PRICING WEATHER DERIVATIVES

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by

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Abstract

Weather Derivatives were first introduced in the USA in 1997 and their creation was driven by the need of companies whose revenues were related to weather fluctuations to hedge against the risk of unwanted weather conditions. Weather Derivatives belong to a different class of derivatives as their underlying asset (weather) is not tradable and this leads to ordinary pricing models (such as Black and Scholes formula) not being applicable. A lot of paperwork was directed towards the pricing of these products and the modeling of the daily average temperature which characterizes the majority of the traded instruments. In this thesis we analyze a suggested model which describes the evolution of temperature which is expressed as a sum of a deterministic and a stochastic part, and discuss 3 different approaches for pricing weather options and weather futures: Pricing under an Equivalent Martingale Measure, Arbitrage Free Pricing and Actuarial Pricing. Then we present implementations for each of the models on temperature call options and compare their results; for the implementation we use Monte Carlo simulations.

Key words: Weather Options, Temperature, Ornstein-Uchlenbeck Process, Risk Neutral Probability Measure, Arbitrage Free Pricing, Actuarial Pricing, Monte Carlo Simulations.

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Chapter 1

Introduction to Weather Derivatives

1.1 About Weather Derivatives

Weather has always been a considerable risk factor for many economic sectors. According to a recent study published in the Bulletin of the American Meteorological Society [1] ,the cost of weather risk for the U.S.A. may annually add up to 485 billion dollars, which represents approximately 3.4% of annual GDP.

Weather derivatives are financial instruments whose income depends on the evolution of an underlying meteorological index. Their creation aimed to protect companies against the risk of weather-related losses. The underlying variables can be, i.e., temperature, humidity, rain or snowfall (measured in a specific location), with the most common underlying variable among them to be temperature. They differ from other derivatives because these underlying variables have no direct value to price the weather derivative.

ΠΑΝΕΠΙΣΤΗΜΙΟ ΠΕΙΡΑΙΑ

Many kinds of businesses are subject to weather risk, such as energy producers and consumers, agricultural industry, retail sales, leisure industry, construction, transportation, etc. Before the advent of weather derivatives, companies had only insurance contracts as a tool to protect themselves against unexpected weather conditions. But an insurance contract can provide coverage only against catastrophic damage (floods, hurricanes, etc.) and only if there is evidence from the company that it has suffered financial losses due to these disasters. In case of less extreme adverse weather conditions there was no protection for these companies, and their earnings were left to the mercy of the weather. Someone could say that weather derivatives are the logical expansion of insurance contracts of that type. On the other hand, Weather Derivatives can be used even by investors whose business is not threatened by weather fluctuations, either for hedging or from speculation, as they help also to the diversification of their investment portfolio since these contracts are not correlated with the financial assets.

1.2 Evolution of the Market of Weather Derivatives

The sector that drove primarily the demand for weather derivatives was the energy sector as energy prices are highly correlated to weather. Energy traders Aquila, Enron, and Koch Industries executed the first weather derivative transactions in 1997 [2].

The market started with a big jump as the winter of 1997-98 was the winter of “El Niño” which was one of the strongest of such events on record. This phenomenon received huge publicity in the American press and many companies wanted to hedge their seasonal weather risk due to the risk of significant earnings decline. Another factor that boosted the demand for weather derivatives was the deregulation of the U.S. energy industry. However, the effects of unpredictable seasonal weather patterns had previously been absorbed and managed within a regulated monopoly environment. But with deregulation, the various participants in the process of producing, marketing, and delivering energy to U.S. households and businesses were left to confront weather as a new and significant risk for them.

Thanks to the above factors, the market for weather derivatives expanded rapidly and contracts started to be traded Over-The-Counter (OTC) as individually negotiated contracts. Later, in order to increase the market size and limit the credit risk which the contracts involved, the Chicago Mercantile Exchange (CME) started an electronic market place for weather derivatives in 1999.

In spite of the interest aroused by the weather derivatives, their development was not so rapid and significant as it was hoped. Several reasons can be accounted to this fact. For instance, the departure of the main traders such as Enron, Aquila and El Paso from the market has lowered the number of transactions, also participants are limited mainly to energy companies, a fact that does not promote the liquidity of the market, and additionally the distrust of investors for the weather products that still deem too risky. But the main obstacle to climate market expansion is the difficulty in evaluating weather derivatives which has as a consequence the high price of the contracts (the seller tends to fix a high bonus to compensate for the difficult exercise of evaluation).

Still more trades are executed in CME. According to the PricewaterhouseCoopers' National Economic Consulting Group industry survey, published on the Weather Risk Management Association's website (2003-2004), [3], initially the weather derivatives market developed OTC reaching 2.5 billion US dollars in 2000-2001 and 4.3 billion in 2001-2002. Thereafter it grew slowly in 2003-2004 (4.7 billion), but then jumped to 9.7 billion in 2004-2005 and to 45.2 billion in 2005-2006. This sudden increase in trading volumes was due to CME's activity, which started trading weather derivatives in 1999 and by 2005 had gained a leading position.

The weather derivatives market was then severely hit by the 2007 crisis, when in a single year trade volumes fell to 19.2 billion dollars and continued dropping. To date, the industry reached 11.8 billion in the last year achieving an average growth of 20% from 2010 and the OTC market grew by 30%, a fact that indicates it increased its volume of trade from 1.9 to 2.4 billion dollars; see also (S. Miller, 2011), [4].

At the current time the country with the biggest volume of trade in weather derivatives is certainly the U.S.A., although weather derivatives are now spreading all over the world.

1.3 The Weather Contracts

Weather derivatives can be usually structured as swaps, futures, forwards and call/put options based on different underlying weather indices. The most popular ones are temperature related. The reason for this is the abundance of historical temperature data and the demand for a weather product coming from end-users with temperature exposure. The most commonly used weather indices in the market are the heating and cooling degree-days (see Definition 1.2), rain and snowfall.

1.3.1 Temperature Indices

Subsequently we define temperature, which will be used from now on in this form.

Definition 1.1 Temperature: Given a specific weather station, let Tj^{max} and Tj^{min} denote the maximal and minimal temperatures (in degrees of Celsius) measured on day i. We define the temperature for day i as

$$\frac{Tj^{max} + Tj^{min}}{2}.$$

As mentioned above, one important underlying variable for weather derivatives is the degree-day. This quantity is defined below.

Definition 1.2 Degree-days: Let T_i be the temperature for day i. We define the heating degree-days [1], HDD_i on that day, as the number of degrees by which the day's average temperature is below the base temperature (18°C or 65° Fahrenheit and in some warmer climates 75° Fahrenheit) and the cooling degree-days [2], CDD_i, as the number of degrees by which the day's average temperature is over the base temperature. In particular, we have:

$$HDD_i \equiv max[18 - T_i, 0],$$

$$CDD_i \equiv max[T_i - 18, 0],$$

respectively.

Most temperature based weather derivatives are based on the accumulation of $HDDs$ or $CDDs$ during a certain period, usually one calendar month or a winter/summer period. Typically the HDD season includes winter months from November to March and the CDD season is from May to September. April and October are often referred to as the "shoulder months". In this thesis we will only study the degree-days indices, because they are most often used.

[1] The name heating degree days originates from the US energy sector. The reason is that if the temperature is below 18° C people tend to use more energy to heat their homes

[2] The name cooling degree days originates from the US energy sector. The reason is that if the temperature is above 18° C people start turning their air conditioners on, for cooling.

Definition 1.3 Degree-days Indices: The HDD and CDD indexes are the number of $HDDs$ and $CDDs$, respectively, over a period of n days:

$$I_n^H = \sum_{i=1}^{n} HDD_i \tag{1.1}$$

and

$$I_n^C = \sum_{i=1}^{n} CDD_i. \tag{1.2}$$

1.3.2 Weather Swaps

Swaps are contracts in which two parties exchange risks during a predetermined period of time. In most swaps, payments are made between the two parties, with one side paying a fixed price, and the other paying a variable price. Unlike to interest rate swaps, which usually have several swap dates, in one type of weather swap that is often used, there is only one date when the cashflows are "swapped". Furthermore, weather swaps appear as either forward or future contracts. They are traded without a premium and have payoff that is linearly depended on some weather index. Prices are quoted in terms of the strike price and the level of the index. Swaps as forwards (which are mostly capped[3] and so are not strictly linear) are traded in the OTC for a very wide range of locations and indices. Swaps as futures (with no caps) are traded in the CME for monthly and seasonal contracts on several locations.

[3] In order to limit the maximum payout by any of the counterparties, the contracts are usually "capped". i.e. only a maximum amount of payout can change hands.

Example 1.4 A chain of wine bars wants to buy coverage to protect itself against bad weather, a fact that would reduce its sales. So it goes on a deal with the following terms: If the temperature falls below 24°C on Thursdays or Fridays between June and September, the company will receive a payment. The payments are fixed at £15.000 per day, up to a maximum limit of £100.000 in total for the whole period.

1.3.3 Weather Options

Weather options are contracts which are exchanged for a premium. Most OTC weather options for end users with weather risk are structured because there is no weather swap market in the location of the users. There are two types of options: calls and puts.

The buyer of a HDD call, for example, pays the seller a premium at the beginning of the contract. In return, if the number of $HDDs$ (HDD index) for the contract period is greater than the predetermined strike level the buyer will receive a payout. The size of the payout is determined by the strike and the tick size. The tick size is the amount of money that the holder of the call receives for each degree-day above the strike level for the period. Often the option has a cap on the maximum payout unlike, for example, traditional options on stocks.

A generic weather option can be formulated by specifying the following parameters: the contract type (call/put), the contract period (i.e. Jan

2012), the underlying index (HDD/CDD), the location (official weather station from which the temperature data are obtained), the strike level, the tick size and the maximum payout (if there is any).

A call option allows an investor to protect himself against the high index levels (I_n^H or I_n^C) and a put option allows a company to hedge against the low index levels. A call option gives to the buyer the following amount (payoff) at the expiration date T of the contract:

$C(T) = \delta max[I_n^H - K, 0]$, during the winter period, and
$C(T) = \delta max[I_n^C - K, 0]$, during the summer period.

A put option provides to the buyer the following amount :
$P(T) = \delta max[K - I_n^H, 0]$, during the winter period, and
$P(T) = \delta max[K - I_n^C, 0]$, during the summer period,
where δ is the tick size which represents the value of one degree-day and K is the strike level.

The form of the payoffs is due to the fact that these contracts give the right but not the obligation to buy (in the case of the call options) or to sell (in the case of the put options) the index (I_n^H or I_n^C) at the expiration date of the contract. This privilege requires the buyer to pay a premium to enter one of these contracts.

Example 1.5 A heating oil retailer wants to protect his company against a loss of a turnover due to an overly warm winter will sell an HDD call option. When a contract is signed, the retailer receives a premium equal to the price of the call option. If at the end of the contract period the actual index I_n^H is below the strike level the company will receive nothing. Otherwise, he will have to pay to the buyer the amount of $\delta(I_n^H - K)$. So, if the winter is particularly warm, the retailer keeps the premium of the call and that is a supplement from his lowered revenues. If the winter is very cold on the other hand, he will be able to finance the payout of the option by means of his increased revenues. Therefore, the retailer has reduced his company's exposure to weather risk. Alternatively, he could buy an HDD put. So, he would pay the price in advance, but if the winter turned to be cold enough, this small loss would have been balanced by his high revenues. If the winter was warm, then he would receive the payoff of the put.

As we mentioned earlier, usually calls and puts have a cap on the maximum payoff. This means that weather options are call and put spreads instead of calls and puts in traditional sense:

$$c(T) = min[cap, \delta max[0, I_n^H - K]]$$

$$c(T) = min[cap, \delta max[0, K - I_n^H]]$$

Similarly we denote the CDD option spreads. In the following example we have a CDD put option traded between Air conditioning Ltd which has the long position and ABC bank the short position on the contract.

