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OPTIMAL CAPITAL ALLOCATION BETWEEN INSURANCE COMPANIES

By

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ΠΑΝΕΠΙΣΤΗΜΙΟ ΠΕΙΡΑΙΩΣ

ΤΜΗΜΑ ΣΤΑΤΙΣΤΙΚΗΣ ΚΑΙ ΑΣΦΑΛΙΣΤΙΚΗΣ ΕΠΙΣΤΗΜΗΣ

ΜΕΤΑΠΤΥΧΙΑΚΟ ΠΡΟΓΡΑΜΜΑ ΣΠΟΥΔΩΝ ΣΤΗΝ ΑΝΑΛΟΓΙΣΤΙΚΗ ΕΠΙΣΤΗΜΗ ΚΑΙ ΔΙΟΙΚΗΤΙΚΗ ΚΙΝΔΥΝΟΥ

ΒΕΛΤΙΣΤΕΣ ΚΑΤΑΝΟΜΕΣ ΚΕΦΑΛΑΙΩΝ ΜΕΤΑΞΥ ΑΣΦΑΛΙΣΤΙΚΩΝ ΕΤΑΙΡΙΩΝ

Γεώργιος Π. Κασαπίδης

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Dedicated to the first mathematician
that I met, my Godfather

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Optimal capital allocation between insurance companies

Keywords: risk measures, robust, coherent risk measures, value – at – risk, expected shortfall, truncated tail – value – at – risk, quota – share, stop – loss, premium principles, optimal reinsurance, risk transfers, reinsurance arrangements, risk adjusted value, expected policyholder deficit, single claims liabilities.

Abstract

The level of capital to be held by an insurance company is essential and should be defined, so that with high probability the company can meet its obligations. Thus the determination of a target level of capital under an appropriate risk measure is essential. In this thesis we study the effect of risk transfer between insurance companies on this target level of required capital. This thesis is based on two papers, aiming to present optimal capital allocation – risk transfers methods, between insurance companies, under certain capital requirements – thus risk measures. Further based on a certain theory and on a stochastic model of single claims development, we study the impact of risk transfer arrangements between insurance companies numerically and hopefully we present interesting conclusions about optimal risk transfer in this model environment.

Βέλτιστες κατανομές κεφαλαίων μεταξύ ασφαλιστικών εταιριών.

Λέξεις κλειδιά: risk measures, robust, coherent risk measures, value – at – risk, expected shortfall, truncated tail – value – at – risk, quota – share, stop – loss, premium principles, optimal reinsurance, risk transfers, reinsurance arrangements, risk adjusted value, expected policyholder deficit, single claims liabilities.

Περίληψη.

Το ύψος του κεφαλαίου που απαιτείται να κρατήσει μια ασφαλιστική εταιρία είναι αναγκαίο να καθοριστεί, έτσι ώστε με μία μεγάλη πιθανότητα η εταιρία να είναι ικανή να ανταποκριθεί στις υποχρεώσεις της. Έτσι ο καθορισμός του αναγκαίου κεφαλαίου, στο πλαίσιο αξιολόγησης μέσω ενός κατάλληλου μέτρου κινδύνου είναι αναγκαίος. Σε αυτή την εργασία μελετούμε την επίδραση της μεταφοράς κινδύνων μεταξύ ασφαλιστικών εταιριών, με βάση τα απαραίτητα και καθορισμένα ύψη κεφαλαίων, που απαιτούνται για την βιωσιμότητα των εταιριών. Η εργασία αυτή βασίζεται σε δύο επιστημονικές εργασίες, με στόχο να παρουσιάσουμε βέλτιστες κατανομές κεφαλαίων – δηλαδή μεθόδους μεταφοράς των κινδύνων, μεταξύ ασφαλιστικών εταιριών, υπό ορισμένες κεφαλαιακές απαιτήσεις – μέσω ορισμένων μέτρων κινδύνων. Περαιτέρω με βάση τη συγκεκριμένη θεωρία και ένα στοχαστικό μοντέλο εμφάνισης ζημιών, μελετούμε την επίδραση των μεταφορών των κινδύνων μεταξύ των ασφαλιστικών εταιριών αριθμητικά και ευελπιστούμε να παρουσιάσουμε ενδιαφέροντα συμπεράσματα, όσον αφορά τη βέλτιστη μεταφορά του κινδύνου σε αυτό το μοντέλο.

Contents

Chapter 1

Introduction.

1.1 Background.

An insurance group must take the right decisions, when it's about risk transfers. Risk transfers can be organized either between different entities of the group or between the insurance group and a single or multiple reinsurers. We could say that the allocation of risks within the insurance group, but between different entities that are under different regulation, is a form of reinsurance.

Reinsurance is insurance for insurers. In reinsurance contracts the insurer transfers part of his risks to the reinsurer. There is a trade – off between the part of the risk, which is retained by the insurer and the premium which is paid to the reinsurer. An insurance company needs to have profit and loss volatility and the needed capital requirements. Thus an insurance company uses risk measures in order to have diversification of the risks and so a balanced portfolio. As well, an insurance company needs to be able for all scenarios, to meet regulators capital requirements. Thus an insurance company might decide to optimize capital needs and earnings volatility via risk transfer arrangements.

So due to the fact that insurance portfolios are mostly unbalanced, with reinsurance the claims experience of the retained insurance portfolio can be organized in such a way that a sharp and short – term rise in loss incidence can be dampened. While the insurer's portfolio becoming more balanced, the reinsurer's portfolio becoming more unbalanced, as he accepts peaks. To achieve balance to his portfolio and thus business success the Reinsurer uses tools, such as the underwriting policy, to establish a large portfolio which is evenly spread, in terms of markets, clients, lines of business, types of risk and types of treaty.

In the following paragraph we denote the types and the arrangements of reinsurance.

1.2 Types of reinsurance.

Reinsurance can be classified, according to the number of risks that are insured. It can be classified as a facultative reinsurance or treaty reinsurance.

The facultative reinsurance is provided in respect of an individual risk, which the primary insurer either cannot or does not want to bear in full itself. Usually it used where only a few risks need reinsurance cover or where these do not fit or only partially fit into treaty reinsurance. Usually under facultative reinsurance are insured the peaks of an insurance portfolio, peak risks in the market or unusual risks to an insurance company.

The treaty reinsurance can be obligatory or automatic reinsurance. Is being used where is a large enough portfolio of risks, so that there is a worthwhile saving in terms of work through automatic reinsurance. Furthermore the potential reinsurance portfolio should be adequately balanced so that an individual risk cannot substantially affect the overall result.

They are two types of reinsurance's arrangements, the proportional and the non – proportional, which can be either under facultative or treaty reinsurance.

In proportional reinsurance the reinsurer accepts a certain percentage of a risk or the insurance contracts ceded to the treaty. In this way the reinsurer participates to the same percentage share in the liabilities, premiums and claims that arise under a specific insurance contract. Proportional treaty reinsurance is subject to a treaty limit, to ensure that the reinsurer does not bear an unlimited amount of liability and loss. This means that, if part of insurance exceeds the treaty limit, this part is not covered under the reinsurance treaty, thus it must be reinsured elsewhere.

In non – proportional reinsurance the reinsurer's participation refers only to the loss affecting a single risk (facultative reinsurance) or the risks covered under the reinsurance treaty. Under the non – proportional reinsurance, the reinsurer does not accept a share of the liabilities and consequently the same share of premiums and claims. Instead the reinsurer participate only individual losses or loss amounts which exceed an agreed limit. This limit is known as deductible or loss retention. The reinsurer accepts the portion of the loss in excess of the deductible, this portion in turn being limited by the reinsurance cover, also known as the cover amount.

So adding the deductible to the reinsurance cover gives the upper limit of cover under the reinsurance treaty. In this arrangement the price, that the primary insurer has to pay for the reinsurance cover (the reinsurance premium), must be calculated separately and agreed between the parties of the treaty.

They are three treaty types in proportional reinsurance:

- Quota share reinsurance. Is the treaty where all the insurance contracts have to be ceded to the treaty if they meet the terms and conditions of the treaty or if they are part of the insurance portfolio defined in the treaty. The allocation to the treaty in terms of amount, is restricted by the treaty limit. If the liability (sum insured) under insurance exceeds the treaty limit, then this excess amount is not reinsured under the quota share treaty and it must be reinsured facultatively. The liabilities, premiums and claims are apportioned between the reinsured and the reinsurer within the treaty limit based on the percentage set out in the treaty.

- Surplus reinsurance. This treaty is structured in such a way that the reinsured retains in full those insurance contracts for which the liability does not exceed an agreed amount. These contracts are not ceded to the reinsurance treaty. The reinsured bears all the losses from these reinsured contracts and receive the entire premium. The insurance contracts for which the liability is exceeds the retention line, must be ceded to the treaty and the amount which can be ceded is limited by the treaty limit. With the surplus treaty there is no fixed – percentage formula for apportioning the shares between the reinsured and the reinsurer, so the reinsurer's share must be calculated separately for each insurance contract. Once the ratio between the retention and the allocation to the treaty has been determined for an insurance contract, the reinsured and reinsurer will participate to the same ratio in the liabilities, premiums and claims under this contract.

- Facultative – obligatory reinsurance. The reinsured is free to choose whether it wishes to cede an insurance contract to the treaty and if so, how much up to the treaty's limit. The allocation to the treaty also constitutes the share of the reinsurer. The reinsurer is obliged to accept the portion allocated to the treaty and for each individual allocation, also participates with an identical percentage in the liabilities, premiums and claims.

In non – proportional reinsurance they are also three treaty types:

- Excess of loss per risk (XL/R) reinsurance. An individual loss on an insured risk triggers the reinsurance cover if it exceeds the deductible.
- Excess of loss per event reinsurance. The reinsurance cover relates to the overall claims in an insurance portfolio caused by one and the same event. All the losses caused by the event are added to form an overall claim amount.

- Stop loss reinsurance. The reinsurance cover applies to the total of all losses per unit of time. The reinsurer provides cover for the case that the total of the losses within a portfolio in the course of a year exceeds a certain amount. The reinsurance cover is not a fixed amount, but a ratio of losses to premium income.

1.3 Mathematical background.

1.3.1 Risk measures.

In this section we review different methods to measure risk. We present risk measures such as the value at risk (VaR), Expected Shortfall (ES), Truncated Tail Value at Risk ($TrTVaR$) and we examine their properties, such as the translation invariance, subadditivity, positive homogeneity, monotonicity and relevance. If one of the above risk measures satisfies all of the previous mentioned properties, then it is considered to be a coherent risk measure. In general a risk measure is a function that assigns a non – negative real number to a risk. Thus a risk measure quantifies the danger of X. So small values produced by a risk measure, tell us that the danger produced by X, may cause a small amount of losses, and thus is relatively safe. Large values tell us that the danger produced by X, may cause a large amount, thus likely if happens our position will be riskiness. These values are risk capital of a portfolio, the extra amount of capital that we should keep us a buffer to a portfolio, to minimize the possibility to become insolvent in the case of facing a big loss.

Also premium principles are examples of possible risk measures. If we suppose that a portfolio of an insurance company is exposed to a possible loss X, a premium calculation P gives us the minimum amount $P(X)$ that the re/insurer should charge the insured for taking this risk. Some examples of insurance premium principles are the followings:

i. Expected Value Principle : $P(Z) = (1+\theta)E(Z),\ for\ \theta > 0$
ii. Standard Deviation Principle : $P(Z) = E(Z) + \beta\sqrt{V(Z)},\ for\ \beta > 0$
iii. Mixed Principle : $P(Z) = E(Z) + \beta V(Z)/E(Z),\ for\ \beta > 0$
iv. Mean Value Principle : $P(Z) = \sqrt{E(Z^2)} = \sqrt{\left(E(Z)\right)^2 + V(Z)}$
v. Wang's Principle : $P(Z) = \int_0^\infty [\Pr(Z \geq t)]^p, for\ 0 < p < 1$
vi. Variance Principle : $P(Z) = E(Z) + \beta V(Z), for\ \beta > 0$

vii. Covariance Principle : $P(Z) = E(Z) + 2\beta V(Z) - \beta CoV(Z,S), for\ \beta > 0\ and\ S\ a\ random\ variable$

viii. Exponential Principle : $P(Z) = \frac{1}{\beta}\log\big(E\big(e^{\beta Z} \big)\big), for\ \beta > 0$

ix. Zero Utility Principle : $P(Z)\ such\ that\ U(w_0) = E(U(w_0 + P(Z) - Z))$

Where Z a random variable that represents the amount of the claims paid by the reinsurer, $U(w_0)$ a utility function of the reinsurer's wealth, $U'(w_0) > 0$ and w_0 is the reinsurer's initial wealth.

Definition 1.1. A risk measure is a functional ρ mapping a risk X to a non – negative real number ρ(X), representing the extra cash which has to be added to X to make it acceptable.

Definition 1.2. A risk measure ρ is said to be coherent when it satisfies the following axioms:

- Translation Invariance : $\rho(X + c) = \rho(X) + c, \forall\ random\ variable\ X$ and for any scalar $c \epsilon \mathbb{R}$
- Subadditivity : $\rho(X + Y) \leq \rho(X) + \ \rho(Y), \forall\ random\ variable\ X, Y$
- Positive Homogeneity : $\rho(cZ) = c\rho(Z), \forall\ random\ variable\ X, Y$ and for any $c \geq 0$, where c is a constant.
- Monotonicity : $\forall\ random\ variable\ X, Y$ with $X \leq Y$, we have $\rho(X) \leq \ \rho(Y)$

Each one of the above properties has a meaning. The translation invariance means that if there is any increase in the liability by a deterministic amount c of a holder's risky position, then it should lead to a same increase in the capital. By the subadditivity we understand if a risk can be reduced by diversification. Positive homogeneity is often associated with independence with respect to the monetary unit used and last, the monotonicity tells us that the capital that is needed as a cushion against the loss X is always smaller than the one is needed for the loss Y, where always Y is greater than X.

These four desirable properties for the coherence of risk measures have been presented and justified first from Artzner et al. (1999). Later we will notice that Expected Shortfall is a coherent risk measure since it justifies all the above properties, but Value at Risk and the Truncated Tail Value at Risk are not, because the subadditivity property is violated.

It is useful to denote since VaR is a α – quantile risk measure the following definition about quantiles as Artzner et al. (1999) presented it.

Definition 1.3. Quantiles. Given $\alpha \in (0 , 1)$ the number q is an α – quantile of the random variable X under the probability distribution P if one of the three equivalent properties below is satisfied:

I. $P[X \leq q] \geq a \geq P[X < q]$
II. $P[X \leq q] \geq a \text{ and } P[X < q] \geq 1 - a$
III. $F_X(q) \geq a \text{ and } F_X(q -)$, with $F_X(q -) = lim_{x \to q, x<q}(F_X(x))$, where F_X is the cumulative distribution function of X.

Definition 1.4. The VaR of a generic loss variable Z at a confidence level $\alpha, VaR_a(Z)$ represents the minimum amount of capital that makes an insurance company to be solvent at least a% of the time. The mathematical formulation is given by

$$VaR_a(Z) = \inf\{z \geq z_0 : \Pr(Z \leq z) \geq \alpha\}$$

, where $z_0 = \sup\{z \in \Re : \Pr(Z \leq z) = 0\}$ represents the left end point of the distribution of Z.

Value at Risk represents the minimum amount of capital that will not exceed by the loss X with probability α. Has been criticized for its incomplete allowance for the risk of extreme events beyond confidence level α , which also leads to a violation of the required subadditivity property, so to be coherent. We can see that under VaR the risk of a portfolio can be larger than the sum of the single risks that are the components of it. Moreover VaR does not take into account the severity of an incurred damage event. Unrecognized violations of VaR subadditivity can have serious consequences for risk models. First they can provide a false sense of security, so that a financial institution may not be adequately hedged. Second it can lead a financial institution to make a suboptimal investment choice, if VaR or a change in VaR, is used for identifying the risk in alternative investment choices. In general VaR does not behave nicely with respect to the addition of risks, even independent ones, thereby creating severe aggregation problems and the use of VaR does not encourage and indeed, sometimes prohibits diversification because VaR does not take into account the economic consequences of the events, the probabilities of which it controls (see Coherent Measures of Risk, Artzner et al. 1999)

The Expected Shortfall (ES) has been proposed (Artzner et al, 1999) as an alternative and more realistic than VaR, risk measure. While VaR focuses on a particular point of loss distribution, the ES at confidence level, evaluates the expected loss amount incurred under the worst $100(1-a)\%$ loss scenarios of X. Expected Shortfall is more sensitive to the shape of the loss distribution in the tail of the distribution and a coherent risk measure, since it satisfies all the needed for the coherence properties.

Definition 1.5.

The mathematical formulation for the Expected Shortfall is:

$$ES_a(Z) = \frac{1}{1-a}\int_a^1 VaR_s(Z)ds = VaR_a(Z) + \frac{1}{1-a}E\big(Z - VaR_a(Z)\big)_+$$

, where $(z)_+ = \max\{z, 0\}$.

Clearly in general $VaR_a(Z) \leq ES_a(Z)$.

At this point we need to speak about robust risk measures, so to connect the above mentioned theory about risk measures with the following. If a risk measure is able to accept any possible lack of model's information and it is impassible to any possible small changes in the data, then this risk measure is a robust one.

So a more robust and sensitive risk measure was introduced by Cont et al. (2010) and Kou et al. (2011) as an alternative to the ES risk measure and named as the Truncated Tail Value at Risk ($TrTVaR$).

The truncated tail value at risk is defined to be the average of VaR levels across a range of loss probabilities. This risk measure unites the ES and VaR and as a compound of them, is the result that part of the tail behavior is measured by $TrTVaR$, so it has similar properties to VaR. $TrTVaR$ is a robust and a no coherent risk measure and unlike VaR is more tail sensitive. It can accommodate model misspecification and it is insensitive to small changes in the data. It is defined as follows:

Definition 1.6. The Truncated Tail Value at Risk is the average of VaR levels across a range of loss probabilities. The mathematical formulation for it, is

$$TrTVaR_{\alpha_1,\alpha_2}(Z) = \frac{1}{\alpha_1 - \alpha_2}\int_{\alpha_2}^{\alpha_1} VaR_s(Z)\,ds = \frac{(1-a_1)ES_{\alpha_1}(Z) - (1-a_2)ES_{\alpha_2}(Z)}{\alpha_2 - \alpha_1}$$

Where the average of VaR levels across a range of loss probabilities is the

$$\phi(u) = \frac{1}{a_1 - a_2} I_{a_1 < u < a_2}$$

and since the ϕ is not decreasing we see now why the $TrTVaR$ is a no coherent risk measure.

1.3.2 Mathematical introduction of risk transfer's problem, within a reinsurance contract.

In a reinsurance contract there is always someone who wants to mitigate part of the risk that his portfolio might has, by transferring a portion of it and there is another who is willing to accept this risk, with the proper reward. So there is the primary insurer (the cedent) and the reinsurer.

Let $X \geq 0$ be the total loss amount incurred during the duration of an insurance contract. We will denote as $F(\cdot)$ the distribution function and as $\bar{F}(\cdot) = 1 - F(\cdot)$ the survival function and the right end – point $x_F := \inf\{z \epsilon \Re : F\ (z) = 1\}$ of the loss distribution can be either finite or infinite.

As we previously said $X = I\ [\ X\ ] + R\ [\ X\ ]$ is the total loss amount while the insurance contract is in duration. Within this insurance contract $I\ [\ X\ ]$ is the amount that the insurer will pay and $R\ [\ X\ ]$ is the amount that the reinsurer has agreed to pay under a certain reinsurance arrangement, when the entire loss exceeds the insurer's amount. We denote as $P\ (R\ [\ X\ ])$ the reinsurer's premium, an extra cost for the insurer since it is a fare for the risk that will be transfer to the reinsurer. We assume that the premium satisfies $P\ (R\ [\ X\ ]) \geq E\ (R\ [\ X\ ])$, otherwise this would lead the reinsurer to insolvency. The reinsurer's premium that is used, is the expected value principle and so it has the format of $P\ (R\ [\ X\ ]) = (\ 1 + \rho\ )E(\ R[\ X\ ]\ )$, where $\rho > 0$ is the security loading factor. Other premium principles could also be used, but principles that are functions of the risk quantile.

In the presence of reinsurance the insurer is now concerned with the risk exposure $L(R\,[\,X\,]) = I\,[\,X\,] + P(R\,[\,X\,])$, that represents the total insurer's loss in the presence of reinsurance.

This leads to our problem, by what way we will be able to minimize the $\varphi_I(\,L(R\,[\,X\,])\,)$ and the proper way to identify the optimal arrangement for the insurer regarding which is the best possible situation to face risk. Where $\varphi_I$represent a measure of the risk taken by the insurer.

From the property of the translation invariance this can leads as to the minimization of the following equation:

$$\varphi_I(\,L(R\,[\,X\,]\,) = \varphi_I(\,I\,[\,X\,]\,) + \,P(R\,[\,X\,]\,)$$

Where I and R are Lipschitz functions meaning that $|\,I(y) - I(x)| \leq |y - x|$ and $|\,R(y) - R(x)| \leq |y - x|$, $\forall x, y \geq 0$.

A proper set of feasible contracts is given by

$$F = \{\,R(\,\cdot\,) : I(\,x\,) = x - R(\,x\,)\;and\;R(\,x\,), are\;non\;decreasing\;functions\}$$

Also we need to mention the definition of co – monotonicity which Dhaene et al. (2002) first mentions about it. They defined comonotonicity of a set of n – vectors in $\mathbb{R}^n$. They denote a n – vector, for example ($x_1, x_2,\ \ .\ .\ .\ , x_n$) as $\underline{x}$ and for two vectors $\underline{x}$ and $\underline{y}$ that fulfill the notation $\underline{x} \leq\ \underline{y}$, then it is true that $x_i \leq\ y_i, \forall\ i \in \mathbb{N}$.

Definition 1.7. The set $A \subseteq \mathbb{R}^n$ is said to be comonotonic if for any $\underline{x}, \underline{y} \in A$, either $\underline{x} \leq\ \underline{y}$ or $\underline{y} \leq\ \underline{x}$ holds.

