UNIVERSITY OF PIRAEUS

Asymptotic Expansions of Econometric Estimators in Time Series Models

by

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Contents

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I, DIMITRA KYRIAKOPOULOU, declare that this thesis titled, 'ASYMPTOTIC EXPANSIONS OF ECONOMETRIC ESTIMATORS IN TIME SERIES MODELS' and the work presented in it are my own. I confirm that:

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"ASYMPTOTIC EXPANSIONS OF ECONOMETRIC ESTIMATORS IN TIME SERIES MODELS" a thesis prepared by DIMITRA KYRIAKOPOULOU in partial fulfillment of the requirements for the degree, DOCTOR OF PHILOSOPHY.

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"I have frequently been questioned, especially by women, of how I could reconcile family life with a scientific career. Well, it has not been easy."

Marie Curie

UNIVERSITY OF PIRAEUS

Abstract

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Techniques for approximating probability distributions like the Edgeworth expansion have a long history in time series models. The purpose of this thesis is to give a detailed study of the asymptotic properties of the Moving Average (MA) and the Exponential GARCH (EGARCH) models. Extending the results in Sargan (1976) [80] and Tanaka (1984) [87], we derive the asymptotic expansions of the distribution, the bias and the mean squared error of the MM and QML estimators of the first order autocorrelation and the MA parameter for the MA(1) model. It turns out that the asymptotic properties of the estimators depend on whether the mean of the process is known or estimated. A comparison of the moment expansions, either in terms of bias or MSE, reveals that there is not uniform superiority of neither of the estimators, when the mean of the process is estimated. This is also confirmed by simulations. In the zero-mean case, and on theoretical grounds, the QMLEs are superior to the MM ones in both bias and MSE terms. The results are important for bias correction and increasing the efficiency of the estimators. Next, we derive the bias approximations of the ML and QML estimators of the EGARCH(1,1) parameters and we check our theoretical results through simulations. With the approximate bias expressions up to O(1/T), we are then able to correct the bias of all estimators. To this end, a Monte Carlo exercise is conducted and the results are presented and discussed. We conclude that, for given sets of parameters values, the bias correction works satisfactory for all parameters. The results for the bias expressions can be used to formulate the approximate Edgeworth distribution of the estimators. Moreover, the asymptotic properties of EGARCH models are still largely unexplored and are considered difficult tasks (see e.g. Straumann and Mikosch, 2006) [83]. There is still no complete answer to the following questions: under which conditions do EGARCH processes have bounded first and second order variance derivatives? And under which conditions is the expectation of the supremum norm of the second order log-likelihood derivative finite, in a neighborhood around the true parameter value? These questions are important because the existence of such moment bounds permits the establishment of large sample statistical properties, such as the asymptotic normality of the QMLEs.

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Chapter 1

Introduction

1.1 A Very Brief History of the Edgeworth Expansion in Time Series

Techniques for approximating probability distributions like the Edgeworth expansion have a long history in econometrics[1]. Unambiguously, one of the most important papers in the related literature is the paper by Sargan (1976) [80], which discusses how in practice we can improve on the asymptotic approximations and proposes the use of the Edgeworth expansion to approximate the marginal distribution of an econometric estimator and improve the use of asymptotic limits in significance testing. A wide variety of econometric estimators can be regarded as functions of the sample data first and second moments. If these functions are reasonably well behaved, it is possible to make a Taylor series expansion about the moments and obtain an approximation of the econometric estimator up to the terms that involve those derivatives from the Taylor expansion. The technical issues of the Edgeworth expansion will be discussed briefly below so that the reader will get familiarized with the notion of the expansion and how it is derived.

Another very useful paper in the time series context is the paper by Phillips (1977a) [74], who derives the Edgeworth series expansions of the finite sample distributions of

[1] The reader is referred to the introduction of the second chapter for a detailed list of the papers written about the Edgeworth expansion in the context of time series models.

the least squares estimator and the associated t ratio test statistic in a first order stochastic difference equation, that is an autoregressive process of first order. Turning our attention to some more complicated cases, the paper by Tanaka (1984) [87] in the mid of eighties, considers the Edgeworth expansion for the distributions of estimators derived by the Maximum Likelihood method in the context of Autoregressive Moving Average (ARMA) models and develops a technique for obtaining the first order Edgeworth type asymptotic expansion for the joint as well as marginal and conditional distributions. Quite recently, Kakizawa (1999) [59] derives valid Edgeworth expansions for the standardized and Studentized versions of some estimators in first order autoregression without Gaussianity.

To close this less technical section, Edgeworth expansions have been developed for various fields: for example, weak dependence (Gotze and Hipp 1983 [45]), Gaussian ARMA structures (Taniguchi 1987 [89]), generalized autoregressive conditional heteroskedasticity (Linton 1997 [65]), Whittle estimation for long-memory Gaussian time series (Lieberman et al (2003) [63] and Andrews and Lieberman 2005 [3]), linear regression processes with long-memory errors (Aga 2011 [1]).

1.1.1 The Edgeworth Expansion and its related formulae

The Edgeworth expansion has been traditionally confined to the independent and identically distributed (i.i.d.) situation (e.g. Bhattacharya and Rao 1976 [17]). The analysis that follows is based on the analysis of some manuscripts regarding the Edgeworth expansion, see for example, Barndorff-Nielsen and Cox 1989 [12]). Let an estimator $\widehat{\theta}$ and if $\sqrt{T}\left(\widehat{\theta}-\theta_0\right)$ is asymptotically normally distributed with zero mean and variance σ^2, where T is the sample size, then in a great many cases of practical interest the distribution function of $\sqrt{T}\left(\widehat{\theta}-\theta_0\right)$ may be expanded as a power series in $\frac{1}{\sqrt{T}}$[2], that is:

$$P\left\{\sqrt{T}\left(\widehat{\theta}-\theta_0\right)\leq x\right\}=\Phi\left(x\right)+\frac{1}{\sqrt{T}}p_1\left(x\right)\phi\left(x\right)+\frac{1}{T}p_2\left(x\right)\phi\left(x\right)+\ldots+\frac{1}{T^{j/2}}p_j\left(x\right)\phi\left(x\right)+\ldots$$

[2]In probability and statistics applications, the quantity becoming large is usually the sample size or an amount of information.

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is the normal density function and $\Phi(x) = \int\limits_{-\infty}^{x} \phi(u)\, du$ is the cumulative normal distribution. The functions p_j are polynomials with coefficients depending on cumulants of $\widehat{\theta} - \theta_0$. The term of order $\frac{1}{\sqrt{T}}$ corrects the basic normal approximation for the main effect of skewness, while the term of order $\frac{1}{T}$ corrects for the main effect of kurtosis and the secondary effect of skewness. The expansion only rarely converges as an infinite series; it is only available as an asymptotic series, or an asymptotic expansion, meaning that if the series is stopped after a given number of terms, then the remainder is of smaller order than the last term that has been included, i.e.:

$$P\left\{\sqrt{T}\left(\widehat{\theta} - \theta_0\right) \leq x\right\} = \Phi(x) + \frac{1}{\sqrt{T}} p_1(x)\phi(x) + \ldots + \frac{1}{T^{j/2}} p_j(x)\phi(x) + o\left(T^{-j/2}\right).$$

The Edgeworth expansion is a true asymptotic expansion of the probability density function of the statistic of interest, as the error of the approximation, which is defined as the difference between the approximation and the true distribution, is controlled. This means that the Edgeworth expansion has the property as an asymptotic expansion that truncating the series after a finite number of terms provides an approximation to a given function. This is an advantage of such type of expansions. The Edgeworth expansion is also considered as an improvement to the central limit theorem and this will be clarified at a later point in this chapter.

We are not concerned with the convergence of the infinite series as $j \to \infty$ for fixed T. We are interested, for fixed j, in the accuracy of the approximation, which tends to increase as the sample size increases and the higher order approximations are asymptotically more accurate than the lower ones. A convergent series is not always useful, because convergence is a concept relating to the behavior of the terms in the series at the tail end, as $j \to \infty$. That a series converges says nothing about how rapidly the terms will decrease in magnitude. When the terms are decreasing rapidly, if we sum just the first few terms and we know that the error incurred is of the order of the next term, we can get a good estimate of the sum. This is why asymptotic series, even when divergent, are practically useful.

The higher order approximations can be viewed as corrected normal approximations. The Edgeworth approximations tend to be most accurate near the mean, rather that at the tails of the distribution. This is especially true of the higher order approximations. The Edgeworth expansion is similar to the Taylor series expansion, except that instead of expanding the value of a generic function around some particular point, an Edgeworth expansion involves approximating a sample size dependent distribution function in powers of $\frac{1}{\sqrt{T}}$.

At this point, it is useful to state below the connection between the cumulants and the moments. If μ_i is the i^{th} raw moment and k_i is the i^{th} cumulant then

$$\begin{aligned} k_1 &= \mu_1 \\ k_2 &= \mu_2 - k_1^2, \\ k_3 &= \mu_3 - 3k_1k_2 - k_1^3, \\ k_4 &= \mu_4 - 4k_1k_3 - 3k_2^2 - 6k_2k_1^2 - k_1^4. \end{aligned}$$

It is also interesting to discuss the differences between the Edgeworth expansion and the normal approximation. For this scope, let the Edgeworth expansion be given by

$$G_T(x) = \Phi(x) + \frac{p_1(x)}{\sqrt{T}} + \frac{p_2(x)}{T} + O\left(T^{-3/2}\right),$$

whereas the Normal approximation is

$$G_T(x) = \Phi(x) + O\left(T^{-1/2}\right). \tag{1.1}$$

The first order asymptotic theory is based on the central limit theorem. Both relations above are right, in the sense that each approximation is correct through its own order. The first order asymptotic theory 1.1 yields an error that converges to zero, but at a relatively slow rate, since if we multiply that error by $\sqrt{T}$, the resulting product will not generally equal zero. In other words, the product of $\sqrt{T}$ and the error has a stable limiting distribution. The first order asymptotic theory ignores all higher order terms, whereas using an estimator that accounted for some of these terms in the Edgeworth expansion would entail greater accuracy. Technically, there is no guarantee that including these terms increases accuracy in every sample. Correcting for the terms in $\frac{1}{\sqrt{T}}$ and $\frac{1}{T}$ actually moves us farther away from the true value of $G_T(x)$. However, for large enough T, this event generally becomes ever less likely and Monte Carlo evidence generally suggests that higher order corrections do help with small T.

The polynomials that appear in the formula of the Edgeworth expansion are the so-called Hermite polynomials, for which we have that

$$(-1)^k \frac{d^k}{dx^k}\phi(x) = H_k(x)\phi(x)$$

where

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= x, \\ H_2(x) &= x^2 - 1, \\ H_3(x) &= x^3 - 3x, \end{aligned}$$

and so on. By differentiating we obtain

$$\frac{d}{dx}\left[H_k(x)\phi(x)\right] = -H_{k+1}(x)\phi(x).$$

The first order Edgeworth expansion is given by

$$G(x) = \Phi(x) - \phi(x)\left\{\frac{\gamma H_2(x)}{6\sqrt{T}}\right\} + O\left(T^{-1}\right).$$

For a symmetric distribution, the asymmetry term γ is zero and the usual central limit theorem approximation $\Phi(x)$ is already first order accurate. It is possible to stop with the first correction term, having an error of order $\frac{1}{T}$ and this is indeed useful if the main aspect of nonnormality of concern is skewness. The second order Edgeworth expansion is given by

$$G(x) = \phi(x)\left\{1 + \frac{\gamma H_3(x)}{6\sqrt{T}} + \frac{\tau H_4(x)}{24T} + \frac{\gamma^2 H_6(x)}{72T}\right\} + O\left(T^{-3/2}\right).$$

This expansion shows that the error of the leading term, i.e. the standard normal density, is $O\left(T^{-1/2}\right)$ in general, provided that γ is different from zero. This fact suggests that convergence to normality is relatively slow, especially in the tails of the distribution.

Suppose $X \sim G(x)$ and $\psi_X(t)$ denotes the characteristic function of X, for which

$$\psi_X(t) = E\exp\{itX\} = \int_{-\infty}^{\infty} e^{itx} dG(x).$$

In a Taylor series expansion, $\psi_X(t) \sim 1 + \sum_{n=1}^{\infty} \frac{\mu_n}{n!}(it)^n$. If $\int_{-\infty}^{\infty} |\psi_X(t)| < \infty$, then $g(x) = G'(x)$ exists and

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\psi_X(t)\,dt.$$

The cumulant generating function is then given by

$$\log\psi_X(t) \sim \sum_{n=1}^{\infty} \frac{k_n}{n!}(it)^n.$$

Another useful result is the relation between the Hermite polynomials and the normal density function, that is:

For any positive integer k,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}e^{-t^2/2}(it)^k\,dt &= \frac{(-1)^k}{2\pi}\frac{d^k}{dx^k}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}e^{-t^2/2}dt \\
&= (-1)^k\frac{d^k}{dx^k}\phi(x) \\
&= H_k(x)\,\phi(x),
\end{aligned}$$

where $e^{-t^2/2}$ is the characteristic function for a standard normal distribution.

Overall, after deriving the characteristic function of our statistic of interest and taking the logarithm of this and expanding in a Taylor series, we have an asymptotic expansion of the cumulant generating function. Applying the inversion theorem and using the relation concerning the Hermite polynomials we obtain a series expansion of the cumulant generating function.

1.2 Contents of the Thesis

This thesis makes important contributions to the existing literature of the Edgeworth expansion in time series models. We extend the results of Sargan (1976) [80] insofar that we allow more general assumptions on the first and second order cumulants of some estimators; this is the case of the MA(1) model that is analyzed in *Chapter 2*, but also notice that some of these cumulants are not zero in other cases as well (see e.g. Ogasawara 2006 [72] and Bao 2009 [10]). Despite the seemingly simple case of the MA(1) process and in view of the great complexity of the computations in its context, even in its lower order, it seems worthwhile to consider how satisfactory the approximations derived from the Edgeworth expansion prove to be for the MA(1) model. Sargan's significant paper (1976) [80] gives us an insight as to whether the approach may be useful in more complicated time series models. In fact, the Edgeworth expansion is indeed an important tool for approximating the distribution of econometric estimators, but in our context the need to relax the assumptions made by Sargan was vital, in order to also incorporate the estimators for the MA(1) model. Thus, the extension of Sargan's results, which is presented in a different subsection in Chapter 2, is an essential one and constitutes an important contribution in the related theory. A further extension of Tanaka (1984) includes $O\left(n^{-1}\right)$ terms in the Edgeworth expansions of the QMLEs. We also apply these extensions to derive moment expansions for all estimators. In that way, the MA model is analyzed in an extent that contributes to the family of the linear time series models and the asymptotic properties of their estimators.

Another source of motivation was the fact that there is no satisfactory asymptotic theory for the maximum likelihood parameter estimates in Nelson's model (1991) [71], the EGARCH$(1,1)$. The EGARCH model is used extensively in applied financial work due to the fact that it captures the negative dynamic asymmetries noticed in many financial series, i.e. the so-called leverage effects. Immediately, there are important questions which might be raised: What is the approximate distribution and the finite-sample properties of the parameter estimates in the EGARCH model? This important topic is discussed in every detail in *Chapter 3*. We extend the results of Linton (1997) [65] as we consider a non symmetric model in the family of the GARCH processes and we present, for the first time, analytic results of the derivatives and their expected values. One of the main contributions made in this context was the conditions explored for the second-order stationarity of these derivatives to hold. Moreover, the important gap in the related literature that many authors have noticed (see e.g. Straumann and Mikosch 2006 [83]) is the following research question: How does one derive the limiting properties of the QMLE, for example the asymptotic normality of the estimators in the EGARCH model? We also contribute in this area by extending the results of Straumann and Mikosch (2006) [83] in such manner as to study in *Chapter 4* the higher order dependence that exists in the EGARCH$(1,1)$ process and provide moment inequalities that permit the establishment of the asymptotic normality of the QMLEs.

1.3 Structure of the Thesis

In what follows, we provide a short description of the individual chapters of this thesis, which constitute three independent discussion papers:

Chapter 2. Utilizing an extension of the result in Sargan (1976) [80], we develop the second order Edgeworth expansions of all estimators of the first order autocorrelation and the MA parameter for the MA(1) model. Employing these expansions, we derive second order Nagar type expansions. We compare all estimators in terms of bias and mean squared error, complemented by a simulation exercise.

Chapter 3. We present analytic derivatives of the log-likelihood function and their expected values and we investigate under which conditions there is second-order stationary solution to the log-variance derivatives in the EGARCH$(1,1)$ model by Nelson (1991) [71]. We also develop the bias approximations for all estimators and we make a simulation exercise in order to check the adequacy of our theoretical results and be able to proceed with the bias correction of the estimators. The approximate skewness is also computed, as well as the Edgeworth-type distributions.

Chapter 4. We provide higher-order moment conditions resulting from the analysis of the QMLE in the EGARCH$(1,1)$ model. We proceed with the asymptotic theory and we also present our main Theorem, i.e. the asymptotic normality of the QMLEs in Nelson's model. We mainly obtain tractable sufficient conditions that guarantee the integrability of the supremum norms of the log-variance derivatives, in a neighborhood around the true parameter vector. This work comprises an extension of the work by Straumann and Mikosch (2006) [83].

Chapter 5. We conclude this thesis and discuss briefly some further research plans.

Chapter 2

Edgeworth and Moment Approximations: The Case of MM and QML Estimators for the MA(1) Models

2.1 Introduction

Techniques for approximating probability distributions like the Edgeworth expansion have a long history in econometrics.[1] In time series models, starting with Phillips (1977a) [74], there is a fair amount of papers dealing with Edgeworth expansions in autoregressive or mixed models; see e.g. Tanaka (1983 [86], 1984 [87]) and Kakizawa (1999) [59]. However, there are relatively few papers concerning the limiting distribution of estimators of the Moving Average (MA) parameters and their properties. Durbin (1959) [39] proposes an estimator for the parameter of the MA(1) model that can reach the asymptotic efficiency of the Maximum Likelihood Estimator (MLE). Tanaka (1984) [87] develops a technique for the first order Edgeworth expansion of the normal MLEs for autoregressive moving-average (ARMA) models and presents the first order expansion

[1] Nagar (1959) [70], Sargan (1974) [79], Phillips (1977b) [75], Sargan and Satchell (1986) [78] and Ogasawara (2006) [72] to quote only a few papers. Rothenberg (1986) [77] gives a review on the asymptotic techniques employed in econometrics. For a book treatment of Edgeworth expansions see e.g. Hall (1992) [50], Barndorff-Nielsen and Cox (1989) [12], and Taniguchi and Kakizawa (2000) [88].

of the MLE for the MA(1) model having a known or an unknown intercept.[2] Dropping normality and developing a Nagar type expansion, Bao and Ullah (2007) [9] present the second order bias and Mean Square Error (MSE) of the Quasi MLE (QMLE) for the MA(1) but without mean and they do not develop a valid Edgeworth expansion.

In this paper we develop the second order Edgeworth expansions of two estimators of θ, the MA parameter, and ρ, the first order autocorrelation, of the following MA(1) model with mean, MA($1|\mu$) say,

$$y_t = \mu + u_t + \theta u_{t-1}, \qquad t = ..., -1, 0, 1, ..., \quad |\theta| < 1, \quad u_t \overset{iid}{\backsim} (0, \sigma^2),$$

where θ is the true parameter value. The asymptotic distribution of the estimators of θ and ρ depends on whether the mean is estimated, or it is known and not estimated. In the latter case, we set $\mu = 0$ without loss of generality, and we are using MA(1) to denote the model.

The first estimator is the popular Quasi Maximum Likelihood Estimator (QMLE). Its expansion is based on techniques developed in Linnik and Mitrofanova (1965) [64] and Mitrofanova (1967) [69] (see also Linton 1997 [65] and Corradi and Iglesias 2008 [27]) and applied in Tanaka (1984) [87].[3] We denote the QMLEs as $\widetilde{\theta}$ and $\widetilde{\mu}$, for the MA($1|\mu$) model, and $\widetilde{\theta}_0$ when we consider the MA(1) one. Employing $\widetilde{\theta}$ and $\widetilde{\theta}_0$ we can evaluate the QMLEs of ρ and ρ_0, denoted by $\widetilde{\rho} = \frac{\widetilde{\theta}}{1+\widetilde{\theta}^2}$ and $\widetilde{\rho}_0 = \frac{\widetilde{\theta}_0}{1+\widetilde{\theta}_0^2}$, respectively (for the expansion of $\widetilde{\rho}_0$ only, see Ali 1984 [2]).

