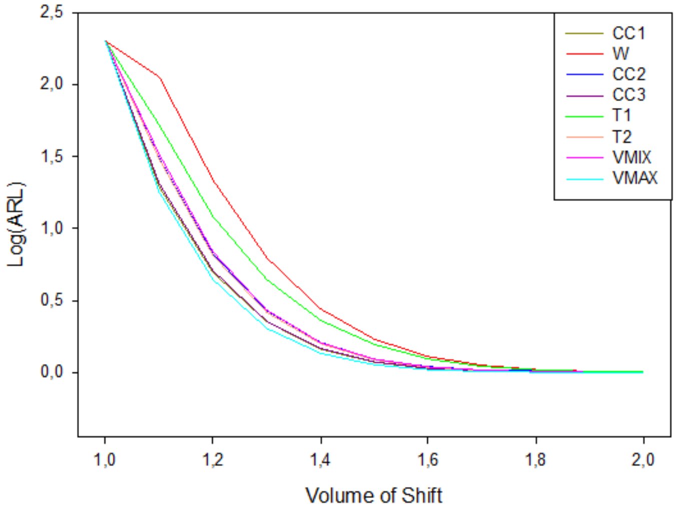


FIGURE 5.18: For n=20 and Shift in Two Variances



5.2.1.4 Scenario with $\rho = 0.5$

The forth scenario assumes that the correlation between the variables is 0.5 meaning that the variables have a moderate positive correlation.

FIGURE 5.19: For n=5 and Shift in One Variance

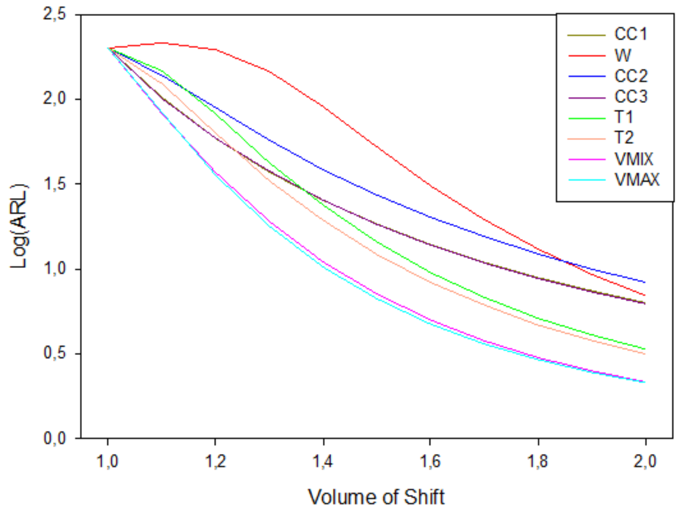
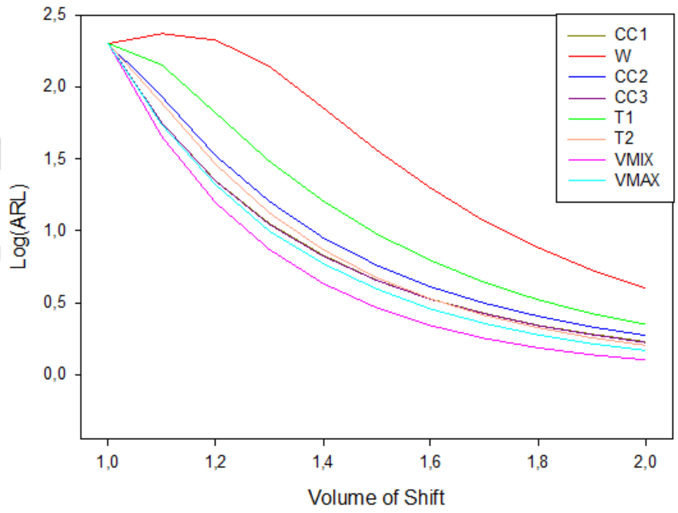


FIGURE 5.20: For n=5 and Shift in Two Variances



For a sample size of 10, the following can be derived:

FIGURE 5.21: For n=10 and Shift in One Variance

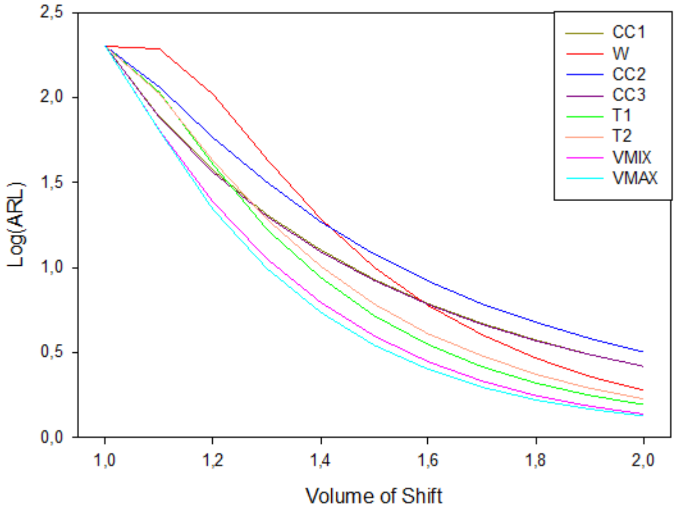
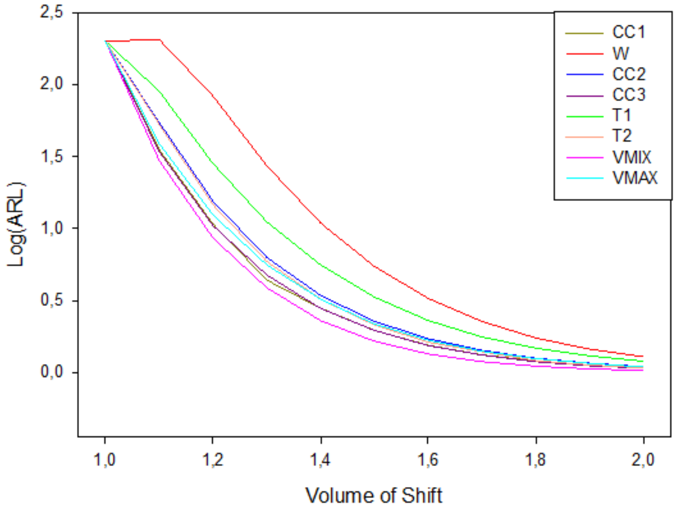


FIGURE 5.22: For n=10 and Shift in Two Variances



By having a sample size of 20 the following charts have been constructed:

FIGURE 5.23: For n=20 and Shift in One Variance

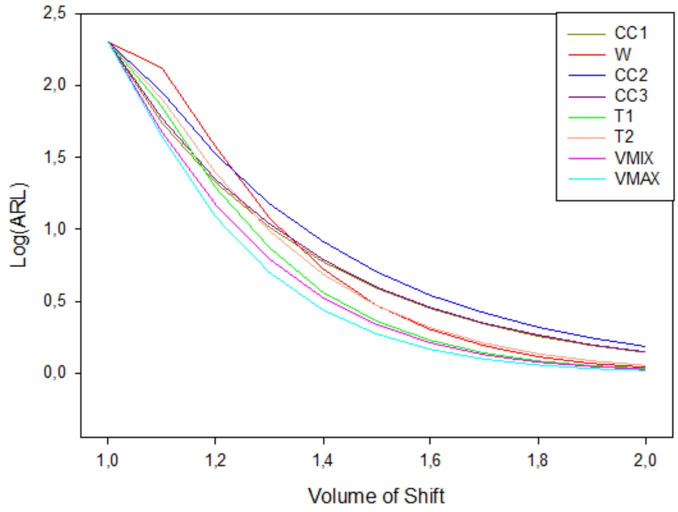
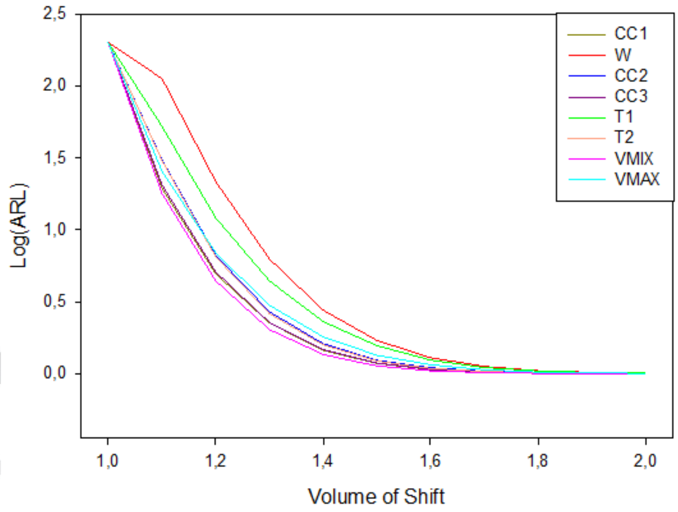


FIGURE 5.24: For n=20 and Shift in Two Variances



5.2.1.5 Scenario with $\rho = 0.75$

The final scenario assumes that the correlation between the variables is 0.75 meaning that the variables have a highly positive correlation.