CDD Put Option	
Current time	January 1, 2012
Location	JFK International Airport, NY
Long Position	Air Conditioning Ltd
Short Position	ABC Bank
Accumulation Period	July 2012
Tick size δ	$10000 per CDD
Strike Price K	550 CDDs
Actual Level	510 CDDs
Payoff at maturity (Long Position)	(550-510)x10000=$400

If this CDD put option had a cap, for example $350.000, the payoff at maturity would be:

$min[cap, \delta max[0, K - CDD]] = min[350,000,\ 10,000(max[0, 550 - 510])] = min[350,000, 400,000] = \$350,000.$

So, in this example, the settlement payoff would have been $350,000 instead of $400.000 if it the put option was uncapped.

For simplicity though, we will not consider the payoff of the spreads, but we will only focus on the payoff of the calls and puts.

1.3.4 Weather Futures and Weather Forwards

Companies can also use weather futures for protection, which are traded on the standardized markets like the Chicago Mercantile Exchange (CME) or by using weather forwards which are traded OTC (Over-the-Counter). Unlike futures, forwards are mostly capped.

The CME contracts are monthly and seasonal ones based on $HDDs$ and $CDDs$. By writing a weather futures, a company can sell or buy the index I_n^H (or I_n^C) according to their hedging strategy. I.e. a company which wants to protect itself against a loss of turnover due to an overly cold winter will buy the index I_n^H. It predetermines the level K of the index I_n^H (the strike level) at which it will buy it at the end of the contract period. On the contrary to the weather options, the company pays no premium to enter into the weather futures or the weather forwards but it has the obligation to buy the index I_n^H at the predetermined level K at the end of the contract. If the actual level of the index I_n^H is above K at maturity, the company will gain the amount $\delta(I_n^H - K)$ since it will buy the index at the level $K < I_n^H$ and will sell it at the level I_n^H. But it will lose the amount $\delta|I_n^H - K|$ if I_n^H is below K, because it has to buy the index I_n^H at level K and will sell it at level I_n^H. In fact, weather futures are cash-settled contracts, which means that there is a daily marking-to-market based upon the index, with the gain or loss applied to the customer's account.

1.4 Pricing Methods for Weather Derivatives

1.4.1 Inadequacy of Black and Scholes Model

Black and Scholes's (B-S) model which was developed in 1973, [5], to price put and call options is still commonly used today. Unfortunately, B-S model is based on certain assumptions that do not apply realistically to weather derivatives.

One of the main assumptions behind the B-S model is that the underlying of the contract (i.e. HDD or CDD for weather derivatives) follows a random walk without mean reversion. More appropriately, this model predicts that the variability of temperature increases with time, so temperature could wander off to any level whatsoever. Another significant reason why the B-S model is inappropriate for modeling weather derivatives is that the model is based on an underlying tradable commodity and weather is a non-tradable quantity (and it cannot be substituted by a linked exchanged security because weather index is poorly correlated with prices of other financial assets). The payoff of a weather option is instead based on a series of weather events, not on the value of the weather. The model also requires possible setting up of a conceptual portfolio with a position in both the options and the security from which the option value is derived. Without the means to trade weather as a security, one cannot build a riskless portfolio.This means that weather derivatives' market is an incomplete market. Therefore, we need to consider the market price of risk [4] in order to obtain unique prices of such contracts.

[4] The market price of risk is the difference between the expected rate of return of the underlying and the riskless interest rate, reported to the quantity of risk measured by the volatility.

As we can see weather derivatives are a different kind of derivative than those analysed by Black and Scholes. This makes weather derivatives' pricing a complicated issue and because of the wideness of the weather derivatives' market, there is a growing scientific interest concerning this subject, and a lot of paper work has been done in order to find the best model to valuate this different asset class.

1.4.2 Some Suggested Models

We will mention some of the suggested models for pricing weather derivatives, and analyze in detail three of them in chapter 3. All suggested models differ more or less on the process used to model the dynamics of temperature over time, on the assumptions made about the market price of weather risk [5] and on the techniques used on pricing.

As far as modeling the dynamics of temperature is concerned, there are either in-sample or out-of-sample approaches. The most usual is in-sample analysis, but there starts to appear some literature on out of sample approaches too. The in-sample approaches rely on historical data. For example, Davis (2001), [6], uses the marginal substitution price ("shadow price") approach to price the derivatives and by modelling HDD days as GBM concludes to explicit expressions for derivative prices, while Alaton (2002), [7], uses some historical data to model temperature as a sum of a mean-reverting seasonal term (deterministic) and a stochastic term (non - deterministic) and solves

[5] Weather is a non-tradable asset and its risk cannot be hedged by other tradable assets.

the equation using an Ornstein-Uhlenbeck process; we analyze this model on Chapter 2. The out of sample approaches rely on weather forecasts (see i.e. Taylor and Buiza (2004). [8]).

One of the models that is used for pricing weather derivatives and is relatively simple on implementation is the Actuarial model. It uses historical data (the so-called Burn Analysis) or Monte Carlo simulation to calculate the conditional expectation of the future payment of weather derivatives in order to evaluate them. We will present a brief analysis of this model in Section 3.3.

Another model is the one of Cao and Wei's (1998, 2004),[9] and [10], who see the problem of pricing weather derivatives from a consumption based perspective; that is, they aim to maximize the expectation of intertemporal utility based on the equilibrium model of Lucas (1978), [11].

In fact there is a wide range of literature concerning the subject of pricing weather derivatives: Brix, Jewson and Zeihmann (2002),[12], Frittelli (2000), [13], Heath, Platen and Schweizer (2001), [14] who all attempt pricing based on no-arbitrage assumptions, which is also analyzed in Section 3.2, and many more. Indicatively we mention some more authors that deal with the subject of pricing weather derivatives: Torró (2003), [15], Benth and Saltyte-Benth (2005), [16] and (2007), [17], Hamisultane (2008)[18], Pirrong and Jermakyan (2001), [19], and many more.

Chapter 2

Modeling the Dynamics of Temperature

As we have already mentioned, temperature derivatives are the most commonly traded. For this reason temperature modeling draws our attention. It should be helpful to adopt a model which describes temperature behavior and thus, many processes have been suggested to model the dynamics of (daily average) temperature. We have chosen to foster one of the proposed models for temperature which appears in Alaton's (2002), [7] paper, as it is a commonly used model (with or without modifications) for temperature modeling, and because we believe that it is a fairly comprehensive model for temperature's movement. Temperature according to the model is expressed by:

- a sine-function,
- an upward trend which derives from the global warming,
- a mean reversion,
- an autoregressive pattern, and
- a seasonal volatility.

In what follows we will build up the model of temperature step-by-step by gradually adding its previously mentioned components and explaining why each of them should enter the model.

2.1 An Expression of Mean Temperature

Mean temperature should be expressed by a sine – function and an upward trend.

- Why sine-function?

The first thing someone observes in temperature movement is strong seasonality. Thus, we guess that this seasonal dependence should be modeled with, for example, some sine-function of the form: $sin(\omega t+\phi)$, where t denotes the time measured in days. We let $t = 1, 2, \ldots$ denote January 1, January 2 and so on. Since we know that the period of the oscillations is one year (neglecting leap years) we have $\omega = 2\pi/365$. The phase angle ϕ enters the function because yearly minimum and maximum mean temperatures are not meant to usually occur at January 1 and July 1 respectively.

- Why positive trend?

Another component of the mean temperature that should be added to the model is a positive trend. Observing data someone can point out a positive trend, which is weak but it does exist. The reason for this slight gradual increase in temperature is the global warming trend all over the world or the urban heating effect. The latter can be either stronger if we obtain data from a highly populated city. To include this weak trend to the model we will assume, as a first approximation, that the warming trend is linear. This is harmless, and we do not need to assume it is polynomial, its effect is weak on the overall dynamics of the mean temperature and only the linear term of this polynomial will dominate.

Up to here, as far as the deterministic part of our model is concerned, the mean temperature T_t^m at time t will have the following form:

$$T_t^m = A + Bt + Csin(\omega t + \phi),$$

where the parameters A, B, C, ϕ have to be chosen so that the curve fits each of the data sets we choose well.

2.2 The Stochastic Part

Temperature is not deterministic as described previously. We shall add a stochastic component to have a better description of temperature's behavior.

Alaton (2002), [7], proposes a stochastic component as he observes in an 8-year (1989-1997) sample of daily average temperature data from Bromma Airport that the quadratic variation $\sigma_t^2 \in \mathbb{R}^+$ of the temperature varies across the different months of the year, but remains nearly constant within each

month. Especially during winter the quadratic variation is much higher than during the rest of the year. Therefore we assume that σ_t is a piecewise constant function with a constant value during each month:
$\sigma_t = \sigma_i, i = 1, 2, ..., 12$, when 1 corresponds to January, 2 to February, etc.
Thus, the stochastic component of temperature (noise), would be $\sigma_t W_t, t \geq 0$, where W_t is a standard Brownian motion.

2.3 Mean Reversion

So far, we have suggested a general form for mean temperature and a noise factor that lets temperature deviate from its mean randomly. But, if we let this deviation unconstrained for too long, we will get unrealistic results about the values it can undertake. Therefore, we shall also add a mean reverting component to our model.

So, putting together all the above assumptions, we get a stochastic differential equation (SDE) that has the following form:

$$dT_t = \alpha(T_t^m - T_t)dt + \sigma_t dW_t, \tag{2.1}$$

where α denotes the speed of mean reversion. The solution is given from a process that is defined as follows:

Definition 2.2 Ornstein-Uchlenbeck process is a stochastic process that describes the speed of a massive Brownian particle under the influence of friction.

The process is stationary, Gaussian, and Markovian, and is the only nontrivial process that satisfies these three conditions, up to allowing linear transformations of the space and time variables. The Ornstein – Uchlenbeck process is a stochastic process that satisfies the following SDE:

$$dX_t = \alpha(\mu - X_t)dt + \sigma dW_t, \tag{2.2}$$

where W_t is a Brownian motion. The constant parameters are:

- $\alpha > 0$ is the rate of mean reversion,
- μ is the long-term mean of the process,
- $\sigma > 0$ is the volatility square-root time of the random fluctuations that are modelled as Brownian motions.

Remark 2.3 Mean-reverting property: If we ignore the random fluctuations in the process due to dW_t, then we see that X_t has an overall drift [1] towards a mean value μ. The process X_t reverts to this mean exponentially, at rate α, with a magnitude in direct proportion to the distance between the current value of X_t and μ. This can be seen by looking at the solution to the ordinary differential equation $dx_t = \alpha(\mu - x_t)dt$, without the dW_T term, which is:

$$x_t = \mu + (x_0 - \mu)e^{-\alpha(t-t_0)}. \tag{2.5}$$

[1] The parameter a is the drift (average change per time unit):

$$a = \frac{\mathbb{E}[x(t+\Delta t) - x(t)]}{\Delta t}. \tag{2.3}$$

.

The parameter b^2 is the variance rate (variance of change per time unit):

$$b^2 = \frac{Var[x(t+\Delta t) - x(t)]}{\Delta t}. \tag{2.4}$$

For this reason the Ornstein – Uchlenbeck process is also called a mean-reverting process.

This is not enough though for our model because as Dornier & Queruel [20] indicate, equation (2.1) does not accomplish mean reversion in the long run, which means that we should add another term to the drift which has the following form:

$$\frac{dT_t^m}{dt} = B + \omega C cos(\omega t + \phi). \tag{2.6}$$

This term will adjust the drift to revert to the mean in the long run. Let's assume the starting point to be T_s and get the following model for temperature:

$$dT_t = \left[\frac{dT_t^m}{dt} + \alpha(T_t^m - T_t) \right] dt + \sigma_t dW_t, t > s. \tag{2.7}$$

In order to solve the SDE of (2.7) we recall next Ito's lemma.