On the other hand, one could equate the sample 1st order autocorrelation, say $\widehat{\rho}$, or $\widehat{\rho}_0$ when there is no mean, with the theoretical one, $\frac{\theta}{1+\theta^2}$, and solve for the unknown parameter. We call these the MM estimators of θ and θ_0, and denote them by $\widehat{\theta}$ and $\widehat{\theta}_0$, respectively (see also Davis and Resnick 1986 [31], p. 556), although strictly speaking they are z-type estimators. Notice that $\widetilde{\rho}$ is the Indirect estimator of ρ, when the true model is an AR(1) and the auxiliary is an MA(1), where the parameter θ is estimated by MM, or by ML in the Constraint Indirect estimation setup (see Calzolari, Fiorentini and Sentana 2004 [22]). On the other hand, $\widehat{\theta}$ is an Indirect estimator of θ when the

[2] From now on we will refer to the up to $n^{-\frac{1}{2}}$ order expansion as first order one and for the up to n^{-1} order as second order expansion, where n is the sample size.

[3] For an alternative methodology based on a Whittle type estimator see Taniguchi (1987) [89], Lieberman et al (2003) [63], and Andrews and Lieberman (2005) [3]. Aga (2011) [1] extending the results of Lieberman et al. (2003) [63], provides an Edgeworth expansion for linear regression process with stationary Gaussian long memory errors.

true model is an MA(1) and the auxiliary is an AR(1) one (see Gourieroux, Monfort and Renault 1993 [47]).

Utilizing an extension of the result in Sargan (1976) [80], presented in Section 2, we develop the second order Edgeworth expansions of $\widehat{\rho}$, $\widehat{\rho}_0$, $\widehat{\theta}$, and $\widehat{\theta}_0$ in Section 3, whereas Section 4 presents the expansions of the QMLEs. Employing these expansions, we derive second order Nagar type expansions of all estimators. Notice that this is the first time that second order Edgeworth and moment expansions of $\widehat{\theta}$, $\widehat{\theta}_0$, $\widetilde{\theta}$, and $\widetilde{\rho}$ appear in the literature. In section 5, the expansions are employed to compare all estimators in terms of bias and MSE. These comparisons are complemented by a simulation exercise. Section 6 concludes. All proofs, rather lengthy and tedious, are collected in Appendix A at the end of the thesis.

2.2 Edgeworth Expansion

Let $\widehat{\varphi}$ be an estimator of φ and

$$\overline{\varphi} = \sqrt{n}\left(\widehat{\varphi} - \varphi\right) = f\left(A_0, A_1, A_2, ..., A_l\right)$$

where f is a function of the statistics A_i, $i = 0, 1, ..., l$, with the following assumptions:

Assumption 1 All the derivatives of f of order 4 and less are continuous, bounded in a neighborhood of $(0, ..., 0)$, such that $f^i = \frac{\partial f}{\partial A_i} \neq 0$ for some $i = 0, 1, ..., l$, and that there are functions h^{ij} and h^{ijk} independent of n such that $f^{ij} = \frac{\partial^2 f}{\partial A_i \partial A_j} = \frac{1}{\sqrt{n}} h^{ij}$, and $f^{ijk} = \frac{\partial^3 f}{\partial A_i \partial A_j \partial A_k} = \frac{1}{n} h^{ijk}$, where all derivatives are evaluated at $(0, ..., 0)$.

The $A_i's$ are functions of the data standardized in such a way so that their cumulants $c_i = cum\left(A_i\right)$, $c_{ij} = cum\left(A_i, A_j\right)$, etc. obey the following assumption:

Assumption 2

$$
\begin{aligned}
c_i &= cum\left(A_i\right) = n^{-\frac{1}{2}} c_i^{(1)} + n^{-1} c_i^{(2)} + o\left(n^{-1}\right), \\
c_{ij} &= cum\left(A_i, A_j\right) = c_{ij}^{(1)} + n^{-\frac{1}{2}} c_{ij}^{(2)} + n^{-1} c_{ij}^{(3)} + o\left(n^{-1}\right), \\
c_{ijk} &= cum\left(A_i, A_j, A_k\right) = n^{-\frac{1}{2}} c_{ijk}^{(1)} + n^{-1} c_{ijk}^{(2)} + o\left(n^{-1}\right), \\
c_{ijkl} &= cum\left(A_i, A_j, A_k, A_l\right) = n^{-1} c_{ijkl}^{(1)} + o\left(n^{-1}\right), \\
c_{ijklm} &= cum\left(A_i, A_j, A_k, A_l, A_m\right) = O\left(n^{-\frac{3}{2}}\right),
\end{aligned}
$$

where $c_i^{(r)}$, $c_{ij}^{(r)}$, $c_{ijk}^{(r)}$ and $c_{ijkl}^{(r)}$ are independent of n, for $r = 1, 2, 3$.

Assumption 3 (Cramer's condition) If the characteristic function of $A = (A_0, A_1, ..., A_l)^{/}$ is $\Psi(z) = \int \exp\left(iz^{/} A\right) dF(A)$, then $\int_{\|z\| > Kn^{\alpha}} |\Psi(z)|\, dz = O\left(n^{\varepsilon - \frac{3}{2}}\right)$ for all $K > 0$, $0 < \alpha < \frac{1}{2}$ and some $\varepsilon < 0$, and where F is the distribution function of A.

These are standard assumptions in the relevant literature (see Chambers 1967 [24], Sargan 1976 [80], and Bhattacharya and Ghosh 1978 [16]). Under these assumptions we present the following Theorem.

Theorem 2.1. *Under Assumptions 1, 2 and 3, the second order Edgeworth expansion of $\overline{\varphi}$ is given by*

$$
P\left(\overline{\varphi} \leq m\right) = \Phi\left(\frac{m}{\omega}\right) - \phi\left(\frac{m}{\omega}\right) \left[\begin{array}{c} \psi_0 + \psi_1\left(\frac{m}{\omega}\right) + \psi_2\left(\frac{m}{\omega}\right)^2 \\ +\psi_3\left(\frac{m}{\omega}\right)^3 + \psi_4\left(\frac{m}{\omega}\right)^4 + \psi_5\left(\frac{m}{\omega}\right)^5 \end{array} \right] + o\left(n^{-1}\right), \quad (2.1)
$$

where

$$
\begin{aligned}
\psi_0 &= \frac{1}{\sqrt{n}} \psi_0^{(1)} + \frac{1}{n} \psi_0^{(2)}, \quad \psi_1 = \frac{1}{\sqrt{n}} \psi_1^{(1)} + \frac{1}{n} \psi_1^{(2)}, \\
\psi_2 &= \frac{1}{\sqrt{n}} \psi_2^{(1)} + \frac{1}{n} \psi_2^{(1)}, \quad \psi_3 = \frac{1}{n} \psi_3^{(2)} \\
\psi_4 &= \frac{1}{12n} \frac{\omega^{(2)}\left(a_1^{(1)} + 3a_3^{(1)}\right)}{\omega^5}, \quad \psi_5 = \frac{1}{72n} \frac{\left(a_1^{(1)} + 3a_3^{(1)}\right)^2}{\omega^6}, \\
\varsigma &= \frac{a_2^{(1)} + 2\left(a_4^{(1)} + 2a_{11}^{(1)}\right)\left(a_1^{(1)} + 3a_3^{(1)}\right) + 4a_6^{(1)} + 12\left(a_8^{(1)} + a_{10}^{(1)}\right) + 3\left(\omega^{(2)}\right)^2}{\omega^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_0^{(1)} &= \frac{1}{6\omega}\left\{3\left(a_4^{(1)}+2a_{11}^{(1)}\right)-\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^2}\right\},\\
\psi_0^{(2)} &= \frac{1}{6\omega}\left\{3\left[a_4^{(2)}+2a_{11}^{(2)}\right]-\frac{a_1^{(2)}+6a_3^{(2)}+\frac{3}{2}\omega^{(2)}\left(a_4^{(1)}+2a_{11}^{(1)}\right)}{\omega^2}+\frac{3}{2}\frac{\omega^{(2)}\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^4}\right\},\\
\psi_1^{(1)} &= \frac{\omega^{(2)}}{2\omega^2},\quad \psi_1^{(2)}=\frac{1}{24\omega^2}\left\{\begin{array}{c}3\left[4\left(a_5^{(1)}+a_7^{(1)}+\omega^{(3)}+2a_{12}^{(1)}\right)+2a_9^{(1)}+\left(2a_{11}^{(1)}+a_4^{(1)}\right)^2\right]\\ -3\varsigma+5\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)^2}{\omega^4}\end{array}\right\},\\
\psi_2^{(1)} &= \frac{\left(a_1^{(1)}+3a_3^{(1)}\right)}{6\omega^3},\quad \psi_2^{(2)}=\frac{1}{6\omega^3}\left[\begin{array}{c}a_1^{(2)}+6a_3^{(2)}+\frac{3}{2}\omega^{(2)}a_4^{(1)}+3\omega^{(2)}a_{11}^{(1)}\\ -3\frac{\omega^{(2)}\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^2}\end{array}\right],\\
\psi_3^{(2)} &= -\frac{1}{72\omega^2}\left[10\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)^2}{\omega^4}-3\varsigma\right],
\end{aligned}
$$

where m is any real number, $\phi(.)$ and $\Phi(.)$ are the standard normal density and distribution functions, and the so called Edgeworth coefficients, $a_j^{(i)}$, for $i=1,2$ and $j=1,...,12$, $\omega^{(i)}$, for $i=1,2,3$, and ω are given in the proof of Theorem 1 in Appendix A.1.

Sargan (1976) [80] assumes that $c_i^{(1)}=c_i^{(2)}=c_{ij}^{(2)}=c_{ij}^{(3)}=c_{ijk}^{(2)}=0$. In this respect, Theorem 1 is a necessary generalization needed in the expansions of all estimators considered in this paper. Notice that some of these cumulants are not zero in other cases as well (see e.g. Ogasawara 2006 [72] and Bao 2009 [10]). Next, we have the following Lemma, which is very useful for the evaluation of the cumulants of $\overline{\varphi}$.

Lemma 2.2. *Under Assumptions 1, 2 and 3, the second order approximate cumulants of $\overline{\varphi}$ are given by*

$$
\begin{aligned}
k_1^{\widehat{\varphi}} &= \frac{a_4^{(1)}+2a_{11}^{(1)}}{2\sqrt{n}}+\frac{a_4^{(2)}+2a_{11}^{(2)}}{2n},\\
k_2^{\widehat{\varphi}} &= \omega^2+\frac{\omega^{(2)}}{\sqrt{n}}+\frac{a_9^{(1)}+2\left(a_5^{(1)}+a_7^{(1)}+\omega^{(3)}+2a_{12}^{(1)}\right)}{2n},\\
k_3^{\widehat{\varphi}} &= \frac{a_1^{(1)}+3a_3^{(1)}}{\sqrt{n}}+\frac{a_1^{(2)}+6a_3^{(2)}}{n},\\
k_4^{\widehat{\varphi}} &= \frac{a_2^{(1)}+4a_6^{(1)}+12\left(a_8^{(1)}+a_{10}^{(1)}\right)+\frac{9}{4}\left(\omega^{(2)}\right)^2}{n}.
\end{aligned}
$$

Furthermore,

$$E\left(\overline{\varphi}^2\right) = k_2^{\widehat{\varphi}} + \frac{\left(2a_{11}^{(1)} + a_4^{(1)}\right)^2}{4n}.$$

The proof of Lemma 1 is also given in Appendix A.2. We can now proceed in finding the expansions of the MM estimators of ρ, μ and θ.

2.3 The Expansions of the MM Estimators

The following analysis is based on Kakizawa (1999) [59]. Given observations $y = (y_0, ..., y_n)'$, the MM estimators of ρ and μ are given by:

$$\widehat{\rho} = \frac{\sum_{t=1}^n \left(y_t - \frac{1}{n}\sum_{t=1}^n y_t\right)\left(y_{t-1} - \frac{1}{n}\sum_{t=1}^n y_{t-1}\right)}{\sum_{t=1}^n \left(y_{t-1} - \frac{1}{n}\sum_{t=1}^n y_{t-1}\right)^2} \quad \textit{and} \quad \widehat{\mu} = \frac{1}{n}\sum_{t=1}^n y_{t-1}.$$

Hence

$$\sqrt{n}\left(\widehat{\rho} - \rho\right) = \frac{\left(1+\theta^4\right)A_1 + \left(1+\theta^2\right)A_2 - \left(1-\theta+\theta^2\right)\frac{1}{\sqrt{n}}\left(A_0\right)^2}{\frac{1}{\sqrt{n}}\left(1+\theta^2\right)^2 A_3 + \frac{1}{\sqrt{n}}2\theta\left(1+\theta^2\right)A_1 - \frac{1}{n}\left(1+\theta^2\right)\left(A_0\right)^2 + \left(1+\theta^2\right)^2\sigma^2}, \tag{2.2}$$

and

$$\sqrt{n}\left(\widehat{\mu} - \mu\right) = \sqrt{n}\left(\frac{1}{n}\sum_{t=1}^n \left(y_{t-1} - \mu\right)\right) = A_0,$$

where

$$A_0 = \frac{1}{\sqrt{n}}\sum_{t=1}^n \left(y_{t-1} - \mu\right), \quad A_1 = \frac{\sum_{t=2}^n u_{t-1}u_{t-2}}{\sqrt{n}},$$

$$A_2 = \frac{1}{\sqrt{n}}\left[\begin{array}{c} \left[\left(y_1-\mu\right)\left(y_0-\mu\right) - \theta\sigma^2\right] + \theta\sum_{t=2}^n u_t u_{t-2} + u_n u_{n-1} - u_1 u_0 \\ -\theta\frac{\theta^2\left(u_0^2-\sigma^2\right) - \theta^2\left(u_{n-1}^2-\sigma^2\right) + \left[\left(y_0-\mu\right)^2 - \left(1+\theta^2\right)\sigma^2\right]}{\left(1+\theta^2\right)} \\ +\frac{1}{n}\left[\left(y_0-\mu\right) - \left(y_n-\mu\right)\right]\left[\left(1+\theta\right)\sum_{t=2}^n u_{t-1} + \theta u_0 - \theta u_{n-1} + \left(y_0-\mu\right)\right] \end{array}\right],$$

$$A_3 = \frac{\sum_{t=2}^n \left(u_{t-1}^2 - \sigma^2\right) + \frac{\theta^2\left(u_0^2-\sigma^2\right) - \theta^2\left(u_{n-1}^2-\sigma^2\right) + \left[\left(y_0-\mu\right)^2 - \left(1+\theta^2\right)\sigma^2\right]}{\left(1+\theta^2\right)}}{\sqrt{n}}.$$

It is now obvious that $\sqrt{n}\left(\widehat{\rho} - \rho\right)$ and $\sqrt{n}\left(\widehat{\mu} - \mu\right)$ are functions of $A_0, ...A_3$, $f\left(A_0, A_1, A_2, A_3\right)$ say, with $f\left(0,0,0,0\right) = 0$. From Appendix A.3, where the cumulants of the $A_i\prime s$ are presented, it is easily seen that Assumption 2 is satisfied, and if $E\left(u_0^{10}\right)$ is finite, we can apply Theorem 1. Notice that most of the second order cumulants of the $A_i\prime s$ include terms of $O\left(n^{-1}\right)$. Hence, the generalization of Sargan (1976) [80] presented in section 2 is a necessary one. Let us now turn our attention to $\widehat{\rho}$.

2.3.1 The Expansion of the MM First Order Autocorrelation

Lemma 2.3. *Under the Assumptions that u_t's are identically and independently distributed, $E\left(u_0^{10}\right) < \infty$, $\left(u_0, u_0^2\right)$ satisfy the Cramer's condition and $\theta \in (-1, 1)$, the second order asymptotic expansion of $P\left(\sqrt{n}\left(\widehat{\rho} - \rho\right) < m\right)$ is given by:*

$$G(m) = \Phi\left(\frac{m}{\omega}\right) - \phi\left(\frac{m}{\omega}\right)\left(\psi_0 + \psi_1 \frac{m}{\omega} + \psi_2\left(\frac{m}{\omega}\right)^2 + \psi_3\left(\frac{m}{\omega}\right)^3 + \psi_5\left(\frac{m}{\omega}\right)^5\right), \quad (2.3)$$

where the polynomial coefficients ψ_i, $i = 0, ..., 5$ are as in Theorem 1 and the Edgeworth coefficients are given in Appendix A.4.

To evaluate the approximate bias, MSE and the cumulants, needed in the sequel, we employ Lemma 1. Letting κ_3 and κ_4 to denote the 3$^{\text{rd}}$ and 4$^{\text{th}}$ order cumulants of u_0, respectively, the cumulants of $\sqrt{n}\left(\widehat{\rho} - \rho\right)$ are:

$$k_1^{\widehat{\rho}} = -\frac{1}{\sqrt{n}}\left(\theta^4 + 2\theta^3 - 2\theta^2 + 2\theta + 1\right)\frac{\theta^2 + \theta + 1}{\left(\theta^2 + 1\right)^3} + o\left(n^{-1}\right),$$

$$k_2^{\widehat{\rho}} = \omega_{\widehat{\rho}}^2 - \frac{1}{n}\left(\xi_{2,1}^{\widehat{\rho}} + \xi_{2,2}^{\widehat{\rho}}\right) + o\left(n^{-1}\right),$$

where $\omega_{\widehat{\rho}}^2 = \frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^4}$, i.e. the asymptotic variance,
$\xi_{2,1}^{\widehat{\rho}} = 2\frac{\left(-4\theta-\theta^2+6\theta^3-12\theta^5+6\theta^7-\theta^8-4\theta^9+\theta^{10}+1\right)(\theta+1)^2}{\left(\theta^2+1\right)^6}$, and $\xi_{2,2}^{\widehat{\rho}} = 4\frac{\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^5}\kappa_3^2 + \frac{\theta^2+4\theta^4-\theta^6+\theta^8+1}{\left(\theta^2+1\right)^4}\kappa_4$,

$$k_3^{\widehat{\rho}} = -\frac{6}{\sqrt{n}}\theta\left(\theta^4 + 1\right)\frac{6\theta^4 + \theta^8 + 1}{\left(\theta^2 + 1\right)^7} + \frac{1}{\sqrt{n}}\frac{\left(1 + \theta^4\right)^3 + \theta^3\left(1 + \theta^2\right)^3}{\left(1 + \theta^2\right)^6}\kappa_3^2 + o\left(n^{-1}\right),$$

$$k_4^{\widehat{\rho}} = \frac{1}{n}\left(\xi_{4,1}^{\widehat{\rho}} + \xi_{4,2}^{\widehat{\rho}} + \xi_{4,3}^{\widehat{\rho}}\right),$$

where $\xi_{4,1}^{\widehat{\rho}} = 6\frac{-1+10\theta^2\left(1+\theta^{16}\right)-30\theta^4\left(1+\theta^{12}\right)+106\theta^6\left(1+\theta^8\right)-129\theta^8\left(1+\theta^4\right)+216\theta^{10}-\theta^{20}}{\left(\theta^2+1\right)^{10}}$,
$\xi_{4,2}^{\widehat{\rho}} = 12\theta\left(\theta^4 + 1\right)\frac{\theta\left(1-\theta^4\right)^2\left(1+\theta^2\right)-10\theta^4\left(1+\theta^4\right)+4\theta^6-2\left(1-\theta^2\right)\left(1-\theta^{10}\right)}{\left(\theta^2+1\right)^9}\kappa_3^2$,
and $\xi_{4,3}^{\widehat{\rho}} = \frac{5\theta^4+4\theta^6+12\theta^8+4\theta^{10}+5\theta^{12}+\theta^{16}+1}{\left(\theta^2+1\right)^8}\kappa_4^2$.

Furthermore, the second order approximate MSE (AMSE) is

$$E\left[\sqrt{n}\left(\widehat{\rho}-\rho\right)\right]^2 = k_2^{\widehat{\rho}} + \frac{1}{n}\left(\theta^4+2\theta^3-2\theta^2+2\theta+1\right)^2 \frac{\left(\theta^2+\theta+1\right)^2}{\left(\theta^2+1\right)^6}. \tag{2.4}$$

It is worth noticing first, that the sign of the asymmetry of the distribution of the errors (κ_3) does not affect the AMSE, i.e. positively and negatively skewed error distributions of the same magnitude have the same effect on the AMSE. Second, the AMSE is a decreasing function of κ_4, for any value of θ in the admissible region. It seems that higher probability of extreme values of the errors increases the accuracy of the estimator. This is not true for the asymmetry parameter κ_3. For positive (negative) values of θ, the AMSE of $\widehat{\rho}$ is a decreasing (increasing) function of κ_3^2. Further, for $\theta = 0$ and under elliptical error distributions, the presented moments are known in the literature (see e.g. Kan and Wang 2010 [60]). Let us now proceed to the expansion of the MM 1$^{\text{st}}$ order autocorrelation when the mean is 0.