Since $I[\,X\,]$ and $R[\,X\,]$ are comonotone (Dhaene et al. 2002), our optimization problem transforms to the following :

$$min_{R \in F}\{ \varphi_I(X) - \varphi_I(R[X]) + P(R[X])\}$$

while

$$\varphi_I(X) = \varphi_I(I[X]) + \varphi_I(R[X])$$

1.3.3 Mathematical introduction of risk transfer's problem, within an insurance group.

Let $X \geq 0$ be the total insurance liabilities for an insurance group, while is consisting of two separate entities. We will denote as $F(\cdot)$ the distribution function and as $\bar{F}(\cdot) =$ $1 - F(\cdot)$ the survival function and the right end – point $x_F := \inf\{z \epsilon \Re : F\ (z) = 1\}$ of the loss distribution can be either finite or infinite.

In this scenario an insurance group aims to allocate the total risk X between its two entities, under the use of appropriate risk transfer arrangements. We denote as

$$X = I_1\ [X] + I_2\ [X]$$

the total insurance liabilities that an insurance group, after risk transfers take place while $I_1\ [X], I_2\ [X]$ are co – monotone random variables and liabilities of the two entities of the insurance group. As we inferred before since they are separate entities, they might be quantified by different risk measures. We denote as φ_1 , φ_2 the risk measures that are used to quantify the entities, so $\varphi_1 \equiv VaR_{a_1}\ or\ \varphi_1 \equiv ES_{a_1}$ and $\varphi_2 \equiv VaR_{a_2}\ or\ \varphi_2 \equiv ES_{a_2}$. Thus the total capital requirements of the first and second entities are $\varphi_1(I_1\ [X])$ and $\varphi_2(I_2\ [X])$.

The risk – adjusted – value of the liabilities, for each of the entities can be defined as

$$RAV((I_k\ [X], \varphi_k, \lambda_k) = E(I_k[X]) + \lambda_k(\varphi_k(I_k\ [X]) - E(I_k[X]))$$

Where $k \in \{1, 2\}$ and $\lambda_k \in [0,1]$ is the cost – of – capital as a percentage of the pure risk capital

$$\varphi_k(\, I_k\, [\, X\,]\,) - E(\, I_k[\, X\,]\,)$$

In general under certain regulatory requirements whenever an insurance company receives liabilities, is necessary to be compensated by the following ways (Asimit et. al 18 Jan 2012):

- The expected value of the future claims
- Funds equal to the cost of raising the necessary regulatory capital to support the liability

The main objective is to minimize the risk – adjusted – value of the liabilities derived by the insurance group, under different risk transfers and capital costs that may derive, to each of the entities. Thus the aim is to minimize under feasible set of risk allocations the following:

$$\begin{aligned} RAV(\,(\,I_1\,[\,X\,], \varphi_1, \lambda_1) + RAV(\,(\,I_1\,[\,X\,], \varphi_1, \lambda_1) \\ &= E(\,X\,) \\ &+ \lambda_1(\,\varphi_1(\,I_1\,[\,X\,]\,) - E(\,I_1[\,X\,]\,) + \lambda_2(\,\varphi_2(\,I_2\,[\,X\,]\,) - E(\,I_2[\,X\,]\,)\ )\ ) \end{aligned}$$

Where

$$\begin{aligned} F = \{\, I_1[\,x\,] + I_2[\,x\,] = x \\ &: I_1[\,x\,]\ and\ I_2[\,x\,], are\ non\ decreasing\ functions\ with\ I_1[\,0\,] \\ &= I_2[\,0\,]\,\} \end{aligned}$$

is the feasible set of allocations.

Since the liabilities are co – monotone and random variables, the above formulation of the risk – adjusted – value and our minimizations problem, is transformed as follows:

$$min_{I_1, I_2 \in F}\{\,(\lambda_1 - \lambda_2) E(\,I_2[\,X\,]\,) + \varphi_2(\,I_2\,[\,X\,]\,) - \lambda_1 \varphi_1(\,I_2\,[\,X\,]\,)\,\}$$

Where

$$\varphi_1(\, I_1\,[\, X\,]\,) \in \{\, VaR_{a_1}(\, I_1\,[\, X\,]\,), ES_{a_1}(\, I_1\,[\, X\,]\,)\}$$

and

$$\varphi_2(\, I_2\,[\, X\,]\,) \in \{\, VaR_{a_2}(\, I_2\,[\, X\,]\,), ES_{a_2}(\, I_2\,[\, X\,]\,)\}$$

At this point it is useful to denote, given that $(\, I_1, I_2) \in F$ I_1 and I_2 are Lipschitz continuous functions meaning that $|\, I_1(y) - I_1(x)| \le |y - x|$ and $|\, I_2(y) - I_2(x)| \le |y - x|$, $\forall x, y \ge 0$.

It is worth to denote that since $I_k(\,\cdot\,)$ is a non – decreasing and continuous function, one important property of VaR is the following:

$$VaR_{a_k}(I_k[\, X\,]\,) = I_k(\, VaR_{a_k}(\, X\,)\,)$$

Necessary assumptions for the above optimization problem are the following (Asimit et. al 18 Jan 2012):

- Insurance liabilities are the only risk that the group is exposed to, in particular the regulatory capital held is invested with no risk.
- The optimization problem remains meaningful even when capital held by the group is higher than the regulatory minimum.
- The capital requirements do not explicitly allow for counter party credit risk arising from the risk transfers considered.

- The optimal risk allocations $I_1[\ X\ ], I_2[\ X\ ]$ must be co – monotone.
- The risk transfer have no impact on the market value of insurance policies by the group and hence the group's profitability.

1.4 Literature review.

Many researches before tried to find a solution to the problem of which is the optimal reinsurance contract. Borch and Arrow were the first trying to find an optimal solution of risk transfer problem. Borch (1960) proved that stop loss treaty minimizes the variance of the retained risk if the reinsurer charges a fixed premium, dependent only on the expected reinsurance claims. Arrow (1963) by maximizing the expected utility as the optimality criterion came to a similar result using the stop loss treaty.

Kaluszka (2005) came to a conclusion that a truncated stop loss is an optimal treaty for models not involving ruin probability and not a quota share or a stop loss treaty. The truncated stop loss treaty is in the following form:

$$R(x) = (x-a)_{+}1(x<\beta) = \begin{cases} 0, & if\ x \leq a \\ x-a, & if\ a < x < b \\ 0, & if\ x \geq b \end{cases}$$

Additionally through several one – period reinsurance models, Kaluszka derived a rule that minimizes the ruin probability of the primer insurer for a fixed reinsurance risk premium. The premium of the reinsurer is calculated according to the economic principle, generalized zero – utility principle and Esscher principle or mean – variance principles. He also mentions some approaches to choosing reinsurance contracts, such as to minimize the ruin probability of the primer insurer in a discrete or a continuous reinsurance contracts Schal (2003), Gaier et al. (2003), or maximizing dividend payments, Pechlivanides (1978), or constructing optimal contracts based on maximizing stability measured by the variance or other functionals, Rantala (1989), Kaluszka (2004c) and last by maximizing the expected utility of the wealth or the return function Hojgaard and Taksar (1998).

In another paper Kaluszka (2000) derives optimal reinsurance under premium principles based on the mean or the variance of the reinsurer's share of the total claim amount. He uses two premium calculation principles the one is the

$$P = E(\, R[X]\,) + \beta VaR(\, R[X]\,)$$

and

$$P = E(\, R[X]\,) + \beta VaR^2(\, R[X]\,)$$

Which in fact are the standard deviation and the variance principle. Later he proves that optimal retention functions are of the form $R^*(x) = a(x-b)_+$, where $0 \leq a \leq 1, b \geq 0$, called as change – loss – reinsurance.

Centeno and Guerra (2010) to confront this problem, chose as optimality criterion the maximization of the adjustment coefficient of the retained risk and assumed that the reinsurance premium is a convex functional. The maximization of the adjustment coefficient is equivalent to minimization of the upper bound of the probability of ultimate ruin provided by the Lundberg inequality. In a revised version they generalize the results obtained previously, by considering that the reinsurance strategies are confined to be pair claim reinsurance and proving that the maximization of the adjustment coefficient is equivalent to solve a two-step problem. The first step is the maximization of the utility of wealth of the retained risk for an exponential utility function, for all positives values of the coefficient of risk aversion. The second step is to solve a single variable equation, the optimal adjustment coefficient equals the coefficient of risk aversion for which the maximal expected value of the utility function is -1.

Cai, Tan, Weng and Zhang (2008) tried to determine the optimal ceded loss functions that minimize Value at Risk (VaR) and Conditional Tail Expectation (CTE) of the total cost in some classes of the of ceded loss functions. They came to a conclusion that depending on the risk measure's level of confidence and safety loading for the reinsurance premium, the optimal reinsurance can be in the forms of stop – loss, quota – share or change – loss.

Cheung (2010) gives alternative ways to analyze and solve the problems of minimizing the VaR and CTE of the total retained loss under the expectation premium principle, that is examined in a recent paper of Cai et al. (2008). The optimal reinsurance problems studied previously, now can be formulated as

$$min_{f \epsilon F} VaR_{T_f(X)}(a)$$

and

$$min_{f \epsilon F} CTE_{T_f(X)}(a)$$

The functions in F who minimize the above objective functions are called optimal ceded functions. He uses a simple geometric argument to show that optimal ceded loss functions must take the form

$$f(x) = c(x - d)_+$$

Since every such function is specified by the slope c and the deductible d, the above transformed optimal reinsurance problems can be solved explicitly by standard calculus method. Cheung's approach is applicable even when the expectation premium principle is replaced by Wang's premium principle and then he analyzes again the value at risk of the total cost minimization process, under Wang's premium principle this time.

Young (1999) proposed axioms for pricing insurance that characterize the premium principle of Wang. It is assumed that insurance prices are given by the expectation with respect to a distorted probability and that the primer insurer is risk averse with respect to maximize their expected utility. It is shown that the resulting premium principle is convex and is being characterized the optimal reinsurance, when premiums are determined by the premium principle and when loss random variable is a mixture of a point mass at zero and a continuous random variable above that.

Grundl and Wandt (2011) compared the shareholder value maximizing capital structure and pricing policy of insurance groups against that of stand – alone insurers. They inferred that groups can utilize intra – group risk diversification by means of capital and risk transfer instruments. They showed that using these instruments enables the group to offer insurance with less default risk and at lower premiums than is optimal for stand – alone insurers.

Gatzert and Schmeiser compared the diversification effect in conglomerates with or without accounting for the altered shareholder value. Further they compared results for a holding company, a parent – subsidiary model, including intra – group retrocession and guarantees. They came in a conclusion that diversification have not effects when are studied under competitive conditions.

1.5 Objective.

The level of the capital to be held by an insurance company is essential and it is defined, so that with high probability the company can meet its liabilities and regulatory obligations. Thus one of the major problems is capital allocation. This term refers to the subdivision of the total capital of the company, between the different legal departments. Therefore it can be viewed as equivalent to the problem of the transfer of funds among insurers and reinsurers.

This paper is based on two basic papers of Asimit et al. that examine risk transfers within a reinsurance company and between an insurance company and reinsurers. Again presenting how they examined the problem of optimal risk transfer and optimal capital allocation for one – period model, for an insurance company using one or multiple reinsurance contracts. Furthermore the aim is to identify the optimal financial position the primer insurer can take, within those contracts. Under certain criteria – such as risk measures, we evaluate the primer insurer's outcomes and rank them. We use some risk measures such as Value at Risk, Expected Shortfall (else Conditional Tail Expectation) and the Truncated Tail-Value-at-Risk to present how the measurement of the potential total insurer's loss is done. Therefore there is one essential decision the insurer must take. He must decide which is the best balance for him, between risk and reward through the optimal use of reinsurance, such as to reduce the adverse of risk and to improve his financial strength and on the same time to minimize the reinsurance premium, which in fact is the additional cost of the reinsurance. Furthermore about the risk transfer in an insurance group, our aim is to present optimal risk transfers under the use of certain risk measures. Again we presenting the way they used a formal setting with two entities, subjected likely to different regulatory requirements.

In the end based on the above mentioned theory and the ways to confront this problem of risk transfer between an insurance company and a reinsurance company (or multiple ones), we present a reinsurance arrangement model we have built and we plot the way it formulates the optimization problems, under different reinsurance contracts, premiums and risk measures.

Chapter 2

Presentation of optimal reinsurance arrangements under certain risk measures.

In this section we present different ways to face the problem of optimal risk transfers for the insurer, towards one or multiple reinsurers with the use of risk measures. Further we present and demonstrate how risk transfer is taking place under a quota share or a stop – loss reinsurance arrangement and under which criteria we choose the optimal one.

2.1 Forms of optimal reinsurance arrangements.

The main objective of this paragraph is to present the optimal solutions for a risk – measure –based reinsurance model under premium principles in general, as Chi and Tan and Weng and Zhang inferred in their papers. We presenting how they determined the optimal coefficient for a quota share or a stop – loss treaty, based on the selection of the premium principle.

2.1.1 Optimal Quota – Share reinsurance.

In quota share reinsurance the cession is the same for the risks in the entire portfolio. We denote the cession as the c, which can take values between$[\,0, 1\,]$. Under the quota share the losses (X) are split as it follows:

$$X_{I_{qs}} = (1 - c)X$$

the amounts that the insurer will retain to his portfolio

and

$$X_{R_{qs}} = cX$$

the amounts that will choose to cede to a reinsurer.

The total cost of the insurer has the formulation $\left(X_{T_{qs}} \right) = X_{I_{qs}} + P(X_{R_{qs}})$.

The insurer seeks to minimize the risk measures that are associated with the total cost. In order to accomplish that, we seek the optimal value of the cession (c) such as to give us the minimum of the following:

$$VaR_a\left(T_{qs}, c^* \right) = min_{c\in[0,1]}\{ VaR_a\left(T_{qs}, c \right) \}$$

Which is the VaR – optimization.

Theorem 2.1. We consider the above VaR optimization, then

i. We assume as $\Pi(\cdot)$ the reinsurance premium that satisfies $\Pi(0) = 0$ and for the constant $c > 0$ by the property of positive homogeneity for the premium we have that $\Pi(cX) = c\Pi(X)$. For these assumptions we have that the optimal quota – share reinsurance is insignificant and the constant c depends on the relative magnitude between $\Pi(X)$ and $S_X^{-1}(a)$ as the it appears at the following:

$$c^* = \begin{cases} 0, & \Pi(X) > S_X^{-1}(a) \\ k \in [0,1], & \Pi(X) = S_X^{-1}(a) \\ 1, & \Pi(X) < S_X^{-1}(a) \end{cases}$$

ii. If $\Pi(cX)$ is convex while $0 \le c \le 1$ then the optimal quota – share reinsurance exists if and only if there is a constant c^* such that

$$\Pi_c'(\, cX \,) - S_X^{-1}(\, a \,) = 0$$

Where $\Pi_c'(\,\cdot\,)$ denotes the partial derivative with respect to c and c^* that satisfies the above equation is the optimal quota – share reinsurance.

2.1.2 Optimal Stop – Loss reinsurance.

In stop – loss reinsurance the retention and the cession, for the insurer takes the following forms, where X is the total loss and d is a positive parameter and known as the retention:

$$X_{I_{sl}} = \begin{cases} X, & X \leq d \\ d, & X > d \end{cases}$$

and

$$X_{R_{sl}} = \begin{cases} 0, & X \leq d \\ X - d, & X > d \end{cases}$$

Similarly as previously, the insurer seeks to minimize the risk measures that are associated with the total cost, in order to find the optimal retention (d). Again we use the following minimization procedures:

For the VaR – optimization

$$VaR_a\big(\, T_{qs}, d^* \,\big) = min_{d \in [0,\infty)}\{\, VaR_a\big(\, T_{qs}, d \,\big)\,\}$$

Theorem 2.2. We consider the above VaR – optimization problem and we make the assumption that $\Pi(\cdot)$ is a premium principle such that $\Pi([X-d]_+)$ is a decreasing in d. Then we have that:

i. The optimal stop – loss reinsurance is insignificant if

   a. $d+\Pi([X-d]_+)$ is an increasing function in d in the interval $[0, S_X^{-1}(a)]$

   b. If there is a constant $d_0 \in (0, S_X^{-1}(a))$ such that $d+\Pi([X-d]_+)$ is increasing at d in the interval $[d_0, S_X^{-1}(a)]$.

   In conclusion the insignificant optimal retention is as it follows:

$$d^* = \begin{cases} 0, & \Pi(X) < S_X^{-1}(a) \\ 0 \ or \ \infty, & \Pi(X) = S_X^{-1}(a) \\ +\infty, & \Pi(X) > S_X^{-1}(a) \end{cases}$$

   If the $\Pi(\cdot)$ satisfies $\lim_{d\to\infty}\Pi([X-d]_+) = 0$ and there is a $d_0 > 0$ such that $d+\Pi([X-d]_+)$ is decreasing at d in the interval $[0, d_0]$ and increasing on $[d_0, \infty]$, then we have that the optimal stop – loss reinsurance exists if and only if:

$$S_X^{-1}(a) > d_0 + \Pi([X-d]_+)$$

ii. Where the optimal retention is d_0 and the corresponding minimum value is

$$min_{d\geq 0}\left\{VaR_a\left(T_{qs}, d\right)\right\} = d_0 + \Pi([X-d]_+)$$

It is worth to mention that in order to find the optimal risk transfer by the insurer's perspective, not only we have to find the optimal retention or cession, such that will give us the minimum measure of the risk, but we have to define also an optimal premium principle as a fair price towards the risk that the reinsurer agrees to take, but still in a price that the insurer agrees to pay.

2.2 Optimal single reinsurance contract.

In this section we show, what A. Asimit, A. Badescu and T. Verdonck inferred in their paper, titled "Optimal Risk Transfer with Multiple Reinsurers" (2012). Then we briefly show how they examine an optimal reinsurance arrangement between an insurer and one or multiple reinsurers, under certain risk measures, such as value – at – risk, expected shortfall and the Truncated Tail Value – at – Risk.

2.2.1 The value – at – risk based optimal decision.

Theorem 2.3. The VaR – based optimal decision that minimizes the insurer's total loss, given that the optimization problem is transformed to

$$min_{R\in F}\{ \varphi_I(X) - \varphi_I(R[X]) + P(R[X])\}$$

, is given by

$$R^*[X] = (X - VaR_{\rho^*}(X))_+ \wedge (VaR_a(X) - VaR_{\rho^*}(X))_+$$

, where $\rho^* = \rho/(1+\rho)$.

So the primer insurer's risk becomes:

$$VaR_a(L(R[X])) = VaR_a(X) \wedge VaR_{\rho^*}(X) + (1 + \rho) \int_{VaR_a(X) \wedge VaR_{\rho^*}(X)}^{VaR_a(X)} \bar{F}(x)dx$$

Proof. See at Appendix.

2.2.2 The Expected Shortfall based optimal decision.

Theorem 2.4.

The ES based optimal decision that solves the $min_{R\in F}\{\varphi_I(X) - \varphi_I(R[X]) + P(R[X])\}$, is given by

$$R^*[X] = \begin{cases} (X - VaR_{\rho^*}(X))_+, & a > \rho^* \\ 0, & a < \rho^* \\ h_1^*(X), & a = \rho^* \end{cases}$$

Where $h_1^*(\cdot)$ is a non – decreasing Lipschitz function with unit constant satisfying $h_1^*(VaR_a(X)) = 0$. So the insurer's risk becomes

$$ES_a(L(R^*[X])) = \begin{cases} VaR_{\rho^*}(X) + (1+p)\int_{VaR_{\rho^*}(X)}^{x_F} \bar{F}(x)dx, & a > \rho^* \\ ES_a(X), & a \leq \rho^* \end{cases}$$

Proof. See at Appendix.

2.3 Optimal multiple reinsurance contract.

In this section A. Asimit et al. examined the VaR – based problem when the insurer makes a reinsurance arrangement between two reinsurers, with different premium principles. Again they approach this problem, through the two stage optimization procedure.

In this case the mathematical form of the reinsurance problem turns to be as the following

$$
\begin{aligned}
min_{(R_1,R_2)\epsilon G} VaR_a(\,L(\,R_1[\,X\,], R_2[\,X\,]\,)\,) \\
&= VaR_a(\,I[\,X\,]\,) + P_1(\,R_1[\,X\,]\,) + P_2(\,R_2[\,X\,]\,), where\ G \\
&= \{(R_1, R_2): I(x) \\
&= x - R_1(x) - R_2(x),\ R_1, R_2\ non - decreasing\ functions.\}
\end{aligned}
$$

The R_1 and R_2 are the risk transfers to the reinsurers 1 and 2 and the $P_1(\,R_1[\,X\,]\,)$ and $P_2(\,R_2[\,X\,]\,)$ are the reinsurance premiums, that the insurer has to pay to the reinsurers 1 and 2, for the allocation of the risk.

It is clear that under their process the risk allocation is being made sequentially and feasible allocations – according to them, should satisfy the following conditions

i. I and R_1 are non – decreasing functions of $X - R_2$
ii. I and R_2 are non – decreasing functions of $X - R_1$
iii. R_1 and R_2 are non – decreasing functions of $X - I$

Under the first and the third above conditions they proposed and gave a proof of it, that I, R_1, R_2 are non – decreasing functions of X and in addition, they are Lipschitz functions with unit constants.

In order to use the two stage optimization problem, they first specify the reinsurance premiums that each reinsurer uses, in order to evaluate somehow the riskiness of the insurer's position and how he defines his own optimal decision. The reinsurance premiums that are taking under consideration are the expected value premium principle and distorted premium principles, that are already mentioned at chapter one. In general any quantile based premium principle can be used into the two stage optimization. In general the insurer will decide about his VaR based optimal decision transferring part of his risk, to a reinsurer that uses the expected value principle and another that uses a distorted premium principle. The expected value premium principle – that the first reinsurer uses, has the following mathematical form

$$P(\,R_1[\,X\,]\,) = (\,1+\rho)E(\,R_1[\,X\,]\,)$$

The distorted premium principle – that the second reinsurer uses, has the following mathematical form

$$P(\,R_2[\,X\,]\,) = \int_0^{x_F} g(\Pr(\,R_2[\,X\,] > x\,)\,)dx$$

with the distortion function being $g\colon [\,0,1\,] \to [\,0,1\,]$ a concave and non – decreasing function, such that $g(0) = 0,\ g(1) = 1\ and\ g(x) \geq x, \forall x$. Assuming that the distortion function is differentiable, we can have the existence of the left and right derivatives as result of the first assumption, that the distortion function is a concave one.