Lemma 2.4 Ito's Lemma: It is a rule used for the calculation of the dynamics of a time-dependent function of a stochastic process (such as Geometric Brownian Motion (GBM)) $x(t)$ which has the form: $x_t = \alpha(x_t, t)dt + b(x_t, t)dz_t$. If $F(x, t)$ is a twice differentiable function then the dynamics of the stochastic process $F(x_t, t)$ are:

$$dF(x_t, t) = \left[\alpha(x_t, t)\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} + \frac{1}{2}b^2(x_t, t)\frac{\partial^2 F}{\partial x^2} \right] dt + \left[(b(x_t, t)\frac{\partial F}{\partial x} \right] dz_t.$$

We now apply the Ito's lemma to $f(T_t, t) = e^{\alpha t}T_t$:

$$\frac{\partial f}{\partial T} = e^{\alpha t}, \frac{\partial f}{\partial t} = \alpha T_t e^{\alpha t}, \frac{\partial^2 f}{\partial T^2} = 0$$

and we get the dynamics of $f(T_t, t)$ as follows:

$$df(T_t,t) = \left[e^{\alpha t}\frac{dT_t^m}{dt} + \alpha e^{\alpha t}T_t^m\right]dt + \sigma_t e^{\alpha t}dW_t. \tag{2.8}$$

Now, integrating from starting point s to t, we get:

$$\begin{aligned} f(T_t,t) - f(T_s,s) &= \int_s^t e^{\alpha\tau}\frac{dT_\tau^m}{d\tau}d\tau + \int_s^t \alpha e^{\alpha\tau}T_\tau^m d\tau + \int_s^t \sigma_\tau e^{\alpha\tau}dW_\tau \\ &= [e^{\alpha\tau}T_\tau^m]_s^t - \int_s^t \alpha e^{\alpha\tau}T_\tau^m d\tau + \int_s^t \alpha e^{\alpha\tau}T_\tau^m d\tau + \int_s^t \sigma_\tau e^{\alpha\tau}dW_\tau \\ &= e^{\alpha t}T_t^m - e^{\alpha s}T_s^m + \int_s^t \sigma_\tau e^{\alpha\tau}dW_\tau \end{aligned}$$

or

$$T_t e^{\alpha t} = e^{\alpha s}T_s + e^{\alpha t}T_t^m - e^{\alpha s}T_s^m + \int_s^t \sigma_\tau e^{\alpha\tau}dW_\tau$$

or

$$T_t = e^{-\alpha(t-s)}T_s + T_t^m - e^{-\alpha(t-s)}T_s^m + \int_t^s \sigma_\tau e^{-\alpha(t-\tau)}dW_\tau$$

So the solution is

$$T_t = (x - T_s^m)e^{-\alpha(t-s)} + T_t^m + \int_s^t e^{-\alpha(t-\tau)}\sigma_\tau dW_\tau, \tag{2.9}$$

where

$$T_t^m = A + Bt + Csin(\omega t + \phi). \tag{2.10}$$

Chapter 3

Review of Pricing Methods

Derivatives either traded in organized exchanges or traded in OTC market need pricing. For example, when it comes for pricing an option we need to calculate the premium paid by the purchaser at the time of the arrangement made with the seller, and when it comes for a future, we need to determine the strike price. In this chapter we will represent three methods of pricing weather derivatives.

3.1 Pricing under an Equivalent Martingale Measure

As mentioned before, the market of weather derivatives is an incomplete market. Therefore, we cannot come up to a unique price for our derivative under the risk neutral probability measure and this factor leads us to the conclusion that we have to consider the market price of risk λ in order to obtain unique prices for such contracts. For simplicity we assume that the market price of risk is constant. We also assume that we are given a risk-free asset with constant interest rate r and a contract that for each degree of Celsius pays one unit of currency (the tick size is 1).

The idea in this model is to eliminate the market price of risk by changing the probability measure to an equivalent martingale measure [1] Q by using the Girsanov's Theorem. Girsanov's theorem is used when we want we want to calculate an expectation under an equivalent measure that eliminates the risk premium. This theorem allows us to change the probability measure by changing the drift of the process. Thus, the expectation will be much easier to be calculated. A statement of Girsanov theorem is the following

Theorem 3.1 Girsanov's theorem:
Let

$$dX_t = b(X_t, t)dt + v(X_t, t)dW_t, \tag{3.1}$$

where W_t is a Brownian motion.
Then, (3.1) can be written as:

$$\begin{aligned} dX_t &= b(X_t, t)dt + v(X_t, t)\left[d\hat{W}_t - \alpha(X_t, t)dt\right] \\ &= \left[b(X_t, t) - \alpha(X_t, t)v(X_t, t)\right]dt + v(X_t, t)d\hat{W}_t, \end{aligned} \tag{3.2}$$

Where

$$d\hat{W}_t = \alpha(X_t, t)dt + dW_t \tag{3.3}$$

is a Brownian motion under the transformed (equivalent) probability measure $d\hat{P} = MdP$, in terms of the assumed martingale

$$M_t = \exp\left[-\frac{1}{2}\int_0^t \alpha^2(X_u, u)du - \int_0^t \alpha(X_u, u)dW_u\right] \tag{3.4}$$

[1] Equivalent measure Q: two probability measures are said to be equivalent, if they share the same null sets (P(A)=0 iff Q(A)=0).

Under this measure, relative asset prices are martingales. Also, we define $M = \frac{d\hat{P}}{dP}$, which is called Radon-Nikodym derivative and it is defined only for equivalent measures. Thus, Girsanov's theorem changes the drift by changing the probability measure. Furthermore it should be stressed that it leaves the volatility unchanged.

We have modeled the dynamics of temperature in Chapter 2 as given by (2.7).

We also define:

$$d\hat{W}_t = \lambda dt + dW_t, \tag{3.5}$$

where $\hat{W}_t$ is a Brownian Motion under the transformed probability measure $dQ \equiv d\hat{P} = MdP$ in terms of

$$M_t = exp\left[-\frac{1}{2}\lambda^2 t - \lambda W_t\right] \tag{3.6}$$

which is a martingale. The relationship (3.5) can be re-written as $dW_t = d\hat{W}_t - \lambda dt$ and substituted in (2.7) gives

$$\begin{aligned} dT_t &= \left[\frac{dT_t^m}{dt} + \alpha(T_t^m - T_t)\right] dt + \sigma_t(d\hat{W}_t - \lambda dt) \\ &= \left[\frac{dT_t^m}{dt} + \alpha(T_t^m - T_t) - \lambda\sigma_t\right] dt + \sigma_t d\hat{W}_t, \end{aligned} \tag{3.7}$$

where now $\hat{W}_t$ is a Q-Weiner process. This was a Girsanov transformation and as we can observe, we may have changed the drift but the volatility of T_t has remained unchanged. So from (2.14) we have that

$$\begin{aligned} Var[T_t|F_s] &= Var\left[\int_s^t e^{-\alpha(t-u)}\sigma_u dW_u|F_s\right] du \\ &= \mathbb{E}\left[\left(\int_s^t e^{-\alpha(t-u)}\sigma_u dW_u\right)^2 |F_s\right] \\ &= \int_s^t \sigma_u^2 e^{-2\alpha(t-u)} du. \end{aligned} \tag{3.8}$$

It is known that the price of a derivative is expressed as a discounted expected value under a martingale measure Q. Thus, we will start by computing the expected value and the variance of T_t under the measure Q. Therefore, thanks to (2.14) we have:

$$\mathbb{E}^P[T_t|F_s] = (T_s - T_s^m)e^{-\alpha(t-s)} + T_t^m. \tag{3.9}$$

So, from (2.14) and (3.5) we get that

$$\mathbb{E}^Q[T_t|F_s] = \mathbb{E}^P[T_t|F_s] - \int_s^t \lambda\sigma_u e^{-\alpha(t-u)} du. \tag{3.10}$$

If we evaluate the integrals in one of the intervals where $\sigma = \sigma_i$ is constant, we get:

$$\begin{aligned} \mathbb{E}^Q[T_t|F_s] &= \mathbb{E}^P[T_t|F_s] - \left[\lambda\sigma_i \frac{e^{-\alpha(t-u)}}{\alpha}\right]_s^t \\ &= \mathbb{E}^P[T_t|F_s] - \frac{\lambda\sigma_i}{\alpha}\left[e^{-\alpha(t-t)} - e^{-\alpha(t-s)}\right] \\ &= \mathbb{E}^P[T_t|F_s] - \frac{\lambda\sigma_i}{\alpha}\left[1 - e^{-\alpha(t-s)}\right] \end{aligned} \tag{3.11}$$

and similarly,

$$Var[T_t|F_s] = \left[\frac{\sigma_i^2 e^{-2\alpha(t-u)}}{2\alpha}\right]_s^t = \frac{\sigma_i^2}{2\alpha}\left[1 - e^{-2\alpha(t-s)}\right]. \tag{3.12}$$

Later we will need to compute the covariance between two different days. For $0 \leq s \leq t \leq u$, using the Ito isometry, the covariance function is given through (2.14) as

$$\begin{aligned}
Cov[T_t, T_u|F_s] &= Cov\left[\int_s^t \sigma_i e^{\alpha(\tau-t)} dW_\tau, \int_s^u \sigma_i e^{\alpha(v-u)} dW_v|F_s\right] \\
&= \mathbb{E}\left[\int_s^t \sigma_i e^{\alpha(\tau-t)} dW_\tau \int_s^u \sigma_i e^{\alpha(v-u)} dW_v|F_s\right] \\
&= \sigma_i^2 e^{-\alpha(t+u)} \mathbb{E}\left[\int_s^t e^{\alpha\tau} dW_\tau \int_s^u e^{\alpha v} dW_v|F_s\right] \\
&= \sigma_i^2 e^{-\alpha(t+u)} \mathbb{E}\left[\int_s^{min(t,u)} e^{2\alpha\tau} d\tau|F_s\right] \\
&= \frac{\sigma_i^2}{2\alpha} e^{-\alpha(t+u)} (e^{2\alpha min(t,u)} - e^{2\alpha s}) \\
&= \frac{\sigma_i^2}{2\alpha} (e^{-\alpha(u+t-2s)} - e^{-\alpha(u+t)}) \\
&= \frac{\sigma_i^2}{2\alpha} (1 - e^{-2\alpha(t-s)}) e^{-\alpha(u-t)} \\
&= e^{-\alpha(u-t)} Var[T_t|F_s],
\end{aligned} \tag{3.13}$$

where the last equation comes from (3.12)

Suppose now that t_1 and t_n denote the first and last day of a month, and we start the process at some time s from the month before $[t_1, t_n]$. To compute the expected value and variance of T_t for $t \in [t_1, t_n]$ in this case, we split the integrals in (3.10) and (3.8) into the two integrals $[s, t_1]$ and $[t_1, t]$ where σ is constant in each one of the σ_i and σ_j respectively. We then get the expected value:

$$\mathbb{E}^Q[T_t|F_s] = \mathbb{E}^P[T_t|F_s] - \frac{\lambda}{\alpha}(\sigma_i - \sigma_j)e^{-\alpha(t-t_1)} + \frac{\lambda\sigma_i}{\alpha}e^{-\alpha(t-s)} - \frac{\lambda\sigma_j}{\alpha} \tag{3.14}$$

and the variance:

$$Var[T_t|F_s] = \frac{1}{2\alpha}(\sigma_i^2 - \sigma_j^2)e^{-2\alpha(t-t_1)} - \frac{\sigma_i^2}{2\alpha}e^{-2\alpha(t-s)} + \frac{\sigma_j^2}{2\alpha}. \tag{3.15}$$

Example 3.1 Pricing an HDD Option: A commonly used weather derivative is an HDD option which, as we have previously mentioned, has heating degree days as an underlying asset. We will now attempt to price an HDD call option, by using the theory described previously.

We have seen that the payoff of an HDD call option is (we assume that ticksize = 1 for simplicity):

$$X = max[I_n^H - K, 0], \tag{3.16}$$

where

$$I_n^H = \sum_{i=1}^{n} max[18 - T_i, 0]. \tag{3.17}$$

The underlying process here is normally distributed, but the maximum factor makes our job to find a pricing formula rather complicated. So we will try to make an approximation.