2.3.1.1 The Zero-mean Expansion

In case that μ is zero, or known and subtracted from the data, we have that

$$\widehat{\rho}_0 = \frac{\sum_{t=1}^n y_t y_{t-1}}{\sum_{t=1}^n y_{t-1}^2}.$$

Hence

$$\sqrt{n}\left(\widehat{\rho}_0-\rho\right) = \frac{\left(1+\theta^4\right)A_1+\left(1+\theta^2\right)A_2}{\left(1+\theta^2\right)^2\frac{1}{\sqrt{n}}A_3+2\theta\left(1+\theta^2\right)\frac{1}{\sqrt{n}}A_1+\left(1+\theta^2\right)^2\sigma^2},$$

where the $A_i\prime s$ are now given by

$$\begin{aligned}
A_1 &= \frac{\sum_{t=2}^n u_{t-1}u_{t-2}}{\sqrt{n}}, \\
A_2 &= \frac{1}{\sqrt{n}}\left[\begin{array}{c}\left(y_1y_0-\theta\sigma^2\right)+\theta\sum_{t=2}^n u_t u_{t-2}+u_n u_{n-1}-u_1u_0 \\ -\theta\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+\theta^2\left(u_0^2-\sigma^2\right)+\left(y_0^2-\left(1+\theta^2\right)\sigma^2\right)}{\left(1+\theta^2\right)}\end{array}\right], \\
A_3 &= \frac{1}{\sqrt{n}}\left[\sum_{t=2}^n\left(u_{t-1}^2-\sigma^2\right)+\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+\theta^2\left(u_0^2-\sigma^2\right)+\left(y_0^2-\left(1+\theta^2\right)\sigma^2\right)}{\left(1+\theta^2\right)}\right].
\end{aligned}$$

Notice that A_1 and A_3 are the same as in the non-zero mean case. However, the term $\frac{1}{n}\left[(y_0-\mu)-(y_n-\mu)\right]\left[(1+\theta)\sum_{t=2}^n u_{t-1}+\theta u_0-\theta u_{n-1}+(y_0-\mu)\right]$ is not included

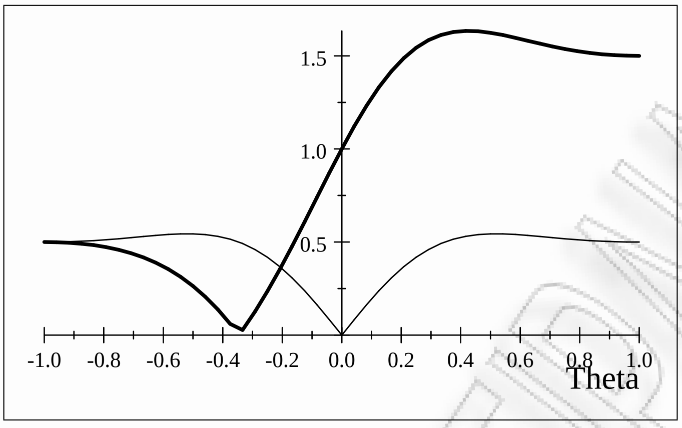

FIGURE 2.1: $|E[n(\widehat{\rho}-\rho)]|$ (thick line) and $|E[n(\widehat{\rho_0}-\rho_0)]|$.

in A_2. Furthermore, $\sqrt{n}(\widehat{\rho_0}-\rho)$ has the same functional form with respect to A_1, A_2 and A_3. Consequently, the derivatives are the same, but now all sums determining the Edgeworth coefficients run from $i=1$ up to 3.

Hence, the asymptotic variance of $\sqrt{n}(\widehat{\rho_0}-\rho)$ is the same as the asymptotic variance of $\sqrt{n}(\widehat{\rho}-\rho)$, i.e. $\omega^2_{\widehat{\rho_0}}=\omega^2_{\widehat{\rho}}=\frac{1+\theta^2+4\theta^4+\theta^6+\theta^8}{(1+\theta^2)^4}$. Further, all Edgeworth coefficients are the same as in the non-zero mean case (see Appendix A.4) apart from $a_4^{(1)}$, $a_5^{(1)}$, $a_7^{(1)}$, and $a_9^{(1)}$, which are also presented in Appendix A.4.

We can now evaluate the bias, the MSE and the cumulants of $\sqrt{n}(\widehat{\rho_0}-\rho)$. The 1$^{\text{st}}$ order cumulant is

$$k_1^{\widehat{\rho_0}}=k_1^{\widehat{\rho}}-\frac{1}{\sqrt{n}}\left(\theta-\theta^2-1\right)\frac{(\theta+1)^2}{\left(\theta^2+1\right)^2}=-\frac{2}{\sqrt{n}}\theta\frac{\theta^4+1}{\left(\theta^2+1\right)^3}+o\left(n^{-1}\right). \qquad (2.5)$$

Comparing the absolute values of the two approximate biases (see Figure 2.1) it is clear that for $\theta\in(-1,-0.2)$ the absolute bias of $\widehat{\rho}$, multiplied by $\sqrt{n}$, is less than the one of $\widehat{\rho_0}$. The opposite is true for $\theta\in(-0.2,1)$.

The AMSE is

$$\begin{aligned}E\left[\sqrt{n}(\widehat{\rho_0}-\rho)\right]^2 &= E\left[\sqrt{n}(\widehat{\rho}-\rho)\right]^2\\ &\quad+\frac{1}{n}\left(1+8\theta-7\theta^2+6\theta^3+8\theta^4+6\theta^5-7\theta^6+8\theta^7+\theta^8\right)\frac{(\theta+1)^2}{\left(\theta^2+1\right)^5}.\end{aligned}$$

Obviously, the sign of the difference between the zero and the non-zero mean case AMSEs depends on the sign of the 8$^{\text{th}}$ degree polynomial. As now the limit of the polynomial is -32, for $\theta \to -1$, and 24, for $\theta \to 1$, it follows that that there are intervals of θ, within $(-1, 1)$, such that the AMSE of $\widehat{\rho_0}$ is lower than the one of $\widehat{\rho}$ and vice versa, for any number of observations, n. However, notice that the asymmetry and kurtosis parameters, κ_3 and κ_4, have the same effect on the AMSE, for any values of θ in the admissible region. Of course, the two AMSEs are equal to the common asymptotic variance $\omega^2_{\widehat{\rho}}$, as $n \to \infty$.

Applying again Lemma 1, we get that the second order cumulant of $\sqrt{n}\left(\widehat{\rho_0} - \rho\right)$ is given by:

$$k_2^{\widehat{\rho_0}} = k_2^{\widehat{\rho}} - \frac{4\theta}{n} \frac{\left[(1-\theta)\left(1-\theta^5\right) + 2\theta^3\right](\theta+1)^2}{\left(\theta^2+1\right)^5} + o\left(n^{-1}\right).$$

As now the Edgeworth coefficients involved in the evaluation of the 3$^{\text{rd}}$ and 4$^{\text{th}}$ order cumulants are the same in the non-zero and the zero mean case (see Lemma 1), i.e. $k_3^{\widehat{\rho_0}} = k_3^{\widehat{\rho}}$ and $k_4^{\widehat{\rho_0}} = k_4^{\widehat{\rho}}$, we can conclude that the non-normality of the estimators of ρ is not affected by the estimation or not of the mean μ, up to $o\left(n^{-1}\right)$. Let us now derive the expansion of the MM mean estimator, $\widehat{\mu}$.

2.3.2 The Expansion of the Mean MM Estimator

As now

$$\sqrt{n}\left(\widehat{\mu} - \mu\right) = f\left(A_0\right) = A_0$$

applying again Theorem 1, it is easy to find the Edgeworth and the polynomial coefficients in the Edgeworth expansion (see Appendix A.4).

It is worth noticing that even in the normality case, i.e. $\kappa_3 = \kappa_4 = 0$, we have that the approximate distribution of $\widehat{\mu}$ is not normal. Furthermore, from Lemma 1, we have that

$$E\left[\sqrt{n}\left(\widehat{\mu} - \mu\right)\right] = 0 + o\left(n^{-1}\right),$$

i.e. $\widehat{\mu}$ is an $o\left(n^{-1}\right)$ unbiased estimator of μ. The AMSE is

$$E\left[\sqrt{n}\left(\widehat{\mu} - \mu\right)\right]^2 = (1+\theta)^2\sigma^2 - \frac{2}{n}\theta\sigma^2 + o\left(n^{-1}\right).$$

This explains the non-normality of the approximation, even if the errors are normally distributed. Let us now turn our attention to the expansion of the MM estimator of θ.

2.3.3 The Expansion of the MM MA Coefficient Estimator

For $|\widehat{\rho}| < 0.5$ the solution for $\widehat{\theta}$ is:

$$\widehat{\theta} = \frac{1-\sqrt{1-4\widehat{\rho}^2}}{2\widehat{\rho}} \quad \textit{and} \quad \widehat{\theta}-\theta = \frac{1-\sqrt{1-4\widehat{\rho}^2}}{2\widehat{\rho}} - \frac{1-\sqrt{1-4\rho^2}}{2\rho} = f\left(\widehat{\rho}\right). \tag{2.6}$$

Hence, given the cumulants of $\sqrt{n}\left(\widehat{\rho}-\rho\right)$ presented in Section 2.3.1, we can apply Theorem 1. The Edgeworth coefficients of $\sqrt{n}\left(\widehat{\theta}-\theta\right)$ are given in Appendix A.5. Applying Lemma 1, we can prove the following Proposition:

Proposition 2.4. *Under the Assumptions of Lemma 2 we have that the 1^{st} order cumulant and the MSE of $\sqrt{n}\left(\widehat{\theta}-\theta\right)$ are*

$$k_1^{\widehat{\theta}} = \frac{1}{\sqrt{n}} \frac{2\theta^2+6\theta^3-2\theta^4+3\theta^5+2\theta^6-\theta^8-\theta^9-1}{\left(1-\theta^2\right)^3} + o\left(n^{-1}\right)$$

and

$$E\left[\sqrt{n}\left(\widehat{\theta}-\theta\right)\right]^2 = \frac{1+\theta^2+4\theta^4+\theta^6+\theta^8}{\left(1-\theta^2\right)^2} + \frac{1}{n}\left(\xi_3^{\widehat{\theta}}+\xi_4^{\widehat{\theta}}\right) + o\left(n^{-1}\right),$$

$$\textit{where } \xi_3^{\widehat{\theta}} = \frac{\begin{array}{c} 1-8\theta-36\theta^2-56\theta^3-93\theta^4-150\theta^5+2\theta^6-192\theta^7+747\theta^8-72\theta^9+3019\theta^{10} \\ +192\theta^{11}+47650\theta^{12}+418\theta^{13}+5421\theta^{14}+352\theta^{15}+2539\theta^{16}+24\theta^{17}+460\theta^{18} \\ -216\theta^{19}-933\theta^{20}-210\theta^{21}-442\theta^{22}-96\theta^{23}-141\theta^{24}-8\theta^{25}+21\theta^{26}+16\theta^{27} \\ +29\theta^{28}+6\theta^{29}+3\theta^{30} \end{array}}{\left(1-\theta^4\right)^6},$$

$$\textit{and } \xi_4^{\widehat{\theta}} = -2\theta\frac{-1+2\theta-5\theta^2+5\theta^3-6\theta^4+2\theta^5-2\theta^6+\theta^7+\theta^8-2\theta^9+\theta^{10}}{\left(1-\theta^2\right)^2(1-\theta)^2}\kappa_3^2 - \frac{1+\theta^2+4\theta^4-\theta^6+\theta^8}{\left(1-\theta^2\right)^2}\kappa_4.$$

Notice first, that the approximate bias of $\widehat{\theta}$ is not affected by the non-normality of the errors, and second that the effect of κ_4 on the AMSE of $\widehat{\theta}$ is the same as the effect on the AMSE of $\widehat{\rho}$, i.e. the AMSE is a decreasing function of κ_4 for all $\theta \in (-1,1)$. However, for positive (negative) values of θ the AMSE of $\widehat{\theta}$ is an increasing (decreasing) function of κ_3^2. This is exactly opposite from the effect that κ_3^2 has on the AMSE of $\widehat{\rho}$. Let us now proceed to the expansion of the MM MA coefficient when the mean is 0.

2.3.3.1 The Zero-Mean Expansion

For the zero mean case, all Edgeworth coefficients are the same as in the non-zero mean one, apart from $\omega^{(3)}$, $a_{11}^{(1)}$ and $a_{12}^{(1)}$, which are given in Appendix A.4. Consequently,

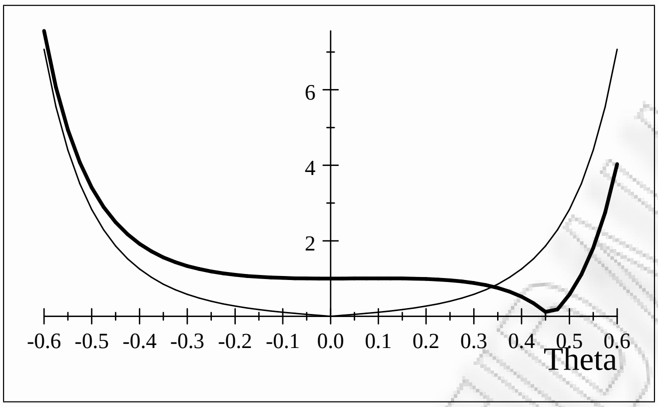

FIGURE 2.2: $\left|E\left[n\left(\widehat{\theta}-\theta\right)\right]\right|$ (thick line) and $\left|E\left[n\left(\widehat{\theta}_0-\theta\right)\right]\right|$

applying Lemma 1 and keeping terms up to order $O\left(n^{-1}\right)$, the approximate bias of $\sqrt{n}\left(\widehat{\theta}_0-\theta\right)$ is

$$k_1^{\widehat{\theta}_0}=k_1^{\widehat{\theta}}+\frac{1}{\sqrt{n}}\left(1+\theta\right)\frac{1-\theta+\theta^2}{1-\theta}=\frac{1}{\sqrt{n}}\theta\frac{1+5\theta^2+2\theta^4+\theta^6-\theta^8}{\left(1-\theta^2\right)^3}. \tag{2.7}$$

Plotting, again, the absolute values of the two approximate biases (multiplied by $\sqrt{n}$), i.e. $\left|E\left(n\left(\widehat{\theta}-\theta\right)\right)\right|$ and $\left|E\left(n\left(\widehat{\theta}_0-\theta\right)\right)\right|$, we observe that for values of θ higher than about 0.3 the approximate bias of $\widehat{\theta}$ is less than the one of $\widehat{\theta}_0$ (see Figure 2.2).

In terms of AMSE we have that, keeping the relevant terms,

$$E\left[\sqrt{n}\left(\widehat{\theta}_0-\theta\right)\right]^2=E\left[\sqrt{n}\left(\widehat{\theta}-\theta\right)\right]^2+\frac{1}{n}\lambda$$

where $\lambda=\dfrac{\begin{array}{c}8\theta+7\theta^2+56\theta^3+65\theta^4+150\theta^5+204\theta^6+192\theta^7+297\theta^8+72\theta^9\\+51\theta^{10}-192\theta^{11}-481\theta^{12}-418\theta^{13}-656\theta^{14}-352\theta^{15}-199\theta^{16}\\-24\theta^{17}+285\theta^{18}+216\theta^{19}+327\theta^{20}+210\theta^{21}+132\theta^{22}+96\theta^{23}\\-\theta^{24}+8\theta^{25}-23\theta^{26}-16\theta^{27}-7\theta^{28}-6\theta^{29}-3\end{array}}{\left(1-\theta^4\right)^6}$, indicating that, first, the non-normality of the errors affects the AMSE of $\widehat{\theta}$ and $\widehat{\theta}_0$ in the same way and second, the asymptotic variance of $\widehat{\theta}$ and $\widehat{\theta}_0$ is the same. However, the sign of λ depends on the sign of the numerator, a polynomial of 29$^{\text{th}}$ degree. As the limit of this polynomial changes sign as $\theta\to\pm0.6$, we can conclude that there are values of θ, in its admissible interval, such that the AMSE of $\widehat{\theta}$ is less than the one of $\widehat{\theta}_0$. Let us now turn our attention to the expansions of the QML estimators of θ, μ and ρ.

2.4 The Expansions of the QML Estimators

In this section we extend the analysis in Tanaka (1984) [87] by dropping normality and including terms of second order in the approximation of the QMLE of the MA($1|\mu$) parameters, θ and μ, say $\widetilde{\theta}$ and $\widetilde{\mu}$.[4] These are the solutions to the following equations:

$$\frac{\partial \ell\left(\widetilde{\theta}\right)}{\partial \theta} = 0 \Rightarrow \frac{1}{\sigma^2}\sum_{t=1}^{n} u_t\left(\theta\frac{\partial u_{t-1}}{\partial \theta} + u_{t-1}\right)\Bigg|_{\theta=\widetilde{\theta}} = 0 \tag{2.8}$$

$$\text{and}$$

$$\frac{\partial \ell\left(\widetilde{\mu}\right)}{\partial \mu} = 0 \Rightarrow \frac{1}{\sigma^2}\sum_{t=1}^{n} u_t\frac{\partial u_t}{\partial \mu}\Bigg|_{\mu=\widetilde{\mu}} = 0,$$

where

$$\ell\left(\theta,\mu\right) = -\frac{n\log(2\pi\sigma^2)}{2} - \frac{\sum_{t=1}^{n} u_t^2}{2\sigma^2} \qquad \text{and} \qquad u_t = y_t - \mu - \theta u_{t-1}.$$

In Appendix A.6 we express $\sqrt{n}\left(\widetilde{\theta}-\theta\right)$ and $\sqrt{n}\left(\widetilde{\mu}-\mu\right)$ as functions of the first, second and third order derivatives of $\ell\left(\theta,\mu\right)$ standardized appropriately and evaluated at the true parameter values. We also present their expectations. In Appendix A.7 we evaluate the needed cumulants of these derivatives, so that Theorem 1 can be applied. Let us now turn our attention to the expansion of $\widetilde{\theta}$.

2.4.1 The Expansion of the QML MA Coefficient Estimator

Lemma 2.5. *Under the Assumptions of Lemma 2, the second order Edgeworth expansion of* $P\left(\sqrt{n}\left(\widetilde{\theta}-\theta\right) < m\right)$ *is given by:*

$$P(m) = \Phi\left(\frac{m}{\omega}\right) - \phi\left(\frac{m}{\omega}\right)\left(\psi_0 + \psi_1\frac{m}{\omega} + \psi_2\left(\frac{m}{\omega}\right)^2 + \psi_3\left(\frac{m}{\omega}\right)^3 + \psi_5\left(\frac{m}{\omega}\right)^5\right),$$

where the coefficients ψ_i, $i = 0, ..., 5$ *are as in Theorem 1 and the Edgeworth coefficients are given in Appendix A.8.*

Applying Lemma 1 we get the first four approximate cumulants, up to $o\left(n^{-1}\right)$, of $\sqrt{n}\left(\widetilde{\theta}-\theta\right)$ as

$$k_1^{\widetilde{\theta}} = \frac{2\theta - 1}{\sqrt{n}},$$

[4] For various approximations of the MLE see Davidson (1981) [30].

$$k_2^{\widetilde{\theta}} = \omega_{\widetilde{\theta}}^2 + \frac{1}{n}(\theta + 6)(2 - \theta) + \frac{1}{n}\xi_2^{\widetilde{\theta}},$$

where $\omega_{\widetilde{\theta}}^2 = 1 - \theta^2$, and $\xi_2^{\widetilde{\theta}} = 2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2 - \left(1-\theta^2\right)\kappa_4$,

$$\begin{aligned} k_3^{\widetilde{\theta}} &= \frac{1}{\sqrt{n}}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2 \quad \textit{and} \\ k_4^{\widetilde{\theta}} &= \frac{1}{n}6\left(1-\theta^2\right)\left(\theta^2+3\right) + \frac{1}{n}\xi_4^{\widetilde{\theta}}, \end{aligned}$$

where $\xi_4^{\widetilde{\theta}} = 12\theta\frac{\theta-\theta^2-2}{1-\theta+\theta^2}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2 + \frac{\left(1-\theta^2\right)^3}{1+\theta^2}\kappa_4^2$.