This leads to the following alternative form of the distorted premium principle $P(\,R_2[\,X\,]\,) = \int_0^1 VaR_s(\,R_2[\,X\,]\,)\Phi(\,s\,)ds$, where $\Phi(\,s\,) = g'(1-s)$.

All the above change our first mathematical form of the reinsurance optimization problem and it becomes

$$min_{(R_1,R_2)\epsilon G} VaR_a(\, I[\,X\,]\,) + (1+\rho)\int_0^1 VaR_s(\, R_1[\,X\,]\,)ds$$
$$+\int_0^1 VaR_s(\, R_2[\,X\,]\,)\Phi(\,s\,)ds$$

Assumption 2.1. Let $g(\,\cdot\,)$ be a distortion function with corresponding function $\Phi(\,\cdot\,)$ such that the equation $\Phi(\,s\,) = 1+\rho$ has just one solution, denoted by s^* with $\rho^* \leq s^*$. And the equation $G(\,1-s\,) = 0$ has a unique solution s^{**}, where $G\,(\,1+\rho)t - g(\,t\,)$.

Given the above assumption they derived a theorem that can give the freedom to the second reinsurer to set its own premium, after the consideration of the entire incurred losses.

Theorem 2.5. Let us assume that the above assumption holds such that $s^{**} \leq a$ and $\Phi(\,s\,) > 0, \forall s$. Then the VaR – based optimal reinsurance contract

$$min_{(R_1,R_2)\epsilon G} VaR_a(\, I[\,X\,]\,) + (1+\rho)\int_0^1 VaR_s(\, R_1[\,X\,]\,)ds$$
$$+\int_0^1 VaR_s(\, R_2[\,X\,]\,)\Phi(\,s\,)ds$$

is as follows $\begin{cases} R_1^*[\,X\,] = (X - VaR_{s**}(\,X\,))_+ \wedge (\,VaR_a(\,X\,) - VaR_{s**}(\,X\,)\,) \\ R_2^*[\,X\,] = X \wedge VaR_{s**}(\,X\,) \end{cases}$

and the corresponding insurer risk becomes

$$VaR_a(\,L(R_1^*[\,X\,], R_2^*[\,X\,]\,) = (1+\rho\,)\int_{VaR_{s**}(\,X\,)}^{VaR_a(\,X\,)} \bar{F}(x)\,dx + \int_0^{VaR_{s**}(\,X\,)} g(\,\bar{F}(x)\,)\,dx$$

Proof. See at Appendix.

Theorem 2.6. The VaR – based insurer risk problem

$$min_{(R_1,R_2)\epsilon G} VaR_a(\ I[\ X\ ]\ ) + (1+\rho)E(\ \ R_1[\ X\ ]\ \ ) + ES_\beta(\ R_2[\ X\ ]\ )$$

, is considered such that max$\{\ \rho^*, \beta\ \} \leq a$. Then its optimal solution has the following composition:

i. Whenever $\beta < \rho^*$

$$R_1^*[\ X\ ] = 0,\ R_2^*[\ X\ ] = \begin{cases} h_2^*(\ X\ ) \wedge a, & X \leq VaR_\beta(\ X\ ) \\ X \wedge VaR_a(\ X\ ) - VaR_\beta(\ X\ ) + a, & X > VaR_\beta(\ X\ ) \end{cases}$$

ii. Whenever $\beta > \rho^*$

$$R_1^*[\ X\ ] = (X - VaR_{\rho^*}(\ X\ ))_+ \wedge \left(\ VaR_a(\ X\ ) - VaR_{\rho^*}(\ X\ )\ \right),$$

$$R_2^*[\ X\ ] = h_3^*(\ X\ ) \wedge b$$

With $a\ \epsilon \left[\ 0, VaR_\beta(\ X\ )\right], \beta\ \epsilon \left[\ 0, VaR_{\rho^*}(\ X\ )\right]$ and h_3^*, h_2^* are non – decreasing Lipschitz functions with unit constants such that

$$h_2^*(0) = h_3^*(0) = 0, h_2^*\left(VaR_\beta(\ X\ )\ \right) = a, h_3^*(\ VaR_\beta(\ X\ )\ ) = b$$

And therefore the corresponding insurer risk becomes

$$VaR_a(\ L(\ R_1[\ X\ ], R_2[\ X\ ]\ )\ ) = \ VaR_\beta(\ X\ ) + \frac{1}{1-\beta}\int_{VaR_\beta(\ X\ )}^{VaR_a(\ X\ )} \bar{F}(x)dx,\ \ if\ \beta < \rho^*$$

And

$$VaR_a(\ L(\ R_1[\ X\ ], R_2[\ X\ ]\ )\ ) = \ VaR_{\rho^*}(\ X\ ) + (\ 1+\rho\ )\int_{VaR_{\rho^*}(\ X\ )}^{VaR_a(\ X\ )} \bar{F}(x)dx,\ \ if\ \beta > \rho^*$$

Proof. See at Appendix.

At the above theorem it is discussed the scenario that the reinsurance premium the second reinsurer charges for the insurer, is in the form of a distorted premium principle, one like the expected shortfall.

2.4 Optimal reinsurance contract under the Truncated Tail Value at Risk.

At the following we show what Asimit et al. (2012) proposed as a different optimal risk transfer that the insurer might take, using this time a more robust risk measure, the truncated tail value at risk. Again they face this problem by solving it, under the two – stage optimization procedure.

Theorem 2.7. If $1+\rho < 1/(a_2 - a_1)$ then the $TrTVaR$ – based optimal reinsurance contract is given by

$$R^*[\,X\,] = \begin{cases} (\,X - VaR_{a^{**}}(\,X\,))_+ \wedge \left(VaR_{a^*}(\,X\,) - VaR_{a^{**}}(\,X\,)\right)_+, & a^* > a_1 \\ (\,X - VaR_{\rho^*}(\,X\,))_+ \wedge \left(VaR_{a_1}(\,X\,) - VaR_{\rho^*}(\,X\,)\right)_+, & a^* < a_1 \end{cases}$$

Where

$$a^* = 1 - \frac{1 - a_2}{1 - (1+\rho)(a_2 - a_1)}\ , a^{**} = \min(\rho^*, a_1)$$

Proof. See at Appendix.

Chapter 3

Presentation of optimal risk transfers in insurance groups, under certain risk measures.

In this section we present, what A. Asimit, A. Badescu and A. Tsanakas inferred in their paper, titled "Optimal Risk Transfer in Insurance Groups" (2012). Then we briefly show how they examine the optimal risk transfers arrangements between two separate legal entities, operating under different regulatory capital requirements and capital costs. This is similar as the previous subject, since the relation of those two legal entities – in an extension, can be assumed to be a relation between a primary insurer and a reinsurer. These capital requirements are for each entity calculated under Value – at – Risk and Expected Shortfall. We later show that the minimization of risk – adjusted – value of the total group's liabilities and the valuation using the cost – of – capital approach, are the optimality criterion that we use. In the absence of capital requirement for the credit risk arising from the risk transfer, we present several optimal risk transfers that achieve capital efficiency.

3.1 The optimization problem for an insurance group.

We present optimal functional forms of risk transfers. We use a setting of two entities of an insurance group that each one of them is subject to different regulatory requirements. The aim is to minimize the risk – adjusted – value of each entity liabilities, when the valuation takes place under the cost – of – capital methodology.

3.1.1 The VaR/VaR setting.

The liabilities of the two entities are both subject to VaR – based capital requirements. So we have that $\varphi_1 \equiv VaR_{a_1}$ and $\varphi_2 \equiv VaR_{a_2}$ and the optimization problem

$$min_{I_1,I_2 \in F}\left\{ (\lambda_1 - \lambda_2)E(\, I_2[\, X\,]\,) + VaR_{a_2}(\, I_2\,[\, X\,]\,) - \lambda_1 VaR_{a_1}(\, I_2\,[\, X\,]\,)\,\right\} \quad (*)$$

In the following theorem the optimal risk transfer is stated, under the above minimization problem.

Theorem 3.1. The optimal solution of $(*)$ is the following:

1. If $\lambda_1 \neq \lambda_2$ then

$$I_2^*[\, X\,] = \begin{cases} \min\{\, X, VaR_{a_1}(X)\}, & \lambda_1 > \lambda_2 \\ (\, X - VaR_{a_1}(X)\,)_+, & \lambda_1 < \lambda_2 \end{cases}$$

2. If $\lambda_1 = \lambda_2$ and $VaR_{a_1}(X) < VaR_{a_2}(X)$ then

$$I_2^*[\, X\,] = \begin{cases} f_2(X), & X > VaR_{a_2}(X) \\ \min\{\, f_1(X), t_1\}, & otherwise \end{cases}$$

3. If $\lambda_1 = \lambda_2$ and $VaR_{a_1}(X) > VaR_{a_2}(X)$ then

$$I_2^*[\, X\,] = \begin{cases} f_4(X), & X > VaR_{a_1}(X) \\ (\, X - VaR_{a_2}(X)\,)_+ + \min\{\, f_3(X), t_2\}, & otherwise \end{cases}$$

Where $f_k(\cdot), k \epsilon \{\, 1, 2, 3, 4\,\}$ are non – decreasing Lipschitz continuous functions with unit constants such that

$$f_1(0) = f_3(0) = 0, f_1\left(VaR_{a_1}(X)\right) = f_2\left(VaR_{a_2}(X)\right) = t_1, f_3\left(VaR_{a_2}(X)\right)$$
$$= t_{2,}\ \ f_4\left(VaR_{a_1}(X)\right) = VaR_{a_1}(X) - VaR_{a_2}(X) + t_2, t_1$$
$$\in \left[\ 0, VaR_{a_1}(X)\ \right],, t_2 \in \left[\ 0, VaR_{a_2}(X)\ \right]$$

Proof. See at Asimit et al. (18 Jan 2012)

3.1.2 The VaR/ES setting.

Now liabilities of the two entities are subject to $VaR\ and\ ES$ – based capital requirements. So we have that $\varphi_1 \equiv VaR_{a_1}$ and $\varphi_2 \equiv ES_{a_2}$ and the optimization problem

$$min_{I_1,I_2 \in F}\left\{\ (\lambda_1 - \lambda_2)E(\ I_2[\ X\ ]\ ) + \lambda_2 ES_{a_2}(\ I_2\ [\ X\ ]\ ) - \lambda_1 VaR_{a_1}(\ I_2\ [\ X\ ]\ )\ \right\} \quad (*)$$

The following theorem is the solution of the above minimization problem and additionally the optimal risk transfer is stated.

Theorem 3.2. The optimal solution of $(*)$ is the following:

1. If $VaR_{a_1}(X) \leq VaR_{a_2}(X)$ then
$$I_2^*[\ X\ ] = \begin{cases} \min\left\{\ X, VaR_{a_1}(X)\right\}, \lambda_1 > \lambda_2 \\ \qquad 0, \qquad\quad \lambda_1 < \lambda_2 \end{cases}$$

2. If $VaR_{a_1}(X) > VaR_{a_2}(X)$ then

$$I_2^*[\,X\,] = \begin{cases} \min\{\,X, VaR_{a_1}(X)\},\ \lambda_1 > \lambda_2 \\ \min\{\,(\,X - VaR_{a_2^{**}}(X)\,)_+, VaR_{a_1}(X) - VaR_{a_2^{**}}(X)\},\ \lambda_1 < \lambda_2 \end{cases}$$

where

$$a_2^{**} = \min\{\,a_1, a_2^*\}\,and\ a_2^* = \frac{\lambda_2 a_2}{\lambda_1(1-a_2)+\lambda_1 a_2}$$

3. If $\lambda_1 = \lambda_2$ and $VaR_{a_1}(X) \le VaR_{a_2}(X)$ then

$$I_2^*[\,X\,] = \min\{\,f_1(X), t_1\}$$

While if $\lambda_1 = \lambda_2$ and $VaR_{a_1}(X) > VaR_{a_2}(X)$ it is

$$I_2^*[\,X\,] = \begin{cases} \min\{\,X, VaR_{a_1}(X)\} - VaR_{a_2}(X) + t_2, & X > VaR_{a_2}(X) \\ f_2(X), & otherwise \end{cases}$$

Where $f_1(\cdot), f_2(\cdot)$ are non – decreasing Lipschitz continuous functions with unit constants such that

$$f_1(0) = f_2(0) = 0, f_1\left(VaR_{a_1}(X)\right) = f_2\left(VaR_{a_2}(X)\right) = t_1, f_2\left(VaR_{a_2}(X)\right)$$
$$= t_{2,}\ \ with\ parameters\ t_1 \in \left[\,0, VaR_{a_1}(X)\,\right], t_2 \in \left[\,0, VaR_{a_2}(X)\,\right]$$

Proof. See at Asimit et al. (18 Jan 2012)

If we reverse the risk measures - based capital requirements to $\varphi_1 \equiv ES_{a_1}$ and $\varphi_2 \equiv VaR_{a_2}$ and the above optimization problem is turning to

$$min_{I_1,I_2 \in F}\{\,(\lambda_1 - \lambda_2)E(\,I_2[\,X\,]\,) + \lambda_2 VaR_{a_2}(\,I_2\,[\,X\,]\,) - \lambda_1 ES_{a_1}(\,I_2\,[\,X\,]\,)\,\}\quad (**)$$

and the solution is the following:

Corollary 3.1. The optimal solution of the above optimization problem is describes as follows:

1. If $VaR_{a_1}(X) \geq VaR_{a_2}(X)$ then

$$I_2^*[\,X\,] = \begin{cases} \text{X}, & \lambda_1 > \lambda_2 \\ (\,X - VaR_{a_2}(X)\,)_+, & \lambda_1 < \lambda_2 \end{cases}$$

2. If $VaR_{a_1}(X) < VaR_{a_2}(X)$ then

$$I_2^*[\,X\,] = \begin{cases} (\,X - VaR_{a_2}(X)\,)_+ + \min\{\,X, VaR_{a_1^{**}}(X)\}, & \lambda_1 > \lambda_2 \\ (\,X - VaR_{a_2}(X)\,)_+, & \lambda_1 < \lambda_2 \end{cases}$$

3. If $\lambda_1 = \lambda_2$ and $VaR_{a_1}(X) \geq VaR_{a_2}(X)$ then

$$I_2^*[\,X\,] = \begin{cases} X - VaR_{a_2}(X) + t_1, & X > VaR_{a_1}(X) \\ \min\{\,f_1(X), t_1\,\}, & otherwise \end{cases}$$

While for $\lambda_1 = \lambda_2$ and $VaR_{a_1}(X) < VaR_{a_2}(X)$ it is

$$I_2^*[\,X\,] = (\,X - VaR_{a_2}(X)\,)_+ + \min\{\,f_2(X), t_2\,\}$$

Where $f_1(\cdot), f_2(\cdot)$ are non – decreasing Lipschitz continuous functions with unit constants such that

$$f_1(0) = f_2(0) = 0, f_1\left(VaR_{a_2}(X)\right) = t_1, f_2\left(VaR_{a_2}(X)\right) = t_2,$$
$$t_1 \in \left[\,0, VaR_{a_1}(X)\,\right],, t_2 \in \left[\,0, VaR_{a_2}(X)\,\right]$$

3.1.2 The ES/ES setting.

At this section we make the assumption that both of liabilities of the two entities are both subject to ES – based capital requirements. The optimization problem transform as follows:

$$min_{I_1,I_2\in F}\left\{ (\lambda_1-\lambda_2)E(\, I_2[\, X\,]\,)+\lambda_2 ES_{\,a_2}(\, I_2\,[\, X\,]\,)-\lambda_1 ES_{a_1}(\, I_2\,[\, X\,]\,)\right\} \quad (***)$$

Like the above settings, the $(***)$ optimization problem has the following solution

Theorem 3.3. Let C be a constant given by:

$$C=\frac{\lambda_2\, a_2}{1-a_2}-\frac{\lambda_1 a_1}{1-a_1}$$

The optimal solution $(***)$ is

1. If $VaR_{a_1}(X)=VaR_{a_2}(X)$ then

$$I_2^*[\,X\,]=\begin{cases} X,\ \ \lambda_1>\lambda_2, C<0 \\ \min\left\{X, VaR_{a_1}(X)\right\},\ \lambda_1>\lambda_2, C<0 \\ \left(X-VaR_{a_1}(X)\,\right)_+, \qquad \lambda_1<\lambda_2, C<0 \\ 0,\ \ \lambda_1<\lambda_2, C>0 \\ (\,X-VaR_{a_1}(X)\,)_+ + \min\{\, f_1(X), t_1\,\},\ \lambda_1=\lambda_2, C<0 \\ \min\{\, f_2(X), t_2\,\}, \qquad \lambda_1=\lambda_2, C>0 \end{cases}$$

Where $f_1(\cdot), f_2(\cdot)$ are non – decreasing Lipschitz continuous functions with unit constants such that

$$f_1(0)=f_2(0)=0, f_1\left(VaR_{a_2}(X)\right)=t_1, f_2\left(VaR_{a_2}(X)\right)=t_2,$$
$$t_1\in\left[\,0, VaR_{a_1}(X)\,\right],, t_2\in\left[\,0, VaR_{a_1}(X)\,\right]$$

2. If $VaR_{a_1}(X) < VaR_{a_2}(X)$ then

$$I_2^*[\,X\,] = \begin{cases} X, \;\; \lambda_1 > \lambda_2, C < 0 \\ \min\left\{\, X, VaR_{a_1^*}(X)\right\},\;\; \lambda_1 > \lambda_2, C > 0 \\ 0, \quad \lambda_1 < \lambda_2 \\ \min\{\, f_3(X), t_3\,\},\;\; \lambda_1 = \lambda_2 \end{cases}$$

Where $a_1^* = \frac{\lambda_1 a_1}{\lambda_1 a_1 + \lambda_2 (1 - a_1)}$ and $f_3(\,\cdot\,)$ is non – decreasing Lipschitz continuous functions with unit constants such that

$$f_3(0) = f_3\left(VaR_{a_1}(X)\right) = t_3 \in \left[\,0, VaR_{a_1}(X)\,\right]$$

3. If $VaR_{a_1}(X) > VaR_{a_2}(X)$ then

$$I_2^*[\,X\,] = \begin{cases} X, \;\; \lambda_1 > \lambda_2 \\ \left(\,X - VaR_{a_2^*}(X)\,\right)_+, \;\; \lambda_1 < \lambda_2, C < 0 \\ 0, \lambda_1 < \lambda_2, C > 0 \end{cases}$$

Where $a_1^* = \frac{\lambda_2 a_2}{\lambda_1 (1 - a_2) + \lambda_2 a_2}$. In addition the $\lambda_1 = \lambda_2$ case is solved by

$$I_2^*[\,X\,] = (\,X - VaR_{a_2}(X)\,)_+ + \min\{\, f_4(X), t_4\}$$

Where $f_4(\cdot)$ is non – decreasing Lipschitz continuous functions with unit constants such that

$$f_4(0) = f_4\left(VaR_{a_2}(X)\right) = t_4 \in \left[\,0, VaR_{a_2}(X)\,\right]$$

Proof. See at Asimit et al. (18 Jan 2012)

3.2 Expected policyholder deficit.

In this section the impact of potential default on policyholder welfare is being under consideration. The impact of risk transfer on policyholder welfare, is quantified, through the examination of the resulting policyholder deficit and is compared to the case where all risk is retained by the first entity. The policyholder deficit can also be seen as an asset transferred from policyholders to shareholders, reflecting the option of the latter to default on their obligations.

Definition 3.1. The expected policyholder deficit for liability Z , which is a random variable and while the available assets are c, which are a fixed number and asset risk is not considered is:

$$EPD(\, Z, c\,) = E[\,(\, Z - c\,)_+\,] = \int_c^{x_F} P(\, Z > z\,)\, dz$$

The capital requirements of the first entity are given by ES_{a_1} and for the second entity by VaR_{a_1}, while $VaR_{a_1} < VaR_{a_2}$. Further we make the assumption that the initial risk X is held by the first entity, so X_1 is the retained risk and the second entity is a subsidiary, so reinsuring the first is providing a contingent payment X_2. So before the risk transfer, the expected policyholder deficit has the following formulation:

$$EPD\big(\, X, ES_{a_1}(\, X\,)\,\big) = E[\,\big(\, X - ES_{a_1}(\, X\,)\,\big)_+\,]$$

The impact of risk transfer on policyholder deficit can be examined as follows:

1. The two entities are differently regulated so the aroused deficits from those entities, are being examined under different regulations. Then the expected policyholder deficit takes the following form:

$$EPD\big(X_1, ES_{a_1}(X_1)\big) + EPD\big(X_2, VaR_{a_2}(X_2)\big) = E\big[(X_1 - ES_{a_1}(X_1))_+\big] + E\big[(X_2 - VaR_{a_2}(X_2))_+\big]$$

the minus of the equation is in matters of the paths that the risk transfer follows. Thus the impact of credit risk arising from the risk transfer is not presented fully. Assuming that all risks are transferred from the first entity to the second, a possible default of the second entity has no impact to the policyholders, that had acquire the policies from the first. A default of the second entity while the first entity has also defaulted is the important scenario for the policyholders. Considering all the above, we come to the conclusion that the risk exposure of the first entity that could lead to a default of the second entity, is given by the following form:

$$\widetilde{X_1} = X_1 + (X_2 - VaR_{a_2}(X_2))_+$$

In the scenario that the credit risk arising from the risk transfer is not fully reflected, by the capital requirement applied to the first entity, the $ES_{a_1}(X_1)$ is reflecting the capital held by the first entity. Thus the expected policyholder deficit is:

$$EPD\big(\widetilde{X_1}, ES_{a_1}(X_1)\big) = E\big[\big(\widetilde{X_1}, ES_{a_1}(X_1)\big)_+\big]$$

2. Including all credit risk in the capital requirement, so the capital that is being held is $ES_{a_1}\big(\widetilde{X_1}\big)$ and the expected policyholder deficit is

$$EPD\big(\widetilde{X_1}, ES_{a_1}(\widetilde{X_1})\big) = E\big[\big(\widetilde{X_1}, ES_{a_1}(\widetilde{X_1})\big)_+\big]$$

3.3 Policyholder deficit arising from optimal risk transfers.

In this section the credit risk in the capital requirement of the first entity, leads to an expected policyholder deficit that is no longer than the one before the risk was transferred. The optimal solutions $I_1^*[\,X\,], I_2^*[\,X\,]$ are assumed to be the shares X_1, X_2 of the aggregated risk X. We will take under consideration two cases, in order to come to a conclusion of different risk allocations.