Under Q and given information at time s, $T_t \sim N(\mu_t, \upsilon_t)$ where μ_t is given by (3.14) and υ_t by (3.15). Let's also assume that the winter in the city from which we obtain the data is very cold so the probability that $max[18 - T_i, 0] = 0$ should be extremely small on a winter day. Therefore, for such a contract we may write:

$$I_n^H = 18n - \sum_{i=1}^{n} T_{t_i}. \tag{3.18}$$

The distribution of this is easier to determine. We know that T_{t_i}, $i = 1, ..., n$ are all samples from an Ornstein-Uhlenbeck process, which is a Gaussian process. This means that also the vector $(T_{t_1}, T_{t_2}, ..., T_{t_n})$ is Gaussian. Since the sum in (3.27) is a linear combination of the elements in this vector, I_n^H is also Gaussian. With this new structure of I_n^H it only remains to compute the first and second moments. We have for $t < t_1$ and (3.18) that

$$\begin{aligned} \mathbb{E}^Q[I_n^H|F_t] &= \mathbb{E}^Q[18n - \sum_{i=1}^{n} T_{t_i}|F_t] \\ &= 18n - \sum_{i=1}^{n} \mathbb{E}^Q[T_{t_i}|F_t] \end{aligned} \tag{3.19}$$

and

$$Var[I_n^H|F_t] = \sum_{i=1}^{n} Var[T_{t_i}|F_t] + 2\sum_{i<j}\sum Cov[T_{t_i}, T_{t_j}|F_t]. \tag{3.20}$$

Now, suppose that we have made the calculation above, and found that

$$\mathbb{E}^Q[I_n^H|F_t] = \mu_n \tag{3.21}$$

and

$$Var[I_n^H|F_t] = \sigma_n^2, \tag{3.22}$$

thus I_n^H is $N(\mu_t, \sigma_n^2)$- distributed. Hence, the price at $t \leq t_1$ of the HDD call option with payoff given by (3.16) is

$$\begin{aligned} c(t) &= e^{-r(t_n - t)}\mathbb{E}^Q\left[max[I_n^H - K, 0]|F_t\right] \\ &= e^{-r(t_n-t)}\int_K^\infty (x-K) f_{I_n^H}(x)dx \\ &= e^{-r(t_n-t)}\left[(\mu_n - K)\Phi(-\alpha_n) + \frac{\sigma_n}{\sqrt{2\pi}}e^{-\frac{\alpha_n^2}{2}}\right] \end{aligned} \tag{3.23}$$

where $\alpha_n = (K - \mu_n)/\sigma_n$ and Φ denotes the cumulative distribution function of the standard normal distribution.

In the same way we can derive a formula for the price of an HDD put option, whose payoff is given by the expression:

$$Payoff = max[K - I_n^H, 0]. \tag{3.24}$$

The price is

$$\begin{aligned} p(t) &= e^{-r(t_n - t)}\mathbb{E}^Q\left[max[K - I_n^H, 0]|F_t\right] \\ &= e^{-r(t_n-t)}\int_0^K (K-x) f_{I_n^H}(x)dx \\ &= e^{-r(t_n-t)}\left[(K - \mu_n)(\Phi(\alpha_n) - \Phi(-\frac{\mu_n}{\sigma_n})) + \frac{\sigma_n}{\sqrt{2\pi}}(e^{-\frac{\alpha_n^2}{2}} - e^{-\frac{1}{2}(\frac{\mu_n}{\sigma_n})^2})\right]. \end{aligned} \tag{3.25}$$

3.2 Arbitrage-free Pricing Method

We assume that no arbitrage [2] is the reasonable basis to build up a theory that will lead us to a "fair" price for a derivative. This is a customary way to conclude to a specific price for many derivative products.

I.e., for pricing an option, we determine a compensation (premium) for the buyer at the time of the arrangement with the seller for the risk he undertakes. On a market without friction and operating continuously [3], the value of an option giving a payoff at maturity is obtained by the creation of a self-financing portfolio composed of the quantities of the underlying and of a riskless asset which will replicate the results of the option at maturity. Concerning that there should be no arbitrage opportunities, the price of the option at time 0 must be equal to the initial cost of the portfolio since they provide the same income at maturity.

The price derived from this operation corresponds to the discounted expectation of the payoff of the option under a risk neutral probability measure Q which is equivalent to the real probability measure. (Cox, Ross and Rubinstein, 1979), [21], introduced the Binomial model which shows that from the equality between the value of the portfolio and the payment of the option at maturity and from downgrading in time, the price of the option at time 0 is determined in a unique way and corresponds to the computation of the expected payment of the option at maturity, discounted by the risk-free rate and

[2] Arbitrage is the opportunity for someone to make profit without undertaking any risk.

[3] We assume that: the market is liquid, there are no transaction costs, investors can borrow and lend at the same time and at the same risk-free rate, every arbitrage opportunity is immediately absorbed by market dynamics, there are no constraints on short - selling of the asset.

taken under the risk-neutral probabilities of the underlying. The limit of the formula for the option price of Cox, Ross and Rubinstein (1979), [21], when the number of periods becomes infinitely large, coincides with the closed form expression that Black-Scholes (1973) provided for this price in continuous time.

For complex options such as weather options, for which the prices cannot have explicit representation because their underlying assets (such as the meteorological index) are not traded on the financial market, no self-financing portfolio can be considered.

As we mentioned in Section 1.4, Black-Scholes formula cannot be applied in weather derivatives as it violates a number of its assumptions. Geman (1999), [22], tried to overcome the obstacle of non-tradability of weather derivatives in order to use the B-S formula. He proposed the substitution of the meteorological index from a contract on energy by stressing the dependence of the energy price with the climate, but Brix, Jewson and Ziemann (2002), [12], demolished this theory by pointing out that the price of gas is more closely linked to the demand than to the temperature. Instead, they proposed to use the weather futures contracts, whose prices are, in their view, highly correlated with the underlying of the weather options which they want to value. With the development of the weather markets this solution seemed feasible, but yet these contracts are not liquid enough to create the self-financing portfolio of perfect replication.

Other strategies though can provide replication of the payoff of the option. These strategies have to do with the way that we choose a risk neutral probability to help us calculate the price. The ideas are discussed below.

- **By maximization of the expected utility of the agent:**

We may choose the quantities of the securities in the portfolio, so as to maximize the expected utility of the agent or in order to reduce the variance of the difference between the value of the portfolio and the result of the option at the end of the period. Authors such as Frittelli (2000), [13], have shown that maximizing the expected utility with a utility function of exponential type gave rise to the calculation of the price of the option. The price is expressed as being the conditional expectation of the payment of the option, discounted by the riskless rate and defined under the risk-neutral distribution of the underlying asset called "minimal entropy martingale measure" [4]; this is because it presents the particularity to minimize the relative entropy or distance of Kullback-Leibler of a probability with regard to the other one defined a priori.

- **By reducing the variance:**

Another way to make the replication possible is to reduce the variance, the conditional expectation of the option price is calculated by using a risk-neutral distribution of the underlying called "variance optimal martingale measure" (Heath, Platen and Schweizer (2001), [14]). It was shown that in an incomplete

[4] is the risk-neutral probability measure that minimizes the entropy difference between the objective probability measure and the risk-neutral measure $\mathbb{Q}$. In incomplete markets, this is one way of choosing a risk-neutral measure (from the infinite number available) so as to still maintain the no-arbitrage conditions.

market (when the payoff of the option is not reachable by a self-financing portfolio), there was a range of arbitrage-free prices for the derivative (Karatzas and Kou (1996)). On the other hand, the price was unique when the market was complete because there was only a single measure of risk-neutral probability; all the other measures were becoming identical with that stemming from the strategy of perfect replication.

These two strategies outlined above are perfectly feasible to treat the problem of liquidity of the weather futures contracts when creating a self-financing portfolio. The question is to know which one to choose. The problem though lies not only in this choice to be made, but it also concerns the difficulty to implement these approaches. To obtain the price of a derivative from one of these strategies, it is advisable to calculate the expectation of its terminal payment by means of the risk-neutral distribution corresponding to the chosen strategy. There are two ways to calculate the expectation under one of these probability measures: either by extracting a risk-neutral distribution from the quoted prices or by solving a partial differential equation (PDE) with a terminal condition whose the conditional expectation of the derivative price is the unique solution. We describe the two ways in a sequel.

3.2.1 Inference of a Risk Neutral Probability

The prices at time t of a weather call option and a futures contract, both written on the HDD index with maturity T, are defined by

$$C(t, I_t^H) = \delta e^{-r(T-t)} \mathbb{E}^Q \left[max(I_T^H - K, 0) | F_t \right] \quad (3.26)$$

$$= \delta e^{-r(T-t)} \int_0^\infty max[I_T^H - K, 0] f_Q(I_T^H) dI_T^H \quad (3.27)$$

and

$$F(t, I_t^H) = \delta \mathbb{E}^Q [I_T^H | F_t] = \delta \int_0^\infty I_T^H f_Q(I_T^H) dI_T^H, \quad (3.28)$$

respectively. Here δ is the tick size, $I_T^H = \sum_{j=1}^n max[18 - T_j, 0]$ is the HDD index on n days of the contract period, T_j is the daily average temperature, r is the risk free rate, K is the strike price, Q is the risk neutral probability (which is not unique here since the market is incomplete), F_t is the available information for temperature up to time t, and f_Q is the risk neutral (or state price) density of the I_T^H index.

As already mentioned, there are several possible risk-neutral distributions. So to calculate a price we need to choose one of them. We can choose a distribution from quoted prices from future contracts that are traded on the CME instead of option prices, as the weather options are traded in the OTC market. For contracts that are written on the meteorological index there should not be any problem, as the risk neutral distribution will be the same.

To infer a risk neutral distribution from the quotations, we should try to find values of the risk neutral density so as to reduce the distances between the price given in (3.42) and the observed market price. This task requires the solution of the following optimization problem:

$$\min_{f_Q} \sum_{t=1}^{M} (F(t, I_t^H) - F_t^{market}),$$

where F_t^{market} is the observed price of the weather futures contract at time t. The objective function though contains a regularization term because the number of unknown densities is superior to the number of equations to be solved. The Jackwerth and Rubinstein (1996) optimization program is recommended here because it does not require a closed-form expression for the theoretical price of the contracts, since it is very difficult or impossible to reduce the price formula of the weather derivatives to an analytical expression; see also Hamisultane (2006), [23].

3.2.2 Resolution of a Partial Differential Equation

The alternative way to compute the price of a weather derivative through the arbitrage free approach is by the resolution of a PDE. Pirrong and Jermakyan (2001), [19], have suggested the calculation of the arbitrage free prices of weather options by including the market prices of risk from the quotations of the weather futures. As we know, in an incomplete market there are as many market prices of risk as risk neutral distributions, because the market price of risk depends on the portfolio strategy one adopts to reduce the risk and simultaneously on the risk neutral distribution associated with the strategy. The market prices of risk λ_t, which have to be inferred from quotations, should minimize the following function:

$$\min_{\lambda_t}(F(t, T_t, I_t^H) - F_t^{market})^2, \tag{3.29}$$

where $F(t, T_t, I_t^H)$ is the theoretical price of the weather derivative, and F_t^{market} the observed price in the market.

Note 3.2 In fact the objective function of (3.29) requires a regularization term because there are more unknown market prices of risk than quotations and thus the minimization problem cannot be solved. This is because prices are quoted only during working days, where λ_t must be computed continuously especially when we use numerical methods to solve the PDE for the theoretical price that appears later. However the presence of a regularization term should increase the computation time considerably, so for our convenience we assume the prices during non-working days remain the same as the last ones quoted. This way, there is no inequality anymore between the unknown parameters and the number of equations. The solution is acceptable because the weather contract prices are not very volatile.

Consider now for example a futures contract based on the HDD index. The terminal condition of this future contract will be

$$F(T, T_T, I_T^H) = I_T^H,$$

where T is the time to maturity. We recall from (3.17) that the formula for the HDD index, which considered in continuous time should be of the form

$$I_t^H = \int_0^t max(18 - T_s, 0)ds$$

and thus, has the following dynamics

$$dI_t^H = max(18 - T_t, 0)dt. \tag{3.30}$$

Also we recall from (3.7) that the dynamics of temperature considering the market price of risk (which emerges as temperature is not a tradable asset), under an equivalent martingale measure Q, are the following:

$$dT_t = \left[\frac{dT_t^m}{dt} + \alpha(T_t^m - T_t) - \lambda_t\sigma_t\right]dt + \sigma_t dW_t^Q. \tag{3.31}$$

We will now present the multivariate Ito's process which we will then apply to the function $F(t, T_t, I_t^H)$.