It is worth noticing that the 3rd approximate cumulant of $\widetilde{\theta}$ is positive even if the errors $u_t's$ are negatively skewed, whereas is symmetrically distributed for symmetric error distribution. Furthermore, $k_4^{\widetilde{\theta}}$ is an increasing function of κ_4^2. Consequently, for either platykurtic or leptokurtic error distribution, the distribution of $\widetilde{\theta}$ becomes platykurtic.

The second order approximate MSE of $\widetilde{\theta}$ is given by

$$E\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^2 = \omega_{\widetilde{\theta}}^2 + \frac{1}{n}\left[-8\theta + 3\theta^2 + 13 + 2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2 - \left(1-\theta^2\right)\kappa_4\right]. \tag{2.9}$$

Notice that the AMSE is a decreasing function of κ_4. This property of $\widetilde{\theta}$ is shared with $\widehat{\rho}$ and $\widehat{\theta}$, as well (see sections 2.3.1 and 2.3.3). Let us now proceed to the expansion of the QML MA coefficient when the mean is 0.

2.4.1.1 The Zero-Mean Expansion

Now for the case that $\mu = 0$, or known and subtracted from the data, we can repeat the procedure of section 2.4.1, appropriately modified (see Appendix A.8). Notice that the derivatives with respect to g_1, w_{11} and q_{111}, and the cumulants of these variables remain the same. Further, as in the expansion of $\widehat{\rho}_0$, all Edgeworth coefficients are the same as in the non-zero mean case apart from $a_4^{(1)}$, $a_5^{(1)}$, $a_7^{(1)}$, and $a_9^{(1)}$, which are presented in Appendix A.8.

In terms of cumulants, from Lemma 1, we have that the first order approximate cumulant, up to $o\left(n^{-1}\right)$, of $\sqrt{n}\left(\widetilde{\theta}_0 - \theta\right)$ is

$$k_1^{\widetilde{\theta}_0} = \frac{1}{\sqrt{n}}\theta, \tag{2.10}$$

which is the same result as in Tanaka (1984) [87], where the 1^{st} order expansion is presented, and Bao and Ullah (2007) [9]. Comparing with the non-zero mean case, it is obvious that estimating the mean increases the absolute approximate bias of the QML estimator of θ for $\theta \in (-1, 0.3)$, whereas for $\theta \in (0.34, 1)$ the approximate bias of $\widetilde{\theta}$ is less than that of $\widetilde{\theta}_0$.

Further, the up to $o\left(n^{-1}\right)$ 2^{nd} order cumulant of $\sqrt{n}\left(\widetilde{\theta}_0 - \theta\right)$ is

$$k_2^{\widetilde{\theta}_0} = k_2^{\widetilde{\theta}} - \frac{4}{n}(1-\theta),$$

whereas the 3^{rd} and 4^{th} order approximate cumulants are the same as the ones of $\sqrt{n}\left(\widetilde{\theta} - \theta\right)$. This can be explained by the fact that these approximate cumulants do not depend on any of the Edgeworth coefficients that change in the zero mean case.

Finally, the second order AMSE of $\sqrt{n}\left(\widetilde{\theta}_0 - \theta\right)$ is

$$E\left[\sqrt{n}\left(\widetilde{\theta}_0 - \theta\right)\right]^2 = E\left[\sqrt{n}\left(\widetilde{\theta} - \theta\right)\right]^2 - \frac{3\theta^2 - 8\theta + 5}{n}.$$

Comparing the above AMSE with the AMSE of $\widetilde{\theta}$ we can conclude that the AMSE of the estimator of θ when we estimate the mean is higher than the one when the mean is zero and not estimated, for all $\theta \in (-1, 1)$. Let us now derive the expansion of the mean QMLE.

2.4.2 The Expansion of the Mean QML Estimator

To find the expansion of $\sqrt{n}(\widetilde{\mu} - \mu)$ we can apply Theorem 1 with appropriate f^i, h^{ij}, h^{ijk}, for $i, j, k = 1, \ldots 4$, and the cumulants in Appendix A.7. The Edgeworth and polynomial coefficients are presented in Appendix A.8. It is worth noticing that the asymptotic variance of $\widetilde{\mu}$ is $(1+\theta)^2 \sigma^2$, the same as the one of $\widehat{\mu}$, and that if the $u_t\prime s$ are normally distributed then the distribution of $\sqrt{n}(\widetilde{\mu} - \mu)$ is normal as well, which is not the case for $\widehat{\mu}$.

Furthermore, from Lemma 1 we have that

$$k_1 = E\left(\sqrt{n}(\widetilde{\mu} - \mu)\right) = 0,$$

i.e. $\widetilde{\mu}$ is an $o\left(n^{-1}\right)$ unbiased estimator. Finally

$$E\left(\sqrt{n}\left(\widetilde{\mu}-\mu\right)\right)^2=\sigma^2\left(1+\theta\right)^2+o\left(n^{-1}\right).$$

2.4.3 The Expansion of the First Order Autocorrelation QMLE

Let us define the QMLE of ρ as

$$\widetilde{\rho}=\frac{\widetilde{\theta}}{1+\widetilde{\theta}^2}.$$

In Appendix A.9 we present the Edgeworth coefficients of the second order approximation of the distribution of $\sqrt{n}\left(\widetilde{\rho}-\rho\right)$. To find the approximate bias and AMSE of $\sqrt{n}\left(\widetilde{\rho}-\rho\right)$, up to $o\left(n^{-1}\right)$, we can apply Lemma 1 and get

$$k_1^{\widetilde{\rho}}=-\frac{\left(1-\theta\right)\left(1+2\theta+3\theta^2\right)}{\sqrt{n}}\frac{\left(1-\theta^2\right)}{\left(1+\theta^2\right)^3}$$

and

$$E\left[\sqrt{n}\left(\widetilde{\rho}-\rho\right)\right]^2=\frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4}+\frac{1}{n}\left(\xi_1^{\widetilde{\rho}}+\xi_2^{\widetilde{\rho}}\right), \tag{2.11}$$

where $\xi_1^{\widetilde{\rho}}=\frac{\left(10\theta+62\theta^2-4\theta^3-65\theta^4-14\theta^5+24\theta^6+7\right)\left(1-\theta^2\right)^2}{\left(1+\theta^2\right)^6}$ and $\xi_2^{\widetilde{\rho}}=4\theta\frac{\theta-\theta^2-\theta^3+\theta^4-2}{\left(\theta+1\right)\left(-\theta+\theta^2+1\right)^2}\frac{\left(1-\theta^2\right)^4}{\left(1+\theta^2\right)^5}\kappa_3^2-\frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4}\kappa_4$. We next concentrate on the expansion in the zero-mean case.

2.4.3.1 The Zero-Mean Expansion

For the zero mean case, all Edgeworth coefficients are the same as in the non-zero mean one, apart from $\omega^{(3)}$, $a_{11}^{(1)}$ and $a_{12}^{(1)}$ (see Appendix A.9). Consequently, applying Lemma 1 and keeping terms up to order $O\left(n^{-1}\right)$, we can find the approximate bias of $\sqrt{n}\left(\widetilde{\rho}_0-\rho\right)$ as

$$k_1^{\widetilde{\rho_0}}=k_1^{\widetilde{\rho}}+\frac{1}{\sqrt{n}}\frac{\left(1-\theta\right)^2\left(1+\theta\right)}{\left(1+\theta^2\right)^2}. \tag{2.12}$$

It is obvious that the absolute values of the approximate bias of $\widetilde{\rho}_0$ is less than the one of $\widetilde{\rho}$.

In terms of AMSE we have that, keeping relevant terms,

$$\begin{aligned} E\left[\sqrt{n}\left(\widetilde{\rho}_0-\rho\right)\right]^2 &= E\left[\sqrt{n}\left(\widetilde{\rho}-\rho\right)\right]^2 \\ &\quad -\frac{1}{n}\left(14\theta+87\theta^2-12\theta^3-70\theta^4-26\theta^5+31\theta^6+14\right)\frac{\left(1-\theta^2\right)^2}{\left(1+\theta^2\right)^6}. \end{aligned}$$

This is different from the non-zero mean case. However, notice that the asymmetry and kurtosis parameters, κ_3 and κ_4, have the same effect on the AMSE, for any values of θ in the admissible region. In fact, the AMSE of $\widetilde{\rho}_0$ is always lower than the one of $\widetilde{\rho}$ for all $\theta \in (-1, 1)$. Of course, for higher values of n the two AMSEs collapse to the common asymptotic variance. Let us now proceed with the comparisons between all estimators.

2.5 Comparing the Estimators

To compare all estimators in terms of bias and MSE we run a simulation exercise. We draw a random sample of $n \in \{50, 200\}$ observations from a non-central Student-t distribution with non-centrality parameter $\eta \in \{-1, 1\}$ and $\nu \in \{11, 20\}$ degrees of freedom. Notice that for these values of η and ν we have that $\kappa_3 \in \{\pm 0.400, \pm 0.17\}$ and $\kappa_4 \in \{1.250, 0.42\}$. For each random sample, we generate the MA($1|\mu$) process y_t for $\theta \in \{-0.9, -0.8, ..., 0.9\}$, $\mu = 5.0$ and $\sigma^2 = 1.0$. We evaluate $\widehat{\rho}$ and if the estimate is in the $(-0.5, 0.5)$ interval we estimate all estimators, otherwise we throw away the sample and draw another one. This will introduce some bias in the estimation of the biases and the MSEs of the estimators, for which the closer θ is at the boundary of the admissible space the fiercer it will be. Furthermore, this will probably affect more the estimation of bias and MSE of the MM estimator of θ, as the maximization of the quasi likelihood is not restricted in any way. For each retained sample we evaluate the MM ($\widehat{\rho}$, $\widehat{\theta}$, and $\widehat{\mu}$), the QML ($\widetilde{\theta}$, $\widetilde{\rho}$ and $\widetilde{\mu}$) and the feasibly bias corrected estimators, i.e. when the estimated value of θ is employed for bias correction, employing the approximate bias formulae of previous sections (see Iglesias and Phillips 2008 [56], as well). We set the number of replications to 20000.

We will present the results for $n \in \{50, 200\}$, $\eta = 1$ and $\nu \in \{11, 20\}$, as the results with $\eta = -1$ and $\nu \in \{11, 20\}$ are almost identical to the reported ones.

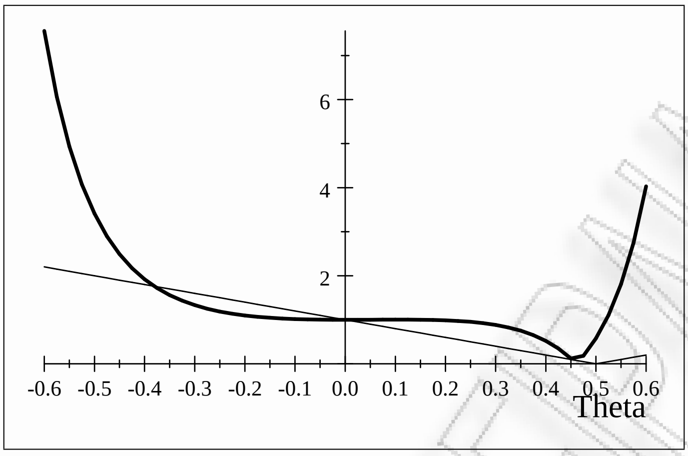

FIGURE 2.3: $\left|E\left[n\left(\widehat{\theta}-\theta\right)\right]\right|$ (thick line) and $\left|E\left[n\left(\widetilde{\theta}-\theta\right)\right]\right|$

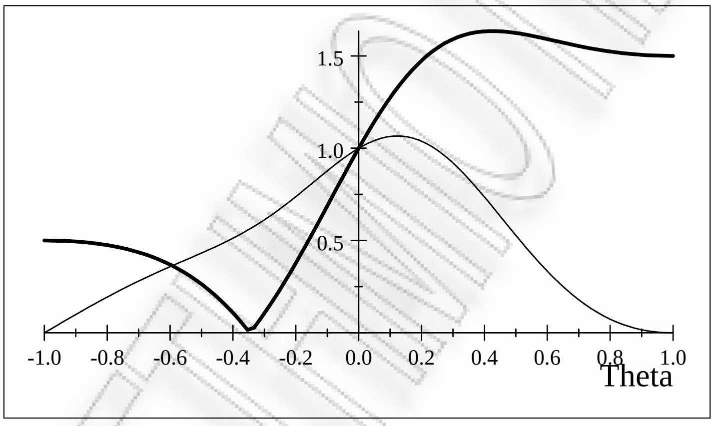

FIGURE 2.4: $\left|E\left[n\left(\widehat{\rho}-\rho\right)\right]\right|$ (thick line) and $\left|E\left[n\left(\widetilde{\rho}-\rho\right)\right]\right|$

2.5.1 Bias of the Estimators

On $o\left(n^{-1}\right)$ approximations grounds, it is apparent that, when μ is estimated, there are areas of the admissible region of θ that the MM estimators of either θ or ρ are less (approximately) biased than the QMLEs (see Figure 2.3 and Figure 2.4). For example, for $-.3 \leq \theta \leq 0$, both $\widehat{\theta}$ and $\widehat{\rho}$ are less biased than $\widetilde{\theta}$ and $\widetilde{\rho}$, respectively. However, the opposite is true for $\theta \geq 0$.

In terms of the simulation results, the same is more-or-less true for the estimated values of the biases of $\widehat{\theta}$ and $\widetilde{\theta}$ (compare the 3$^{\text{rd}}$ with the 6$^{\text{th}}$ column of Table 2.1, for non-central Student-t with $\nu = 20$, and the same ones in Table 2.2, for $\nu = 11$). However, there are important differences between the two estimators. Regarding the MM estimator, the approximate biases are far away from the estimated ones for values of θ near the ends of the admissible parameter region. In fact, for θ lower than -0.4 (for $n = 50$) and -0.5 (for $n = 200$), the approximate bias continuously underestimates the estimated one. The opposite is true for θ higher than 0.5 for both samples. For $\theta = -0.9$ or $\theta = 0.9$, the under and over estimation is massive, respectively. On the other hand, regarding the QMLE, the estimated bias of $\widetilde{\theta}$ is higher from the approximate one for $\theta < 0.4$, when $n = 50$, and for $\theta < -0.4$, when $n = 200$. In terms of the bias corrected estimators, it is apparent that when the approximate biases are close to the estimated ones, the corrected estimators are, by all terms, unbiased. Furthermore, it seems that the decrease in the degrees of freedom affects the estimated bias of $\widehat{\theta}$ more than that of $\widetilde{\theta}$. This is an indication that the assumption $E\left(u_0^{10}\right)$ exists is more important for the MM estimator of θ than for the QMLE.

For the estimators of ρ (see Tables 2.3 and 2.4), the estimated biases of the feasibly corrected estimators of both estimators $\widehat{\rho}$ and $\widetilde{\rho}$ are less, in absolute value, from the equivalent ones of the estimated biases. Furthermore, the estimated biases of the feasibly corrected $\widehat{\rho}$ are less, in absolute values, than the ones of the feasibly corrected $\widetilde{\rho}$ when $\theta \in [-0.3, 0.0]$ for $n = 50$, and $\theta \in [-0.4, 0.0]$ for $n = 200$, which partly confirms Figure 2.4. It seems that near the ends of the admissible region of θ the approximate bias of $\widetilde{\rho}$ is more accurate as compared with the one of $\widehat{\rho}$, i.e. it is closer to the estimated bias. Finally, the decrease in the degrees of freedom of the distribution of the errors affects the bias results, of both estimators, only marginally.

However, for the zero-mean case notice that the QMLEs of either θ or ρ are less (approximately) biased than the MM ones, for all $\theta \in (-1, 1)$. To see this, compare (2.7) with (2.10), and (2.5) with (2.12), respectively.

Hence, in terms of bias and when μ is estimated, for negative values of θ, but close to 0, the approximation of $\widehat{\theta}$ and $\widehat{\rho}$ work better than those of $\widetilde{\theta}$ and $\widetilde{\rho}$, whereas for $\theta > 0$ or θ close to -1 the QMLEs approximations are better.

2.5.2 MSE of the Estimators

In terms of second order AMSEs, we plot the ones of the two estimators of θ in Figure 2.5 and the corresponding ones of the estimators of ρ in Figure 2.6. Notice that in both graphs we set $n = 20$ and in both cases μ is estimated. It is apparent that there is not

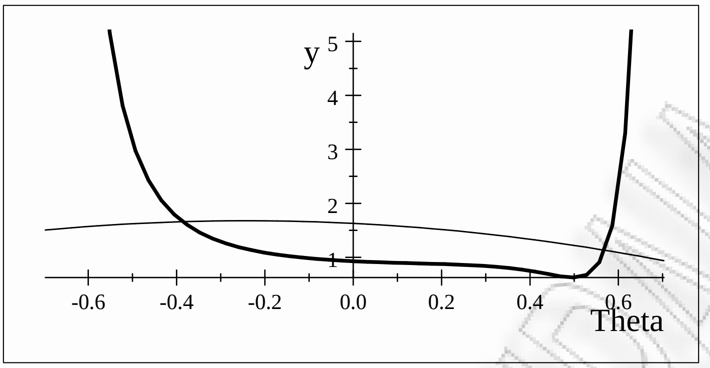

FIGURE 2.5: MSE of $\sqrt{20}\left(\widehat{\theta}-\theta\right)$ (thick) and $\sqrt{20}\left(\widetilde{\theta}-\theta\right)$, for $\kappa_3 = 0.17$ and $\kappa_4 = 0.42$.

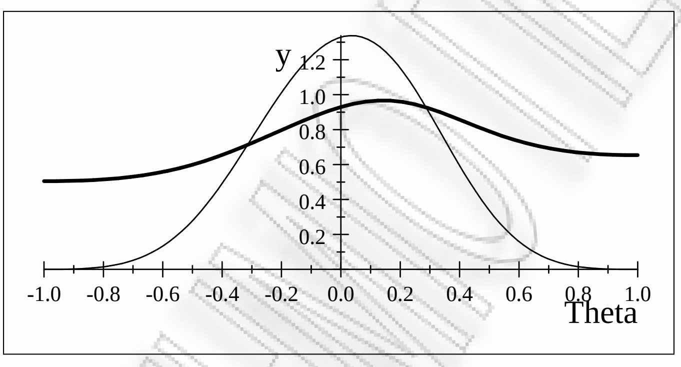

FIGURE 2.6: MSE of $\sqrt{20}\left(\widehat{\rho}-\rho\right)$ (thick) and $\sqrt{20}\left(\widetilde{\rho}-\rho\right)$, for $\kappa_3 = 0.17$ and $\kappa_4 = 0.42$.

uniform superiority of neither the QMLEs nor the MM ones, over the whole range of the admissible values of θ. In fact, it seems that for $\theta \in (-0.3, 0.3)$, and for the above sample size, the MSE of the MM estimators are smaller than the ones of the QMLEs.

These findings can be explained by the following facts: i) the asymptotic variance of $\widetilde{\theta}$, $\mathrm{AV}\left(\widetilde{\theta}\right)$, is less than or equal to $\mathrm{AV}\left(\widehat{\theta}\right)$, a well known result, and the same is true for $\mathrm{AV}(\widetilde{\rho})$ and $\mathrm{AV}(\widehat{\rho})$. In fact, only for $\theta = 0$ $\mathrm{AV}\left(\widetilde{\theta}\right) = \mathrm{AV}\left(\widehat{\theta}\right)$ and $\mathrm{AV}(\widetilde{\rho}) = \mathrm{AV}(\widehat{\rho})$, and we have strict inequality for all other values of θ. ii) For the $\frac{1}{n}$ terms, which do not include κ_3 or κ_4, and for $\theta \in (-0.5, 0.6)$ the term of $E\left[\sqrt{n}\left(\widehat{\theta}-\theta\right)\right]^2$ is lower than the one of $E\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^2$, for any sample size. The same is true for the equivalent terms of the estimators of ρ for $\theta \in (-0.8, 0.5)$. iii) For $\theta \in (-1, 0)$, $E\left[\sqrt{n}\left(\widehat{\theta}-\theta\right)\right]^2$ is a decreasing function of κ_3^2, whereas $E\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^2$, $E\left[\sqrt{n}\left(\widehat{\rho}-\rho\right)\right]^2$ and $E\left[\sqrt{n}\left(\widetilde{\rho}-\rho\right)\right]^2$ are increasing functions of κ_3^2. The opposite is true for $\theta \in (0, 1)$. iv) All MSEs are decreasing functions of κ_4, for $\theta \in (-1, 1)$. However, $E\left[\sqrt{n}\left(\widehat{\theta}-\theta\right)\right]^2$ and $E\left[\sqrt{n}\left(\widehat{\rho}-\rho\right)\right]^2$ are decreasing at a higher rate.