When $\lambda_1 > \lambda_2$ the optimal risk allocations are the followings:

$$X_1 = \min\left\{\,(\,X - VaR_{a_1^*}(X)\,)_+, VaR_{a_2}(X) - VaR_{a_1^*}(X)\right\}$$

$$X_{2=} \min\{\,X, VaR_{a_1^*}(X)\,\} + (\,X - VaR_{a_2^*}(X)\,)_+$$

where $a_1^* = \frac{\lambda_1 a_1}{\lambda_2(1-a_1)+\lambda_1 a_1}$ and λ_1, λ_2 are such that $a_1^* < a_2$.

So the expected policyholder deficits are now presented to the following lemma:

Lemma 3.1. For X_1, X_2 as above, the expected policyholder deficits are in the form

1. $EPD\big(\,X_1, ES_{a_1}(\,X_1\,)\,\big) + EPD\big(\,X_2, VaR_{a_2}(\,X_2\,)\,\big) = \int_{VaR_{a_1^*}(X)}^{x_F} \bar{F}(x)\,dx$
2. $EPD\big(\,\widetilde{X_1}, ES_{a_1}(\,X_1\,)\,\big) = \int_{VaR_{a_1^*}(X)+ES_{a_1}(\,X_1\,)}^{x_F} \bar{F}(x)\,dx$
3. $EPD\big(\,\widetilde{X_1}, ES_{a_1}(\,\widetilde{X_1}\,)\,\big) = \int_{VaR_{a_1^*}(X)+ES_{a_1}(\,\widetilde{X_1}\,)}^{x_F} \bar{F}(x)\,dx$

Where

$$ES_{a_1}(\,X_1\,) = \frac{1}{(1-a_1)} \int_{VaR_{a_1^*}(X)}^{VaR_{a_2}(X)} \bar{F}(x)\,dx$$

and

$$ES_{a_1}\left(\widetilde{X_1} \right) = \frac{1}{(1-a_1)} \int_{VaR_{a_1^*}(X)}^{x_F} \bar{F}(x)\, dx$$

It holds that

$$EPD\left(\widetilde{X_1}, ES_{a_1}\left(\widetilde{X_1} \right) \right) \le \; EPD\left(X_1, ES_{a_1}(X_1) \right)$$

And if

$$VaR_{a_2}(X) \le ES_{a_1}(X)$$

then

$$EPD\left(X_1, ES_{a_1}(X_1) \right) + EPD\left(X_2, VaR_{a_2}(X_2) \right) = EPD\left(\widetilde{X_1}, ES_{a_1}\left(\widetilde{X_1} \right) \right) \ge \; EPD\left(X_1, ES_{a_1}(X_1) \right)$$

Where a_1, a_2 are the confidence levels used in regulatory practice of insurance, satisfying the following condition $VaR_{a_2}(X) \le ES_{a_1}(X)$.

Proof. See at Asimit et al. (18 Jan 2012)

In conclusion from the above lemma, we can derive two results. The first is that while a risk transfer increases, then the expected policyholder deficit increases too, if the credit risk is not enough computed for capital settings 1 and 2. The second is that by allowing credit risk in the capital requirement of the first entity, increases its capital sufficiently so that the expected policyholder deficit is actually reduced in relation to the situation before the risk transfer.

Now when $\lambda_1 < \lambda_2$ the optimal risk allocations are the followings:

$$X_1 = \min\{ X, VaR_{a_2}(X)\}$$

and

$$X_2 = (X - VaR_{a_2}(X))_+$$

Similarly the expected policyholder deficits are now presented to the following lemma:

Lemma 3.2. For X_1, X_2 as above, the expected policyholder deficits are in the form

1. $EPD\big(X_1, ES_{a_1}(X_1) \big) + EPD\big(X_2, VaR_{a_2}(X_2) \big) = \int_{ES_{a_1}(X_1)}^{x_F} \bar{F}(x)\, dx$
2. $EPD\big(\widetilde{X_1}, ES_{a_1}(X_1) \big) = \int_{ES_{a_1}(X_1)}^{x_F} \bar{F}(x)\, dx$
3. $EPD\big(\widetilde{X_1}, ES_{a_1}\big(\widetilde{X_1} \big) \big) = \int_{ES_{a_1}(X_1)}^{x_F} \bar{F}(x)\, dx$

Where

$$ES_{a_1}(X_1) = VaR_{a_1}(X) + \frac{1}{(1 - a_1)} \int_{VaR_{a_1}(X)}^{VaR_{a_2}(X)} \bar{F}(x)\, dx$$

It holds that

$$EPD\big(\widetilde{X_1}, ES_{a_1}\big(\widetilde{X_1} \big) \big) = EPD\big(X_1, ES_{a_1}(X_1) \big)$$

and

$$\begin{aligned} EPD\big(X_1, ES_{a_1}(X_1) \big) + EPD\big(X_2, VaR_{a_2}(X_2) \big) &= EPD\big(\widetilde{X_1}, ES_{a_1}\big(\widetilde{X_1} \big) \big) \\ &\geq EPD\big(X, ES_{a_1}(X_1) \big) \end{aligned}$$

Proof. See at Asimit et al. (18 Jan 2012)

Again in conclusion from the above lemma, we can derive two results. The first is that as the risk transfer increases the expected policyholder increases too, in the case that the credit risk is not enough computed for capital settings 1 and 2. The second is that by allowing credit risk in the capital requirement of the first entity, the expected policyholder deficit remains unchanged in relation to the situation before the risk transfer.

3.4 When credit risk interferes in the capital requirements and the optimality of risk transfer.

In this section we examine the case that the risk transfer is not anymore optimal in relation to the previous regulatory requirement. To confront this issue we present optimal risk transfers, when the first entity includes all of the default risk of the second separate legal entity.

We shall denote the recovery rate as $1 - \gamma$, which is the percentage of the exposure to the second entity that will recovered in the case of default and given that we suppose that the second entity of the insurance group, has the role to reinsure the first one and since in our previous settings, the assets available be paid as regulatory expenses on the second entity, we make the assumption that γ is really close to 1.

Thus we have the following mathematical formulation for our problem . First we denote the risk to the first entity, which includes the arising credit risk

$$g_1[\,X,\gamma\,] = \; I_1[\,X\,] + \gamma(\,I_2[\,X\,] - I_2\big[\,VaR_{a_2}\,(\,X\,)\big]\,)_+$$

and our optimization problem is now the following:

$$min_{I_1,I_2\in F}\{ E(g_1[X,\gamma] + I_2[X]) + \lambda_1(ES_{\ a_1}(g_1[X,\gamma]) - E(g_1[X,\gamma] \ )) \\ + \lambda_2(VaR_{a_2}(I_2 \, [X]) - E(I_2[X])) \}$$

and the solution of the above is the following theorem.

Theorem 3.4. Denote $\gamma_1 = \frac{(\lambda_2 + \lambda_1 a_1)/(1-a_1)}{(1+\frac{\lambda_1 a_1}{1-a_1})}$ and a_1^* as the corollary 2.1. We make the assumption that $VaR_{a_1}\,(X) \leq VaR_{a_2}\,(X)$, then the optimal solutions are:

1. If $\lambda_1 > \lambda_2$ then

$$I_2^*[X] = \begin{cases} (X - VaR_{a_2}(X))_+ + \min\{ X, VaR_{a_1^*}(X), & 0 \leq \gamma < 1 \\ \min\{ X, VaR_{a_1^*}(X), & \gamma_1 < \gamma \leq 1 \end{cases}$$

2. If $\lambda_1 < \lambda_2$ then

$$I_2^*[X] = \begin{cases} (X - VaR_{a_2}(X))_+, & 0 \leq \gamma < \gamma_1 \\ 0, & \gamma_1 < \gamma \leq 1 \end{cases}$$

where $I_1^*[X] = X - I_2^*[X]$.

Proof. See at Asimit et al. (18 Jan 2012)

It is quite clear that the value of γ is very important for the produced results. Small values of $\gamma < \gamma_1$ give us as results, optimal risk the same as Corollary 2.1. Large values of $\gamma_1 \leq \gamma$ means a lower recovery given default. The optimal risk transfer changes, and the tail risk is no more transferred from the first entity (ES_{a_1} −regulated) to the second entity (VaR_{a_2} −regulated).

Chapter 4

The analytical presentation of a reinsurance model between an insurer & one or two reinsurers.

In this section, first we present two models of Thomas Cayè's Thesis titled Single liability claims stochastic modeling and applications. These two models produce claims and thus are appropriate to do the assumption, that they could be the portfolio of a re/insurance company and upon it, to build and examine something new. Thus later, based on the first model and the on the above mentioned theory, we developed a model of a reinsurance arrangement, between an insurance company and one or two reinsurance companies.

4.1 Presentation of Thomas Caye's model one and four.

First the following assumptions are inferred and are valid for both of the models. It assumed that the behaviors of different claims (the reporting and payment delay, payment duration, severity) are independent and further different random variables simulating the behavior of a claim, are independent of each other.

4.1.1 Model one.

The first model captures the variation in the severity of a claim, in the reporting and payment delays, while the payment is paid in one payment. The portfolio is considered built at time -1. The accidents happened at time $t = -1$ & $t = 0$, during the contract year. We have to mention that the reporting and payment delay can be zero. In this case we suppose that the first report and the first payment happened in year t = 0.

- The number of accidents happening during the contract year, N is Poisson distributed with parameter λ, $N \sim Poi(\lambda)$
- The reporting times $(T_i), i \epsilon [1, N]$ are independently Poisson distributed with parameter μ, $T_i \sim Poi(\mu), \forall\, i \epsilon [1, N]$
- The claims are settled in one full payment and the time delays $(P_i), i \epsilon [1, N]$, between the reporting and the payment are independently identically Poisson distributed with parameter ν, $P_i \sim Poi(\nu), \forall\, i \epsilon [1, N]$
- The severities $(X_i), i \epsilon [1, N]$ of the claims are independently and identically distributed. They follow a log – normal distribution with mean S_0 and log – variance σ_x $X_i \sim LN\left(\log(S_0) - \frac{{\sigma_x}^2}{2}, {\sigma_x}^2\right), \forall\, i \epsilon [1, N]$
- Because reporting and payment time delays are independent, we have that the total payment times $(D_i), i \epsilon [1, N]$ are Poisson distributed with parameter $\mu + \nu$.

4.1.2 Model four.

Even though we do not build a similar reinsurance arrangement for this model, it is interesting to present it. The fourth model contains IBNeR claims. It is worth to mention at this moment the definition of an IBNeR claim.

Definition 3.1. IBNeR are the claims that incurred but the reported claims are not enough. The amount of the loss varies and changes during the development years of a claim.

Take for example a car accident. At the beginning you may have a fix amount to pay for the damages that incurred to the cars, but later one passenger may find out that he was injured and he may claim an amount of money, for his health or for other costs.

In this model the IBNeR claims come from the expected total claim cost variability during the claim development. Variability to the loss initial expectation is added in order IBNeR claims to appear in the model. It is assumed that the first guess on the claim severity is not biased. The $(\,(\epsilon_j^i{}_{j\epsilon[1,d_i]}\,)_{i\epsilon[1,N]}$ is an i.d.d. family of log – normally distributed random variables with mean 1 and variance σ_ϵ^2, $\epsilon_j^{(i)} \sim LN\left(-\frac{\sigma_\epsilon^2}{2}, \sigma_\epsilon^2\right), \forall (i,j) \in [1,N]x[1,d_i]$. The payment starts at time $t = T_i + P_i$, with a fraction $\frac{1}{d_i}$ of the final amount, the rest to pay $\widetilde{R_1^i} = \left(\frac{d_i - 1}{d_i}\right) * X_0^{(i)}$ is then updated, multiplied by $\epsilon_1^{(i)}$ and becomes $R_1^i = \left(\frac{d_i - 1}{d_i}\right) * X_0^{(i)} * \epsilon_1^{(i)}$, where $X_0^{(i)}$ is the initial loss expectation $X_i = X_0^{(i)}$ and $(X_t^{(i)})_{t\geq 0}$ is a stochastic process, constant with respect to time and $\epsilon_0^{(i)} = 1, \forall i \in [1,N]$. At time $t = T_i + P_i + 1$ a fraction $\frac{1}{d_i - 1}$ of the rest to pay is paid by

$$Y_2^i = \frac{1}{d_i - 1} * R_1^i = \frac{1}{d_i} * \epsilon_1^{(i)} * X_0^{(i)}$$

The rest to pay $\widetilde{R_2^i} = \left(\frac{d_i - 2}{d_i}\right) * \epsilon_1^{(i)} * X_0^{(i)}$ is then updated $R_2^i = \left(\left(\frac{d_i - 2}{d_i}\right) * \epsilon_1^{(i)} * X_0^{(i)}\right) * \epsilon_2^{(i)}$. By induction we have that at time period $t = T_i + P_i + k \leq T_i + P_i + d_i - 1$ the payment is

$$Y_{k+1}^i = 1_{\{k \leq d_i - 1\}} \prod_{l=1}^{j} \epsilon_l^{(i)} (X_0^{(i)}/d_i)$$

And the updated rest to pay is

$$R_{k+1}^i = 1_{\{k \leq d_i - 1\}} \frac{d_i - k - 1}{d_i} \prod_{l=1}^{j} \epsilon_l^{(i)} X_0^{(i)}$$

And the ultimate amount to pay for the claim as known at time $t = T_i + P_i + k$ is

$$X_t^{(i)} = X_0^{(i)}(\sum_{j=0}^{k}\prod_{l=1}^{j}\epsilon_l^{(i)} + (d_i - k - 1)\prod_{l=1}^{k+1}\epsilon_l^{(i)})/d_i$$

4.1.3 Graphical representation of the optimization problems.

In this section we present graphical representations of the two step optimization problem that Asimit et al. (2012) proposed in their paper. In the model of the single reinsurance arrangement we will use the expected value principle in the form

$$P = (1+\rho)EL, \qquad for\ \rho = 0.05$$

and instead of the usual expected shortfall, we will use the conditional excess expected shortfall formed like it follows

$$EES_a(X) = ES_a(X) - E(X)$$

Our original portfolios come from the model one of the above paragraph, the one is the reporting process of the claims and the other is the payment process. We will present under the use of proportional and non – proportional reinsurance and more precise under the use of quota – share, stop – loss, excess – of – loss reinsurance, how the optimization procedures for the insurer are formed, while he evaluates his position by the use of certain risk measures.

To be more precise, the insurer decides which portion of his portfolio will retain and which wishes to cede. This action is taking place under one (or multiple) reinsurance arrangement(s).

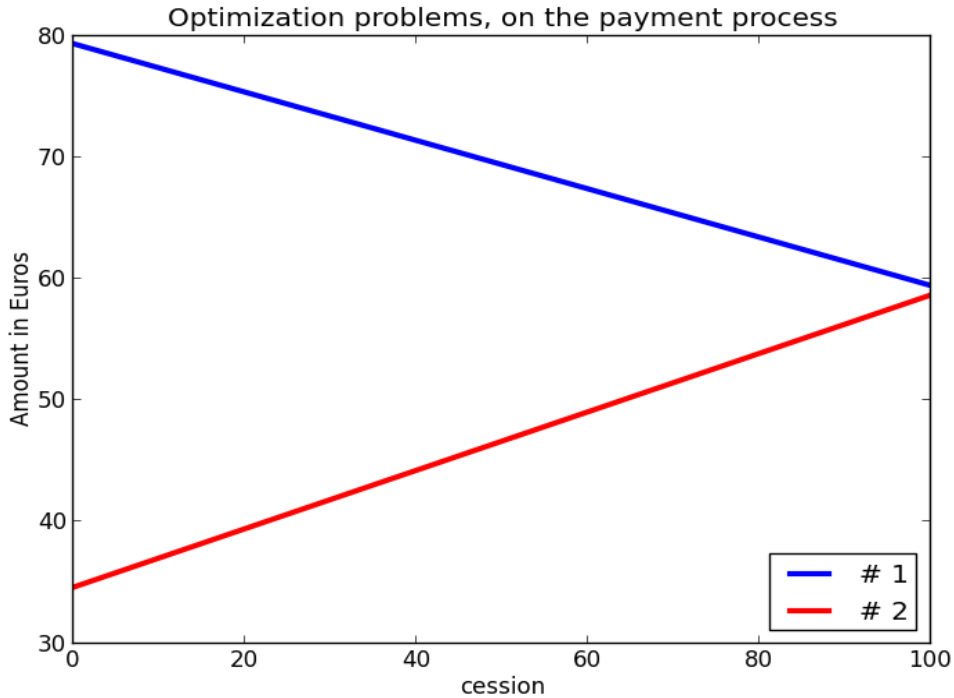
We present under different reinsurance arrangements how the final optimization problem

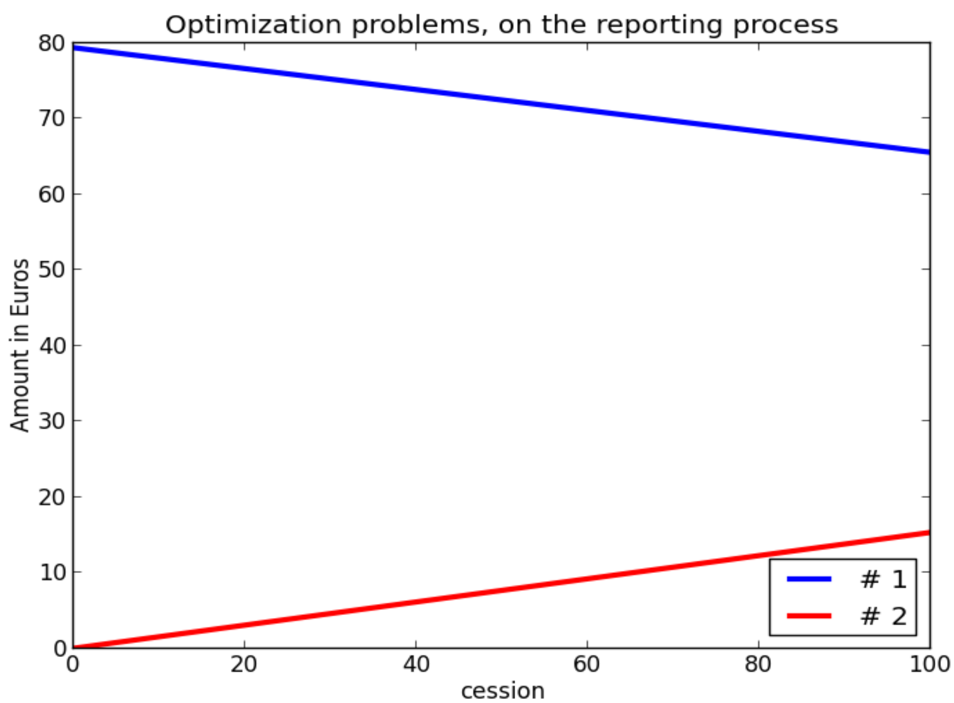
$$min_{R \in F}\{ \varphi_I(X) - \varphi_I(R[X]) + P(R[X])\}$$

is graphically formulated, with the use of certain risk measures and the expected value premium principle, while the insurer has already decided the portion of the risk transfer that wants to cede.

So the aim is to present all the possible optimal arrangements that the insurer may choose, under different each time portions of the original portfolio. The role of the portion, in a quota – share reinsurance, plays the cession and for an excess – of – loss or a stop – loss plays the deductible.

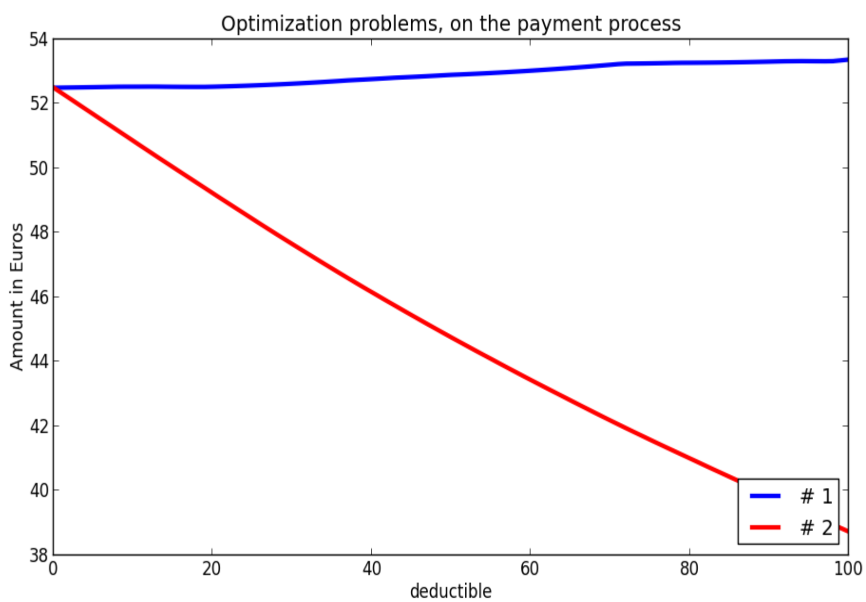
At the following plots, we present the optimal arrangement that lays for the insurer's portfolios, which in fact are the reporting and payment processes of the incurred claims. This optimal arrangement is taking place under the use of a quota – share reinsurance arrangement, while the portion of the ceded portfolio is between zero and one hundredth percent.