Lemma 3.4 Multivariate Ito's Lemma: The dynamics of a stochastic process $F(t, X_t, Y_t)$, where F is a twice differentiable function and X_t and Y_t are stochastic processes with dynamics $dX_t = A_t dt + B_t dW_t$ and $dY_t = \Gamma_t dt + \Delta_t dW_t$ respectively, are

$$dF(t, X_t, Y_t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX_t + \frac{\partial F}{\partial Y}dY_t + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}dX_t dX_t + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}dY_t dY_t + \frac{\partial^2 F}{\partial X \partial Y}dX_t dY_t;$$

here, $dX_t dX_t = B_t^2 dt$, $dY_t dY_t = \Delta_t^2 dt$ and $dX_t dY_t = B_t \Delta_t dt$

The theoretical price of the future contract should be equal to the discounted expected value of the terminal condition of the contract calculated under the equivalent martingale measure Q, given the available information set F_t; that is

$$F(t, T_t, I_t^H) = \mathbb{E}^Q[I_t^H | F_t]$$

From the tower property of the conditional expectation one may observe that $\mathbb{E}^Q[I_t^H|F_t]$ is a Q-martingale and so is the left hand side of the last equality, whose dynamics are computed by the multivariate Ito's Lemma as

$$dF(t,T_t,I_t^H) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial T}dT_t + \frac{\partial F}{\partial I^H}dI_t^H + \frac{1}{2}\frac{\partial^2 F}{\partial T^2}dT_t dT_t +$$

$$+\frac{1}{2}\frac{\partial^2 F}{\partial I^H}dI_t^H dI_t^H + \frac{\partial^2 F}{\partial T \partial I^H}dT_t dI_t^H, \tag{3.32}$$

but from (3.30) and (3.31) $dI_t^H dI_t^H = 0$, $dT_t dI_t^H = 0$ and $dT_t dT_t = \sigma_t^2 dt$, so

$$dF(t,T_t,I_t^H) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial T}dT_t + \frac{\partial F}{\partial I^H}dI_t^H + \frac{1}{2}\frac{\partial^2 F}{\partial T^2}\sigma_t^2 dt \tag{3.33}$$

According next to (3.30), (3.31) and (3.33), (3.32) becomes:

$$dF(t,T_t,I_t^H) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial T}\left[\left[\frac{dT_t^m}{dt} + \alpha(T_t^m - T_t) - \lambda_t\sigma_t\right) dt + \sigma_t dW_t^Q\right] +$$

$$+\frac{\partial F}{\partial I^H}max(18 - T_t, 0)dt + \frac{1}{2}\frac{\partial^2 F}{\partial T^2}\sigma_t^2 dt =$$

$$\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial T}\left[\frac{dT_t^m}{dt} + \delta(T_t^m - T_t) - \lambda_t\sigma_t\right] + \frac{\partial F}{\partial I^H}max(18 - T_t, 0) + \frac{1}{2}\frac{\partial^2 F}{\partial T^2}\sigma_t^2\right) dt + \sigma_t dW_t^Q \tag{3.34}$$

But, as we mentioned above the process $F(t, T_t, I_t^H)$ is also a martingale, so, its drift should be equal to 0. Thus, the PDE for the theoretical price $F(t, T_t, I_t^H)$, is the following:

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial T}\left[\frac{dT_t^m}{dt} + \alpha(T_t^m - T_t) - \lambda_t \sigma_t\right] + \frac{\partial F}{\partial I^H} max(18 - T_t, 0) + \frac{1}{2}\frac{\partial^2 F}{\partial T^2}\sigma_t^2 = 0.$$

This is about a weather future. Respectively for a weather option the PDE has the form:

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial T}\left[\frac{dT_t^m}{dt} + \alpha(T_t^m - T_t) - \lambda_t \sigma_t\right] + \frac{\partial F}{\partial I^H} max(18 - T_t, 0) + \frac{1}{2}\frac{\partial^2 F}{\partial T^2}\sigma_t^2 = r_t F$$

The implementation of this method will be analyzed in Section 4.5.

3.3 The Actuarial Pricing Method

The actuarial method for pricing weather derivatives is proposed by several authors such as Brix, Jewson and Ziemann (2002), [12] , Augros and Moreno (2002), [24] and Bay et al (2003), [25] . In practice, weather derivatives are usually valued by the actuarial method, because of the simplicity of its implementation. Also, it can be applied in any type of weather derivative, regardless of its liquidity, whether it's feasible to obtain quotations for it or whether we can substitute it for another relative asset.

According to the actuarial pricing method, the value of a weather derivative corresponds to the conditional expectation of its future payment, calculated under the real probability plus a charge depending on a risk measure which is usually the volatility. Thus, the actuarial prices of the weather call/put options and futures on the HDD index at time t are expressed as follows (Hamisultane, 2008, [18]) :

$$C_{Actuarial}(t, T_t, I_t^H) = \delta e^{-r(T-t)} \left(\mathbb{E}[max(I_T^H - K, 0)|F_t] + \lambda \sigma_{max(I_T^H - K, 0)} \right), \tag{3.35}$$

$$P_{Actuarial}(t, T_t, I_t^H) = \delta e^{-r(T-t)} \left(\mathbb{E}[max(K - I_T^H, 0)|F_t] + \lambda \sigma_{max(K - I_T^H, 0)} \right), \tag{3.36}$$

$$F_{Actuarial}(t, T_t, I_t^H) = \delta(\mathbb{E}[I_T^H|F_t] + \lambda \sigma_{I_T^H}) \tag{3.37}$$

where δ is the tick size, K is the strike price (for the options), r is the risk-free rate, F_t denotes the available information about temperature until time t, and time T represents the maturity date of the contracts. Also, $\lambda \sigma_{max(I_T^H - K, 0)}, \lambda \sigma_{max(K - I_T^H, 0)}$ and $\lambda \sigma_{I_T^H}$ are called safety loading (Hamisultane, 2006, [26]) and designate the risk premiums, where $\sigma_{max(I_T^H - K, 0)}, \sigma_{max(K - I_T^H, 0)}$ and $\sigma_{I_T^H}$ denote the volatility of payoffs of the options and futures respectively.

The actuarial method is based on the law of large numbers. The latter indicates that as the number of samples increases, we obtain a more and more reliable estimation of the true expected value of the observed phenomenon. The expectation under the real probability can be computed in either of the two following ways:

- by using historical data (Burn Analysis), or
- by using the technique of Monte Carlo simulation.

Burn Analysis: In this approach, we accumulate degree days of a year, and then we determine the payoff of the derivative for this specific year, and then we repeat the process for other years. Then, the average of annual payoffs will represent the expected price of the derivative.

Monte Carlo simulation technique: In this approach instead of historical prices for temperature, we use a model of daily average temperature to generate a set of paths and for each of these paths we construct the HDD index which is used to calculate the payoff. Then, the average of the payoffs from all the generated paths will represent the expected price of the derivative.

The Growth Optimal Portfolio (GOP)*: Platen and West (2004), [27], settled the link between the arbitrage free methods and the actuarial method via the notion of "Growth Optimal Portfolio (GOP)" and in that way justified the use of the actuarial method on weather derivatives. GOP is a self-financing portfolio that maximizes the expected logarithmic utility from terminal wealth. Under certain conditions, this GOP can be considered as a numeraire portfolio which converts the variables expressed in units of this numeraire into martingales whatever probability measure is used. Therefore, they write the option price in units of the GOP at time 0 as follows:

$$\hat{C}(0,t) = \mathbb{E}(\hat{H}_t|F_0), \tag{3.38}$$

where $\hat{H}_t = \frac{H_t}{S_t^{(\pi)}}$ is the payoff of the option in units of the GOP and S_t^{π} represents the GOP at time t. The price of the option not in terms of the GOP is given by:

$$C(0,t) = S_0^{(\pi)}\hat{C}(0,t) \tag{3.39}$$

and

$$C(0,t) = \mathbb{E}\left(\frac{S_0^{(\pi)}}{S_t^{(\pi)}}H_t|F_0\right). \tag{3.40}$$

By defining the discrete time Radon-Nikodym derivative as

$$\frac{\partial Q}{\partial P} = \Lambda_t = \frac{\hat{S}_t^{(0)}}{\hat{S}_0^{(0)}} = \frac{S_t^{(0)}}{S_t^{(\pi)}}\frac{S_0^{(\pi)}}{S_0^{(0)}}, \tag{3.41}$$

where $\hat{S}_t^{(0)} = \frac{S_t^{(0)}}{S_t^{(\pi)}}$ stands for the domestic savings account in units of the GOP, they demonstrate that this price is formulated as the discounted expectation of the payoff under the probability Q, i.e.

$$C(0,t) = \frac{S_0^{(0)}}{S_t^{(0)}}\mathbb{E}^Q(H_t|F_0). \tag{3.42}$$

If the payoff of the option is independent of the GOP, they show that it can be expressed as an actuarial price, i.e.

$$C(0,t) = S_0^{(\pi)}\mathbb{E}\left(\frac{1}{S_t^{(\pi)}}H_t|F_0\right)\mathbb{E}(H_t|F_0) \tag{3.43}$$

and

$$C(0,t) = P(0,t)\mathbb{E}(H_t|F_0). \tag{3.44}$$

where $P(0,t) = S_0^{(\pi)}\mathbb{E}\left(\frac{1}{S_t^{(\pi)}}H_t|F_0\right)$ corresponds to the price of a zero coupon bond at time 0.

Approximating the GOP by the MCSI World index, they show that the weather index of Sydney is very uncorrelated with this index and therefore that the weather derivatives should be priced by the actuarial approach.

Chapter 4

Computation of the Weather Derivative Prices

4.1 Simulation of Temperature Trajectories

We recall from chapter 2 the model that gives us the dynamics of temperature

$$dT_t = \left[\frac{dT_t^m}{dt} + \alpha(T_t^m - T_t) - \lambda\sigma\right] dt + \sigma_t dW_t,$$

where $T_t^m = A + Bt + Csin(\omega t + \phi)$

In order to simulate trajectories of this process, we need to discretize this equation. Discretizing dT_t to a time interval $\delta = T_j - T_{j-1}$, should give us

$$\delta T = T_j - T_{j-1} = \delta T^m + \alpha(T_{j-1}^m - T_{j-1})\delta t - \lambda\sigma\delta t + \sigma\epsilon\sqrt{\delta t} \tag{4.1}$$

where $\epsilon_{j=1}^{N-1}$ are independent standard normally distributed random variables.

Now assuming that time interval δt corresponds to 1 day, (4.1) can be written as:

$$T_j = T_{j-1} + \delta T^m + \alpha(T_{j-1}^m - T_{j-1}) - \lambda\sigma + \sigma\epsilon$$

or

$$T_j = (1-\alpha)T_{j-1} - (1-\alpha)T_{j-1}^m + T_j^m + (\epsilon - \lambda)\sigma$$

or

$$T_j = (1-\alpha)(T_{j-1} - T_{j-1}^m) + T_j^m + (\epsilon - \lambda)\sigma,$$

where $T_{j-1}^m = A+B(j-1)+Csin(\omega(j-1)+\phi)$ and $T_j^m = A+Bj+Csin(\omega j+\phi)$

Inserting the following values for the parameters of the equation: $T_0 = 0,\ A = 6,\ B = 6\cdot 10^{-5},\ C = 10.4,\ \alpha = 0.23,\ \omega = \frac{2\pi}{365},\ \phi = -2,\ \sigma = 3.4,$ and for a time period of 3500 days (nearly 9,5 years), we get the plot in figure 1:

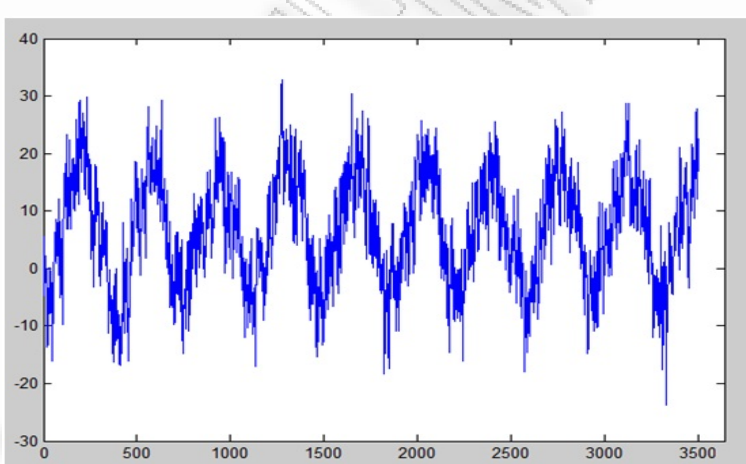


Figure 4.1: Simulation of Temperature Trajectories.