In terms of the simulations, it is immediately obvious that the AMSEs are close to the estimated ones for the MM estimator of θ (see Tables 2.5 and 2.6) in the middle range of values of θ, and are massively higher than the estimated ones at the two ends of the admissible range. On the other hand, the estimated MSEs of $\widetilde{\theta}$ are almost always underestimated by the approximate ones over the whole interval of θ. The underestimation is worse for values of θ less than -0.6 and higher than 0.6. For $n = 50$, the estimated MSE of $\widehat{\theta}$ is less than the one of $\widetilde{\theta}$ for $\theta \in (-0.1, 0.1)$, partially confirming Figure 2.5. The estimated MSEs of the bias corrected $\widetilde{\theta}$ are less than the ones of $\widehat{\theta}$ for all values of θ apart for $\theta = 0$, and this is true for both sample sizes. By decreasing the degrees of freedom of the error distribution, the estimated MSEs are lower for $\widehat{\theta}$ and higher for $\widetilde{\theta}$ (compare the 3^{rd} and 6^{th} columns of Table 2.5 with the respective ones of Table 2.6). This is in agreement with the approximate results for $\widehat{\theta}$ but not for $\widetilde{\theta}$. Finally, apart from the central part of the admissible range of θ, the MSE of the corrected $\widetilde{\theta}$ is almost always less than the one of $\widehat{\theta}$.

The estimated MSEs of $\widehat{\rho}$ are close to the AMSE ones (closer for $n = 200$ than for $n = 50$) and they are more so for $\theta \in (-0.6, 0.6)$ (see Table 2.7 and Table 2.8). The same is true for the MSEs of $\widetilde{\rho}$. Comparing the MSEs of $\widehat{\rho}$ with those of $\widetilde{\rho}$, for $\nu = 20$ and for both sample sizes, it is apparent that the estimated MSEs of $\widehat{\rho}$ are less than those of $\widetilde{\rho}$, for $\theta \in (-0.1, 0.1)$ partially confirming Figure 2.6. The same is true for the MSEs of the two estimators, for 11 degrees of freedom. The biased corrected $\widetilde{\rho}$ has, more or less, a smaller MSE than the corrected $\widehat{\rho}$ and for both samples.

Hence, to conclude this section, we can say that in terms of MSE and for small sample size, the QML method is more efficient for the estimation of θ and ρ only for the interval $(-1.0, -0.6) \cup (0.0, 1.0)$.

2.6 Conclusions

This chapter, by extending the results in Sargan (1976) [80] and Tanaka (1984) [87], derives the asymptotic expansions of the MM and QML estimators of the 1^{st} order autocorrelation, the mean parameter and the MA parameter for the MA(1) model. The necessity of Sargan's extension rests on the fact that the 1^{st} and 2^{nd} order cumulants of some estimators include also terms of $O\left(n^{-\frac{1}{2}}\right)$ and $O\left(n^{-1}\right)$. First, the second order

Edgeworth and Nagar-type expansions of the MM estimators are derived in a more general setup of Sargan (1976) [80] and second, the first order expansions in Tanaka (1984) [87] are extended to include terms of second order for the QML ones. It is worth noticing that the second order approximate bias of all estimators is not affected by the non-normality of the errors. A comparison of the expansions, either in terms of approximate bias or AMSE, reveals that there is not uniform superiority of neither of the estimators of θ and ρ, something which is also confirmed by the simulation results. Furthermore, it seems that the approximations work well for the middle rage of the admissible values of θ, whereas when θ takes values near the two ends, -1 and $+1$, the approximation are very poor with the MM approximations being affected more than the QMLE ones. Finally, the approximate bias and AMSE of the estimators depend on whether the mean of the process is known or estimated. In the zero-mean case, and on approximate grounds, the QMLEs of θ and ρ are superior to the MM ones in both approximate bias and AMSE terms.

The results can be utilized to provide finer approximations of the distributions of the estimators, as compared to the asymptotically normal ones. In fact, the bias results were employed to correct the up to $O\left(n^{-1}\right)$ bias of the estimators. It turned out that the feasibly corrected $\widetilde{\rho}$ is, almost always, less biased than $\widehat{\rho}$, for the whole interval of θ, without considerable alteration of its MSE. This indicates that the presented expansion works well for as small sample size as 50. On the other hand, the approximation of $\widehat{\theta}$ works well only for values of θ close to 0, with even as much as 200 observations. The presented approximations of $\widetilde{\theta}$ and $\widetilde{\rho}$ are somewhere in the middle, i.e. work well for a large interval of values of θ. Furthermore, in the Indirect Inference literature, our results constitute an application of the general results in Arvanitis and Demos (2009) [6].

The analysis presented here can be extended to any ARMA$(p, q|\mu)$ model. However, the algebra involved is becoming extremely tedious even for small values of p and q. Furthermore, one could consider the stochastic process $y_t = \mu + u_t + \theta_s u_{t-s}$, where $s = 1, 2, \ldots$. For specific values of s, this class of models could capture seasonal effects, e.g. for quarterly data $s = 4$, for monthly data $s = 12$, etc. (see e.g. Ghysels and Osborn 2001 [44]). In this case, the cumulants, at least up to 2^{nd} order, of the various statistics employed in sections 3 and 4 will become functions of s, complicating further the evaluations of the Edgeworth coefficients and the moments of the estimators.

Another interesting issue could be the expansion of the estimators as the parameter θ reaches the boundary of the admissible region, i.e. when $\theta \to \pm 1$ (in this respect see Andrews 1999 [5], and Iglesias and Linton 2007 [54]). Furthermore, along the lines of Durbin (1959) [39] and Gourieroux et al. (1993) [47], the properties of the MM estimators can be improved by considering the expansions not only of the first order autocorrelation but higher order ones. Finally, one could, utilizing the presented expansions, consider adjusted Box-Pierce tests along the lines of Kan and Wang (2010) [60], or develop asymptotic expansions of the error variance estimators, as well, and consider expansions of various tests, e.g. Wald etc. We leave these issues for future research.

This chapter is available as a Discussion Paper of Demos and Kyriakopoulou (2008) [34].

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	THETA MM			THETA QML		
	$\mathbf{n = 50}$, $u_t \overset{iid}{\sim}$ *non − central Student − t* with **20** *df*, *and non − centrality* = 1					
Theta	**Approx. Bias**	**Est. Bias**	**Bias Feas.**	**Approx. Bias**	**Est. Bias**	**Bias Feas.**
-0.9	-119.9542	2.6921	69.8340	-0.3960	-0.3219	-0.0355
-0.8	-12.4383	1.8054	157.3671	-0.3677	-0.2655	-0.0002
-0.7	-3.0324	1.0435	188.8647	-0.3394	-0.1749	0.0686
-0.6	-1.0691	0.4293	30.4886	-0.3111	-0.1310	0.0917
-0.5	-0.4825	0.0414	228.7782	-0.2828	-0.1381	0.0646
-0.4	-0.2715	-0.1237	5.1714	-0.2546	-0.1601	0.0231
-0.3	-0.1884	-0.1315	0.3206	-0.2263	-0.1631	0.0002
-0.2	-0.1553	-0.1129	0.0167	-0.1980	-0.1481	-0.0051
-0.1	-0.1437	-0.0998	0.0070	-0.1697	-0.1260	-0.0035
0	-0.1414	-0.0958	0.0055	-0.1414	-0.1037	-0.0016
0.1	-0.1419	-0.0944	0.0047	-0.1131	-0.0817	-0.0001
0.2	-0.1397	-0.0902	-0.0011	-0.0849	-0.0608	0.0004
0.3	-0.1249	-0.0781	-0.1714	-0.0566	-0.0434	-0.0025
0.4	-0.0737	-0.0995	-1.3525	-0.0283	-0.0454	-0.0245
0.5	0.0818	-0.2445	-53.2987	0.0000	-0.0691	-0.0677
0.6	0.5699	-0.5946	-49.9888	0.0283	-0.0959	-0.1140
0.7	2.3447	-1.1736	-475.2946	0.0566	-0.1032	-0.1411
0.8	11.3308	-1.9416	-99.9750	0.0849	-0.1070	-0.1648
0.9	117.4888	-2.8222	-179.6870	0.1131	-0.1636	-0.2403
	$\mathbf{n = 200}$, $u_t \overset{iid}{\sim}$ *non − central Student − t* with **20** *df*, *and non − centrality* = 1					
-0.9	-59.9771	3.0855	146.5577	-0.1980	-0.0446	0.1538
-0.8	-6.2191	1.8673	312.9910	-0.1838	-0.0733	0.1113
-0.7	-1.5162	0.8553	1457.2564	-0.1697	-0.0694	0.1010
-0.6	-0.5345	0.1583	23.1202	-0.1556	-0.0807	0.0757
-0.5	-0.2412	-0.1320	13.1220	-0.1414	-0.1141	0.0284
-0.4	-0.1357	-0.1449	0.1865	-0.1273	-0.1280	0.0006
-0.3	-0.0942	-0.1009	0.0095	-0.1131	-0.1183	-0.0040
-0.2	-0.0776	-0.0802	0.0017	-0.0990	-0.1041	-0.0041
-0.1	-0.0718	-0.0731	0.0002	-0.0849	-0.0891	-0.0033
0	-0.0707	-0.0716	-0.0005	-0.0707	-0.0735	-0.0020
0.1	-0.0709	-0.0716	-0.0010	-0.0566	-0.0581	-0.0009
0.2	-0.0698	-0.0701	-0.0021	-0.0424	-0.0435	-0.0006
0.3	-0.0624	-0.0611	-0.0083	-0.0283	-0.0300	-0.0015
0.4	-0.0368	-0.0365	-0.0953	-0.0141	-0.0209	-0.0065
0.5	0.0409	-0.0505	-8.2296	0.0000	-0.0315	-0.0312
0.6	0.2849	-0.3287	-61.1723	0.0141	-0.0623	-0.0758
0.7	1.1723	-0.9893	-603.7911	0.0283	-0.0833	-0.1108
0.8	5.6654	-1.9529	-1046.4400	0.0424	-0.0903	-0.1318
0.9	58.7444	-3.1666	-304.7226	0.0566	-0.1710	-0.2259

Table 2.1: Biases of the MA Coefficient Estimators under non-central Student-t with 20 degrees of freedom

	THETA MM			THETA QML		
$\mathbf{n=50}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with $\mathbf{11}$ df, $and\ non-centrality = 1$						
Theta	**Approx. Bias**	**Est. Bias**	**Bias Feas.**	**Approx. Bias**	**Est. Bias**	**Bias Feas.**
-0.9	-119.9542	2.3323	508.2411	-0.3960	-0.9820	-0.5467
-0.8	-12.4383	1.6906	3440.5556	-0.3677	-0.9269	-0.5221
-0.7	-3.0324	1.11025	194.5709	-0.3394	-0.6700	-0.3038
-0.6	-1.0691	0.61611	87.0193	-0.3111	-0.4516	-0.1224
-0.5	-0.4825	0.2364	915.3231	-0.2828	-0.3228	-0.0270
-0.4	-0.2715	-0.0088	46.2209	-0.2546	-0.2782	-0.0125
-0.3	-0.1884	-0.1279	2.2324	-0.2263	-0.2601	-0.0234
-0.2	-0.1553	-0.1583	10.1938	-0.1980	-0.2402	-0.0326
-0.1	-0.1437	-0.1565	14.6839	-0.1697	-0.2102	-0.0320
0	-0.1414	-0.1491	-0.0003	-0.1414	-0.1741	-0.0257
0.1	-0.1419	-0.1451	-0.0948	-0.1131	-0.1399	-0.0212
0.2	-0.1397	-0.14757	-1.5566	-0.0849	-0.1129	-0.0235
0.3	-0.1249	-0.1737	-8.4322	-0.0566	-0.1005	-0.0399
0.4	-0.0737	-0.2663	-15.5777	-0.0283	-0.1058	-0.0733
0.5	0.0818	-0.4777	-13.7759	0.0000	-0.1177	-0.1130
0.6	0.5699	-0.8195	-193.7762	0.0283	-0.1170	-0.1406
0.7	2.3447	-1.2831	-94.4679	0.0566	-0.0833	-0.1365
0.8	11.3308	-1.8498	-96.4860	0.0849	-0.0234	-0.1074
0.9	117.4888	-2.4850	-1230.0230	0.1131	0.0168	-0.0970
$\mathbf{n=200}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with $\mathbf{11}$ df, $and\ non-centrality = 1$						
-0.9	-59.9771	3.0777	2082.6805	-0.1980	-0.0479	0.1506
-0.8	-6.2191	1.8795	126.6579	-0.1838	-0.0718	0.1128
-0.7	-1.5162	0.8624	67.4715	-0.1697	-0.0666	0.1038
-0.6	-0.5345	0.1755	146.7298	-0.1556	-0.0757	0.0806
-0.5	-0.2412	-0.1280	16.4453	-0.1414	-0.1133	0.0292
-0.4	-0.1357	-0.1445	0.9975	-0.1273	-0.1251	0.0034
-0.3	-0.0942	-0.1004	0.0101	-0.1131	-0.1163	-0.0020
-0.2	-0.0776	-0.0793	0.0027	-0.0990	-0.1022	-0.0022
-0.1	-0.0718	-0.0720	0.0014	-0.0849	-0.0873	-0.0016
0	-0.0707	-0.0702	0.0010	-0.0707	-0.0719	-0.0004
0.1	-0.0709	-0.0700	0.0006	-0.0566	-0.0566	0.0006
0.2	-0.0698	-0.0681	-0.0001	-0.0424	-0.0421	0.0008
0.3	-0.0624	-0.0580	-0.0043	-0.0283	-0.0286	0.0000
0.4	-0.0368	-0.0297	-104.2074	-0.0141	-0.0199	-0.0055
0.5	0.0409	-0.0571	-2.3114	0.0000	-0.0342	-0.0339
0.6	0.2849	-0.3269	-26.6365	0.0141	-0.0653	-0.0788
0.7	1.1723	-0.9924	-43.5185	0.0283	-0.0813	-0.1088
0.8	5.6654	-1.9865	-52.7870	0.0424	-0.0926	-0.1341
0.9	58.7444	-3.1849	-111.6753	0.0566	-0.1725	-0.2274

Table 2.2: Biases of the MA Coefficient Estimators under non-central Student-t with 11 degrees of freedom

	RHO MM			RHO QML		
	$\mathbf{n = 50}$, $u_t \overset{iid}{\sim}$ *non − central Student − t* with **20** *df*, *and non − centrality* = 1					
Theta	**Approx. Bias**	**Est. Bias**	**Bias Feas.**	**Approx. Bias**	**Est. Bias**	**Bias Feas.**
-0.9	0.0699	0.5784	0.5426	-0.0140	0.0290	0.0357
-0.8	0.0672	0.5235	0.4886	-0.0274	0.0135	0.0301
-0.7	0.0617	0.4277	0.3951	-0.0397	0.0103	0.0365
-0.6	0.0522	0.2981	0.2697	-0.0507	0.0111	0.0458
-0.5	0.0373	0.1621	0.1413	-0.0611	-0.0018	0.0406
-0.4	0.0153	0.0565	0.0486	-0.0725	-0.0282	0.0224
-0.3	-0.0150	0.0017	0.0133	-0.0866	-0.0527	0.0079
-0.2	-0.0534	-0.0328	0.0041	-0.1043	-0.0713	0.0012
-0.1	-0.0972	-0.0628	0.0032	-0.1241	-0.0868	-0.0016
0	-0.1414	-0.0928	0.0026	-0.1414	-0.1001	-0.0041
0.1	-0.1801	-0.1194	0.0021	-0.1504	-0.1075	-0.0059
0.2	-0.2085	-0.1396	0.0017	-0.1468	-0.1068	-0.0072
0.3	-0.2250	-0.1550	-0.0015	-0.1301	-0.0987	-0.0093
0.4	-0.2309	-0.1890	-0.0300	-0.1041	-0.0939	-0.0206
0.5	-0.2297	-0.2625	-0.1023	-0.0747	-0.0914	-0.0368
0.6	-0.2250	-0.3687	-0.2091	-0.0472	-0.0808	-0.0444
0.7	-0.2197	-0.4822	-0.3233	-0.0253	-0.0595	-0.0387
0.8	-0.2154	-0.5719	-0.4136	-0.0104	-0.0402	-0.0305
0.9	-0.2129	-0.6231	-0.4651	-0.0024	-0.0321	-0.0285
	$\mathbf{n = 200}$, $u_t \overset{iid}{\sim}$ *non − central Student − t* with **20** *df*, *and non − centrality* = 1					
-0.9	0.0350	0.5627	0.5341	-0.0070	0.0119	0.0187
-0.8	0.0336	0.4860	0.4581	-0.0137	0.0068	0.0201
-0.7	0.0308	0.3535	0.3271	-0.0198	0.0078	0.0272
-0.6	0.0261	0.1948	0.1715	-0.0253	0.0041	0.0291
-0.5	0.0187	0.0700	0.0527	-0.0305	-0.0144	0.0158
-0.4	0.0077	0.0131	0.0061	-0.0362	-0.0332	0.0027
-0.3	-0.0075	-0.0079	-0.0001	-0.0433	-0.0439	-0.0009
-0.2	-0.0267	-0.0271	-0.0008	-0.0521	-0.0540	-0.0023
-0.1	-0.0486	-0.0487	-0.0012	-0.0620	-0.0641	-0.0030
0	-0.0707	-0.0705	-0.0016	-0.0707	-0.0723	-0.0031
0.1	-0.0901	-0.0896	-0.0017	-0.0752	-0.0764	-0.0028
0.2	-0.1043	-0.1037	-0.0017	-0.0734	-0.0747	-0.0028
0.3	-0.1125	-0.1122	-0.0017	-0.0650	-0.0673	-0.0032
0.4	-0.1154	-0.1191	-0.0052	-0.0521	-0.0576	-0.0057
0.5	-0.1148	-0.1537	-0.0399	-0.0373	-0.0537	-0.0158
0.6	-0.1125	-0.2564	-0.1436	-0.0236	-0.0517	-0.0271
0.7	-0.1098	-0.3964	-0.2847	-0.0127	-0.0410	-0.0273
0.8	-0.1077	-0.5122	-0.4012	-0.0052	-0.0260	-0.0200
0.9	-0.1065	-0.5876	-0.4770	-0.0012	-0.0194	-0.0176

TABLE 2.3: Biases of First Order Autocorrelation Estimators under non-central Student-t with 20 degrees of freedom