FIGURE 5.25: For n=5 and Shift in One Variance

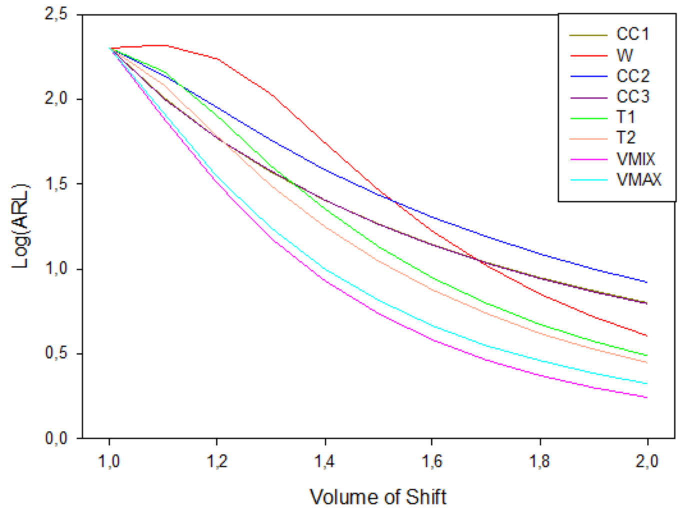
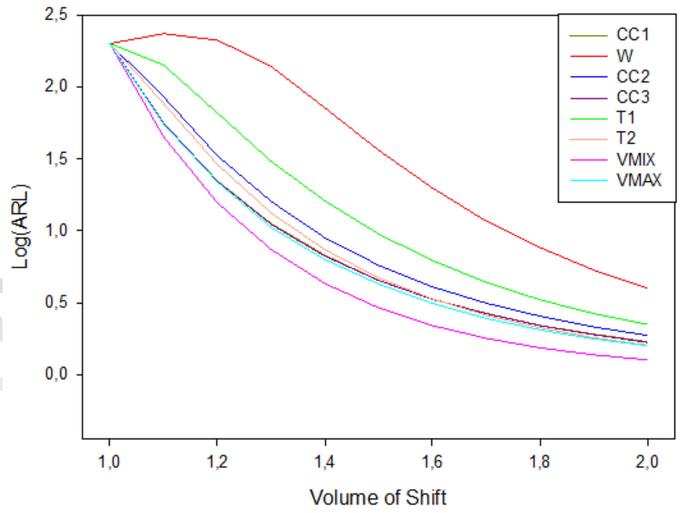


FIGURE 5.26: For n=5 and Shift in Two Variances



For a sample size of 10, the following can be derived:

FIGURE 5.27: For n=10 and Shift in One Variance

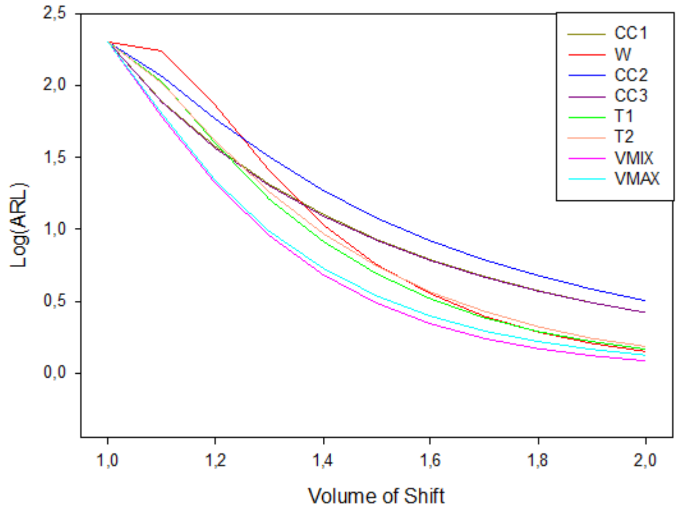
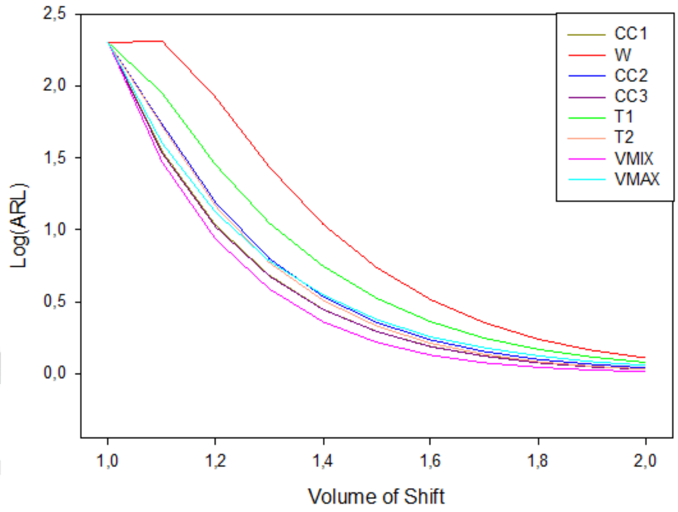