Figure 2 below shows a diagram of real data from Helsinki (Finland) for 3,500 consecutive days as well as the simulation (from 01.01.2003 to 31.07.2012)

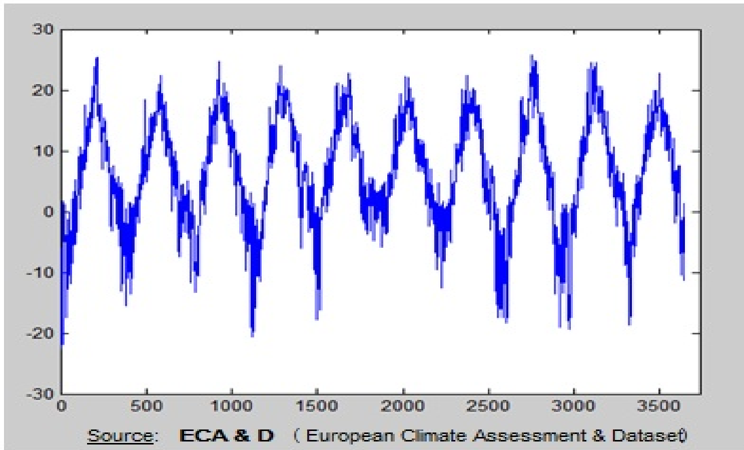


Figure 4.2: Daily mean temperatures during 2003 - 2012 (Helsinki, Finland).

4.2 Computation of the Theoretical Price of an HDD call (and put) option under an Equivalent Martingale Measure

Recall from chapter 2 that the payoff of the HDD call option is:

Payoff $= max[I^H - K, 0]$ (we use tick size = 1 for simplicity)

where $I^H = \sum_{i=1}^{n} max[18 - T_t, 0]$.

As already mentioned in chapter 3, the presence of the maximum function, does not let us to determine the price of the derivative now at time t=0 (discounted expected payoff under the risk neutral probability) without considerable error.

So we make the simplifying assumption that temperature will not exceed 18°C during the periods that winter contracts (HDD) are written. This is assumption is reasonable and be used without spoiling the results, as long as we refer to a contract during very cold months (i.e. December,January). With the index now being expressed as: $I^H = 18n - \sum_{i=1}^n T_t$, we compute the prices of a call (and put) via the following equations:

$$c(0) = e^{-rt_n}((\mu_n - K)\Phi(-\alpha_n) + \frac{\sigma_n}{\sqrt{2\pi}} e^{\frac{-\alpha_n^2}{2}},$$

$$p(0) = e^{-rt_n}[(K - \mu_n)\left(\Phi(\alpha_n) - \Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right) + \frac{\sigma_n}{\sqrt{2\pi}}\left(e^{-\frac{\alpha_n^2}{2}} - e^{-\frac{1}{2}(\frac{\mu_n}{\sigma_n})^2}\right),$$

where $\alpha_n = \frac{K-\mu_n}{\sigma_n}$, $\mu_n = \mathbb{E}^Q[I^H]$, $\sigma_n^2 = Var[I^H]$ and Φ denotes the cumulative standard normal distribution.

In Section 4.6 we can see different prices for an HDD call option that this method gives for different strike prices K, T_0 and times to maturity t_n.

4.3 Computation of the price of an HDD call (and put) option using Monte Carlo Simulations

In this section, we will use Monte Carlo simulations to compute the price of an HDD call(and put) option. We start by simulating a number of trajectories ($NRepl$) of the temperature (section 4.1) for a given period of time t_n, starting from today's temperature T_0, and then accumulate each of these trajectories in order to construct the HDD index $I_i^H, i = 1, \ldots, NRepl$ for each of them respectively. Now we can determine the $NRepl$ payoffs at time t_n:

$max[I_i^H - K, 0]$ for the HDD call option and $max[K - I_i^H, 0]$ for the HDD put option. Then we approximate the expected value (which is under the risk neutral probability) by the arithmetic average. Note that using this technique we did not need to make any simplifying assumption about the distribution of I^H as it is was crucial for the theoretical price computation.

In Section 4.6 we can see different prices for an HDD call option that this method gives for different strike prices K, T_0 and times to maturity t_n.

4.4 Computation of the Actuarial Price of an HDD Call (and put) option by Monte Carlo Simulations

In this section, we will approximate the expectation of an HHD weather futures and an HDD weather call option actuarial price via Monte Carlo simulation. The HDD call and put option prices given by the actuarial approach (from chapter 3) are:

$$c(0) = e^{-rt_n} \left(\mathbb{E}^P \left[max[I^H - K, 0] \right] + \kappa \sigma_{payoff} \right),$$

$$p(0) = e^{-rt_n} \left(\mathbb{E}^P \left[max[K - I^H, 0] \right] + \kappa \sigma_{payoff} \right),$$

σ_{payoff} denotes the volatility of the payoffs and it is calculated as the standard deviation of the payoffs.

In Section 4.6 we can see different prices for an HDD call option that this method gives for different strike prices K, T_0 and times to maturity t_n.

4.5 Computation of the price of an HDD call option by resolving a PDE

Recall the PDE for the HDD call option from chapter 3 :

$$\frac{\partial F}{\partial t}+\frac{\partial F}{\partial T}\left[\frac{dT_t^m}{dt}+\alpha(T_t^m-T_t)-\lambda_t\sigma_t\right]+\frac{\partial F}{\partial I^H}max(18-T_t,0)+\frac{1}{2}\frac{\partial^2 F}{\partial T^2}\sigma_t^2=rF \tag{4.2}$$

where $T_t^m = A + Bt + Csin(\omega t + \phi)$.

This PDE can be solved by the numerical method of finite differences. This requires the construction of a uniform grid (of equally spaced points); it also requires the replacement of the continuous derivatives of the above equation by a discrete operator (here this operator will be truncated Taylor series). The operator can be forward, backward or central difference. For example for the partial derivative with respect to time, we get respectively:

- Forward difference: $\frac{\partial F}{\partial t}=\frac{F^g_{i,j+1}-F^g_{i,j}}{\delta t}$
- Backward difference: $\frac{\partial F}{\partial t}=\frac{F^g_{i,j}-F^g_{i,-1j}}{\delta t}$,
- Central difference: $\frac{\partial F}{\partial T}=\frac{F^g_{i,j+1}-F^g_{i,j-1}}{2\delta T}$,
- Second order central difference: $\frac{\partial^2 F}{\partial T^2}=\frac{F^g_{i,j+1}-2F^g_{i,j}+F^g_{i,j-1}}{(\delta T)^2}$,

where i, j, and g, correspond to the temperature variable T, time variable t and index variable IH respectively.

In this case we will use a grid of M x N x G points, where M,N and G correspond to the number of points for the above variables T, t and IH respectively. Also the discretization steps that will be used are: $i\delta T$,for the temperature, $j\delta t$ for the time and $g\delta I$ for the index variable, where $i = 0, 1, \ldots, M$, $j = 0, 1, \ldots, N$ and $g = 0, 1, \ldots, G$. The operator will be used in such a way that the terms will be gathered to lead us to a resolution scheme: We can either use the explicit, implicit or Semi-Implicit (Crank Nicolson) scheme

- The Explicit scheme uses a backward difference for time derivatives and central difference and second order central difference for space derivatives.
- The Implicit scheme uses a forward difference for time derivatives and central difference and second order central difference for space derivatives.
- The semi-implicit scheme (Crank-Nicolson method) is a combination of the implicit scheme at and the explicit scheme at $n + 1$ (however, the method itself is not simply the average of those two methods, as the equation has an implicit dependence on the solution)

For simplicity of the implementation we will use the explicit scheme; however it has to be used under restrictions for the size of the grid to avoid the oscillations of the solutions. For better results we recommend the implicit scheme which is also more effective than the Crank Nicholson's method (Hamisultane, 2008), [18].

Remark 4.3.1: Due to the fact that the diffusion term $\frac{\partial^2 F}{\partial IH}$ does not exist in the PDE, in order to avoid the oscillations of the solutions it is considered that the index value remains constant between the days of observation (and thus, the component $\frac{\partial F}{\partial IH}$ for which there is no corresponding diffusion term disappears).

So discretizing the equation (4.1) according to the explicit scheme, we have (where $i = 1, \ldots, M$ for variable T, $j = 1, \ldots, N$ for variable t and $g = 1, \ldots, G$ for variable g)

$$\frac{F^g_{i,j} - F^g_{i,j-1}}{\delta t} + \left(\frac{\delta T^m_j}{\delta t} + \alpha(T^m_{j-1} - i\delta T) - \lambda\sigma\right)\frac{F^g_{i+1,j} - F^g_{i-1,j}}{2\delta T} +$$

$$+\frac{1}{2}\sigma^2\frac{F^g_{i+1,j} - 2F^g_{i,j} + F^g_{i-1,j}}{(\delta T)^2} = rF^g_{i,j}$$

or

$$F^g_{i,j-1} = F^g_{i,j} + rF^g_{i,j} - \delta t\left(\frac{\delta T^m_j}{\delta t} + \alpha(T^m_{j-1} - i\delta T) - \lambda\sigma\right)\frac{F^g_{i+1,j}}{2\delta T} +$$

$$+\left(\frac{\delta T^m_j}{\delta t} + \alpha(T^m_{j-1} - i\delta T) - \lambda\sigma\right)\delta t\frac{F^g_{i-1,j}}{2\delta T} - \frac{1}{2}\sigma^2\delta t\frac{F^g_{i+1,j}}{(\delta T)^2} +$$

$$+\frac{F^g_{i,j}}{(\delta T)^2}\sigma^2\delta t - \frac{1}{2}\sigma^2\delta t\frac{F^g_{i-1,j}}{(\delta T)^2}$$

or

$$F_{i,j}^{g} = rF_{i,j}^{g} + \frac{1}{2}\delta t\left(\left(\frac{\delta T_j^m}{\delta t} + \alpha(T_{j-1}^m - i\delta T) - \lambda\sigma\right)\frac{1}{2\delta T} - \frac{\sigma^2}{(\delta T)^2}\right)F_{i-1,j}^{g} +$$

$$+\left(1 + \frac{1}{(\delta T)^2}\sigma^2\delta t + r\right)F_{i,j-1}^{g} + \frac{F_{i,j}^{g}}{(\delta T)^2}\sigma^2\delta t - \frac{1}{2}\sigma^2\delta t\frac{F_{i-1,j}^{g}}{(\delta T)^2} +$$

$$+\frac{\delta t}{2\delta T}\left(\frac{\delta T_j^m}{\delta t} + \alpha(T_{j-1}^m - i\delta T) - \lambda\sigma\right)F_{i-1,j}^{g}$$

So, the discretized equation has now the form

$$F_{i,j-1}^{g} = \alpha_{i,j}F_{i-1,j}^{g} + b_{i,j}F_{i,j}^{g} + c_{i,j}F_{i+1,j}^{g} \tag{4.3}$$

where

$$\Lambda_{i,j} = \frac{\delta T_j^m}{\delta t} + \alpha(T_{j-1}^m - i\delta T) - \lambda\sigma,$$

$$\alpha_{i,j} = \frac{1}{2}\delta t\left(-\frac{1}{\delta T}\Lambda_{i,j} + \frac{\sigma^2}{(\delta T)^2}\right),$$

$$b_{i,j} = 1 - \left(\frac{\sigma^2}{(\delta T)^2} + r\right)\delta t$$

and

$$c_{i,j} = \frac{1}{2}\delta t\left(\frac{\Lambda_{i,j}}{\delta T} + \frac{\sigma^2}{(\delta T)^2}\right).$$

The terminal conditions are of great importance because they are the starting point in making the grid. So the boundary conditions are:

ΠΑΝΕΠΙΣΤΗΜΙΟ ΠΕΙΡΑΙΑ

$F^g_{i,N} max[g\delta I^H - K, 0]$, $\forall i$ and $\forall g$. At the expiration of the contract, the cumulative index I, reaches its highest level.