	RHO MM			RHO QML		
$\mathbf{n=50}$, $u_t \overset{iid}{\sim}$ *non − central Student − t* with **11** *df*, *and non − centrality* = 1						
Theta	**Approx. Bias**	**Est. Bias**	**Bias Feas.**	**Approx. Bias**	**Est. Bias**	**Bias Feas.**
-0.9	0.0699	0.5935	0.5529	-0.0140	0.0998	0.0956
-0.8	0.0672	0.5553	0.5163	-0.0274	0.0647	0.0743
-0.7	0.0617	0.4851	0.4493	-0.0397	0.0404	0.0670
-0.6	0.0522	0.3848	0.3547	-0.0507	0.0281	0.0705
-0.5	0.0373	0.2634	0.2431	-0.0611	0.0152	0.0716
-0.4	0.0153	0.1398	0.1353	-0.0725	-0.0164	0.0533
-0.3	-0.0150	0.0353	0.0546	-0.0866	-0.0601	0.0240
-0.2	-0.0534	-0.0390	0.0124	-0.1043	-0.1007	-0.0008
-0.1	-0.0972	-0.0949	-0.0058	-0.1241	-0.1348	-0.0188
0	-0.1414	-0.1396	-0.0114	-0.1414	-0.1585	-0.0292
0.1	-0.1801	-0.1780	-0.0145	-0.1504	-0.1732	-0.0368
0.2	-0.2085	-0.2129	-0.0219	-0.1468	-0.1785	-0.0441
0.3	-0.2250	-0.2539	-0.0447	-0.1301	-0.1777	-0.0547
0.4	-0.2309	-0.3129	-0.0939	-0.1041	-0.1723	-0.0684
0.5	-0.2297	-0.3959	-0.1727	-0.0747	-0.1580	-0.0775
0.6	-0.2250	-0.4851	-0.2606	-0.0472	-0.1300	-0.0740
0.7	-0.2197	-0.5661	-0.3414	-0.0253	-0.0979	-0.0636
0.8	-0.2154	-0.6258	-0.4011	-0.0104	-0.0750	-0.0565
0.9	-0.2129	-0.6590	-0.4345	-0.0024	-0.0724	-0.0619
$\mathbf{n=200}$, $u_t \overset{iid}{\sim}$ *non − central Student − t* with **11** *df*, *and non − centrality* = 1						
-0.9	0.0350	0.5606	0.5320	-0.0070	0.0117	0.0185
-0.8	0.0336	0.4873	0.4594	-0.0137	0.0067	0.0200
-0.7	0.0308	0.3568	0.3304	-0.0198	0.0082	0.0277
-0.6	0.0261	0.2008	0.1776	-0.0253	0.0058	0.0308
-0.5	0.0187	0.0725	0.0553	-0.0305	-0.0139	0.0164
-0.4	0.0077	0.0144	0.0075	-0.0362	-0.0312	0.0048
-0.3	-0.0075	-0.0066	0.0012	-0.0433	-0.0421	0.0010
-0.2	-0.0267	-0.0256	0.0007	-0.0521	-0.0520	-0.0003
-0.1	-0.0486	-0.0473	0.0003	-0.0620	-0.0623	-0.0011
0	-0.0707	-0.0691	-0.0001	-0.0707	-0.0707	-0.0014
0.1	-0.0901	-0.0883	-0.0004	-0.0752	-0.0751	-0.0016
0.2	-0.1043	-0.1025	-0.0005	-0.0734	-0.0739	-0.0020
0.3	-0.1125	-0.1109	-0.0005	-0.0650	-0.0668	-0.0027
0.4	-0.1154	-0.1178	-0.0040	-0.0521	-0.0575	-0.0056
0.5	-0.1148	-0.1564	-0.0425	-0.0373	-0.0553	-0.0174
0.6	-0.1125	-0.2564	-0.1437	-0.0236	-0.0531	-0.0285
0.7	-0.1098	-0.3976	-0.2859	-0.0127	-0.0406	-0.0269
0.8	-0.1077	-0.5192	-0.4081	-0.0052	-0.0264	-0.0205
0.9	-0.1065	-0.5911	-0.4804	-0.0012	-0.0196	-0.0179

TABLE 2.4: Biases of First Order Autocorrelation Estimators under non-central Student-t with 11 degrees of freedom

	THETA MM				THETA QML			
	$\mathbf{n = 50}$, $u_t \overset{iid}{\sim}$ $non-central\ Student-t$ with **20** df, $and\ non-centrality = 1$							
Theta	**As. Var.**	**Theor. MSE**	**Est. MSE**	**MSE Feas.**	**As. Var.**	**Theor. MSE**	**Est. MSE**	**MSE Feas.**
-0.9	149.4822	45798.7215	9.2603	3.0017×10^{6}	0.1900	0.6248	1.6791	1.5143
-0.8	28.6136	484.4521	5.3034	1.4480×10^{8}	0.3600	0.7706	1.3503	1.2291
-0.7	10.0950	33.1730	3.1428	2.6871×10^{8}	0.5100	0.8973	1.0593	0.9926
-0.6	4.7409	6.7677	2.3402	2.3989×10^{6}	0.6400	1.0051	0.9527	0.9069
-0.5	2.7014	2.8776	2.1187	8.8767×10^{8}	0.7500	1.0238	0.9517	0.8999
-0.4	1.7958	1.7794	1.9282	95072.049	0.8400	1.0827	1.0117	0.9476
-0.3	1.3564	1.3302	1.5380	400.5876	0.9100	1.1250	1.0847	1.0162
-0.2	1.1355	1.1157	1.2447	1.1304	0.9600	1.1505	1.1306	1.0648
-0.1	1.0309	1.0120	1.0824	1.0504	0.9900	1.1594	1.1455	1.0849
0	1.0000	0.9717	1.0316	1.0170	1.0000	1.1517	1.1420	1.0865
0.1	1.0309	0.9779	1.0612	1.0457	0.9900	1.1272	1.1188	1.0680
0.2	1.1355	1.0315	1.1868	1.1654	0.9600	1.0861	1.0779	1.0316
0.3	1.3564	1.1500	1.4654	128.0959	0.9100	1.0284	1.0163	0.9742
0.4	1.7958	1.3781	1.8267	3558.4082	0.8400	0.9541	0.9270	0.8889
0.5	2.7014	1.8718	2.1320	2.5005×10^{7}	0.7500	0.8633	0.8386	0.8054
0.6	4.7409	3.7244	2.5099	1.6114×10^{7}	0.6400	0.7559	0.7542	0.7285
0.7	10.0950	20.8664	3.5297	2.6532×10^{9}	0.5100	0.6319	0.6671	0.6504
0.8	28.6136	402.3640	5.8394	3.7682×10^{7}	0.3600	0.4913	0.6040	0.5963
0.9	149.4822	44033.2160	10.0036	6.0477×10^{7}	0.1900	0.3340	0.6926	0.6972
	$\mathbf{n = 200}$, $u_t \overset{iid}{\sim}$ $non-central\ Student-t$ with **20** df, $and\ non-centrality = 1$							
-0.9	149.4822	11561.7921	12.4627	5.0189×10^{7}	0.1900	0.2980	0.6478	0.6566
-0.8	28.6136	142.5732	6.5330	8.7613×10^{8}	0.3600	0.4568	0.5673	0.5631
-0.7	10.0950	15.8645	3.8782	3.8452×10^{10}	0.5100	0.5965	0.6426	0.6353
-0.6	4.7409	5.2476	3.2167	9.8300×10^{5}	0.6400	0.7171	0.7293	0.7141
-0.5	2.7014	2.7454	2.8204	1.6813×10^{6}	0.7500	0.8185	0.8226	0.7942
-0.4	1.7958	1.7917	2.0722	25.0913	0.8400	0.9007	0.9177	0.8834
-0.3	1.3564	1.3498	1.4596	1.3486	0.9100	0.9637	0.9835	0.9503
-0.2	1.1355	1.1305	1.1697	1.1387	0.9600	1.0076	1.0270	0.9960
-0.1	1.0309	1.0262	1.0414	1.0296	0.9900	1.0324	1.0516	1.0229
0	1.0000	0.9929	1.0014	0.9952	1.0000	1.0379	1.0555	1.0293
0.1	1.0309	1.0177	1.0304	1.0240	0.9900	1.0243	1.0390	1.0150
0.2	1.1355	1.1095	1.1417	1.1271	0.9600	0.9915	1.0028	0.9810
0.3	1.3564	1.3048	1.3960	1.3386	0.9100	0.9396	0.9489	0.9292
0.4	1.7958	1.6914	1.9229	21.9595	0.8400	0.8685	0.8761	0.8583
0.5	2.7014	2.4940	2.7137	3.7218×10^{5}	0.7500	0.7783	0.7795	0.7639
0.6	4.7409	4.4867	3.2293	2.1971×10^{7}	0.6400	0.6690	0.6760	0.6644
0.7	10.0950	12.7878	4.1397	5.6913×10^{9}	0.5100	0.5405	0.5753	0.5693
0.8	28.6136	122.0512	6.9133	1.5682×10^{10}	0.3600	0.3928	0.4505	0.4509
0.9	149.4822	11120.4157	13.0629	8.0211×10^{8}	0.1900	0.2260	0.3895	0.4041

TABLE 2.5: MSEs of the MA Coefficient Estimators under non-central Student-t with 20 degrees of freedom

	THETA MM				THETA QML			
	$\mathbf{n=50}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **11** df, $and\ non-centrality=1$							
Theta	**As. Var.**	**Approx. MSE**	**Est. MSE**	**MSE Feas.**	**As. Var.**	**Approx. MSE**	**Est. MSE**	**MSE Feas.**
-0.9	149.4822	45796.7977	6.7796	3.1337×10^{9}	0.1900	0.6223	4.8244	3.8564
-0.8	28.6136	484.0577	4.2198	4.3013×10^{11}	0.3600	0.7427	4.6397	3.7568
-0.7	10.0950	33.0270	2.6322	4.3913×10^{8}	0.5100	0.8496	3.7673	3.1505
-0.6	4.7409	6.6975	1.8382	3.5795×10^{7}	0.6400	0.9402	2.8431	2.4473
-0.5	2.7014	2.8377	1.5646	2.6715×10^{10}	0.7500	1.0145	2.1713	1.9058
-0.4	1.7958	1.7535	1.5122	3.3288×10^{7}	0.8400	1.0724	1.7860	1.5748
-0.3	1.3564	1.3113	1.4555	5542.5185	0.9100	1.1139	1.5719	1.3869
-0.2	1.1355	1.1004	1.3234	3.4207×10^{6}	0.9600	1.1389	1.4998	1.3301
-0.1	1.0309	0.9985	1.1855	8.5623×10^{6}	0.9900	1.1473	1.4464	1.2934
0	1.0000	0.9591	1.1132	1.0635	1.0000	1.1391	1.3999	1.2629
0.1	1.0309	0.9653	1.1329	148.5750	0.9900	1.1143	1.3414	1.2186
0.2	1.1355	1.0182	1.2347	47993.0271	0.9600	1.0730	1.2802	1.1687
0.3	1.3564	1.1350	1.3926	1.8318×10^{6}	0.9100	1.0155	1.1987	1.0970
0.4	1.7958	1.3605	1.5400	1.5703×10^{6}	0.8400	0.9419	1.1269	1.0336
0.5	2.7014	1.8514	1.7182	6.3816×10^{5}	0.7500	0.8522	1.0895	1.0041
0.6	4.7409	3.7094	2.1307	9.0900×10^{8}	0.6400	0.7466	1.0673	0.9908
0.7	10.0950	20.9308	3.0811	3.0584×10^{7}	0.5100	0.6247	1.1649	1.0859
0.8	28.6136	403.3154	4.8125	6.5571×10^{7}	0.3600	0.4864	1.4213	1.3209
0.9	149.4822	44059.5724	7.5403	3.2970×10^{10}	0.1900	0.3315	1.9286	1.7865
	$\mathbf{n=200}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **11** df, $and\ non-centrality=1$							
-0.9	149.4822	11561.3110	12.4152	5.8496×10^{10}	0.1900	0.2974	0.6451	0.6527
-0.8	28.6136	142.4745	6.5224	1.1308×10^{8}	0.3600	0.4557	0.5528	0.5494
-0.7	10.0950	15.8279	3.9224	8.6988×10^{6}	0.5100	0.5949	0.6340	0.6278
-0.6	4.7409	5.2300	3.2107	1.8720×10^{8}	0.6400	0.7151	0.7278	0.7142
-0.5	2.7014	2.7354	2.8316	1.4926×10^{6}	0.7500	0.8161	0.8262	0.7980
-0.4	1.7958	1.7852	2.0949	9949.5922	0.8400	0.8981	0.9222	0.8886
-0.3	1.3564	1.3450	1.4791	1.3617	0.9100	0.9610	0.9909	0.9579
-0.2	1.1355	1.12672	1.1855	1.1542	0.9600	1.0047	1.0355	1.0047
-0.1	1.0309	1.0228	1.0562	1.0443	0.9900	1.0293	1.0614	1.0328
0	1.0000	0.9897	1.0161	1.0100	1.0000	1.0348	1.0663	1.0400
0.1	1.0309	1.0145	1.0458	1.0397	0.9900	1.0211	1.0509	1.0268
0.2	1.1355	1.1061	1.1591	1.1446	0.9600	0.9883	1.0153	0.9934
0.3	1.3564	1.3010	1.4205	1.3530	0.9100	0.9364	0.9616	0.9417
0.4	1.7958	1.6870	1.9883	2.1583×10^{8}	0.8400	0.8655	0.8869	0.8689
0.5	2.7014	2.4889	2.6981	5142.8915	0.7500	0.7756	0.7847	0.7691
0.6	4.7409	4.4829	3.2454	3.7048×10^{6}	0.6400	0.6666	0.6833	0.6717
0.7	10.0950	12.8039	4.1560	1.9465×10^{6}	0.5100	0.5387	0.5780	0.5719
0.8	28.6136	122.2890	6.9785	3.1515×10^{6}	0.3600	0.3916	0.4568	0.4572
0.9	149.4822	11127.0047	13.1387	2.3224×10^{7}	0.1900	0.2254	0.3954	0.4100

TABLE 2.6: MSEs of the MA Coefficient Estimators under non-central Student-t with 11 degrees of freedom

	RHO MM				RHO QML			
	$\mathbf{n = 50}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **20** df, $and\ non-centrality = 1$							
Theta	**As. Var.**	**Approx. MSE**	**Est. MSE**	**MSE Feas.**	**As. Var.**	**Approx. MSE**	**Est. MSE**	**MSE Feas.**
-0.9	0.5028	0.5046	0.5528	0.5240	0.0006	0.0019	0.0043	0.0055
-0.8	0.5126	0.5135	0.5121	0.4894	0.0064	0.0125	0.0126	0.0151
-0.7	0.5327	0.5319	0.4569	0.4447	0.0269	0.0431	0.0402	0.0445
-0.6	0.5676	0.5637	0.4383	0.4413	0.0766	0.1094	0.0985	0.1050
-0.5	0.6224	0.6130	0.4896	0.5097	0.1728	0.2272	0.1990	0.2075
-0.4	0.6998	0.6819	0.6155	0.6515	0.3273	0.4031	0.3580	0.3686
-0.3	0.7957	0.7667	0.7486	0.7964	0.5338	0.6225	0.5763	0.5906
-0.2	0.8945	0.8556	0.8596	0.9148	0.7563	0.8461	0.8094	0.8277
-0.1	0.9710	0.9303	0.9348	0.9910	0.9324	1.0190	0.9912	1.0089
0	1.0000	0.9717	0.9692	1.0176	1.0000	1.0917	1.0656	1.0743
0.1	0.9710	0.9679	0.9531	0.9858	0.9324	1.0398	1.0058	0.9989
0.2	0.8945	0.9209	0.8922	0.9052	0.7563	0.8764	0.8338	0.8118
0.3	0.7957	0.8455	0.8011	0.7949	0.5338	0.6485	0.6054	0.5758
0.4	0.6998	0.7627	0.6816	0.6532	0.3273	0.4174	0.3840	0.3551
0.5	0.6224	0.6894	0.5697	0.5113	0.1728	0.2307	0.2173	0.1947
0.6	0.5676	0.6341	0.5191	0.4243	0.0766	0.1069	0.1095	0.0955
0.7	0.5327	0.5971	0.5373	0.4061	0.0269	0.0395	0.0458	0.0394
0.8	0.5126	0.5753	0.5864	0.4270	0.0064	0.0103	0.0152	0.0130
0.9	0.5028	0.5645	0.6280	0.4526	0.0006	0.0013	0.0050	0.0043
	$\mathbf{n = 200}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **20** df, $and\ non-centrality = 1$							
-0.9	0.5028	0.5032	0.5272	0.5010	0.0006	0.0009	0.0021	0.0026
-0.8	0.5126	0.5128	0.4731	0.4526	0.0064	0.0080	0.0094	0.0104
-0.7	0.5327	0.5325	0.4187	0.4081	0.0269	0.0310	0.0333	0.0352
-0.6	0.5676	0.5666	0.4350	0.4367	0.0766	0.0848	0.0853	0.0880
-0.5	0.6224	0.6201	0.5460	0.5588	0.1728	0.1864	0.1818	0.1848
-0.4	0.6998	0.6954	0.6749	0.6947	0.3273	0.3463	0.3403	0.3442
-0.3	0.7957	0.7884	0.7760	0.8006	0.5338	0.5560	0.5499	0.5561
-0.2	0.8945	0.8848	0.8689	0.8968	0.7563	0.7787	0.7735	0.7822
-0.1	0.9710	0.9608	0.9415	0.9696	0.9324	0.9541	0.9516	0.9603
0	1.0000	0.9929	0.9714	0.9953	1.0000	1.0229	1.0222	1.0266
0.1	0.9710	0.9702	0.9484	0.9638	0.9324	0.9593	0.9583	0.9547
0.2	0.8945	0.9011	0.8809	0.8860	0.7563	0.7863	0.7842	0.7729
0.3	0.7957	0.8082	0.7903	0.7860	0.5338	0.5625	0.5607	0.5458
0.4	0.6998	0.7155	0.6939	0.6819	0.3273	0.3499	0.3496	0.3359
0.5	0.6224	0.6392	0.5808	0.5571	0.1728	0.1873	0.1890	0.1789
0.6	0.5676	0.5842	0.4808	0.4329	0.0766	0.0842	0.0889	0.0828
0.7	0.5327	0.5488	0.4689	0.3900	0.0269	0.0301	0.0352	0.0324
0.8	0.5126	0.5283	0.5145	0.4106	0.0064	0.0074	0.0101	0.0093
0.9	0.5028	0.5182	0.5697	0.4494	0.0006	0.0008	0.0024	0.0022

TABLE 2.7: MSEs of the First Order Autocorrelation Estimators under non-central Student-t with 20 degrees of freedom

	RHO MM				RHO QML			
	$\mathbf{n=50}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **11** df, $and\ non-centrality=1$							
Theta	**As. Var.**	**Approx. MSE**	**Est. MSE**	**MSE Feas.**	**As. Var.**	**Approx. MSE**	**Est. MSE**	**MSE Feas.**
-0.9	0.5028	0.5004	0.5794	0.5585	0.0006	0.0013	0.0236	0.0223
-0.8	0.5126	0.5090	0.5505	0.5361	0.0064	0.0105	0.0260	0.0307
-0.7	0.5327	0.5269	0.5074	0.5052	0.0269	0.0391	0.0560	0.0677
-0.6	0.5676	0.5581	0.4731	0.4890	0.0766	0.1034	0.1283	0.1470
-0.5	0.6224	0.6067	0.4812	0.5205	0.1728	0.2200	0.2469	0.2723
-0.4	0.6998	0.6747	0.5521	0.6171	0.3273	0.3970	0.4146	0.4455
-0.3	0.7957	0.7582	0.6823	0.7713	0.5338	0.6213	0.6332	0.6685
-0.2	0.8945	0.8456	0.8205	0.9255	0.7563	0.8538	0.8734	0.9108
-0.1	0.9710	0.9188	0.9220	1.0291	0.9324	1.0375	1.0707	1.1013
0	1.0000	0.9591	0.9691	1.0612	1.0000	1.1191	1.1615	1.1727
0.1	0.9710	0.9550	0.9604	1.0223	0.9324	1.0704	1.1167	1.0988
0.2	0.8945	0.9083	0.8990	0.9214	0.7563	0.9035	0.9542	0.9079
0.3	0.7957	0.8340	0.7986	0.7769	0.5338	0.6679	0.7241	0.6610
0.4	0.6998	0.7524	0.6893	0.6209	0.3273	0.4283	0.4912	0.4272
0.5	0.6224	0.6803	0.6198	0.5017	0.1728	0.2351	0.3023	0.2502
0.6	0.5676	0.6260	0.6035	0.4388	0.0766	0.1077	0.1663	0.1325
0.7	0.5327	0.5899	0.6297	0.4258	0.0269	0.0390	0.0799	0.0621
0.8	0.5126	0.5687	0.6663	0.4344	0.0064	0.0098	0.0342	0.0263
0.9	0.5028	0.5584	0.6920	0.4449	0.0006	0.0011	0.0199	0.0156
	$\mathbf{n=200}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **11** df, $and\ non-centrality=1$							
-0.9	0.5028	0.5022	0.5236	0.4974	0.0006	0.0008	0.0022	0.0027
-0.8	0.5126	0.5117	0.4722	0.4516	0.0064	0.0075	0.0095	0.0105
-0.7	0.5327	0.5313	0.4213	0.4106	0.0269	0.0300	0.0333	0.0352
-0.6	0.5676	0.5652	0.4389	0.4404	0.0766	0.0833	0.0859	0.0887
-0.5	0.6224	0.6185	0.5487	0.5614	0.1728	0.1846	0.1832	0.1862
-0.4	0.6998	0.6935	0.6802	0.7002	0.3273	0.3447	0.3435	0.3475
-0.3	0.7957	0.7863	0.7845	0.8094	0.5338	0.5557	0.5554	0.5618
-0.2	0.8945	0.8823	0.8796	0.9080	0.7563	0.7806	0.7809	0.7899
-0.1	0.9710	0.9579	0.9541	0.9828	0.9324	0.9587	0.9606	0.9696
0	1.0000	0.9898	0.9851	1.0095	1.0000	1.0298	1.0322	1.0368
0.1	0.9710	0.9670	0.9619	0.9779	0.9324	0.9669	0.9687	0.9653
0.2	0.8945	0.8980	0.8933	0.8988	0.7563	0.7931	0.7940	0.7827
0.3	0.7957	0.8053	0.8014	0.7975	0.5338	0.5674	0.5687	0.5537
0.4	0.6998	0.7130	0.7032	0.6914	0.3273	0.3526	0.3546	0.3408
0.5	0.6224	0.6369	0.5834	0.5591	0.1728	0.1884	0.1910	0.1807
0.6	0.5676	0.5822	0.4825	0.4345	0.0766	0.0844	0.0900	0.0837
0.7	0.5327	0.5470	0.4675	0.3884	0.0269	0.0299	0.0351	0.0323
0.8	0.5126	0.5267	0.5188	0.4134	0.0064	0.0073	0.0102	0.0094
0.9	0.5028	0.5167	0.5732	0.4522	0.0006	0.0008	0.0026	0.0024

TABLE 2.8: MSEs of the First Order Autocorrelation Estimators under non-central Student-t with 11 degrees of freedom

Chapter 3

Bias Correction of ML and QML Estimators in the EGARCH(1,1) Model

3.1 Introduction

The last years there has been a substantial interest in deriving the asymptotic properties of econometric estimators in time series models. Although there is an important and growing literature that deals with the asymptotics of the Generalized Autoregressive Conditional Heteroskedastic (GARCH) models, either in terms of consistency and asymptotic normality of the estimators or in terms of the finite-sample theory, the asymptotic properties of the estimators in the Exponential GARCH (EGARCH) process of Nelson (1991) [71] have not been fully explored. Comparing to the GARCH process, the advantages of the EGARCH model are well-known, with the main one being the fact that the model captures the negative dynamic asymmetries noticed in many financial series, i.e. the so-called leverage effects.