FIGURE 5.28: For n=10 and Shift in Two Variances



By having a sample size of 20 the following charts have been constructed:

FIGURE 5.29: For n=20 and Shift in One Variance

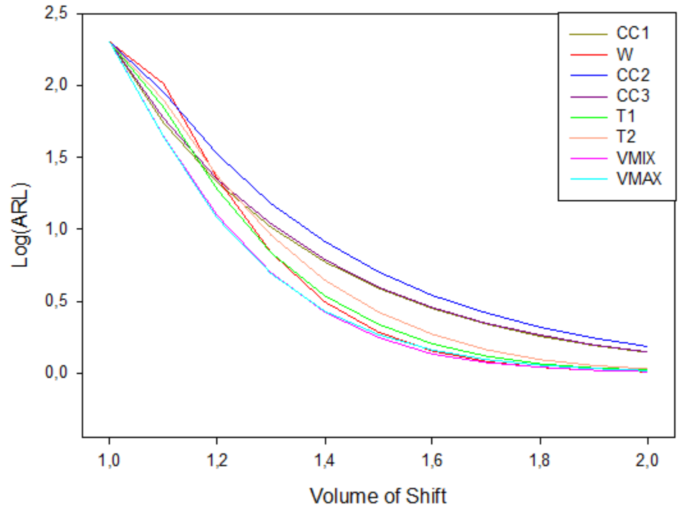
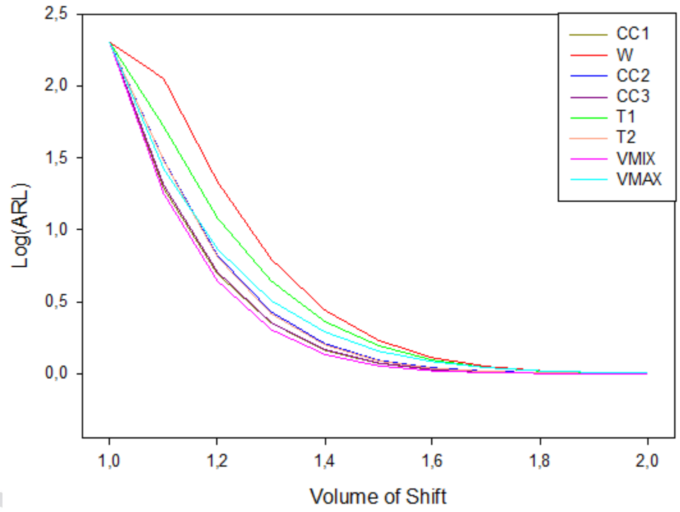


FIGURE 5.30: For n=20 and Shift in Two Variances



5.2.2 Comparing Bivariate Control Charts

In this section the comparison of the charts in the different scenarios will be presented.