$F^g_{0,j} = e^{-r(T-j\delta t)}(G\delta I^H - K)$, $\forall j$ and $\forall g$. When temperature reaches 0° C (or a lowest level), at some time t it is highly possible to have maintained this low level until time t and keep being this low until maturity time (this assumption comes from the stationarity property of Ornstein-Uchlenbeck process that temperature follows).

$F^g_{M,j} = 0$ $\forall j$ and $\forall g$. Similarly, when the temperature reaches a high level, at time t its highly possible to maintain this high level until maturity, so the cumulative index will be equal to 0 at maturity.

In the equation (4.3), we have 3 known values $F^g_{i-1,}$, $F^g_{i,j}$, $F^g_{i+1,j}$ which are linked to one unknown value $F^g_{i,j-1}$. These known values are given by the terminal conditions at maturity. To solve equation (4.3) we have to go backwards in time (from $j = N-1, .., 0$).

4.6 Results

In Table 1 we get 5 prices for 5 different strike prices K respectively of an HDD call option with $T_0 = 0°C$, risk free rate r $= 5\%$ and time to maturity $t_n = 48$ days; with parameters for the temperature: $T_0 = 0$, $A = 6$, $B = 6 \cdot 10^{-5}$, $C = 10.4$, $\alpha = 0.23$, $\omega = 2\pi/365$, $\phi = -2$, $\sigma = 3.4$, $\lambda = 0.08$ except from the actuarial method in which we use $\lambda = 0$ for the temperature and $\kappa_{payoff} = 0.08$. We also use 10000 replications for the Monte Carlo simulations.

Table 1

Prices for an HDD Call Option (To=0, r=5%,tn=48) given from the 3 methods for different strike prices and their corresponding differences

Strike Price K	Theoretical method (MM)	Monte Carlo method (MM)	Actuarial method	PDE Fin Diff Expl Method	TH-MC difference	TH-ACT difference	TH-PDE difference	MC-ACT difference	MC-PDE difference	ACT-PDE difference
480	56,233	55,963	55,972	58,32	0,27	0,261	-2,087	-0,009	-2,357	-2,348
530	51,697	51,454	51,543	52,707	0,243	0,154	-1,01	-0,089	-1,253	-1,164
560	48,976	48,804	48,953	49,343	0,172	0,023	-0,367	-0,149	-0,539	-0,39
600	45,347	45,101	45,0213	44,858	0,246	0,3257	0,489	0,0797	0,243	0,1633
650	40,812	40,533	40,483	39,26	0,279	0,329	1,552	0,05	1,273	1,223

In Table 2 we get 4 prices for 4 different To respectively of an HDD call option with strike price 560, risk free rate r $= 5\%$ and time to maturity $t_n = 48$ days; with parameters for the temperature: $A = 6$, $B = 6 \cdot 10 - 5$, $C = 10.4$, $\alpha = 0.23$, $\omega = 2\pi/365$, $\phi = -2$, $\sigma = 3.4$, $\lambda = 0.08$ except from the actuarial method in which we use $\lambda = 0$ for the temperature and $\kappa_{payoff} = 0.08$. We also use 10000 replications for the Monte Carlo simulations.

Table 2

Prices for an HDD Call Option (K=560,r=5%,tn=48) given from the 3 methods for different To and their corresponding differences						
To	Theoretical method (MM)	Monte Carlo method (MM)	Actuarial method	TH-MC difference	TH-ACT difference	MC-ACT difference
5	47,222	46,813	46,419	0,409	0,803	0,394
10	45,467	44,838	45,015	0,629	0,452	-0,177
15	43,713	42,77	43,071	0,943	0,642	-0,301
20	41,96	41,075	41,019	0,885	0,941	0,056

In Table 3 we get 4 prices for 4 different times to maturity t_n respectively of an HDD call option with $T_o = 0°C$, strike price 560 and risk free rate r = 5%; with parameters for the temperature: $T_o = 0,\ A = 6,\ B = 6 \cdot 10^{-5},\ C = 10.4,\ \alpha = 0.23,\ \omega = 2\pi/365,\ \phi = -2,\ \sigma = 3.4,\ \lambda = 0.08$ except from the actuarial method in which we use $\lambda = 0$ for the temperature and $\kappa_{payoff} = 0.08$. We also use 10000 replications for the Monte Carlo simulations.

Table 3

Prices for an HDD Call Option (To=0,K=560,r=5%) given from the 3 methods for different To and their corresponding differences						
Time to Maturity tn	Theoretical method (MM)	Monte Carlo method (MM)	Actuarial method	TH-MC difference	TH-ACT difference	MC-ACT difference
30	29,389	27,494	26,229	1,895	3,16	1,265
35	41,803	41,555	41,768	0,248	0,035	-0,213
40	48,315	47,834	49,221	0,481	-0,906	-1,387
45	49,816	49,759	51,688	0,057	-1,872	-1,929

Remark 4.6.1: We observe that all the prices that the different models give from the simulations are relatively close; that means that the models are consistent with each other. However, the PDE approach shows inconsistency when as we raise the grid's density. That is explained by the instability of the explicit scheme. As it is proposed in section 4.5, someone could implement the method using the implicit scheme for more accurate results.

Chapter 5

Conclusion

As soon as weather derivatives appeared as a financial instrument, a growing literature started towards the determining of a "fair" pricing model for them. As weather is not a tradable asset, this task becomes very complicated. Through this study, we have understood that in order to use a model for pricing this derivative it is required to choose a model that describes the evolution of the underlying meteorological index (here is temperature) as well as possible. This is the key to pricing weather derivatives. We applied Alaton's (2002), [7], model for temperature. This model is a relatively simple model for implementation and we have seen that describes well the movement of temperature. However this model makes some simplifying assumptions(i.e. volatility is not actually constant within the days of a month as assumed in the model); since temperature's behavior is an even more complicated procedure to simulate than this model assumes it is, someone should consider some more complicated models for temperature to get even better results: A suggested model for temperature is one that uses stochastic volatility (see Benth and Saltyte-Benth (2005, 2007), [16], [17], and a model that combines in-sample and out-of-sample analysis to model the daily average temperature (see Taylor and Buiza (2004), [8]).

We analyzed three methods used for pricing weather derivatives: Pricing under an Equivalent Martingale Measure which we implemented in an HDD call option by two different ways, Arbitrage free pricing method also implemented in an HDD option using the PDE approach, and Actuarial method implemented using Monte Carlo simulations. The four different ways of implementation are consistent to each other, as the prices they give are very close when we insert the same parameters to the different algorithms that correspond to each one of them. Theoretical prices tend to be slightly higher than those that Monte Carlo simulations and Actuarial Method give. Monte Carlo simulation method gives slightly lower prices than these of the Arbitrage free Method. As far as the PDE approach is concerned, in comparison with the other methods, it gives close prices for a certain range of sizes of the grid; when we try to increase the grid's density, the method crushes because of the instability of the explicit scheme of finite differences that is used for the implementation.

As for the parameters we insert to the model of temperature in order to compute the prices for the HDD call option, we have used estimated prices from Alaton 2002, [7]). These estimations come from 40 years daily average temperature in different locations across Sweden. For the parameter that correspond to the market price of risk (here weather risk), we use the value $\lambda = 0.08$. As we have previously mentioned in the study, this value can be inferred by quoted prices of corresponding call options in the market by solving some optimization problem. This is usually not feasible when it comes to weather options, as they are usually traded in the OTC market.

Chapter 6

List of algorithms used for the computation of prices (Matlab)

6.1 List of Algorithms

(1) Monte Carlo simulation of Temperature Path trajectories:

```
function [ TPaths ] = TemperaturePaths
( To,A,B,C,a,omega,phi,sigma,NSteps,NRep,lamda )
TPaths=zeros(NRep,1+NSteps);
TPaths(:,1)=To;
for ti=1:NRep
    for tj=1:NSteps
        TPaths(ti,tj+1)=A+B*(tj+1)+C*sin(omega*(tj+1)+phi)+
        (1-a)*(TPaths(ti,tj)-(A+B*tj+C*sin(omega*tj+phi)))+
        sigma*randn-lamda*sigma;
    end
end
end
```

(2) Theoretical Price of an HDD Call option (and an HDD Put option) under an Equivalent Martingale Measure.

```
function [ CallPrice,PutPrice ] =
EquivalentMMTheor( To,K,r,tn,A,B,C,a,omega,phi,lamda,sigma )
Tmto=A+C*sin(phi);
Sum1=0;
Sum2=0;
Sum3=0;
Sum4=0;
for ti=1:tn
    Tmti=A+B*ti+C*sin(omega*ti+phi);
    EPTti=(To-Tmto)*exp(-a*ti)+Tmti;
    EQTti=EPTti-(lamda*sigma/a)*(1-exp(-a*ti));
    VarTti=(sigma^2/2*a)*(1-exp(-2*a*ti));
    for tj=ti+1:tn
        CovTtiTtj=exp(-a*(tj-ti))*VarTti;
        Sum3=Sum3+CovTtiTtj;
    end
    Sum1=Sum1+EQTti;
    Sum2=Sum2+VarTti;
    Sum4=Sum4+Sum3;
end
EQHn=18*tn-Sum1;
VarHn=Sum2+2*Sum4;
an=(K-EQHn)/sqrt(VarHn);
CallPrice=exp(-r*tn)*((EQHn-K)*normcdf(-an)+
sqrt(VarHn/2*pi)*exp(-an^2/2))
PutPrice=exp(-r*tn)*((K-EQHn)*(normcdf(an)-
normcdf(-EQHn/sqrt(VarHn)))+sqrt(VarHn/2*pi)*
(exp(-an^2/2)-exp((-1/2)*EQHn^2/VarHn)))
end
```

(3)Pricing of an HDD Call option (and an HDD Put option) by Monte-Carlo simulation

```
function [ CallPrice,PutPrice ] = EquivalentMMMC
( To,A,B,C,a,omega,phi,sigma,NSteps,NRep,K,r,lamda )
DiscPayoffsHnCall=zeros(NRep,1);
DiscPayoffsHnPut=zeros(NRep,1);
for ti=1:NRep
    Path=TemperaturePaths(To,A,B,C,a,omega,phi,sigma,NSteps,1,lamda);
    Hn(ti)=sum(max(18-Path(1:NSteps),0));
    DiscPayoffsHnCall(ti)=exp(-r*NSteps)*max((Hn(ti)-K),0);
    DiscPayoffsHnPut(ti)=exp(-r*NSteps)*max((K-Hn(ti)),0);
end
CallPrice=mean(DiscPayoffsHnCall)
PutPrice=mean(DiscPayoffsHnPut)
end
```

(4) Pricing of an HDD Call option (and an HDD Put option) by Actuarial Method

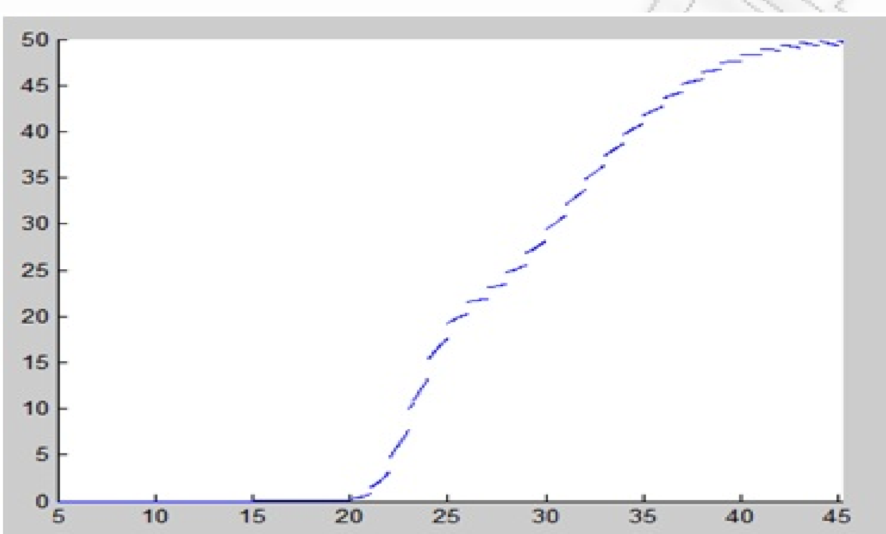
```
function [ CallPrice,PutPrice ] = ActuarialMC
( To,A,B,C,a,omega,phi,NSteps,NRep,K,r,k,sigma )
Hn=zeros(NRep,1);
DiscPayoffsHnCall=zeros(NRep,1);
DiscPayoffsHnPut=zeros(NRep,1);
for ti=1:NRep
    Path=TemperaturePaths(To,A,B,C,a,omega,phi,sigma,NSteps,1,0);
    Hn(ti)=sum(max(18-Path(1:NSteps),0));
    s=std(Hn);
    volatilitypayoff=s/sqrt(NSteps/365);
    DiscPayoffsHnCall(ti)=exp(-r*NSteps)*max((Hn(ti)-K),0);
    DiscPayoffsHnPut(ti)=exp(-r*NSteps)*max((K-Hn(ti)),0);
end
CallPrice=mean(DiscPayoffsHnCall+k*volatilitypayoff)
PutPrice=mean(DiscPayoffsHnPut+k*volatilitypayoff)
end
```