The asymptotic aspects of the conditionally heteroskedastic models have been discussed under many different considerations, in order to analyze the statistical properties of these estimators. Since the important work of Engle (1982) [40] and that of Bollerslev (1986) [20], who introduced the Autoregressive Conditional Heteroskedasticity (ARCH) and Generalized ARCH models, respectively, a huge amount of literature on the asymptotics has appeared in short time. Weiss (1986) [90] proved Consistency and Asymptotic Normality (CAN) of the maximum likelihood estimators in ARCH models, assuming normal distribution of the errors and imposing a rather restrictive condition that the data have bounded fourth moments, excluding in that way from the proof many other interesting conditionally heteroskedastic models. Quite parallel, Lee and Hansen (1994) [61] and Lumsdaine (1996) [66] relaxed the condition which Weiss imposed and they looked at the consequences of the possible failure of the normality assumption on the errors, providing conditions under which CAN exist in the GARCH$(1,1)$ specification (for multivariate frameworks see e.g. Jeantheau, 1998 [57]; Comte and Lieberman, 2003 [26]).

The finite sample properties of the QML estimators in the first order GARCH model are investigated through an asymptotic expansion of the Edgeworth type, as Linton (1997) [65] developed[1] in which he also provided the higher-order bias of the estimators. Furthermore, Iglesias and Linton (2007) [54] derive the second-order asymptotic theory of the quasi-maximum likelihood estimator in stationary and nonstationary GARCH models, when constraints are imposed and they correct the first- and second-order bias of the estimator. Nowadays, many researchers work on the asymptotic behavior of these estimators, with unceasing interest.

Until the influential work of Nelson (1991) [71], the conditional heteroskedastic models that had been developed could not explain the asymmetry effects, indicating that alternative models might be suitable for financial applications. Turning our attention to asymmetric GARCH models, and more specifically to the EGARCH model which has become a popular model in applied financial work, very little is known about its statistical properties. Although we are endowed with the moment structure investigated by He, Terasvirta and Malmsten (2002) [52], the limiting properties of the maximum likelihood estimators in the EGARCH models do not exist in the literature. The interest in consistency and asymptotic normality results of EGARCH has been growing and the problem of the theoretical properties not yet been explored await for an answer; see, for example, Straumann and Mikosch (2006) [83][2]. The finite sample properties of the maximum likelihood and quasi-maximum likelihood estimators of the EGARCH$(1,1)$

[1] The validity of the Edgeworth expansions in the GARCH model is established in the paper of Corradi and Iglesias (2008) [27].

[2] In a recent paper, Zaffaroni (2009) [91] estimates the EGARCH parameters with Whittle methods and the asymptotic distribution theory of these estimators is established.

process using Monte Carlo methods have been examined in the paper of Deb (1996) [33][3]. He used, however, response surface methodology in order to examine the finite sample bias and other properties in interest, by summarizing the results of a wide array of experiments.

In this chapter we derive the bias approximations of the Maximum Likelihood (ML) and Quasi-Maximum Likelihood (QML) Estimators of the EGARCH$(1,1)$ parameters and we check our theoretical results through simulations. With the approximate bias expressions, we are then able to correct the bias of all estimators. To this end, a Monte Carlo exercise is conducted and the results are presented and discussed. We provide two types of the bias correction mechanism in order to decide for the bias reduction in practice for the popular model of Nelson. It is the first time that analytically the higher order biases appear in this literature for a nonlinear model like the EGARCH one and these results can now be used as to be incorporated into the relative analysis of other similar specifications, see e.g. Iglesias and Linton (2007) [54]. We conclude that, for given sets of parameters values, the bias correction works satisfactory for all the parameters. The results for the approximate bias expressions can be used in order to formulate the approximate Edgeworth distribution of the estimators.

The organization of this chapter is as follows: Section 3.2 presents the model and estimators. Section 3.3 deals with the main results of our analysis. First, analytic derivatives and their expected values are presented. Second, conditions for stationarity of the log-variance derivatives are investigated. In the sequel, the theoretical bias approximations of the Maximum Likelihood and Quasi Maximum Likelihood Estimators are calculated and the simulation results for the bias correction of the estimators are presented. Finally, Section 3.4 concludes. All proofs, rather lengthy, are collected in the Appendix B. Let us now turn our attention to the definition of the EGARCH$(1,1)$ model and the estimators.

3.2 The Model and Estimators

Let us consider the following model, where the observed data $\{y_t\}_{t=1}^{T}$ are generated by the EGARCH$(1,1)$ process, see Nelson (1991) [71], in which the conditional variance,

[3]Perez and Zaffaroni (2008) [73] compare the finite sample properties of the MLE and Whittle estimators, in terms of bias and efficiency, in the EGARCH model and its long-memory version.

h_t, depends on both the size and the sign of the lagged residuals:

$$\begin{aligned} y_t &= \mu + u_t, \qquad t = 1, ...T, \qquad \text{where} &(3.1)\\ u_t &= z_t\sqrt{h_t}, \quad z_t \sim iidD\,(0,1) \\ \ln(h_t) &= \alpha + \theta z_{t-1} + \gamma g\,(z_{t-1}) + \beta \ln(h_{t-1}), \qquad \text{where} &(3.2)\\ g\,(z_t) &= |z_t| - E\,|z_t|\,. \end{aligned}$$

The process $\{u_t\}$ is a real-valued discrete time stochastic process (the error process) and h_t is a positive with probability one $\mathcal{A}_{t-1}$-measurable function (the conditional variance), where $\mathcal{A}_{t-1}$ is the sigma-algebra generated by the past values of z_t, i.e. $\{z_{t-1}, z_{t-2}, z_{t-3}, ...\}$. The function $g\,(z_t)$ is a well-defined function of z_t. The process h_t is not observed and thus is constructed via recursion using the estimating values of the parameters and a proper initial value for the conditional variance. To allow for the possibility of nonnormality in the conditional distribution of $\{y_t\}$, we assume that the $\{z_t\}$ are independently and identically distributed (i.i.d.) with zero mean and unit variance. We do not impose any symmetric distributional property, however the proofs automatically become very complicated. The conditional variance is constrained to be non-negative by the assumption that the logarithm of h_t is a function of past $z_t's$. Comparing to the relative analysis, Nelson's paper was the first which models the conditional variance as a function of variables which are not solely squares of the observations.

Note from (3.2) that $\ln(h_t)$ constitutes a causal AR(1) process with mean $\alpha/(1-\beta)$ and error sequence $[\theta z_{t-1} + \gamma(|z_{t-1}| - E\,|z_{t-1}|)]$. The unique stationary solution to (3.2), provided that $|\beta| < 1$, is given by its almost sure (a.s.) representation, provided that $\gamma \geq |\theta|$[4]:

$$\begin{aligned} \ln(h_t) &= \alpha(1-\beta)^{-1} + \sum_{k=0}^{\infty} \beta^k \left(\theta z_{t-1-k} + \gamma g\,(z_{t-1-k})\right) \Rightarrow \\ \ln(h_t) &\geq (\alpha - \gamma E\,|z_t|)\,(1-\beta)^{-1} \quad \text{a.s.} \end{aligned}$$

The conditional variance responds asymmetrically to rises and falls in stock price, which is believed to be important for example in modelling the behavior of stock returns. It

[4] This means that the configurations mimic the stylized fact that a shock always leads to increased volatility.

is an important stylized fact for many assets. The coefficients $(\theta+\gamma)$ and $(\theta-\gamma)$ (if $z_t \geq 0$ and $z_t < 0$, respectively) show the asymmetry in response to positive and negative y_t. The parameter θ is referred to as the leverage parameter, which shows the effect of the sign of y_t. The term $\gamma\left[|z_t| - E|z_t|\right]$ represents a magnitude effect. Formulae for the higher order moments of u_t are given in Nelson (1991) [71]. The parameter α can be made a function of time (α_t) to accommodate the effect of any non-trading periods of forecastable effects.

The unconditional mean and variance of y_t is:

$$E\left(y_t\right) = \mu,$$

and

$$Var\left(y_t\right) = \exp\left(\frac{\alpha}{1-\beta}\right)\prod_{i=0}^{\infty} E\left[\exp\left[\beta^i\left(\theta z_0 + \gamma g\left(z_0\right)\right)\right]\right],$$

which, under normality of the errors, becomes the following result:

$$Var\left(y_t\right) = \exp\left(\frac{\alpha - \gamma\sqrt{\frac{2}{\pi}}}{1-\beta}\right)\prod_{i=0}^{\infty}\left[\exp\left(\frac{\beta^{2i}\left(\gamma^*\right)^2}{2}\right)\Phi\left(\beta^i\gamma^*\right) + \exp\left(\frac{\beta^{2i}\delta^2}{2}\right)\Phi\left(\beta^i\delta\right)\right],$$

where $\gamma^* = \gamma + \theta$, $\delta = \gamma - \theta$ and $\Phi\left(k\right)$ is the value of the cumulative standard Normal evaluated at k, i.e. $\Phi\left(k\right) = \int_{-\infty}^{k}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx$.

Proof. The proof of the unconditional variance is given in the Appendix B.1. □

To estimate the parameters of the model in (3.1) and (3.2), we employ the quasi-maximum likelihood estimation. Maximum likelihood is the procedure which is most often used in estimating the parameters in time series models, but for most applications it is very difficult to justify the conditional normality assumption. Therefore, the log-likelihood function may be misspecified. However, we can still obtain estimates by maximizing a Gaussian quasi-log-likelihood function and under the auxiliary assumption of an i.i.d. distribution for the standardized innovations $z_t's$. The estimators which are derived by this maximization problem are the so-called Quasi Maximum Likelihood Estimators (QMLEs). The fact that we maximize a quasi-log-likelihood is justified by the evidence that distributions of asset returns are often thick tailed and as a consequence the normality assumption is violated.

An important and really interesting feature of our model is that the assumption of the block diagonality of the information matrix no longer holds. This is also the case for the ARCH-M model and the asymmetric model of the Augmented ARCH (see Bera and Higgins, 1993 [13], p. 349; also Bollerslev, Engle and Nelson, 1994 [19], p. 2981). This implies that the off-diagonal blocks involving partial derivatives with respect to both mean and variance parameters are not null matrices, while this is the case in other GARCH-type models. Below we present analytic proofs of this argument in the context of the EGARCH$(1,1)$ model and these results disaccord with Malmsten (2004) [67], even if the distribution of the innovations is symmetric, which implies that $Ez^3 = 0$.

In the EGARCH$(1,1)$ model, there is no explicit expression of the probability density of the vector $(y_1, ..., y_T)'$ since the distribution of $(h_1, ..., h_T)'$ is not known. To overcome this difficulty, we consider an approximate conditional log-likelihood instead. Some assumptions are also required for the initial values of the conditional variance h_t, which should be drawn from the stationary distribution, and the squared standardized residuals z_t^2. Assuming that $z_0 = 0$ and $\ln(h_0) = \frac{\alpha}{1-\beta}$, we obtain a good approximation to the conditional Gaussian log-likelihood, as follows:

$$\begin{aligned}\ell(\mu,\alpha,\theta,\beta,\gamma|\, z_0,h_0) &= -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln(h_t) - \sum_{t=1}^{T}\frac{(y_t-\mu)^2}{2h_t} = \\ &= -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln(h_t) - \frac{1}{2}\sum_{t=1}^{T}z_t^2. \qquad (3.3)\end{aligned}$$

Notice that h_t and z_t are both functions of $\boldsymbol{\omega}$ and μ, where $\boldsymbol{\omega} = (\alpha,\theta,\beta,\gamma)^{/}$, i.e. the vector of unknown log-variance parameters, so that both are functions of $\boldsymbol{\varphi} = \left(\boldsymbol{\omega}^{/},\mu\right)^{/}$, which represents the vector of all unknown parameters. The first order conditions are recursive and consequently do not have explicit solutions.

The likelihood function is derived as though the errors are conditionally normal and is still maximized at the true parameters. Having specified the log-likelihood function, the quasi maximum likelihood estimator is then defined as

$$\widehat{\varphi_T} = \arg\max_{\varphi\in\Theta}\frac{1}{T}\sum_{t=1}^{T}\ell(\varphi). \qquad (3.4)$$

The parameter space is of the form

$$\Theta = \mathbb{R} \times [0,1) \times D,$$

where

$$D = \left\{ (\theta, \gamma)' \in \mathbb{R}^2 \mid \theta \in \mathbb{R}, \gamma \geq |\theta| \right\}.$$

Let us proceed with the main results of our analysis, beginning with the analytic derivatives of the log-likelihood function and their expected values.

3.3 The Main Results

3.3.1 Analytic derivatives and their expected values

In this section we present analytic derivatives[5] of the log-likelihood function and their expected values, which are needed in the sequel to evaluate the asymptotic bias of the QMLEs and to calculate the cumulants of the Edgeworth distribution. It is of great importance to mention that there are no such analytic results in the related literature of the finite sample theory, and it is especially this feature that makes this analysis to differ from the previous one, that of Linton (1997) [65], who studied the case of the GARCH$(1,1)$ model. Let us first proceed with the derivatives of the log-likelihood function and their analytic representation.

Following henceforth the notation employed in Linton (1997) [65], i.e. $h_{t;\circ} = \frac{\partial \ln(h_t)}{\partial \circ}$ and so on, the derivatives of the log-likelihood function with respect to all parameters are:

[5] Fiorentini, Calzolari and Panattoni (1996) [41] argue that the computation of analytic derivatives of the log-likelihood is essential, as the computational benefit of their use is really substantial for estimation purposes.

First with respect to the mean parameter,

$$
\begin{aligned}
\mathcal{L}_{\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}},\\
\mathcal{L}_{\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu}-\sum_{t=1}^{T}\left(\frac{1}{h_t}+2\frac{z_t}{\sqrt{h_t}}h_{t;\mu}+\frac{1}{2}z_t^2h_{t;\mu}^2\right),\\
\mathcal{L}_{\mu\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu,\mu}+3\sum_{t=1}^{T}\frac{1}{h_t}h_{t;\mu}\\
&\quad -3\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\left(h_{t;\mu,\mu}-h_{t;\mu}^2\right)-\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(3h_{t;\mu}h_{t;\mu,\mu}-h_{t;\mu}^3\right)
\end{aligned}
$$

while for $i,j,k\in\{\alpha,\theta,\gamma,\beta\}$ the derivatives are:

$$
\begin{aligned}
\mathcal{L}_{i} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i},\\
\mathcal{L}_{ij} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;i}^2,\\
\mathcal{L}_{ijk} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j,k}-\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(3h_{t;i}h_{t;j,k}-h_{t;i}^3\right).
\end{aligned}
$$

The cross derivatives are given by the following expressions:

$$
\begin{aligned}
\mathcal{L}_{i\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,\mu}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;i}h_{t;\mu}-\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}h_{t;i},\\
\mathcal{L}_{i\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,\mu,\mu}-2\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\left(h_{t;i,\mu}-h_{t;i}h_{t;\mu}\right)\\
&\quad -\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(2h_{t;\mu}h_{t;i,\mu}-h_{t;i}h_{t;\mu}^2+h_{t;i}h_{t;\mu,\mu}\right)+\sum_{t=1}^{T}\frac{1}{h_t}h_{t;i},\\
\mathcal{L}_{ij\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j,\mu}-\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\left(h_{t;i,j}-h_{t;i}h_{t;j}\right)\\
&\quad -\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(h_{t;j}h_{t;i,\mu}-h_{t;j}h_{t;i}h_{t;\mu}+h_{t;i,j}h_{t;\mu}+h_{t;i}h_{t;j,\mu}\right).
\end{aligned}
$$

Note that the log-likelihood derivatives are expressions of the log-variance derivatives, $h_{t;\circ}$, where the latter are given in the Appendix B.6. The expected values of the log-likelihood derivatives are also given in the Appendix B.2.

The cross-products of the log-likelihood derivatives are:

for $i, j \in \{\alpha, \theta, \gamma, \beta\}$,

$$
\begin{aligned}
\mathcal{L}_i \mathcal{L}_{ij} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i}\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j}-\frac{1}{2}\sum_{t=1}^{T}z_t^2 h_{t;i}^2\right),\\
\mathcal{L}_i \mathcal{L}_{j\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i}\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;j,\mu}-\frac{1}{2}\sum_{t=1}^{T}z_t^2 h_{t;j}h_{t;\mu}-\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}h_{t;i}\right),\\
\mathcal{L}_i \mathcal{L}_{\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i}\left[\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu}-\sum_{t=1}^{T}\left(\frac{1}{h_t}+2\frac{z_t}{\sqrt{h_t}}h_{t;\mu}+\frac{1}{2}z_t^2 h_{t;\mu}^2\right)\right],\\
\mathcal{L}_\mu \mathcal{L}_{ij} &= \left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\right)\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j}-\frac{1}{2}\sum_{t=1}^{T}z_t^2 h_{t;i}^2\right),\\
\mathcal{L}_\mu \mathcal{L}_{j\mu} &= \left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\right)\left(\begin{array}{c}\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;j,\mu}-\frac{1}{2}\sum_{t=1}^{T}z_t^2 h_{t;j}h_{t;\mu}\\ -\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}h_{t;i}\end{array}\right),\\
\mathcal{L}_\mu \mathcal{L}_{\mu\mu} &= \left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\right)\left[\begin{array}{c}\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu}\\ -\sum_{t=1}^{T}\left(\frac{1}{h_t}+2\frac{z_t}{\sqrt{h_t}}h_{t;\mu}+\frac{1}{2}z_t^2 h_{t;\mu}^2\right)\end{array}\right].
\end{aligned}
$$

The expectations of the cross-products are given in the Appendix B.4.

Let us turn our attention to the conditions for stationarity of the log-variance derivatives.

3.3.2 Conditions for stationarity of the log-variance derivatives

In this section we investigate under which conditions there is a second-order stationary solution to the log-variance derivatives, needed for the existence and the evaluation of the log-likelihood derivatives, and hence in order to calculate the bias expressions of the QMLEs. The existence, stationarity and ergodicity of the second order derivatives of the conditional variance are necessary if someone studies the validity in a Taylor series expansion of the first order derivatives of the log-likelihood.