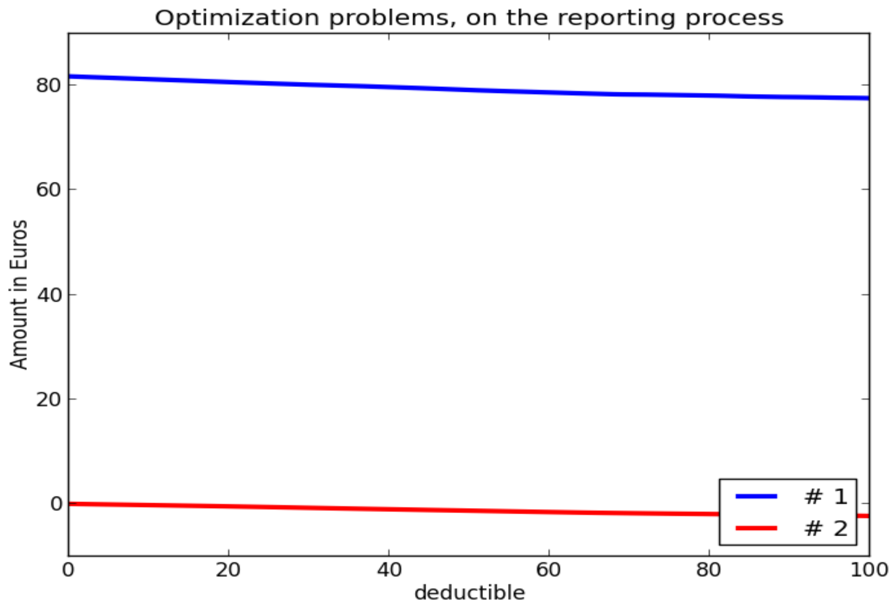

Optimization problems, on the payment process
Amount in Euros
cession
1
2
80
70
60
50
40
30
0
20
40
60
80
100


Optimization problems, on the reporting process
Amount in Euros
cession
1
2
80
70
60
50
40
30
20
10
0
0
20
40
60
80
100

The #1 and #2 are different optimal arrangements per different cession. The #1 is under the value – at – risk and the #2 is under the expected shortfall. We can see that while the loading factor of the premium is $\rho = 0.05$ and the cession is zero or takes values near to zero the optimal arrangements have approximately the value of the measured original portfolio under the value at risk or the expected shortfall. As the portion of the ceded portfolio grows the optimal arrangements tend to be near to the value of the charged premium. So we can see that optimal arrangements for the insurer are better to be chosen whenever he cedes 40% - 60% of his portfolio, always due to the fact that there is always an interpretation of the risk measures that he uses and the premium that he pays.

Now follows the graphical presentation of the optimal arrangements for the insurer, while he cedes his portfolios with the use of the stop – loss reinsurance, this time while the deductible takes the values from zero to one, examining the aggregated claims per unit of time, for the portfolios.



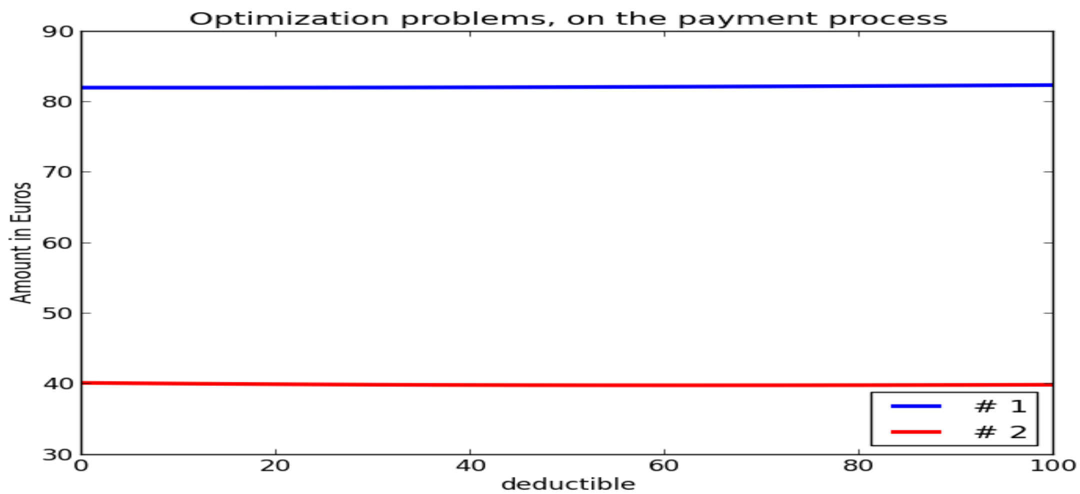


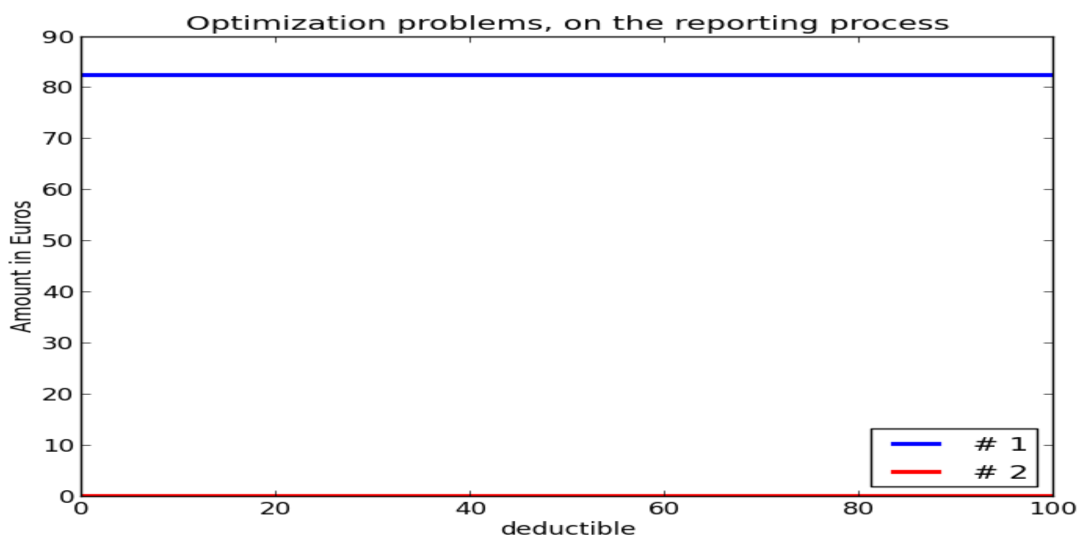
At the payment pattern of the first optimization problem under the value at risk as a risk measure, we see that independently of the value of the deducible our optimal arrangement for the problem is approximately the same. This happens due to the fact that the value at risk for the ceded losses and the premium that the insurer pays for those losses, are again approximately the same, for all the possible deductibles.

Further we see that in the case that the expected shortfall is used as a risk measure at the optimization problem, the optimal arrangements follow a decreasing path. Although it begins by the same value as the VaR optimization problem begins, we see the more he cedes the more premium he pays, thus although the graph is decreasing is not necessary the optimal way to transfer capitals.

The reporting pattern of the first optimization problem under the above mentioned risk measures, remains relatively stable as the deductible changes values. This happens because in both cases the capital that is in risk for the ceded portfolio has approximately the same value for all the different values that the deductible could take, with the capital that he retains and the premium that he has to pay.

Under the use of excess – of – loss reinsurance arrangement, the ceded portfolios come from the examination between the deductible - that takes the values from zero to one - and each claim per unit of time. This time the graphical presentations of the optimal arrangements for the insurer are:





We notice that both portfolios - that is actually the payment and the reporting patterns of the incurred claims - the optimal arrangements for the insurer under different deductibles, are exactly the same as he would not choose to reinsure the portfolios, since the optimal arrangements have always as a value the value at risk or the expected shortfall of the original portfolio.

We have built a reinsurance arrangement under two quota – share treaties, in the case that the insurer chooses to reinsure his portfolio under two reinsurers. We present how optimal arrangements take place, when first the insurer chooses to ceded the 25% or the 50% or the 80% of his portfolio. Under this first ceded part of the portfolio, we examine how the second quota share arrangement takes place while this time the cession is between 0% and 50% of the retained original portfolio.

The mathematical formulation for a multiple reinsurance contract is as follows:

$$min_{(R_1[X]R_2[X])\in G} VaR_a(L(R_1[X], R_2[X])) = VaR_a(I[X]) + P_1(R_1[X]) + P_2(R_2[X])$$

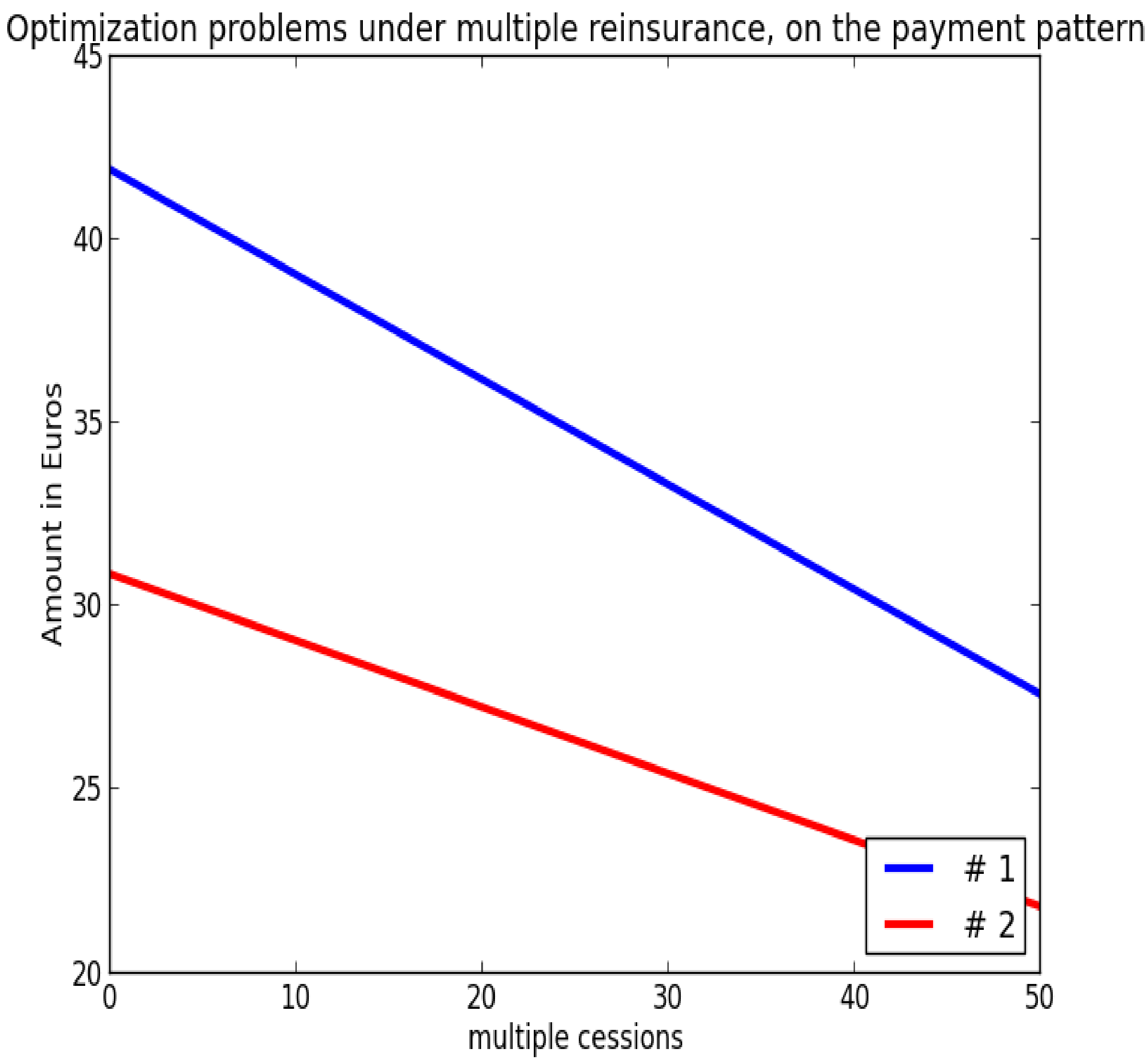
while the premium principles are measured different in the second reinsurance contract compare to the first one. This time for an optimization problem based in value at risk or in expected shortfall measure, we use the following premiums:

$$P = EL(I[X]) + \gamma * VaR(R[X])$$

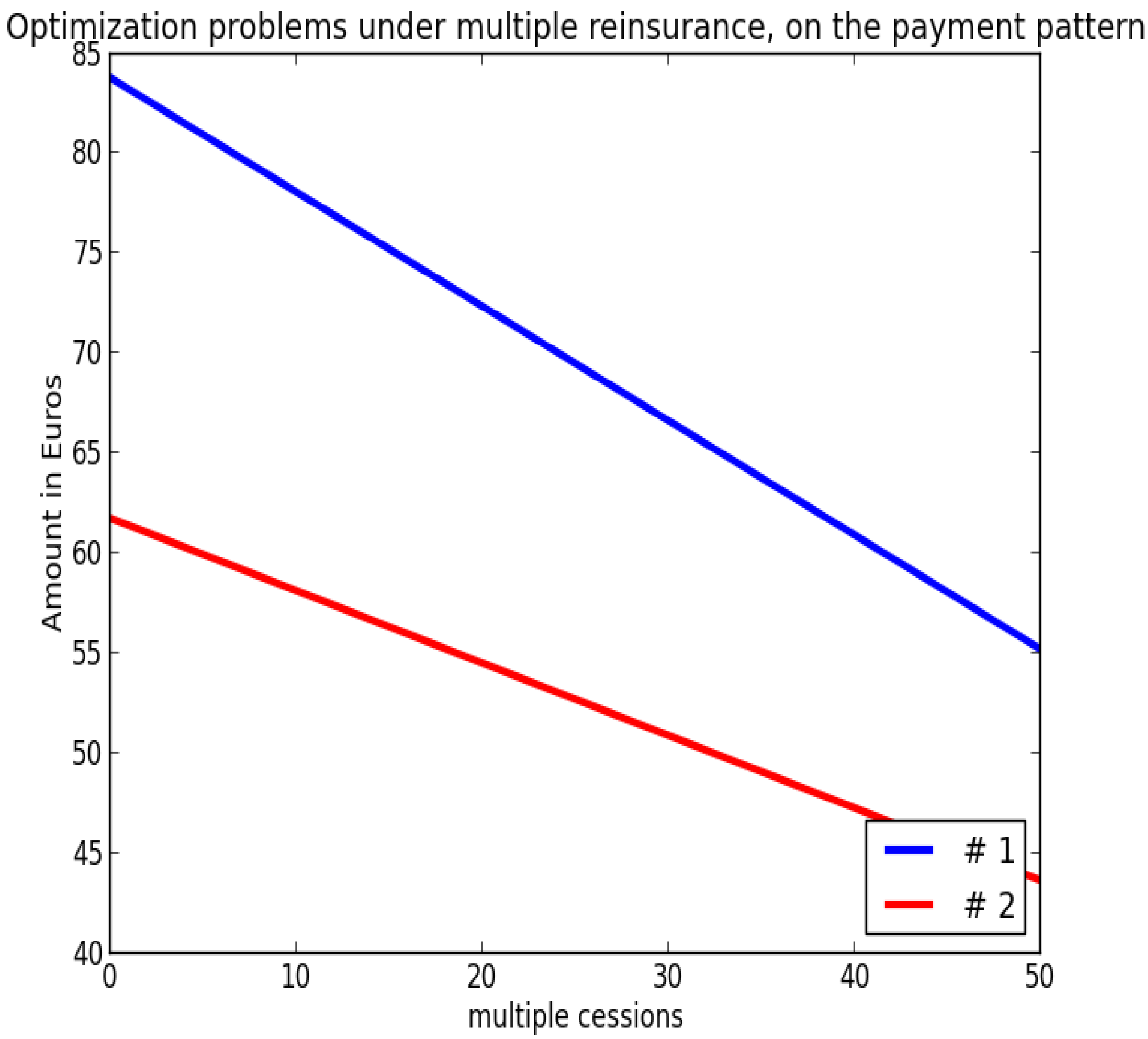
and

$$P = EL(I[X]) + \gamma * ES(R[X])$$

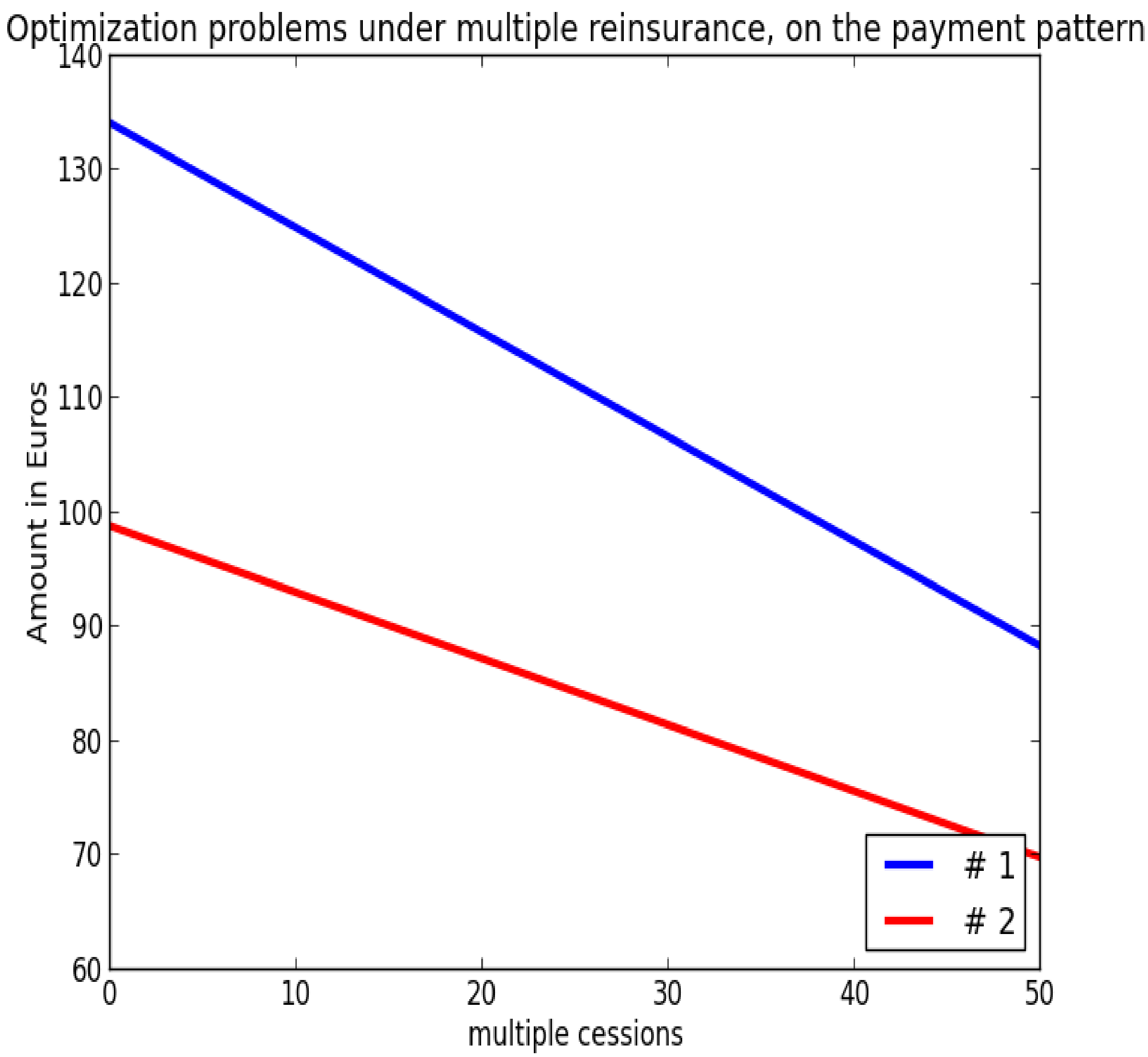
While the first ceded part is 25% of the original portfolio, the optimal arrangement is in the following form:



Now while the first ceded part is 50% of the original portfolio, the optimal arrangement is in the following form:



Finally while the first ceded part is 80% of the original portfolio, the optimal arrangement is in the following form:



Given the above graphs we could say that the insurer the more he chooses to cede, the more he ends to pay for the premiums. Thus the best solution for him would be to choose the first reinsurance arrangement to be with a cession of 25% of his original portfolio and the next one cession with the other reinsurer to be at most at level of the 20% .

Appendix

In this section we present the parts of the code that we built in order to plot the above graphical presentations of the reinsurance arrangement.