For the first scenario with correlation between the variables equal to -0.75 it seems that the VMIX chart performs better regardless the sample size and the shift. If the researcher has estimated from the data that there is a high negative correlation between the two variables of interest, then the VMIX chart should be applied for monitoring the process' dispersion. It can also be seen that the VMAX chart performs best for a shift in one variable regardless the volume of the shift. More specifically as the sample size increases the VMAX chart approximates the performance of the VMIX chart. VMAX chart can also be selected for monitoring the process if the sample size is small (n=*5*) if the shift occurs in both variables. In contrary to these charts, the W chart is the worst chart for monitoring the process especially if a shift in both variables takes place. When there is a shift only in the dispersion of one variable, then the W chart is not able to detect the shift if it is of low volume. For big shifts the performance of the chart rapidly improves. Also, as the sample size increases, W chart becomes better and its performance is similar to the VMIX and VMAX chart. So the W chart should be selected for detecting a shift in one variable, for shifts bigger than $1.7\sigma^2$ if the sample size is 10 or for shifts bigger than $1.3\sigma^2$ if the sample size is 20. T2 chart should be considered for shifts in both variables regardless the sample size but preferably for shifts over $1.4\sigma^2$. The T2 chart can also be selected for shift in one variable with a small sample size because it has the third best performance recorded. CC3 and CC1 charts have identical performance and should only be considered for shift in both variables regardless the sample size. They have the second best performance for large sample sizes ($n = 10$ or 20) and the third best performance for small sample sizes ($n = 5$). T1 chart is able to detect a shift in one variable when the sample size is more than 10. Finally the CC2 chart has the most moderate performance since it does not performs best in any specific sample size.

For the second scenario with correlation between the variables equal to -0.30 it seems that the VMAX chart is the best for detecting a shift in only one variable regardless the sample size and the volume of the shift while VMIX chart outperforms for shift in both variables regardless the sample size and the volume of the shift. It is easy to say that these charts should be preferred in the appropriate situation. As a proposal it should be said that if the researcher knows that there is a moderate negative shift between the two variables, then both VMIX and VMAX charts should be used for detecting shifts (either in one or in both variables) and a signal to the process would be given when the first chart signals. In this scenario, the W chart performs best only in big sample sizes ($n = 20$) and big shift ($> 1.8\sigma^2$) in only one variable. T1 chart in this case seems to have one of the best performances when it comes to a shift in one variable regardless the sample size. For sample size equal to 5 it should be preferred for shifts over $1.4\sigma^2$, for sample size equal to 10 it should be preferred for shifts over 1.3σ and for big sample sizes ($n = 20$) for shifts over $1.2\sigma^2$. T2 chart should be considered for a shift in one variable regardless the shift but for shifts over $1.2\sigma^2$. For a shift in both variables it should be preferred if the sample size is 5 for shifts over $1.6\sigma^2$, for sample size 10 when the shift is over $1.5\sigma^2$ or for sample sizes 20 if the shift is over $1.2\sigma^2$. CC1 and CC3 should be considered only for shift in both variables when sample size is 5 and the volume of the shift is less than $1.6\sigma^2$ or for sample sizes 10 and 20 regardless the shift. Also CC1 and CC3 can be considered for a small shift ($< 1.2\sigma^2$) in one variable regardless the sample size. Again, CC2 is not exceptional in any case so it should not be considered.

For the third scenario with correlation between the variables equal to 0 it seems that VMAX chart is the best chart for detecting shifts in one variable regardless the sample size. For shift in both variables it should be considered only for sample size of 5 because it has the second best performance. VMIX chart is again the best performing chart for shifts in both variables regardless sample size and shift. VMIX can be considered for shifts in one variable for sample size less than 10 because it has the second best performance or for shifts less than $1.4\sigma^2$ if the sample size is 20. From the multivariate charts, T1 can be chosen for detecting shifts over $1.2\sigma^2$ in one variable for sample size over 10 and should not be preferred at all if the shift occurs in both variances simultaneously. In this case T2 can be considered but only if the shifts are over $1.6\sigma^2$ for sample sizes less than 10 and regardless the volume of the shifts if the sample size is 20. CC1 and CC3 can be chosen for shift in both variances regardless the sample size or in one variable when the volume of the shift is less than $1.2\sigma^2$. W and CC2 chart in this scenario are not appropriate because their performance is among the worst.