(5) Pricing of an HDD Call option by PDE Approach

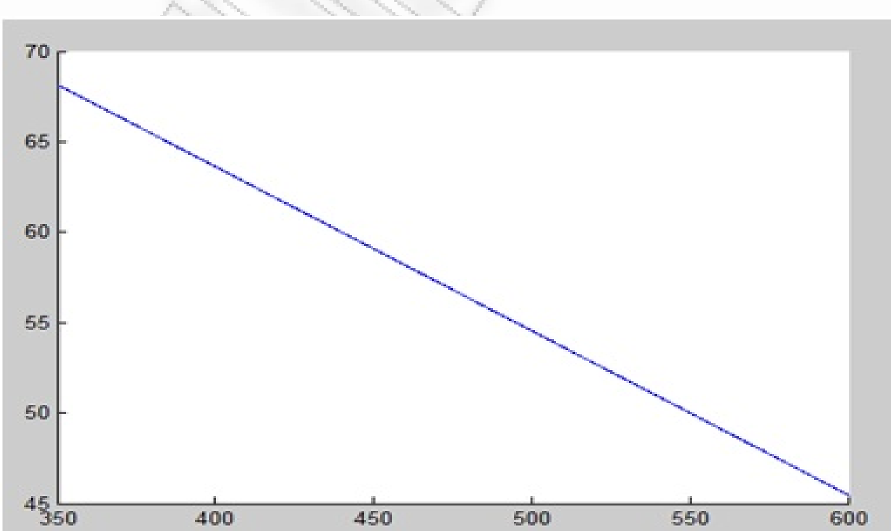
```
function [ Price ] = PDEExplicit( A,B,C,dT,dI,
Imax,N,a,omega,phi,sigma,K,r,Io,To,Tmax,lamda )
M=round(Tmax/dT);
dT=Tmax/M;
dt=1;
G=round(Imax/dI);
dI=Imax/G;
matval=zeros(M+1,N+1,G+1);
vetj=0:N;
%oriakes sunthikes
for g=0:G+1
    matval(:,N+1,g+1)=max(g*dI-K,0);
    matval(1,:,g+1)=exp(-r*dt*(N-vetj))*(G*dI-K);
    matval(M+1,:,g+1)=0;
end
%suntelestes
Lamda=zeros(M+1,N+1);
for i=0:M
    for j=0:N
        Lamda(i+1,j+1)=B+C*omega*cos(omega*dt*j+phi)
        +a*(A+B*j*dt+C*sin(omega*j*dt+phi)-dT*i)-lamda*sigma;
    end
end
a=0.5*dt*(-(1/dT)*Lamda+((sigma^2)/(dT^2))*ones(M+1,N+1));
b=(1-((((sigma/dT)^2)+r)*dt))*ones(M+1,N+1);
c=0.5*dt*((Lamda/dT)+(sigma^2)/(dT^2)*ones(M+1,N+1));
for g=1:G+1
    for j=N:-1:1
        for i=2:M
            matval(i,j,g)=a(i,j).*matval(i-1,j+1,g)
            +b(i,j).*matval(i,j+1,g)+c(i,j).*matval(i+1,j+1,g);
        end
    end
end
X=linspace(0,Tmax,M+1);
Y=linspace(0,Imax,G+1);
[X,Y]=meshgrid(X,Y);
F=zeros(M+1,G+1);
for i=1:M+1
    for g=1:G+1
        F(i,g)=matval(i,1,g);
    end
end
Price=interp2(X,Y,F,To,Io);
end
```

6.2 List of Plots

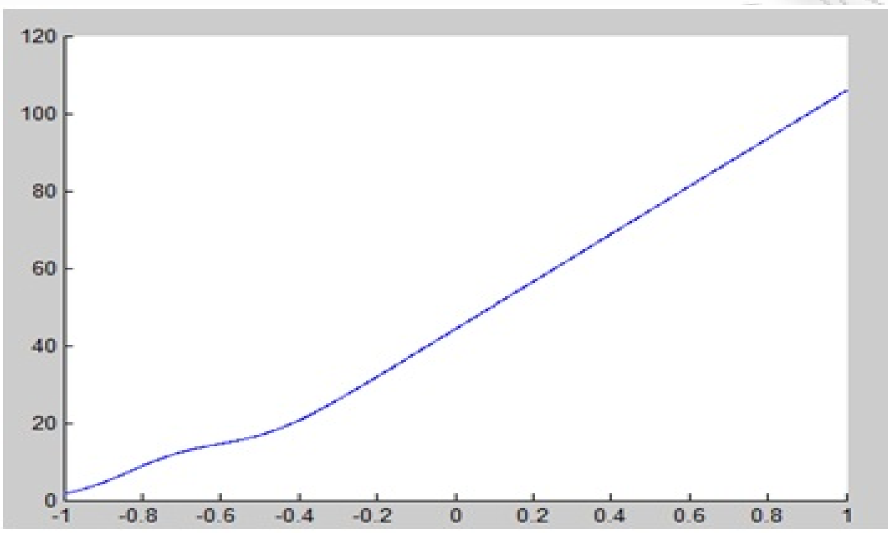
1. Price of an HDD Call Option vs different Times to Maturity T_n (Theoretical Price).



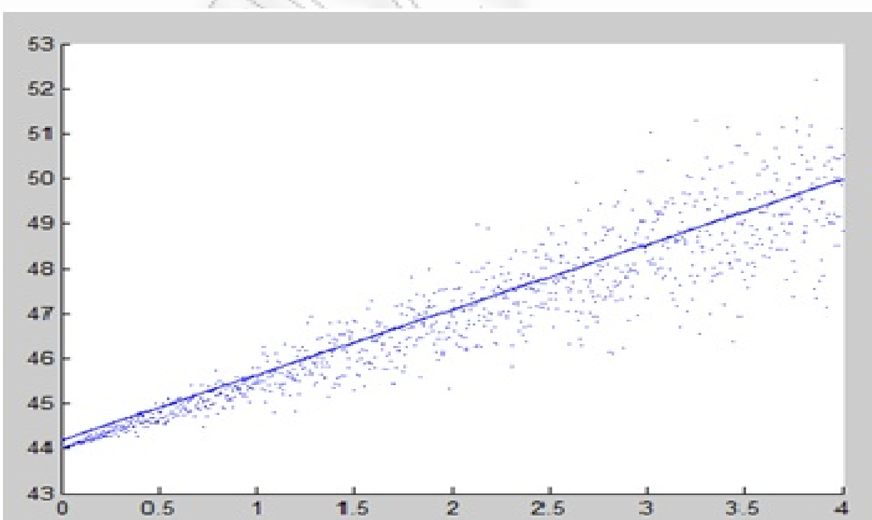
2.Price of an HDD Call Option vs different Strike Prices K (Theoretical Price).



3. Price of an HDD Call Option vs different weather risks λ (Theoretical Price).



4.Price of an HDD Call Option vs different volatilities σ (Theoretical Price and MC Simulations).



Bibliography

[1] Peter H. Larsen P.H. Waldman D.M. Lazo J.K., Lawson M. U.s. economic sensibility to weather variability, 2011.

[2] Considine G. Introduction to weather derivatives, weather derivatives group, aquila energy, 2001. URL http://www.dmeoncmeglobex.net/trading/weather/files/WEA_intro_to_weather_der.pdf.

[3] Gianluca Piovani et al. Weather derivatives market: The italian case. 2012. URL http://www.europeanjournalofsocialsciences.com/ISSUES/EJSS_30_3_01.pdf.

[4] Miller S. Weather builds on derivatives sales in (2011). *Financial Risk Today*, 2011. URL http://www.financialriskstoday.com.

[5] Black F. and Scholes M. The pricing of options and corporate liabilities. *Journal of political economy*, 1973.

[6] Davis M. Pricing weather derivatives by marginal value. *Imperial College London*, December 2000.

[7] Alaton P. On modelling and pricing weather derivatives. 2002. URL http://www.treasury.nl/files/2007/10/treasury_301.pdf.

[8] Taylor J.W. and Buizza R. Comparison of temperature density forecasts from garch and atmospheric models. *Journal of Forecasting*, 22, 2004.

[9] Cao M. and Wei J. Pricing weather derivatives: An equilibrium approach. *Weather derivatives valuation and market price of weather risk*, 1998.

[10] Cao M. and Wei J. Weather derivatives valuation and market price of weather risk. *Journal of futures market*, 24, 2004.

[11] Lucas R.E. Jr. Asset prices in an exchange economy. *Econometrica*, 46, 1978.

[12] Jewson S. Zeihmann C. Brix A. Weather derivative modelling and valuation: A statistical perspective. *Climate risk and the weather market, financial risk management with weather hedges*, 2002.

[13] Frittelli M. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical finance*, 10, 2000.

[14] Schweizer M. Heath D., Platan E. Numerical comparison of local risk minimization and mean - variance hedging. *Handbook in mathematical finance: option pricing, interest rates and risk management, Cambridge University Press*, 2001.

[15] Valor E. Torró H., Meneu V. Single factor stochastic models with seasonality applied to underline weather derivatives variables. *Journal of risk finance*, 4, 2003.

[16] Benth F.E. and Saltyte-Benth J. Stochastic modelling of temperature variations with a view towards weather derivatives. *Applied mathematical finance*, 12, 2005.

[17] Benth F.E. and Saltyte-Benth J. The volatility of temperature and pricing of weather derivatives. *Quantitative finance*, 7, 2007.

[18] Hamisultane H. Which method for pricing weather derivatives ?, 2008. URL http://halshs.archives-ouvertes.fr/docs/00/35/58/56/PDF/wpaper0801.pdf.

[19] Pirrong C. and Jermakyan M. The price of power: The valuation of power and weather derivatives. 2001. URL http://papers.ssrn.com/sol3/papers.cfm?abstract_id=240815.

[20] Dornier F. and Queruel M. Caution to the wind, weather risk special report. *Energy and power risk management risk magazine*, 2000.

[21] Cox S, Ross A., and Rubinstain M. Option pricing a simplified approach. *Journal of financial economics*, 1979.

[22] Geman H. The bermudan triangle: Weather, electricity and issurance derivatives. *Insurance derivatives from exotic options to exotic underlying*, 1999.

[23] Hamisultane H. Extracting information from the market to price the weather derivatives, 2006. URL http://papers.ssrn.com/sol3/papers.cfm?abstract_id=907445.

[24] Augros J.C. and Moreno M. Les dérivés financiers et d'assurance. *Economica*, 2002.

[25] Bay X. Carraro L. Roustant O., Laurent J.P. Model risk in the pricing of weather derivatives. *Working paper*, 2003.

[26] Hamisultane H. Pricing the weather derivatives in the presence of long memory in temperatures, 2006. URL http://halshs.archives-ouvertes.fr/docs/00/08/87/00/PDF/weathderiv_longmemory.pdf.

[27] Platen E. and West J. Fair pricing of weather derivatives. *Working paper*, 2004.