The main code that produces the single claims liabilities, transformed from the original Caye's code, is the following:

```
import numpy as np

def claims_function(l=50, mu=10, nu=10, numSim=1000, S0=1, sigma=1):

    N = np.random.poisson( l,[numSim,1] )

    T = []

    P = []

    D = []

    X = []

    TmaxList = []

    for k in range(numSim):

        Tmax = 0

        a = list(np.sort(np.random.poisson(mu,int(N[k]))))

        b = list(np.random.poisson(nu,int(N[k])))

        T.append(a)

        P.append(b)

        c = [sum(pair) for pair in zip(a,b)]

        D.append(c)
```

```
        X.append( np.random.lognormal(-sigma*sigma/2.+np.log(S0), sigma, int(N[k]) ) )

        if N[k]!= 0:

            if Tmax < max( D[k] ):

                Tmax = max( D[k] )

        TmaxList.append(Tmax)

    Srep = []

    Spay = []

# Set up of the 3d list of zeros

    for i in range(numSim):      # i ... number of simulations

        simulationRep = []   # Create a list for the reported amounts that are delayed, for
each simulation

        simulationPay = []   # Create a list for the payment amounts that are delayed, for
each simulation

        j = int(N[i])            # j number of claims

            for k in range( j ):      # k   ... counts the number of claims

                dur = TmaxList[i]      # dur ... duration of the indivudal claim we're looking at,
same per simulation

            cf1 = np.zeros( dur + 1 )   # set up the cash flows, per claim for each simulation

            cf2 = np.zeros( dur + 1 )

                simulationRep.append(cf1)   # append each cash flow to the simulationRep list

                simulationPay.append(cf2)   # append each cash flow to the simulationPay list


            Srep.append(simulationRep)   # append each simulationRep list to the list of the
Stochastic Process of the Reported Amounts
```

```
            Spay.append(simulationPay)    # append each simulationPay list to the list of the Stochastic Process of the Payment Amounts


# Fill in the 3d list of zeros with the reported and the payments amounts - decummulated

   for i in range(numSim):   # i ... number of simulations

      for s in range( int(N[i]) ):   # s ... number of claims

         val = X[i][s]

         Srep[i][s][ T[i][s] ] = val

         Srep[i][s][ D[i][s] ] = -val

         if D[i][s] != 0:

            for l in range( D[i][s], 2 * D[i][s], D[i][s] ):#, max(D[i]) + 1):

               value = X[i][s]

               Spay[i][s][l] += value

         else:

            value = X[i][s]

            Spay[i][s][0] += value

# ---> sum of each element per simulation of the Stochastic Process of the Reported or Payments Amounts

   AgrSrep = []         # create an empty list of the stochastic process of the Srep

   AgrSpay =[]

   w = 0                # counter

   for sim_data in Srep:      # loops the simulations of the first "creation" of the Srep

      new_AgrSimRep = []
```

```
        t = np.zeros(TmaxList[w] + 1)    # Create an empty array for each simulation of the first "creation" of the Srep

        for data in sim_data:        # takes the cash flow of each accident of each simulation from the first "creation" of the Srep

            for i in range( len(data) ):

                t[i] += data[i]         # sums all the cash flows of all accidents of each simulation from the first "creation" of the Srep

        new_AgrSimRep.append(t)         # appends the previous sum to a new simulation

        AgrSrep.append(new_AgrSimRep)   # appends the new simulation to the final - and right - Srep

        w += 1

    q = 0

    for sim_data2 in Spay:    # loops the simulations of the first "creation" of the Srep

        new_AgrSimPay = []

        r = np.zeros(TmaxList[q] + 1)   # Create an empty array for each simulation of the first "creation" of the Srep

        for data2 in sim_data2:     # takes the cash flow of each accident of each simulation from the first "creation" of the Srep

            for i in range( len(data2) ):

                r[i] += data2[i]    # sums all the cash flows of all accidents of each simulation from the first "creation" of the Srep

        new_AgrSimPay.append(r)      # appends the previous sum to a new simulation

        AgrSpay.append(new_AgrSimPay)     # appends the new simulation to the final - and right - Srep

        q += 1

return AgrSrep, AgrSpay,Srep, Spay, TmaxList, N
```

The VaR is formulated as the following function:

```
def VaR(ptf,l): #layer
    TotalLossPerSim=[]
    for el in ptf:
        for claims in el:
            TotalLossPerSim.append( sum(claims) ) #instead of claims[-1], because by this structure of The Spay, now the last element of the cashflow of an accident in a simulation, is not the sum of all the previous claims of the year.
    TotalLossPerSim.sort()
    NumOfBigLosses = l * len(ptf)
    if NumOfBigLosses < 1:
        var=TotalLossPerSim[0]
    else:
        var=( TotalLossPerSim[int( len(ptf) - NumOfBigLosses ) - 1] + TotalLossPerSim[int( len(ptf) - NumOfBigLosses )] ) /2
    return(var)
```

The expected shortfall is in the form:

```
# l argument is the layer
def calc_shortfall(ptf,l):
    UltimateLossPerSim = []
    for el in ptf:
        for claims in el:
            UltimateLossPerSim.append( sum(claims) )
    UltimateLossPerSim.sort()
    NumOfBigLosses = l * len(ptf)
    if NumOfBigLosses == 0:
        ExpShortfall=sum( UltimateLossPerSim[-int(NumOfBigLosses): ] ) / len(ptf)
    else:
        AverageOfLosses = sum(UltimateLossPerSim[-int(NumOfBigLosses):]) /
float(NumOfBigLosses)
        ExpShortfall = AverageOfLosses - sum(UltimateLossPerSim) / len(ptf)
    return(ExpShortfall)
```

The Aggregated form of the stochastic processes for the reporting and payment patterns, are given by the following function:

```
import numpy as np

def aggregate_ptf(ptf):

    AgrPtf = []

    w = 0

    for sim_data in ptf:

        new_AgrPtf = []

        for data in sim_data:

            t = np.zeros(len(data))

            for i in range( len(data) ):

                t[i] += data[i]

        new_AgrPtf.append(t)

        AgrPtf.append(new_AgrPtf)

        w += 1

    return AgrPtf
```

The expected losses of a portfolio are given by:

```
import numpy as np

def calc_expected_loss(ptf):
    SumLastElem=0.
    for el in ptf:
        for claim in el:
            SumLastElem += sum( claim )
    EL = SumLastElem / len( ptf )
    return EL
```

The excess – of – loss reinsurance arrangement is given by the following function:

```
def excess_of_loss( ptf, d, l ):
    ptf_stays = []
    ptf_ceded = []
    for sim in ptf:
        new_sim_ptf_stays = []
        new_sim_ptf_ceded = []
        for claim in sim:
            new_claim_stays = []
            new_claim_ceded = []
            for i in range( len(claim) ):
                if claim[i] <= d and claim[i] > 0:
```

```
            new_claim_stays.append(claim[i])
            new_claim_ceded.append(0)
        elif claim[i] >= d and claim[i] <= l:
            new_claim_stays.append(d)
            new_claim_ceded.append(claim[i] - d)
        elif claim[i] >= d and claim[i] > l:
            new_claim_stays.append(d + claim[i] - l)
            new_claim_ceded.append(l - d)
        elif claim[i] < 0:
            if abs(claim[i]) <= d and abs(claim[i]) > 0:
                new_claim_stays.append(claim[i])
                new_claim_ceded.append(0)
            elif abs(claim[i]) > d and abs(claim[i]) <= l:
                new_claim_stays.append(-d)
                new_claim_ceded.append(claim[i] + d)
            elif abs(claim[i]) > d and abs(claim[i]) >l:
                new_claim_stays.append(claim[i] + l - d)
                new_claim_ceded.append(-l + d)
    new_sim_ptf_stays.append(new_claim_stays)
    new_sim_ptf_ceded.append(new_claim_ceded)
ptf_stays.append(new_sim_ptf_stays)
ptf_ceded.append(new_sim_ptf_ceded)
```

```
    return ptf_stays, ptf_ceded


def calc_xlr(ptf1, ptf2, d, l):
    a, b = excess_of_loss(ptf1,d, l)
    c, d = excess_of_loss(ptf2, d, l)
    return a, b, c, d
```

The quota – share reinsurance arrangement is given by:

```
def qs(ptf, cession,treaty_limitPTF):
    ptf_ceded = []
    ptf_stays = []
    a = 0    #counter --> purpose : to give the appropriate treaty limit
    for sim in ptf:
        for claim in sim:
            new_simulation_ceded = []
            new_simulation_retained = []
            claim_ceded = []
            claim_retained = []
            for i in claim:
                if i > 0 and i <= treaty_limitPTF[a]:
                    claim_ceded.append( i * cession )
```

```
                elif i > 0 and i > treaty_limitPTF[a]:

                    claim_ceded.append( treaty_limitPTF[a] * cession )

                elif i < 0 and abs(i) > treaty_limitPTF[a]:

                    claim_ceded.append( -(treaty_limitPTF[a] * cession) )

            new_simulation_ceded.append( claim_ceded )

            for j in claim:

                if  j > 0 and j <= treaty_limitPTF[a] :

                    claim_retained.append( j * (1.-cession) )

                elif j > 0 and j > treaty_limitPTF[a]:

                    claim_retained.append( treaty_limitPTF[a] * (1 - cession) + (j -
treaty_limitPTF[a]) )

                elif j < 0 and abs(j) > treaty_limitPTF[a]:

                    claim_ retained.append( -( (treaty_limitPTF[a] * (1 - cession) ) + (j -
treaty_limitPTF[a]) ) )

            new_simulation_retained.append(claim_retained)

            a += 1

        ptf_ceded.append(new_simulation_ceded)

        ptf_stays.append(new_simulation_retained)

    return ptf_stays, ptf_ceded
```

Another formulation of the quota –share that we use, which this time splits the claims accordingly the portion of the cession that the insurer decides to ceded from his portfolio is:

```
def deAgrQS(ptf, cession):
    ptf_ceded = []
    ptf_retained = []
    for sim in ptf:
        sim_retained = []
        sim_ceded = []
        for claim in sim:
            retained_part = []
            ceded_part = []
            for i in range( len(claim) ):
                retained_part.append( ( 1- cession )*claim[i] )
                ceded_part.append(cession*claim[i])
            sim_retained.append(retained_part)
            sim_ceded.append(ceded_part)
        ptf_retained.append(sim_retained)
        ptf_ceded.append(sim_ceded)
    return ptf_retained, ptf_ceded
```

The stop – loss reinsurance arrangement is given by:

```
def stop_loss(ptf, deductible):
    ptf_stays = []
    ptf_ceded = []
    for sim in ptf:
        sim_stays = []
        sim_ceded = []
        for aggrclaims in sim:
            aggrclaims_stays = []
            aggrclaims_ceded = []
            for el in aggrclaims:
                if el > 0 and el <= deductible:
                    aggrclaims_stays.append(el)
                    aggrclaims_ceded.append(0)
                elif el > deductible:
                    aggrclaims_stays.append(deductible)
                    aggrclaims_ceded.append( el - deductible )
                elif el < 0:
                    if abs(el)> 0 and abs(el) <= deductible:
                        aggrclaims_stays.append(el)
                        aggrclaims_ceded.append(0)
```

```
            elif abs(el) > deductible:
                aggrclaims_stays.append(-deductible)
                aggrclaims_ceded.append( el + deductible)
        sim_stays.append(aggrclaims_stays)
        sim_ceded.append(aggrclaims_ceded)
    ptf_stays.append(sim_stays)
    ptf_ceded.append(sim_ceded)
return (ptf_stays, ptf_ceded)
```

And the formulation for a double quota – share reinsurance arrangement, between two reinsurers is:

```
import deAgregatedQS as dAgQS
import VaR as var
import ExpectedShortfall as es
import mean
import matplotlib as m
```

```
def Multi_QS(AgrSrep_stays, AgrSpay_stays, AgrSpay_ceded, AgrSrep_ceded, layer, b, d, p, gamma, cession, premium_1, premium_1_rep):

# Set of lists for the reporting pattern ...

    OptimizationMultiReProblem1 = []

    OptimizationMultiReProblem2 = []

    OptimizationMultiReProblem_rep1 = []

    OptimizationMultiReProblem_rep2 = []

    ces2 = 0

    while ces2 <= 50:

        cession2 = float( float(ces2)/ 100 )

        AgrSrep_stays2, AgrSrep_ceded2 = dAgQS.deAgrQS(AgrSrep_stays, cession2)

        AgrSpay_stays2, AgrSpay_ceded2 = dAgQS.deAgrQS(AgrSpay_stays, cession2)

# For the Payment pattern ---->

        ee = var.VaR(AgrSpay_stays2,layer)

        gg = es.calc_shortfall(AgrSpay_stays2,layer)

        ELretain = el.calc_expected_loss(AgrSpay_stays2)

# For the ceded part 1 ---->

        premium_2_ = ELretain + gamma * var.VaR(AgrSpay_ceded2,layer)
```

```
premium_3_ = ELretain+ gamma * es.calc_shortfall(AgrSpay_ceded2,layer)

o = ee + premium_2_ + premium_1

OptimizationMultiReProblem1.append(o)

oo = gg + premium_3_ + premium_1

OptimizationMultiReProblem2.append(oo)

# For the Reports pattern ----->

eee = var.VaR(AgrSrep_stays2,layer)

ggg = es.calc_shortfall(AgrSrep_stays2,layer)

ELretain_rep = el.calc_expected_loss(AgrSrep_stays2)

# For the ceded part 1 ---->

premium_2_rep = ELretain_rep + gamma * var.VaR(AgrSrep_ceded2,layer)

premium_3_rep = ELretain_rep+ gamma * es.calc_shortfall(AgrSrep_ceded2,layer)

w = eee + premium_2_rep + premium_1_rep

OptimizationMultiReProblem_rep1.append(w)

ww = ggg+ premium_3_rep + premium_1_rep

OptimizationMultiReProblem_rep1.append(ww)

ces2 += 1
```

PLOTS FOR THE PAYMENT PATTERN

```
 m.pyplot.plot(OptimizationMultiReProblem1, color='b', linewidth=3.0, label = '# 1',hold=True)

   m.pyplot.plot(OptimizationMultiReProblem2, color='r', linewidth=3.0, label = '# 2',hold=True)

   m.pyplot.xlabel('multiple cessions')

   m.pyplot.ylabel('Amount in Euros')

   m.pyplot.title('Optimization problems under multiple reinsurance, on the payment pattern')

   m.pyplot.legend(loc='lower right')

   m.pyplot.show()
```

PLOTS FOR THE REPORTING PATTERN

```
   m.pyplot.plot(OptimizationMultiReProblem_rep1, color='b', linewidth=3.0, label = '# 1')

   m.pyplot.plot(OptimizationMultiReProblem_rep2, color='r', linewidth=3.0, label = '# 2')

   m.pyplot.xlabel('multiple cessions')

   m.pyplot.ylabel('Amount in Euros')
```

```
    m.pyplot.title('Optimization problems under multiple reinsurance, on the reported pattern')

    m.pyplot.legend(loc='lower right')

    m.pyplot.show()

    return
```

The part of the code that behaves accordingly to what the user inputs is:

```
import code

import QuotaShare as QS

import ExcessOfLoss as xlr

import AggregatePTFs as agg

import sys

import VaR as var

#import ExpectedLoss as el

import ExpectedShortfall as es

#import AverageLag as al

#import pprint

import StopLoss as st

import mean

import matplotlib.pyplot as plt

import matplotlib as m

#import StoringFile as sf

#import deAgregatedQS as dAgQS
```

```
import DoubleQSRe as qsqs

def run_code_with_user_input():

    #pp = pprint.PrettyPrinter(indent=1, width=1000, depth=None, stream=None)

    AgrSrep, AgrSpay, Srep, Spay,TmaxList, N = code.claims_function(l = 50,mu = 10, nu = 10, numSim = 1000, S0 = 1, sigma = 1)

    #sf.StoringFile( AgrSrep, AgrSpay) # ... code for storing

    ptf = 1 # ... Select Reinsurance program, input 1 for Quota Share, input 2 for Stop - Loss, input 3 for Excess - of - Loss per risk.

    Multiple_Re = True # ... True for a double Quota Share Re-treaty, False for a single one.

    quantile=0.99

    layer=1-quantile

    #EL = mean.arithmetic_mean(AgrSpay)

    VaRorig = var.VaR(AgrSpay,layer)

    ESorig = es.calc_shortfall(AgrSpay, layer)

    VaRorig_rep = var.VaR(AgrSpay,layer)

    ESorig_rep = es.calc_shortfall(AgrSrep, layer)

    if ptf==1:

        treaty_limit = []
```

```
    for sim in AgrSpay: # making one list for the treaty limits  per ptf/simulation
        for claim in sim:
            treaty_limit.append( 8 )
# SET OF THE LISTS FOR THE PAYMENT PATTERN
    lista_R = [] #list for the retained part of the VaR
    lista_C = [] #list for the ceded part of the VaR
    List_R = [] #list for the retained part of the ES
    List_C = [] #list for the ceded part of the ES
    ListOfEls = [] #list of the different EL-because of the different cessions-ceded
    ListOfEls_stays = []
    OptimizationProblem1 = []
    OptimizationProblem2 = []
# SET OF THE LISTS FOR THE REPORTING PATTERN
    lista_rep_C = []
    lista_rep_R = []
    List_rep_C = []
    List_rep_R = []
    ListOfEls_rep = []
```

```
ListOfEls_rep_stays = []

OptimizationProblem1_rep = []

OptimizationProblem2_rep = []

p = 0.05

gamma = 0.05

ces = 0

while ces <= 100 :

    cession = float( float(ces) / 100 )

    print cession

    AgrSrep_stays, AgrSrep_ceded = QS.qs(AgrSrep, cession,treaty_limit)

    AgrSpay_stays, AgrSpay_ceded = QS.qs(AgrSpay, cession,treaty_limit)

    a = var.VaR(AgrSpay_ceded,layer)

    lista_C.append(a) # A list of VaR's for the ceded amounts, that are refered to different cessions and so to different portfolios (Spay)

    b = var.VaR(AgrSpay_stays,layer)

    lista_R.append(b) # A list of VaR's for the retained amounts, that are refered to different cessions and so to different portfolios (Spay).

    c = es.calc_shortfall(AgrSpay_ceded,layer)
```

```
        List_C.append(c)

        d = es.calc_shortfall(AgrSpay_stays,layer)

        List_R.append(d)

        expLoss_pay_ceded = mean.arithmetic_mean(AgrSpay_ceded)

        ListOfEls.append(expLoss_pay_ceded)

        expLoss_pay_retained = mean.arithmetic_mean(AgrSpay_stays)

        ListOfEls_stays.append(expLoss_pay_retained)

        premium_1 = ( 1 + p ) * expLoss_pay_ceded

        #premium_2 = EL + gamma * var.VaR(AgrSpay_ceded,layer)

        #premium_3 = EL+ gamma * es.calc_shortfall(AgrSpay_ceded,layer)

        t = VaRorig - a + premium_1

        tt = ESorig - c + premium_1

        OptimizationProblem1.append(t)

        OptimizationProblem2.append(tt)

# For the Reporting pattern the OptSolutions

        e = var.VaR(AgrSrep_ceded,layer)

        lista_rep_C.append(e)
```

```
f = var.VaR(AgrSrep_stays,layer)

lista_rep_R.append(f)

h = es.calc_shortfall(AgrSrep_ceded,layer)

List_rep_C.append(h)

k = es.calc_shortfall(AgrSrep_stays,layer)

List_rep_R.append(k)

expLoss_rep_ceded = mean.arithmetic_mean(AgrSrep_ceded)

ListOfEls_rep.append(expLoss_rep_ceded)

expLoss_rep_retained = mean.arithmetic_mean(AgrSrep_stays)

ListOfEls_rep_stays.append(expLoss_rep_retained)

premium_1_rep = ( 1 + p ) * expLoss_rep_ceded

#premium_2_rep = EL + gamma * var.VaR(AgrSrep_ceded,layer)

#premium_3_rep = EL+ gamma * es.calc_shortfall(AgrSrep_ceded,layer)

r = VaRorig_rep - e + premium_1_rep

rr = ESorig_rep - h + premium_1_rep

OptimizationProblem1_rep.append(r)

OptimizationProblem2_rep.append(rr)
```

```
if Multiple_Re == True:

    if ces == 80 or ces == 60 or ces == 50 or ces ==35 or ces ==25 or ces ==15:

        qsqs.Multi_QS(AgrSrep_ceded, AgrSpay_ceded, AgrSpay_ceded,
AgrSrep_ceded, layer, b, d, p, gamma, cession)

ces += 1

# PLOTS FOR THE PAYMENT PATTERN

m.pyplot.plot(OptimizationProblem1, color='b', linewidth=3.0, label = '# 1')

m.pyplot.plot(OptimizationProblem2, color='r', linewidth=3.0, label = '# 2')

m.pyplot.xlabel('cession')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Optimization problems, on the payment process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()

m.pyplot.plot(lista_C, color='r', linewidth=3.0, label = 'Retained')

m.pyplot.plot(lista_R, color='b', linewidth=3.0, label = 'Ceded')

m.pyplot.xlabel('cession')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Value at Risk for the Retained & Ceded ptf, on the payment process')
```

```
m.pyplot.legend(loc='lower right')

m.pyplot.show()

m.pyplot.plot(List_R, color='r', linewidth=3.0, label = 'Retained')

m.pyplot.plot(List_C, color='b', linewidth=3.0, label = 'Ceded')

m.pyplot.xlabel('cession')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Expected Shortfall for the  Retained & Ceded ptf, on the payment process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()

m.pyplot.plot(ListOfEls,color='r', linewidth=3.0, label = 'Ceded' )

m.pyplot.plot(ListOfEls_stays, color='b', linewidth=3.0, label = 'Retained')

m.pyplot.xlabel('cession')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Expected Losses for the Retained & Ceded ptf, on the payment process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()
```

```
# PLOTS FOR THE REPORTING PATTERN

    m.pyplot.plot(OptimizationProblem1_rep, color='b', linewidth=3.0, label = '# 1')

    m.pyplot.plot(OptimizationProblem2_rep, color='r', linewidth=3.0, label = '# 2')

    m.pyplot.xlabel('cession')

    m.pyplot.ylabel('Amount in Euros')

    m.pyplot.title('Optimization problems, on the reporting process')

    m.pyplot.legend(loc='lower right')

    m.pyplot.show()

    m.pyplot.plot(lista_rep_C, color='r', linewidth=3.0, label = 'Retained')

    m.pyplot.plot(lista_rep_R, color='b', linewidth=3.0, label = 'Ceded')

    m.pyplot.xlabel('cession')

    m.pyplot.ylabel('Amount in Euros')

    m.pyplot.title('Value at Risk for the Retained & Ceded ptf, on the reporting process')

    m.pyplot.legend(loc='lower right')

    m.pyplot.show()

    m.pyplot.plot(List_rep_C, color='r', linewidth=3.0, label = 'Retained')

    m.pyplot.plot(List_rep_R, color='b', linewidth=3.0, label = 'Ceded')

    m.pyplot.xlabel('cession')
```

```
        m.pyplot.ylabel('Amount in Euros')
        m.pyplot.title('Expected Shortfall for the  Retained & Ceded ptf, on the reporting process')
        m.pyplot.legend(loc='lower right')
        m.pyplot.show()
        m.pyplot.plot(ListOfEls_rep,color='r', linewidth=3.0, label = 'Ceded' )
        m.pyplot.plot(ListOfEls_rep_stays, color='b', linewidth=3.0, label = 'Retained')
        m.pyplot.xlabel('cession')
        m.pyplot.ylabel('Amount in Euros')
        m.pyplot.title('Expected Losses for the Retained & Ceded ptf, on the reporting process')
        m.pyplot.legend(loc='lower right')
        m.pyplot.show()
        exit()
    elif ptf == 2:
# SET OF THE LISTS FOR THE PAYMENT PATTERN
        lista_R = [] #list for the retained part of the VaR
        lista_C = [] #list for the ceded part of the VaR
```

```
    List_R = [] #list for the retained part of the ES

    List_C = [] #list for the ceded part of the ES

    ListOfEls = [] #list of the different EL-because of the different cessions-ceded

    ListOfEls_stays = []

    OptimizationProblem1 = []

    OptimizationProblem2 = []

# SET OF THE LISTS FOR THE REPORTING PATTERN

    lista_rep_C = []

    lista_rep_R = []

    List_rep_C = []

    List_rep_R = []

    ListOfEls_rep = []

    ListOfEls_rep_stays = []

    OptimizationProblem1_rep = []

    OptimizationProblem2_rep = []

    p = 0.05

    gamma = 0.05

    mirrordeductible = 0
```

```
    while mirrordeductible <= 100 :

        deductible = float( float(mirrordeductible) / 100 )

        print deductible

        AgrSrep_stays, AgrSrep_ceded = st.stop_loss( AgrSrep, deductible )

        AgrSpay_stays, AgrSpay_ceded = st.stop_loss( AgrSpay, deductible )

# For the Ceded pattern the OptSolutions

        a = var.VaR(AgrSpay_ceded,layer)

        lista_C.append(a) # A list of VaR's for the ceded amounts, that are refered to
different cessions and so to different portfolios (Spay)

        b = var.VaR(AgrSpay_stays,layer)

        lista_R.append(b) # A list of VaR's for the retained amounts, that are refered to
different cessions and so to different portfolios (Spay).

        c = es.calc_shortfall(AgrSpay_ceded,layer)

        List_C.append(c)

        d = es.calc_shortfall(AgrSpay_stays,layer)

        List_R.append(d)

        expLoss_pay_ceded = mean.arithmetic_mean(AgrSpay_ceded)

        ListOfEls.append(expLoss_pay_ceded)
```

```
        expLoss_pay_retained = mean.arithmetic_mean(AgrSpay_stays)

        ListOfEls_stays.append(expLoss_pay_retained)

        premium_1 = ( 1 + p ) * expLoss_pay_ceded

        #premium_2 = EL + gamma * var.VaR(AgrSpay_ceded,layer)

        #premium_3 = EL+ gamma * es.calc_shortfall(AgrSpay_ceded,layer)

        t = VaRorig - a + premium_1

        tt = ESorig - c + premium_1

        OptimizationProblem1.append(t)

        OptimizationProblem2.append(tt)

# For the Reporting pattern the OptSolutions

        e = var.VaR(AgrSrep_ceded,layer)

        lista_rep_C.append(e)

        f = var.VaR(AgrSrep_stays,layer)

        lista_rep_R.append(f)

        h = es.calc_shortfall(AgrSrep_ceded,layer)

        List_rep_C.append(h)

        k = es.calc_shortfall(AgrSrep_stays,layer)

        List_rep_R.append(k)
```

```
        expLoss_rep_ceded = mean.arithmetic_mean(AgrSrep_ceded)

        ListOfEls_rep.append(expLoss_rep_ceded)

        expLoss_rep_retained = mean.arithmetic_mean(AgrSrep_stays)

        ListOfEls_rep_stays.append(expLoss_rep_retained)

        premium_1_rep = ( 1 + p ) * expLoss_rep_ceded

        #premium_2_rep = EL + gamma * var.VaR(AgrSrep_ceded,layer)

        #premium_3_rep = EL+ gamma * es.calc_shortfall(AgrSrep_ceded,layer)

        r = VaRorig_rep - e + premium_1_rep

        rr = ESorig_rep - h + premium_1_rep

        OptimizationProblem1_rep.append(r)

        OptimizationProblem2_rep.append(rr)

        mirrordeductible += 1

# PLOTS FOR THE PAYMENT PATTERN

    m.pyplot.plot(OptimizationProblem1, color='b', linewidth=3.0, label = '# 1')

    m.pyplot.plot(OptimizationProblem2, color='r', linewidth=3.0, label = '# 2')

    m.pyplot.xlabel('deductible')

    m.pyplot.ylabel('Amount in Euros')

    m.pyplot.title('Optimization problems, on the payment process')
```

```
m.pyplot.legend(loc='lower right')

m.pyplot.show()

m.pyplot.plot(lista_C, color='r', linewidth=3.0, label = 'Retained')

m.pyplot.plot(lista_R, color='b', linewidth=3.0, label = 'Ceded')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Value at Risk for the Retained & Ceded ptf, on the payment process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()

m.pyplot.plot(List_R, color='r', linewidth=3.0, label = 'Retained')

m.pyplot.plot(List_C, color='b', linewidth=3.0, label = 'Ceded')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Expected Shortfall for the  Retained & Ceded ptf, on the payment process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()
```

```
m.pyplot.plot(ListOfEls,color='r', linewidth=3.0, label = 'Ceded' )

m.pyplot.plot(ListOfEls_stays, color='b', linewidth=3.0, label = 'Retained')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Expected Losses for the Retained & Ceded ptf, on the payment process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()

# PLOTS FOR THE REPORTING PATTERN

m.pyplot.plot(OptimizationProblem1_rep, color='b', linewidth=3.0, label = '# 1')

m.pyplot.plot(OptimizationProblem2_rep, color='r', linewidth=3.0, label = '# 2')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Optimization problems, on the reporting process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()

m.pyplot.plot(lista_rep_C, color='r', linewidth=3.0, label = 'Retained')

m.pyplot.plot(lista_rep_R, color='b', linewidth=3.0, label = 'Ceded')
```

```
m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Value at Risk for the Retained & Ceded ptf, on the reporting process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()

m.pyplot.plot(List_rep_C, color='r', linewidth=3.0, label = 'Retained')

m.pyplot.plot(List_rep_R, color='b', linewidth=3.0, label = 'Ceded')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Expected Shortfall for the  Retained & Ceded ptf, on the reporting process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()

m.pyplot.plot(ListOfEls_rep,color='r', linewidth=3.0, label = 'Ceded' )

m.pyplot.plot(ListOfEls_rep_stays, color='b', linewidth=3.0, label = 'Retained')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')
```

```
        m.pyplot.title('Expected Losses for the Retained & Ceded ptf, on the reporting process')
        m.pyplot.legend(loc='lower right')
        m.pyplot.show()
        exit()
    elif ptf == 3:
        l = 1.2
        p = 0.05
        gamma = 0.05
# SET OF THE LISTS FOR THE PAYMENT PATTERN
        lista_R = [] #list for the retained part of the VaR
        lista_C = [] #list for the ceded part of the VaR
        List_R = [] #list for the retained part of the ES
        List_C = [] #list for the ceded part of the ES
        ListOfEls = [] #list of the different EL-because of the different cessions-ceded
        ListOfEls_stays = []
        OptimizationProblem1 = []
        OptimizationProblem2 = []
```

```
# SET OF THE LISTS FOR THE REPORTING PATTERN

    lista_rep_C = []

    lista_rep_R = []

    List_rep_C = []

    List_rep_R = []

    ListOfEls_rep = []

    ListOfEls_rep_stays = []

    OptimizationProblem1_rep = []

    OptimizationProblem2_rep = []

    mirrordeductible = 0

    while mirrordeductible <= 100 :

        deductible = float( float(mirrordeductible) / 100 )

        print deductible

        Srep_stays, Srep_ceded, Spay_stays, Spay_ceded = xlr.calc_xlr(Srep, Spay, deductible, l)

        AgrSrep_stays, AgrSrep_ceded, AgrSpay_stays, AgrSpay_ceded= agg.calc_aggregate_ptf(Srep_stays, Srep_ceded, Spay_stays, Spay_ceded)
```