The correlation in the fourth scenario is moderately positive ($\rho = 0.5$). It seems that the VMAX chart should be chosen for a shift in one variable regardless the sample size or for shift in both variables when the sample size is small ($n = 5$) because the performance is the second best. VMIX chart performs best when the shift occurs in both variables regardless the sample size and has the second best performance for a shift in one variable. W chart though improves its performance as the volume of the shift gets bigger for one variable, can only be chosen for a big sample size ($n = 20$) and volume of the shift $> 1.6\sigma^2$. T1 chart has a really good performance when it comes to a shift in one variable regardless the sample size. It best performs for a shift over $1.4\sigma^2$ when the sample size is 5, for a shift over $1.3\sigma^2$ when the sample size is 10 and for a shift over $1.2\sigma^2$ for sample size equal to 20. T2 performs really good for shifts in both variables when the sample size is 5 and the shift is over $1.6\sigma^2$, hen the sample size is 10 and the shift is over $1.5\sigma^2$ and when the sample size is 20 and the shift is over $1.2\sigma^2$. Also it performs well for sample size of 5 and shift over $1.2\sigma^2$ in one variable. Again CC1 and CC3 perform really good in shifts in both variables regardless the sample size and for a small shift ($< 1.2\sigma^2$) in one variable regardless the sample size. Finally CC2 has not a special case in which it should be preferred.

In the final scenario the correlation was set to be 0.75 meaning a strong positive correlation. In this scenario the same results apply as in the case of $\rho = -0.75$.

5.2.3 Summary

In this chapter, the comparisons of some presented charts took place. First of all, the scenarios of the comparisons were determined which involved different sample sizes, different correlations between the variables and of course different shifts in one or both variances. For achieving this, the CCs were simulated and their control limits were computed for achieving an in-control ARL equal to 200. The comparison of the charts, showed that in general VMIX and VMAX perform beter than the other CCs. Also the CC1 and CC3 CCs perform really good for detecting a shift in both variables. T1 and T2 seem to have a good performance depending on the scenario and W chart, was really good for detecting big shifts in one variable when there is a big negative correlation involved. It should be mentioned that the simulation program used was *Wolfram Mathematica 9* and the graphs were created with *Systat SigmaPlot 12.5*.

Chapter 6

Discussion and Scope for Further Research

6.1 Discussion

In this master thesis the theory behind statistical process control was initially presented. After that, a variety of univariate, bivariate and multivariate control charts was presented which was the result of an extensive review of a great number of articles based on control charts both univariate and multivariate. Finally some specific control charts were compared for determining the most efficient control chart for any given scenario which involves a bivariate process.

Although much progress has been made in the field of statistical process control, more research must be considered for improving this area of interest. In the following section further research and proposals will be discussed.

6.2 Further Research and Proposals

As already mentioned, multivariate process control according to Jackson (1991), should provide four simple information to the researcher:

- an answer to the question: "Is the process in control?",
- an overall "Type I error",
- the relation between the variables should be taken into account and
- an answer to the question "What variable causes the problem?" if the process is out-of-control.

Regarding the first information that should be obtained from multivariate statistical process control, it should be mentioned that more research should be done for constructing additional CCs based on different statistical quantities. That will result in a bigger variety of options to the researcher with some of them more efficient than other. As a proposal to the researchers occupied in constructing CCs is to consider using the multivariate coefficient of variation as a quantity measuring the dispersion in a multivariate level. From the literature it seems that four (4) different quantities have been proposed as multivariate coefficients of variation and should be considered as a potential statistical quantity for proposing a new chart.

For the fourth information that always should be gained from monitoring a process must be said that it is relatively new in literature and only small steps have been made regarding the dispersion in a multivariate level. It is crucial to say that this area must be expanded because while the main objective of the practitioner is to monitor the process, more important is to know exactly what is wrong with the process and especially with which variable when the whole system does not perform as it should be. More weight should be given on interpreting and controlling the variability because if the practitioner manages to control the dispersion then the target of the process will not fluctuate.

While Jacksons' list is right on what information should be obtained from multivariate process control, it is probably incomplete and should be expanded by one information that should be provided to the researcher. The researcher should be in position to answer to the question "Am I using the optimal way to monitor the process?". The reason is really simple. If the best CC is not used for the scenario encountered then it is not sure for the researcher to know in any given time the process is in control (first information from Jackson) leading to ignorance on what is wrong and in which variable (fourth information from Jackson).

A large-scale research should be done with main objective to determine the best option for every scenario that can be encountered. This painful study should include scenarios for different number of variables, different sample sizes, different correlations, different shifts in variances, meaning not only shift in one variance or in two simultaneously but also in more with not the same volume of shift. Also scenarios with different in control ARL should be included. The work done in this thesis will continue and will revolve on this job for CCs that monitor the dispersion of a multivariate process.

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