For the Ceded pattern the OptSolutions

a = var.VaR(AgrSpay_ceded,layer)

lista_C.append(a) # A list of VaR's for the ceded amounts, that are refered to different cessions and so to different portfolios (Spay)

b = var.VaR(AgrSpay_stays,layer)

lista_R.append(b) # A list of VaR's for the retained amounts, that are refered to different cessions and so to different portfolios (Spay).

c = es.calc_shortfall(AgrSpay_ceded,layer)

List_C.append(c)

d = es.calc_shortfall(AgrSpay_stays,layer)

List_R.append(d)

expLoss_pay_ceded = mean.arithmetic_mean(AgrSpay_ceded)

ListOfEls.append(expLoss_pay_ceded)

expLoss_pay_retained = mean.arithmetic_mean(AgrSpay_stays)

ListOfEls_stays.append(expLoss_pay_retained)

premium_1 = (1 + p) * expLoss_pay_ceded

#premium_2 = EL + gamma * var.VaR(AgrSpay_ceded,layer)

#premium_3 = EL+ gamma * es.calc_shortfall(AgrSpay_ceded,layer)

```
t = VaRorig - a + premium_1

tt = ESorig - c + premium_1

OptimizationProblem1.append(t)

OptimizationProblem2.append(tt)

# For the Reporting pattern the OptSolutions

e = var.VaR(AgrSrep_ceded,layer)

lista_rep_C.append(e)

f = var.VaR(AgrSrep_stays,layer)

lista_rep_R.append(f)

h = es.calc_shortfall(AgrSrep_ceded,layer)

List_rep_C.append(h)

k = es.calc_shortfall(AgrSrep_stays,layer)

List_rep_R.append(k)

expLoss_rep_ceded = mean.arithmetic_mean(AgrSrep_ceded)

ListOfEls_rep.append(expLoss_rep_ceded)

expLoss_rep_retained = mean.arithmetic_mean(AgrSrep_stays)

ListOfEls_rep_stays.append(expLoss_rep_retained)

premium_1_rep = ( 1 + p ) * expLoss_rep_ceded
```

```
        #premium_2_rep = EL + gamma * var.VaR(AgrSrep_ceded,layer)

        #premium_3_rep = EL+ gamma * es.calc_shortfall(AgrSrep_ceded,layer)

        r = VaRorig_rep - e + premium_1_rep

        rr = ESorig_rep - h + premium_1_rep

        OptimizationProblem1_rep.append(r)

        OptimizationProblem2_rep.append(rr)

        mirrordeductible += 1

# PLOTS FOR THE PAYMENT PATTERN

    m.pyplot.plot(OptimizationProblem1, color='b', linewidth=3.0, label = '# 1')

    m.pyplot.plot(OptimizationProblem2, color='r', linewidth=3.0, label = '# 2')

    m.pyplot.xlabel('deductible')

    m.pyplot.ylabel('Amount in Euros')

    m.pyplot.title('Optimization problems, on the payment process')

    m.pyplot.legend(loc='lower right')

    m.pyplot.show()

    m.pyplot.plot(lista_C, color='r', linewidth=3.0, label = 'Retained')

    m.pyplot.plot(lista_R, color='b', linewidth=3.0, label = 'Ceded')

    m.pyplot.xlabel('deductible')
```

```
m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Value at Risk for the Retained & Ceded ptf, on the payment process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()

m.pyplot.plot(List_R, color='r', linewidth=3.0, label = 'Retained')

m.pyplot.plot(List_C, color='b', linewidth=3.0, label = 'Ceded')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Expected Shortfall for the  Retained & Ceded ptf, on the payment
process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()

m.pyplot.plot(ListOfEls,color='r', linewidth=3.0, label = 'Ceded' )

m.pyplot.plot(ListOfEls_stays, color='b', linewidth=3.0, label = 'Retained')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Expected Losses for the Retained & Ceded ptf, on the payment
process')
```

```
m.pyplot.legend(loc='lower right')

m.pyplot.show()
```

```
# PLOTS FOR THE REPORTING PATTERN

m.pyplot.plot(OptimizationProblem1_rep, color='b', linewidth=3.0, label = '# 1')

m.pyplot.plot(OptimizationProblem2_rep, color='r', linewidth=3.0, label = '# 2')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Optimization problems, on the reporting process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()



m.pyplot.plot(lista_rep_C, color='r', linewidth=3.0, label = 'Retained')

m.pyplot.plot(lista_rep_R, color='b', linewidth=3.0, label = 'Ceded')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Value at Risk for the Retained & Ceded ptf, on the reporting process')
```

```
m.pyplot.legend(loc='lower right')

m.pyplot.show()



m.pyplot.plot(List_rep_C, color='r', linewidth=3.0, label = 'Retained')

m.pyplot.plot(List_rep_R, color='b', linewidth=3.0, label = 'Ceded')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')

m.pyplot.title('Expected Shortfall for the  Retained & Ceded ptf, on the reporting process')

m.pyplot.legend(loc='lower right')

m.pyplot.show()



m.pyplot.plot(ListOfEls_rep,color='r', linewidth=3.0, label = 'Ceded' )

m.pyplot.plot(ListOfEls_rep_stays, color='b', linewidth=3.0, label = 'Retained')

m.pyplot.xlabel('deductible')

m.pyplot.ylabel('Amount in Euros')
```

```
    m.pyplot.title('Expected Losses for the Retained & Ceded ptf, on the reporting process')

    m.pyplot.legend(loc='lower right')

    m.pyplot.show()


if __name__ == '__main__':

    run_code_with_user_input()

    sys.exit(1)
```

The following part is a collection of the exact proofs of the theorems mentioned at the previous chapters, as they are written at the based scientific papers for this thesis. This is only for the convenience of the reader, so not to lose time searching the wright one reference.

Proof of the Theorem 2.7.

We have to minimize over the set F the following function

$$
\begin{aligned}
&TrTVaR_{a_1,a_2}(\,L\,(R\,[\,X\,]\,)\,)\\
&\qquad = TrTVaR_{a_1,a_2}(\,X\,) - TrTVaR_{a_1,a_2}(\,R\,[\,X\,]\,) + (1+\rho)E(\,R\,[\,X\,]\,)\\
&\qquad = TrTVaR_{a_1,a_2}(\,X\,) + (1+\rho)\int_0^{a_1} R(\,VaR_s(\,X\,)\,)ds + (1+\rho\\
&\qquad - \frac{1}{\alpha_1-\alpha_2})\int_{a_1}^{a_2} R(\,VaR_s(\,X\,)\,)ds + (1+\rho)\int_{a_2}^{1} R(\,VaR_s(\,X\,)\,)ds
\end{aligned}
$$

Using the two – stage procedure we have to minimize the followings.

$$
\begin{aligned}
&min_{R\epsilon F}(1+\rho)\int_0^{a_1} R(\,VaR_s(\,X\,)\,)ds + (1+\rho\\
&\qquad - \frac{1}{\alpha_1-\alpha_2})\int_{a_1}^{a_2} R(\,VaR_s(\,X\,)\,)ds + (1+\rho)\int_{a_2}^{1} R(\,VaR_s(\,X\,)\,)ds
\end{aligned}
$$

This is subject to $R(\,VaR_s(\,X\,)\,) = \xi_1, R(\,VaR_s(\,X\,)\,) = \xi_2$, while $(\xi_1, \xi_2) \in C_1$ is a vector of constants with,

$$C_1 := \{ 0 \leq \xi_2 - \xi_1 \leq VaR_{a_2}(X) - VaR_{a_1}(X) , 0 \leq \xi_1 \leq VaR_{a_1}(X), 0 \leq \xi_2 \leq VaR_{a_2}(X)\}$$

, where $1 + \rho - \frac{1}{\alpha_1 - \alpha_2}$ is assumed to be negative and likewise some arguments of the proofs of the theorems 2.3 and 2.4, we have that the solution comes by

$$R_1^* [X; \ \xi_1; \ \xi_2] := \big(X - VaR_{a_1}(X) + \xi_1 \big)_+ \wedge \xi_2$$

Taking in consideration the

$$ES_a(Z) = \frac{1}{1-a}\int_a^1 VaR_s(Z)ds = VaR_a(Z) + \frac{1}{1-a}E\big(Z - VaR_a(Z)\big)_+$$

, where $(z)_+ = \max\{z, 0\}$.

With the

$$TrTVaR_{\alpha_1,\alpha_2}(Z) = \frac{1}{\alpha_1 - \alpha_2}\int_{\alpha_2}^{\alpha_1} VaR_s(Z)ds = \frac{(1-a_1)ES_{\alpha_1}(Z) - (1-a_2)ES_{\alpha_2}(Z)}{\alpha_2 - \alpha_1}$$

And given that

$$E(R_1^* [X; \ \xi_1; \ \xi_2]) = \int_0^{\xi_2} \Pr(R_1^* [X; \ \xi_1; \ \xi_2] > x)\, dx = \int_{VaR_{a_1}(X)-\xi_1}^{VaR_{a_1}(X)+\xi_2-\xi_1} \Pr(X > x)\, dx$$

we can see that minimizing over the set C_1 the

$$H_4(\xi_1, \xi_2) = (1+\rho)\int_{VaR_{a_1}(X)-\xi_1}^{VaR_{a_1}(X)+\xi_2-\xi_1} \Pr(X > x)\, dx - \frac{1}{a_2 - a_1}\Big((1 - a_1)\xi_1 - (1 - a_2)\xi_2 + \int_{VaR_{a_1}(X)-\xi_1}^{VaR_{a_1}(X)+\xi_2-\xi_1} \Pr(X > x)\, dx$$

We can see that

$$\frac{dH_4}{d\xi_2} = \frac{1-a_2}{a_2-a_1} + (\,1+\rho-\frac{1}{a_2-a_1})\overline{F}(VaR_{a_1}(X)+\xi_2-\xi_1 < 0$$

if and only if

$$F\big(\,VaR_{a_1}(X)+\xi_2-\xi_1\,\big) < \; a^* \equiv \;\big(\,VaR_{a_1}(X)+\xi_2-\xi_1\,\big) < \; VaR_{a^*}(X)$$

Where for $a^* < a_2$ is always true.

Assuming now that $a^* > a_1$ and given that $\xi_2 \epsilon$ [$\xi_1, \xi_1 + VaR_{a_2}(X) - VaR_{a_1}(X)$] then for any fixed $\xi_1 \in$ [$0, VaR_{a_1}(X)$] we see that if $\xi_2 \in$ [$\xi_1, \xi_1 + VaR_{a^*}(X) - VaR_{a_1}(X)$ then $\frac{dH_4}{d\xi_2} < 0$ and $\frac{dH_4}{d\xi_2} \geq 0$ if

$$\xi_2 \in [\;\xi_1 + VaR_{a^*}(X) - VaR_{a_1}(X) * \xi_1 VaR_{a_1}(X) + VaR_{a_2}(X) - VaR_{a_1}(X)\,]$$

So

$$H_4(\xi_1,\xi_2) \geq \; H_4\left(\xi_1, \xi_1 + VaR_{a^*}(X) - VaR_{a_1}(X)\right)$$

$$= (1+\rho)\int_{VaR_{a_1}(X)-\xi_1}^{VaR_{a^*}(X)} \Pr(\,X > x)\,dx - \xi_1 + K_1$$

where K_1 is a constant with respect to ξ_1. Finally taking the derivative at the right side, with respect to ξ_1, we have the requisite.

Now at the case of $a^* < a_1$, we have that $\frac{dH_4}{d\xi_2} > 0$ is true for any $\xi_1 \in$ [$0, VaR_{a_1}(X)$] and any $\xi_2 \in$ [$\xi_1, \xi_1 + VaR_{a_2}(X) - VaR_{a_1}(X)$].

So

$$H_4(\xi_1,\xi_2) \geq \; H_4(\xi_1,\xi_1) = (1+\rho)\int_{VaR_{a_1}(X)-\xi_1}^{VaR_{a_2}(X)} \Pr(\,X > x)\,dx - \xi_1 + K_2$$

Where K_2 is a constant with respect to ξ_1.

Proof of the Theorem 2.5.

Assuming that $s^* < a$ the first step of the optimization problem is,

$$\begin{cases} min_{(R_1,R_2)\in G} VaR_a(\, I\,[\,X\,]\,) + (1+\rho)\int_0^1 VaR_s(R_1[\,X\,])ds + \int_0^1 VaR_s(R_2[\,X\,])\Phi(s)ds \\ subject\ to\ R_1(VaR_{s^*}(X)) = \ \xi_{11}, R_1\big(VaR_a(X)\big) = \xi_{12} \\ R_2(VaR_{s^*}(X)) = \ \xi_{21}, R_2\big(VaR_a(X)\big) = \xi_{22} \end{cases}$$

Can be solved, where $\xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in D_1$, where

$$D_1 = \begin{cases} 0 \le \xi_{11} \le \xi_{12}, 0 \le \xi_{21} \le \xi_{22}, \xi_{11} + \xi_{21} \le VaR_{s^*}(X), \xi_{12} + \xi_{22} \le VaR_a(X) \\ \xi_{12} - \xi_{11} + \xi_{22} - \xi_{21} \le VaR_a(X) - VaR_{s^*}(X) \end{cases}$$

The solution of the above is given by

$$R_1^*[\,X, \xi\,] = \begin{cases} (\,X - VaR_{s^*}(X) + \xi_{11})_+ \wedge \xi_{11},\ X \le VaR_a(X) - (\xi_{12} - \xi_{11}) - (\xi_{22} - \xi_{21}) \\ (X - VaR_a(X) + (\xi_{22} - \xi_{21}) + \xi_{12})_+ \wedge \xi_{12}\ otherwise \end{cases}$$

And

$$R_1^*[\,X, \xi\,] = \begin{cases} (\,X - VaR_{s^*}(X) + \xi_{11} + \xi_{21})_+ \wedge \xi_{21}, \qquad X \le VaR_a(X) - (\xi_{22} - \xi_{21}) \\ (X - VaR_a(X) + \xi_{22})_+ \wedge \xi_{22}\ otherwise \end{cases}$$

The second step for the optimization problem is the following

$$\begin{aligned} min_{D_1} G_1(\xi) \\ &= VaR_a - \xi_{12} - \xi_{22} \\ &+ \int_{VaR_{s^*}(X) - \xi_{11} - \xi_{21}}^{VaR_{s^*}(X)} g\big(\bar{F}(x)\big)dx + (1+\rho)(\int_{VaR_{s^*}(X) - \xi_{11}}^{VaR_{s^*}(X)} \bar{F}(x)dx \\ &+ \int_{VaR_{s^*}(X) - (\xi_{12} - \xi_{11}) - (\xi_{22} - \xi_{21})}^{VaR_{s^*}(X) - (\xi_{22} - \xi_{21})} \bar{F}(x)dx \end{aligned}$$

Given the assumption 2.1 and the derivative of the G_1 $with\ respect\ to$ ξ_{12} is the same as we would like to minimize

$$G_2(\xi_{11}, \xi_{21}, \xi_{22}) = G_2(\xi_{11}, VaR_a(X) - VaR_{s^*}(X) - (\xi_{22} - \xi_{21}) + \xi_{11}, \xi_{21}, \xi_{22})$$
$$= VaR_{s^*}(X) - \xi_{11} - \xi_{21} + (1 + \rho)\int_{VaR_{s^*}(X)-\xi_{11}}^{VaR_{s^*}(X)-(\xi_{22}-\xi_{21})} \bar{F}(x)dx + \int_{VaR_{s^*}(X)-\xi_{11}-\xi_{21}}^{VaR_{s^*}(X)-\xi_{11}} g(\bar{F}(x))dx + \int_{VaR_{s^*}(X)-(\xi_{22}-\xi_{21})}^{VaR_{s^*}(X)} g(\bar{F}(x))dx$$

Over a set – a region with respect to ξ_{22}, where

$D_2 = 0 \le \xi_{11}, 0 \le \xi_{21}, \xi_{11} + \xi_{21} \le VaR_{s^*}(X), 0 \le \xi_{22} - \xi_{21} \le VaR_a(X) - VaR_{s^*}(X)$ and $\xi_{21} \le \xi_{22} \le VaR_a(X) - VaR_{s^*}(X) + \xi_{21}$, so we have that

$$\frac{dG_2}{d\xi_{22}} = -(1+\rho)\bar{F}\left(VaR_a(X) - (\xi_{22} - \xi_{21}) + g\left(\bar{F}\left(VaR_a(X)\right)\right) - (\xi_{22} - \xi_{21})\right)$$
$$= -G\left(\bar{F}\left(VaR_a(X) - (\xi_{22} - \xi_{21})\right)\right)$$

We can see that $G(0) = 0$ and $G(1) = \rho > 0$ and due to the derivation we can make from Assumption 2.1, that $G(\cdot)$ has a global minimum at $1 - s^*$, we have that $s^{**} < s^*$ and $G(t) \le 0\ \forall\ 0 \le t \le 1 - s^{**}$. Given the last one and the

$$\bar{F}\left(VaR_a(X) - (\xi_{22} - \xi_{21})\right) \le \bar{F}(VaR_{s^*}(X) \le 1 - s^*$$

the above minimization is equivalent to solving

$$min_{D_3} G_3(\xi_{11}, \xi_{21})$$
$$= VaR_{s^*}(X) - \xi_{11} - \xi_{21} + (1+\rho)\int_{VaR_{s^*}(X)-\xi_{11}}^{VaR_a(X)} \bar{F}(x)dx + \int_{VaR_{s^*}(X)-\xi_{11}-\xi_{21}}^{VaR_{s^*}(X)-\xi_{11}} g(\bar{F}(x))dx$$

over the set $D_3 = 0 \le \xi_{11}, 0 \le \xi_{21}, \xi_{11} + \xi_{21} \le VaR_{s^*}(X)$ and

$$\frac{dG_3}{d\xi_{21}} = -1 + g(\bar{F}(VaR_{s^*}(X) - \xi_{11} - \xi_{21})) \le 0$$

and it is reduced to finding the solution of

$$min_{D_1} G_4(\xi_{11}) = G_3(\xi_{11}, VaR_{s^*} - \xi_{11})$$

$$= (1+\rho)\int_{VaR_{s^*}(X)-\xi_{11}}^{VaR_a(X)} \bar{F}(x)dx + (1+\rho)(\int_0^{VaR_{s^*}(X)-\xi_{11}} g(\bar{F}(x))dx$$

and easily we can find that $\frac{dG_4}{d\xi_{11}} = G(\bar{F}(\, VaR_{s^*}(X) - \xi_{11}\,)\,)$.

It is clear that

$$(\bar{F}(\, VaR_{s^*}(X) - \xi_{11}\,) \leq 1 - s^{**} \; if \; and \; only \; if \; VaR_{s^{**}}(X) \leq VaR_{s^*}(X) - \xi_{11}$$

So from all the above we have that $G_4(\cdot)$ is non – decreasing on $[\, 0, \; VaR_{s^*}(X) - VaR_{s^{**}}(X)\,]$ and on $[\, VaR_{s^*}(X) - VaR_{s^{**}}(X), VaR_{s^*}(X)\,]$. Thus the global minimum is attained at the $VaR_{s^*}(X) - VaR_{s^{**}}(X)$, so the case $s^* < a$ is fully explained.

Assuming now that $s^{**} \leq a < s^*$ the first optimization problem is given as before from the

$$\begin{cases} min_{(R_1,R_2)\in G} VaR_a(\, I\, [\, X\,]\,) + (1+\rho)\int_0^1 VaR_s(R_1[\, X\,])ds + \int_0^1 VaR_s(R_2[\, X\,])\Phi(s)ds \\ subject \; to \; R_1(VaR_{s^*}(X)) = \xi_{11}, R_1\big(VaR_a(X)\big) = \xi_{12} \\ R_2(VaR_{s^*}(X)) = \xi_{21}, R_2\big(VaR_a(X)\big) = \xi_{22} \end{cases}$$

Can be solved, where $\xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in D_1$, where

$$D_1 = \begin{cases} 0 \leq \xi_{11} \leq \xi_{12}, 0 \leq \xi_{21} \leq \xi_{22}, \xi_{11} + \xi_{21} \leq VaR_{s^*}(X), \xi_{12} + \xi_{22} \leq VaR_a(X) \\ \xi_{12} - \xi_{11} + \xi_{22} - \xi_{21} \leq VaR_a(X) - VaR_{s^*}(X) \end{cases}$$

The proof continues through similar derivations to the ones previously displayed when $s^* < a$. The derivatives are taken in the same order and yield the following global minimal solution $\xi_{11}^* = \xi_{22}^* = VaR_a(X) - VaR_{s^{**}}(X), \xi_{21}^* = \xi_{22}^* = VaR_{s^{**}}(X)$ which concludes the $s^{**} \leq a < s^*$ scenario.

Proof of the Theorem 2.6.

The optimal solution is being derived through two optimization problems. Initially,

$$\begin{cases} min_{(R_1,R_2)\in G} VaR_a(\,I\,[\,X\,]\,) + (1+\rho)\int_0^1 VaR_s(R_1[\,X\,])ds + \dfrac{1}{1-\beta}\int_0^1 VaR_s(R_2[\,X\,])ds \\ \qquad subject\ to\ R_1(VaR_\beta(X)) = \ \xi_{11}, R_1\big(VaR_a(X)\big) = \xi_{12} \\ \qquad R_2(VaR_\beta(X)) = \ \xi_{21}, R_2\big(VaR_a(X)\big) = \xi_{22} \end{cases}$$

where $\xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in \mathcal{E}_1$, and

$$\mathcal{E}_1 = \begin{cases} 0 \le \xi_{11} \le \xi_{12}, 0 \le \xi_{21} \le \xi_{22}, \xi_{11} + \xi_{21} \le VaR_\beta(X), \xi_{12} + \xi_{22} \le VaR_a(X) \\ \qquad \xi_{12} - \xi_{11} + \xi_{22} - \xi_{21} \le VaR_a(X) - VaR_\beta(X) \end{cases}$$

Assuming that $\beta < \rho^*$ we have that $R_1(\cdot), R_2(\cdot)$ start from 0 on $[0, VaR_\beta(X)]$ until the required upper levels are reached, with amendment that , $R_2(\cdot)$ arrives at ξ_{11} as late as possible. The allocation on the first reinsurer may take any shape as long as it does not attain the ξ_{21} level before the time when $R_1(\cdot)$ commences an ascending behavior. Further both functions increase as slowly as possible on $(VaR_\beta(X), VaR_a(X))$ and the second risk allocation does not reach its upper level before the first one, since $1+\rho > \frac{1}{1-\beta}$. At the end both of the reinsurance counters – parties become risk adverse whenever the total loss exceeds an amount of $VaR_a(X)$. So the solution from the above optimization problem can be formulated as

$$R_1^*[\,X,\xi\,] = \begin{cases} (\,X - VaR_\beta(X) + \xi_{11})_+ \wedge \xi_{11},\ \ X \le VaR_a(X) - (\xi_{12} - \xi_{11}) \\ \qquad (X - VaR_a(X) + \xi_{12}) \wedge \xi_{12}\ otherwise \end{cases}$$

And

$$R_2^*[\,X,\xi\,] = \begin{cases} h_2^*(X,\xi) \wedge \xi_{21}, \qquad X \le VaR_a(X) - (\xi_{12} - \xi_{11}) - (\xi_{22} - \xi_{21}) \\ \qquad (X - VaR_a(X) + \xi_{12} - \xi_{11} + \xi_{22}) \wedge \xi_{22}\ otherwise \end{cases}$$

Where $h_2^*(;\xi)$ is a non – decreasing Lipchitz function with a constant such that $h_2^*(0,\xi) = 0$ and $h_2^*\big(VaR_\beta(X) - \xi_{11}, \xi\big) = \xi_{21}$.

The second step for the optimization is

$$min_{\mathcal{E}_1} G_1(\xi)$$

$$= VaR_a(X) - \xi_{12} - \xi_{22} + (1+\rho)(\int_{VaR_\beta(X)-\xi_{11}}^{VaR_\beta(X)} \bar{F}\,(x)dx$$

$$+ \int_{VaR_\alpha(X)-(\xi_{12}-\xi_{11})}^{VaR_\alpha(X)} \bar{F}\,(x)dx\,) + \xi_{21}$$

$$+ \frac{1}{1-\beta}\int_{VaR_\beta(X)-(\xi_{12}-\xi_{11})-(\xi_{22}-\xi_{21})}^{VaR_\alpha(X)-(\xi_{12}-\xi_{11})} \bar{F}(x)\,dx$$

Note that $\xi_{21} \le \xi_{22} \le VaR_a(X) - VaR_\beta(X) - (\xi_{12} - \xi_{11}) +\ \xi_{21}$ always holds, suggesting that $\mathcal{E}_1$ represents a ξ_{22} – simple region. Given that

$$\frac{dG_1}{d\xi_{22}} = -1 + \frac{1}{1-\beta}\bar{F}(VaR_a(X) - (\xi_{12}-\xi_{11}) - (\xi_{22}-\xi_{21}))$$

$$\le -1 + \frac{1}{1-\beta}\,\bar{F}\left(VaR_\beta(X)\right) \le 0$$

we have that problem is minimized to

$$G_2(\xi_{11},\, \xi_{12},\, \xi_{21}) = G_1\big(\xi_{11},\, \xi_{12},\, \xi_{21},\, VaR_a(X) -\ VaR_\beta(X) - \big(\xi_{12}-\xi_{11}\big) + \xi_{21}\big)$$

$$= VaR_\beta(X) - \xi_{11} + (1$$

$$+ \rho)\int_{VaR_\beta(X)-\xi_{11}}^{VaR_\beta(X)} \bar{F}(x)dx + \int_{VaR_\alpha(X)-(\xi_{12}-\xi_{11})}^{VaR_\alpha(X)-\xi_{11}} \bar{F}(x)dx$$

$$+ \frac{1}{1-\beta}\int_{VaR_\beta(X)}^{VaR_\alpha(X)-(\xi_{12}-\xi_{11})} \bar{F}(x)dx$$

over the set

$\mathcal{E}_2 = 0 \le \xi_{11}, 0 \le \xi_{21}, \xi_{11} + \xi_{21} \le VaR_\beta(X), 0 \le \xi_{12} - \xi_{11} \le VaR_a(X) - VaR_\beta(X),$

where $\mathcal{E}_2$ is a ξ_{12} simple region since $\xi_{11} \le \xi_{12} \le VaR_\beta(X) - VaR_a(X) + \xi_{11}$ holds.

Further $\frac{dG_2}{d\xi_{12}} = \left(1 + \rho + \frac{1}{1-\beta}\right)\bar{F}\big(VaR_a(X) - (\xi_{12} - \xi_{11})\big) \ge 0$

Thus we have a further reduction to

$$min_{\mathcal{E}_3} G_3(\xi_{11}, \xi_{21}) = G_2(\xi_{11}, \xi_{11}, \xi_{21})$$

$$= VaR_\beta(X) - \xi_{11} + (1+\rho)\int_{VaR_\beta(X)-\xi_{11}}^{VaR_\beta(X)} \overline{F}\,(x)dx$$

$$+ \frac{1}{1-\beta}\int_{VaR_\beta(X)}^{VaR_\alpha(X)} \overline{F}(x)\,dx$$

where $\mathcal{E}_3 = \{\, 0 \le \xi_{11}, 0 \le \xi_{21}, \xi_{11} + \xi_{21} \le VaR_\beta(X)\, \}$. For any fixed ξ_{21} the above function is increasing in ξ_{11}. So we have that the minimum is attained at $\xi_{11}^* = 0$ and ξ_{21}^* can take any value from $[\, 0, VaR_\beta(X)\,]$.

The mathematical formulation of the risk allocation is then given by

$$R_1^*[\, X, \xi\,]$$
$$= \begin{cases} (\, X - VaR_\beta(X) + \xi_{11})_+ \wedge \xi_{11}, & X \le VaR_a(X) - (\xi_{12} - \xi_{11}) - (\xi_{22} - \xi_{21}) \\ (X - VaR_a(X) + \xi_{22} - \xi_{21} + \xi_{12}) \wedge \xi_{12} & otherwise \end{cases}$$

And

$$R_2^*[\, X, \xi\,] = \begin{cases} h_3^*(X, \xi) \wedge \xi_{21}, & X \le VaR_a(X) - (\xi_{22} - \xi_{21}) \\ (X - VaR_a(X) + \xi_{22}) \wedge \xi_{22} & otherwise \end{cases}$$

Where $h_3^*(;\xi)$ is a non – decreasing Lipchitz function with a constant such that $h_3^*(0,\xi) = 0$ and $h_3^*\big(VaR_\beta(X) - \xi_{11}, \xi\big) = \xi_{21}$. Easily we can find that the second step optimization problem is given by the following mathematical formulation and by taking the appropriate derivatives in the same order as at previous proofs, we can complete the proof.

$$min_{\mathcal{E}_1} VaR_a(X) - \xi_{12} - \xi_{22} + \xi_{21} + \frac{1}{1-\beta}\int_{VaR_a(X)-(\xi_{22}-\xi_{21})}^{VaR_a(X)} \overline{F}(x)dx$$

$$+ (1+\rho)\Big(\int_{VaR_\beta(X)-\xi_{11}}^{VaR_\beta(X)} \overline{F}(x)dx + \int_{VaR_a(X)-(\xi_{12}-\xi_{11})-(\xi_{22}-\xi_{21})}^{VaR_a(X)-(\xi_{22}-\xi_{21})} \overline{F}(x)dx\,\Big)$$

Proof of the Theorem 2.3.

We need to minimize

$$VaR_a(\,L(R[\,X\,])\,) = VaR_a(\,X\,) - VaR_a(\,R\,[\,X\,]) + (\,1+\rho\,)E(\,R[\,X\,])$$

$$= VaR_a(\,X\,) - R(VaR_a(\,X\,)\,) + (1+\ \rho)\int_0^1 R(VaR_s(\,X\,)\,)\,ds$$

,over F.

At this point we need to denote that the last relation is truth, since $R(\,\cdot\,)$ is a non – decreasing continuous function. The solution of the above optimization problem, is given by analyzing it, in two sub – optimal problems.

The first sub – optimal problem is to minimize the following

$$min_{R\in F}\int_0^1 R(\,VaR_s(\,X\,)\,)ds\,,subject\ to\ R(\,VaR_a(\,X\,)\,) = \xi, where\ 0 \le \xi \le VaR_a(\,X\,).$$

Due to the fact that the function $R(\,\cdot\,)$ satisfies $R(\,0\,) = 0$ and increases slowly to ξ on $[\,0, VaR_a(\,X\,)\,]$ with a stagnant behavior in order to minimize its value at each point.

Also since $R(\,\cdot\,)$ is a Lipschitz function, this means that is not able to increase with a slope greater that one. Thus the above sub – optimized problem is solved by the

$$R^*[\,X,\xi\,] = (\,X - VaR_a(\,X\,) + \xi\,)_+ \wedge \xi$$

Since,

$$E(\,R^*[\,X,\xi\,]\,) = \int_0^{\xi} \Pr(\,R^*[\,X,\xi\,]\ > x\,)\,dx = \int_{VaR_a(\,X\,)-\xi}^{VaR_a(\,X\,)} \Pr(\,X > x)\,dx$$

The second sub – optimal problem is to minimize the following relation :

$$H_1(\xi) = -\xi + (1+\rho)\int_{VaR_a(X)-\xi}^{VaR_a(X)} \bar{F}(x)\,dx$$

over the set $[0, VaR_a(X)]$.

Finding the derivative of $H_1(\xi)$, which is the $H_1'(\xi) = -1 + (1+\rho)\bar{F}(VaR_a(X) - \xi)$, we can see that for $\forall\xi$ that fulfills the relation $VaR_a(X) - VaR_{\rho^*}(X) \geq \xi$, then the above derivative takes non positive values.

This leads us to the conclusion that if $VaR_a(X) \geq VaR_{\rho^*}(X)$, then $H_1(\cdot)$ is minimized at $VaR_a(X) - VaR_{\rho^*}(X)$, that is indeed the

$$R^*[X] = (X - VaR_{\rho^*}(X))_+ \wedge (VaR_a(X) - VaR_{\rho^*}(X))_+$$

$H_1(\cdot)$ has a minimum at zero, when $VaR_a(X) < VaR_{\rho^*}(X)$, so the insurer should choose the $R^*[X,0] = 0$ as the optimal decision.

Proof of the Theorem 2.4.

We want to minimize the following relation

$$ES_a(\, L(\, R^*[\, X\,])\,) = ES_a(\, X\,) - ES_a(\, R[\, X\,]\,) + (1+\rho)E(\, R[\, X\,]\,)$$

From the expected shortfall's definition, which is

$$ES_a(\, X\,) = \frac{1}{1-a}\int_a^1 VaR_s(\, X\,)ds = VaR_a(\, X\,) + \frac{1}{1-a}E\big(\, X - VaR_a(\, X\,)\big)_+$$

and due to the fact that $R(\,\cdot\,)$ is a non – decreasing Lipschitz function, we have that $ES_a(\, L(\, R^*[\, X\,])\,) =$
$ES_a(\, X\,) - \frac{1}{1-a}\int_a^1 R(VaR_s(\, X\,)\,)\ ds + (1+\rho)\int_a^1 R(VaR_s(\, X\,)\,)\ ds$, over the F and that is what we aim to minimize.

Again we solve this problem transforming the original optimization problem to into two sub – optimal problems, that we tend to solve. The first sub – optimal problem is to try to minimize

$$\begin{cases} min_{R\in F}\{\, (1+\rho)\displaystyle\int_a^1 R(VaR_s(\, X\,)\,)\ ds \\ subject\ to\ R(VaR_a(\, X\,)\,) = \xi, \qquad 0 \le \xi \le VaR_a(\, X\,) \end{cases} \qquad (\,*\,)$$

We assume first the case of $a > \rho^* \Longrightarrow 1 + \rho - \frac{1}{1-a} < 0$. The above problem is solved by the $R^*[\, X, \xi\,] = (\, X - VaR_a(\, X\,) + \xi)_+$, knowing that the function $R(\,\cdot\,)$ satisfies $R(\,0\,) = 0$ and increases slowly to ξon $[\,0, VaR_a(\, X\,)\,]$ and increases fast after then.

The second sub – optimal problem that we tend to minimize, over the set $[\,0, VaR_a(\, X\,)\,]$, is the following relation

$$H_2(\xi) = -ES_a(R^*[X,\xi]) + (1+\rho)E(R^*[X,\xi])$$
$$= -ES_a(X) + VaR_a(X) - \xi + (1+\rho)\int_{VaR_a(X)-\xi}^{x_F} \bar{F}(x)\,dx$$

We can see that the above differs from the $H_1(\xi) = -\xi + (1+\rho)\int_{VaR_a(X)-\xi}^{VaR_a(X)} \bar{F}(x)\,dx$, which is defined at the proof of the Theorem 2.1. by just a constant and therefore has the same behavior and so the minimization for $H_2(\cdot)$ can be succeeded at $VaR_a(X) - VaR_{\rho^*}(X)$.

In the case of $a < \rho^* \Longrightarrow 1 + \rho - \frac{1}{1-a} > 0$ and the solution for the first sub – optimal problem $(*)$ is given by $R^*[X,\xi] = (X - VaR_a(X) + \xi)_+ \wedge \xi$. By similar way we can solve the second sub – optimal problem.

In the case of $a = \rho^*$ the set of the possible solutions is given by

$$R^*[X,\xi] = \begin{cases} (X - VaR_{\rho^*}(X) + \xi)_+, & X \le VaR_a(X) \\ h_1^*(X,\xi), & X > VaR_a(X) \end{cases}$$

Where $h_1^*(X,\xi)$ is a Lipschitz function such that $h_1^*(VaR_a(X),\xi) = \xi$.

The second sub – optimal problem to minimize, over $[0, VaR_a(X)]$ is the following

$$H_3(\xi) = \int_0^a (VaR_s(X) - VaR_a(X) + \xi)_+ ds$$
$$= \int_{F(VaR_a(X)-\xi)}^a (VaR_s(X) - VaR_a(X) + \xi)_+ ds$$

$$\forall \xi_1, \xi_2 \in [0, VaR_a(X)]$$

satisfying the

$$0 \le \xi_1 \le \xi_2 \le VaR_a(X)$$

we have that

$$H_3(\xi_1) \leq H_3(\xi) = \int_{F(VaR_a(X)-\xi_1)}^{a} (VaR_s(X) - VaR_a(X) + \xi_2)ds \leq H_3(\xi_2)$$

Because

$$(VaR_s(X) - VaR_a(X) + \xi_2 > 0$$

$$\forall\, F(VaR_a(X) - \xi_2) < s \leq F(VaR_a(X) - \xi_1)$$

the second inequality is true and so $H_3(\cdot)$ has a minimum at zero. So the function that we started with, has the minimum value at

$$ES_a(X) + (1+\rho)\int_0^a (VaR_s(X) - VaR_a(X))_+ ds = ES_a(X)$$

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