We consider the following example:

$$
\begin{aligned}
h_{t;\alpha}h_{t;\alpha\alpha} &= \frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^2+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^3\\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\alpha}\\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha}h_{t-1;\alpha,\alpha}. \qquad (3.5)
\end{aligned}
$$

In order to calculate the expected value of the above expression, we first assume that $E\left(h_{t;\alpha}^2\right), E\left(h_{t;\alpha}^3\right)$ and $E\left(h_{t;\alpha,\alpha}\right)$ exist. Next, define:

$$\begin{aligned} A\left(z_{t-1}\right) &= \frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha}^2 \\ &+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha}^3 \\ &+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha,\alpha}, \end{aligned}$$

and

$$B^2\left(z_{t-1}\right)=\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 .$$

Then,

$$\begin{aligned} h_{t;\alpha} h_{t;\alpha\alpha} &= A\left(z_{t-1}\right)+B^2\left(z_{t-1}\right) h_{t-1;\alpha} h_{t-1;\alpha,\alpha}= \\ &= A\left(z_{t-1}\right)+\sum_{k=1}^{\infty} \prod_{i=0}^{k-1} B^2\left(z_{t-1-i}\right) A\left(z_{t-1-k}\right) . \end{aligned}$$

The infinite sum converges almost surely. To see this, let:

$$S_n=A\left(z_{t-1}\right)+\sum_{k=1}^{n} \prod_{i=0}^{k-1} B^2\left(z_{t-1-i}\right) A\left(z_{t-1-k}\right) .$$

Then we have:

$$\begin{aligned} E\left(S_n\right) &= E\left[A\left(z_{t-1}\right)\right]+\sum_{k=1}^{n} E\left[\prod_{i=0}^{k-1} B^2\left(z_{t-1-i}\right)\right] E\left[A\left(z_{t-1-k}\right)\right]= \\ &= E\left[A\left(z_{t-1}\right)\right]\left[\sum_{k=0}^{n}\left\{E\left[B^2\left(z_{t-1-i}\right)\right]\right\}^k\right] . \end{aligned}$$

Thus, $E\left(\lim_{n\to\infty} S_n\right)=E\left[A\left(z_{t-1}\right)\right]\left\{1-E\left[B^2\left(z_{t-1-i}\right)\right]\right\}^{-1}<\infty$, providing that $E\left[A\left(z_{t-1}\right)\right]<\infty$. In order to ensure the existence of a stationary solution to the (3.5), we should impose the condition that

$$E\left[B^2\left(z_{t-1-i}\right)\right]<1.$$

In a similar manner, the rest stationarity conditions of all log-variance derivatives and products of them follow.

Proposition 3.1. *Given*

a) $\left|\beta_0 - \frac{1}{2}\gamma_0 E\,|z|\right| < 1$

b) $\left|\beta_0^2 + \frac{1}{4}\theta_0^2 + \frac{1}{4}\gamma_0^2 - \gamma_0\beta_0 E\,|z| + \frac{1}{2}\gamma_0\theta_0 E\left(z\,|z|\right)\right| < 1$

and

c) $$\left|\begin{array}{c} \beta_0^3 + \frac{3}{4}\beta_0\theta_0^2 + \frac{3}{4}\beta_0\gamma_0^2 - \frac{1}{8}\theta_0\left(\theta_0^2 + 3\gamma_0^2\right)E\left(z^3\right) - \frac{3}{2}\beta_0^2\gamma_0 E\,|z| \\ +\frac{3}{2}\beta_0\theta_0\gamma_0 E\left(z\,|z|\right) - \frac{1}{8}\gamma_0\left(\gamma_0^2 + 3\theta_0^2\right)E\,|z|^3 \end{array}\right| < 1,$$

then

the second-order stationarity of all log-variance derivatives follows.

Proof. The proof comes immediately from the results in the *Appendices B.3* and *B.7*. □

Let us now proceed with the bias approximations of the QMLEs.

3.3.3 Bias Approximations

In this section we develop the bias approximations for the ML and QML estimators in the EGARCH$(1,1)$[6]. One of the main advantages of developing the bias expressions is to use them as a bias correction mechanism. This is one of the practical applications of the bias approximations. Moreover, these results help to analyze the consequences of introducing restrictions in the log-variance parameters. With these expressions, one can compute the Edgeworth approximate distribution. It is also important to explore the theoretical properties of the estimators so that the statistical inference is possible.

We use a McCullagh (1986) [68] result for the standardized estimator having a stochastic expansion, see in p.209, and taking expectations we end up with the asymptotic bias of the QML estimator. Our next step is to check our bias approximations through simulations. Note that McCullagh's expansion has already been applied in the literature to retrieve the bias in many nonlinear models, such as Linton (1997) [65]. When dealing with nonlinear models, it is very common to have the bias expressions in terms of expectations and applying these expressions for bias correction. At this point, it is important to state briefly the main differences between our analysis and that of Linton. First of all, we generalize the finite-sample analysis of heteroskedastic time series models considering a non-symmetric distribution of the errors. Furthermore, we show that the

[6] Iglesias and Phillips (2002) [55] developed theoretical bias approximations for the MLEs of the parameters in an ARCH(1) model.

block-diagonality of the information matrix does not hold in our case, which implies that there are new terms in the bias expressions of the estimators. This means that we cannot use the results that appear in the literature from the analysis of the GARCH model.

Assumption 3.3.1 *We assume that the errors have bounded J^{th} moments, for some $J > 6$, and we denote by κ_3 and κ_4 their third and fourth order cumulants, where the latter is given by:*

$$\kappa_4 = E\left(z^4 - 3\right).$$

Under the above assumptions, we are now able to present our Theorem which is useful for the evaluation of the bias approximations of all estimators and also to construct the Edgeworth expansions in this setting.

Theorem 3.3.1 *Given that $z_t \sim iidD\,(0,1)$ and non-symmetric, and for $i,j,k \in \{\mu,\alpha,\theta,\gamma,\beta\}$ unless the parameter μ is used separately to underline the difference, the following moments of the log-likelihood derivatives converge to finite limits as $T \to \infty$:*

$$c_{ij} = \frac{1}{T}E\left(\mathcal{L}_{ij}\right) = -\frac{1}{2}\tau_{i,j},$$

$$c_{ijk} = \frac{1}{T}E\left(\mathcal{L}_{ijk}\right) = -\frac{1}{2}\left(\tau_{ij,k} + \tau_{ik,j} + \tau_{jk,i} - \tau_{i,j,k}\right),$$

$$c_{ij,k} = \frac{1}{T}E\left(\mathcal{L}_{ij}\mathcal{L}_{k}\right) = -\frac{1}{4}\left[\tau^{zz}_{k;i,j} - (\kappa_4+2)\left(\tau_{ij,k} - \tau_{i,j,k}\right)\right],$$

$$c_{\mu\mu} = \frac{1}{T}E\left(\mathcal{L}_{\mu\mu}\right) = -\left(\overline{\pi} + \frac{\tau_{\mu,\mu}}{2}\right),$$

$$c_{i\mu\mu} = \frac{1}{T}E\left(\mathcal{L}_{i\mu\mu}\right) = \overline{\pi_i} - \frac{1}{2}\left(\tau_{i,\mu\mu} + 2\tau_{\mu i,\mu} - \tau_{\mu,i,\mu}\right),$$

$$c_{\mu\mu\mu} = \frac{1}{T}E\left(\mathcal{L}_{\mu\mu\mu}\right) = -\frac{1}{2}\left(3\tau_{\mu\mu,\mu} - \tau^3_{\mu}\right) + 3\overline{\pi_{\mu}},$$

$$c_{i\mu,\mu} = \frac{1}{T}E\left(\mathcal{L}_{i\mu}\mathcal{L}_{\mu}\right) = -\frac{1}{4}\left\{\begin{array}{c} 4\overline{\pi_i} - (\kappa_4+2)\left(\tau_{i\mu,\mu} - \tau_{i,\mu,\mu}\right) \\ +\tau^{zz}_{\mu;i\mu} + 2\tau^{zh}_{i,\mu} + 2\kappa_3\left(2\tau^{h}_{i,\mu} - \tau^{h}_{i\mu}\right) \end{array}\right\},$$

$$c_{i\mu,j} = \frac{1}{T}E\left(\mathcal{L}_{i\mu}\mathcal{L}_{j}\right) = -\frac{1}{4}\left\{-(\kappa_4+2)\left(\tau_{i\mu,j} - \tau_{i,j,\mu}\right) + \tau^{zz}_{\mu;i\mu} + 2\kappa_3\tau^{h}_{ij}\right\},$$

$$c_{\mu\mu,i} = \frac{1}{T}E\left(\mathcal{L}_{\mu\mu}\mathcal{L}_{i}\right) = -\frac{1}{4}\left\{-(\kappa_4+2)\left(\tau_{\mu\mu,i} - \tau_{i,\mu,\mu}\right) + \tau^{zz}_{i;\mu\mu} + 4\kappa_3\tau^{h}_{i,\mu}\right\},$$

$$c_{ij,\mu} = \frac{1}{T}E\left(\mathcal{L}_{ij}\mathcal{L}_{\mu}\right) = -\frac{1}{4}\left\{\begin{array}{c} -(\kappa_4+2)\left(\tau_{ij,\mu} - \tau_{i,j,\mu}\right) + \tau^{zz}_{\mu;ij} + 2\tau^{zh}_{i,j} \\ +2\kappa_3\left(2\tau^{h}_{i,j} - \tau^{h}_{ij}\right) \end{array}\right\},$$

$$c_{\mu\mu,\mu} = \tfrac{1}{T}E\left(\mathcal{L}_{\mu\mu}\mathcal{L}_{\mu}\right) = -\tfrac{1}{4}\left\{\begin{array}{c} 8\overline{\pi_{\mu}} - (\kappa_4+2)\left(\tau_{\mu\mu,\mu} - \tau_{\mu,\mu,\mu}\right) \\ +\tau^{zz}_{\mu;\mu\mu} + 2\tau^{zh}_{\mu,\mu} + 2\kappa_3\left(3\tau^{h}_{\mu,\mu} - \tau^{h}_{\mu\mu}\right) \end{array}\right\},$$

where $\tau_i = \frac{1}{T}\sum_{t=1}^{T} E\left(h_{t;i}\right)$, $\tau_{i,j} = \frac{1}{T}\sum_{t=1}^{T} E\left(h_{t;i}h_{t;j}\right)$, $\tau_{ij,k} = \frac{1}{T}\sum_{t=1}^{T} E\left(h_{t;ij}h_{t;k}\right)$

and $\tau_{i,j,k} = \frac{1}{T}\sum_{t=1}^{T} E\left(h_{t;i}h_{t;j}h_{t;k}\right)$.

Also, $\overline{\pi} = \frac{1}{T}\sum_{t=1}^{T} E\left(\frac{1}{h_t}\right)$, *and* $\overline{\pi_i} = \frac{1}{T}\sum_{t=1}^{T} E\left(\frac{1}{h_t}h_{t;i}\right)$,

while $\tau^{zz}_{k;i,j} = \frac{1}{T}\sum\limits_{s<t}\sum E\left[\left(z_s^2-1\right)h_{s;k}h_{t;i}h_{t;j}\right]$, $\tau^{zh}_{i,j} = \frac{1}{T}\sum\limits_{s<t}\sum E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;i}h_{t;j}\right)$,

$\tau^{h}_{i,\mu} = \frac{1}{T}\sum_{t=1}^{T} E\left(\frac{1}{\sqrt{h_t}}h_{t;i}h_{t;\mu}\right)$ *and* $\tau^{h}_{i\mu} = \frac{1}{T}\sum_{t=1}^{T} E\left(\frac{1}{\sqrt{h_t}}h_{t;i,\mu}\right)$.

Proof. Given in the Appendix B.5. □

In order to calculate the bias approximations, we need to find expressions for the c^{ij}, c_{ijk} and $c_{jk,l}$. Let us first consider the case when the mean parameter is supposed to be equal to zero and not estimated. With techniques of McCullagh (1986) [68], the standardized estimators, derived from choosing θ to solve $\mathcal{L}_i(\omega,\mu) = 0$, for $i \in \{\alpha,\theta,\gamma,\beta\}$, have the following stochastic expansion[7]:

$$\sqrt{T}\left\{\widehat{\varphi}_i - \varphi_i\right\} \approx -c^{ij}Z_j + \frac{1}{\sqrt{T}}\left\{c^{ij}c^{kl}Z_{jk}Z_l - c^{ij}c^{kl}c^{mn}c_{j\ln}Z_kZ_m/2\right\} + O_P\left(\frac{1}{T}\right), \quad (3.6)$$

where

$$Z_j = T^{-1/2}\mathcal{L}_j$$

and

$$Z_{jk} = T^{-1/2}\left\{\mathcal{L}_{jk} - E\left(\mathcal{L}_{jk}\right)\right\}$$

are evaluated at the true parameters and are jointly asymptotically normal. Raising pairs of indices signifies components from the matrix inversion.

Taking expectations of the right-hand side in (3.6), we get:

$$E\left[\sqrt{T}\upsilon'\left\{\widehat{\varphi}(\mu) - \varphi\right\}\right] \approx \frac{1}{\sqrt{T}}\upsilon_i c^{ij}c^{kl}\left\{c_{jk,l} + c_{jkl}\left(\kappa_4+2\right)/4\right\},$$

[7] We make use of the summation convention, that is: $c^{ij}Z_j = \sum_j c^{ij}Z_j$, in which repeated indices in an expression are to be summed over.

where υ is the 4×1 parameter vector. If $\kappa_4 = 0$, QML equals ML and then the above formula equals the one of Cox and Snell (1968) [28], i.e.:

$$E\left[\sqrt{T}\upsilon'\left\{\widehat{\varphi}\left(\mu\right)-\varphi\right\}\right] \approx \frac{1}{\sqrt{T}}\upsilon_i c^{ij} c^{kl}\left\{c_{jk,l}+\frac{1}{2}c_{jkl}\right\}.$$

Let us now consider the other case, when the mean parameter is unknown and estimated. Hence, if we incorporate the effects of estimating μ, the stochastic expansions take the following form:

$$\sqrt{T}\left\{\widehat{\varphi}_i\left(\widehat{\mu}\right)-\varphi_i\right\}-\sqrt{T}\left\{\widehat{\varphi}_i\left(\mu\right)-\varphi_i\right\} \approx \frac{1}{\sqrt{T}}\left\{c^{ij}c^{kl}Z_{jk}Z_l - c^{ij}c^{kl}c^{mn}c_{j\ln}Z_k Z_m/2\right\},$$

where now $i, j, k, l \in \{\alpha, \theta, \gamma, \beta, \mu\}$. Taking expectations of the right-hand side, we find the asymptotic bias of the estimators in this case.

In terms of the mean squared error, from (3.6) we have up to $O_P\left(\frac{1}{T}\right)$:

$$E\left[\sqrt{T}\upsilon'\left\{\widehat{\varphi}\left(\mu\right)-\varphi\right\}\right]^2 \approx -\upsilon_i c^{ij}\left(\kappa_4+2\right)/2, \tag{3.7}$$

which is the asymptotic variance. If we let the remainder to be of $O\left(T^{-3/2}\right)$, then the mean squared error is again evaluated by (3.7), with the difference now that there would be added terms of $O\left(T^{-1}\right)$. Of course, as $T \to \infty$, the mean squared error approaches the asymptotic variance. In what follows, we present the simulation results and discuss the bias correction of all estimators.

3.3.4 Simulations

In this section we make a simulation exercise in order to check the adequacy of our theoretical results and be able to proceed with the bias correction of the estimators. We draw a random sample of $T = \{750, 1500, 3000, 5000, 10000, 25000, 50000\}$ observations and 500 observations for initialization, under the assumption of normality. We make 50000 replications for sample sizes up to 10000 and 300000 replications for 25000 and more observations, in order to decrease the Monte Carlo error. The mean parameter is supposed to be equal to zero and hence is not estimated, so the parameter vector is $(\alpha, \beta, \gamma, \theta)'$. We check the performance of the bias correction mechanism for different sets of parameter values and we will present the results for three sets, i.e. $(0.1, 0.9, 0.7, -0.4)$,

$(-0.1, 0.9, 0.6, -0.2)$ and $(0.5, 0.5, 0.8, -0.5)$. The first two sets include values for the parameters that are close to what is observed from the financial data. We multiply the bias by T and not $\sqrt{T}$, i.e. $E\left(T\left(\widehat{\varphi}-\varphi\right)\right)$, as in this way we keep a constant term in the bias expressions that is important to distinguish what happens when we increase the sample size, as the next terms in the expressions will tend to zero, as $T \to \infty$.

The bias correction mechanism is constructed under the specification of two methods. The first one, called first-step correction, is the classical one, in which we estimate the model and we retrieve the estimated parameters. Next, we compute the bias expressions by using the estimates and we are then able to correct the bias of the estimators with the corresponding values of the bias, i.e.

$$\widetilde{\varphi} = \widehat{\varphi} - \frac{1}{T} bias\left(\widehat{\varphi}\right).$$

Notice that there is nothing to prevent the case of $\widetilde{\varphi}$ being outside the admissible area (see also Linton, 1997 [65] as well as Iglesias and Linton, 2007 [54]). In such a case we throw away the random sample and draw a new one.

The second method that we employ, called full-step correction, is a method proposed by Arvanitis and Demos (2010) [7], in which we solve an optimization problem of the form

$$\min_{\varphi} \left\{ \widehat{\varphi} - \varphi - \frac{1}{T} bias\left(\varphi\right) \right\}^2.$$

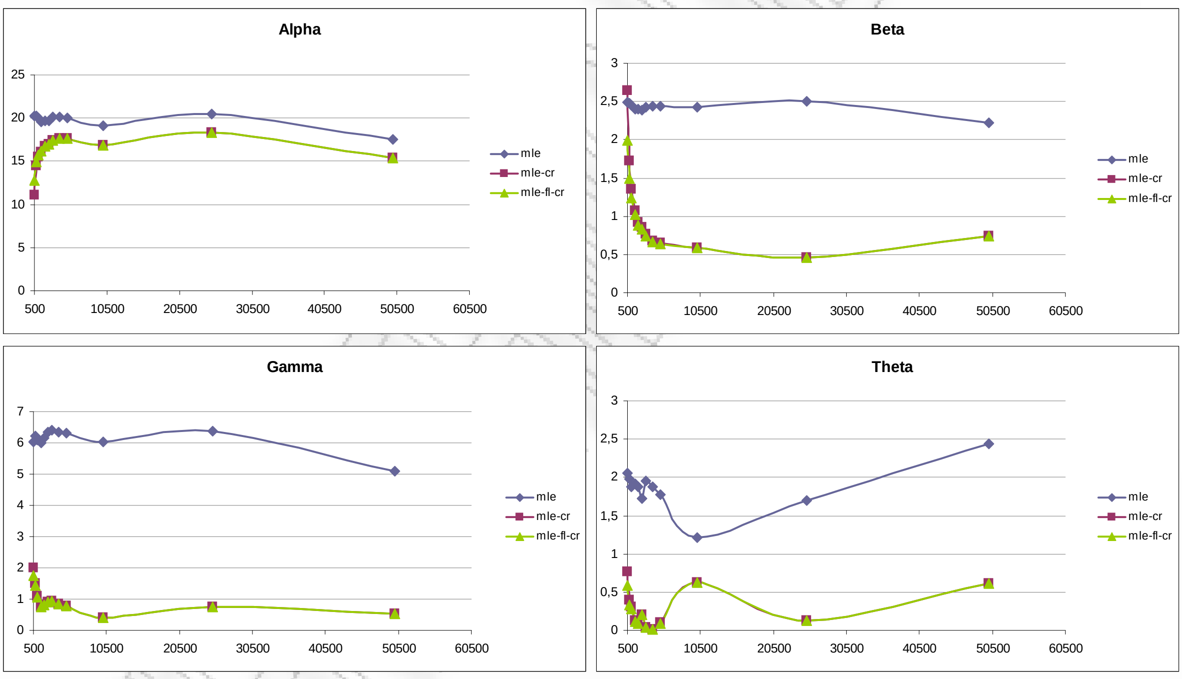
In this respect, this method is a multi-step maximization procedure, using numerical derivatives. This justifies the name of the first method, which is the first step of the multi-step optimization problem. In this way, the second method incorporates the constraints that are imposed on the coefficients and as a consequence the corrected estimate of the EGARCH parameter cannot lie outside the admissible region, i.e. the corrected beta will be less than one in absolute value.

Figures 3.1 and 3.2 represent the bias correction performance under the normality assumption. For the first set of parameter values (Figure 3.1) we see that the bias correction works in all cases and the corrected bias of the MLEs tend to zero, as the sample size increases. For Figure 3.2, the bias correction represents some intervals in which it behaves well, especially for small sample sizes. The case of the beta coefficient is the most ideal in the sense that the bias of the MLE is stabilized in the constant term of its expression, as T increases.

When dropping the normality assumption, we run the simulations under the hypothesis of mixture of normals for standardized random variables (see Figure 3.3 and Figure 3.4). In fact, the errors are drawn from a normal distribution with mean 0.01 and variance 9, with probability 0.1, and with probability 0.9 they are drawn from a normal distribution with mean -0.001 and variance 0.111. In this way, the theoretical mean and variance of the distribution are 0 and 1, respectively. Notice that with these hyperparameter values the theoretical skewness and kurtosis of the random errors are 0.0266 and 24.334 respectively, approximately matching the sample counterparts of most financial data.

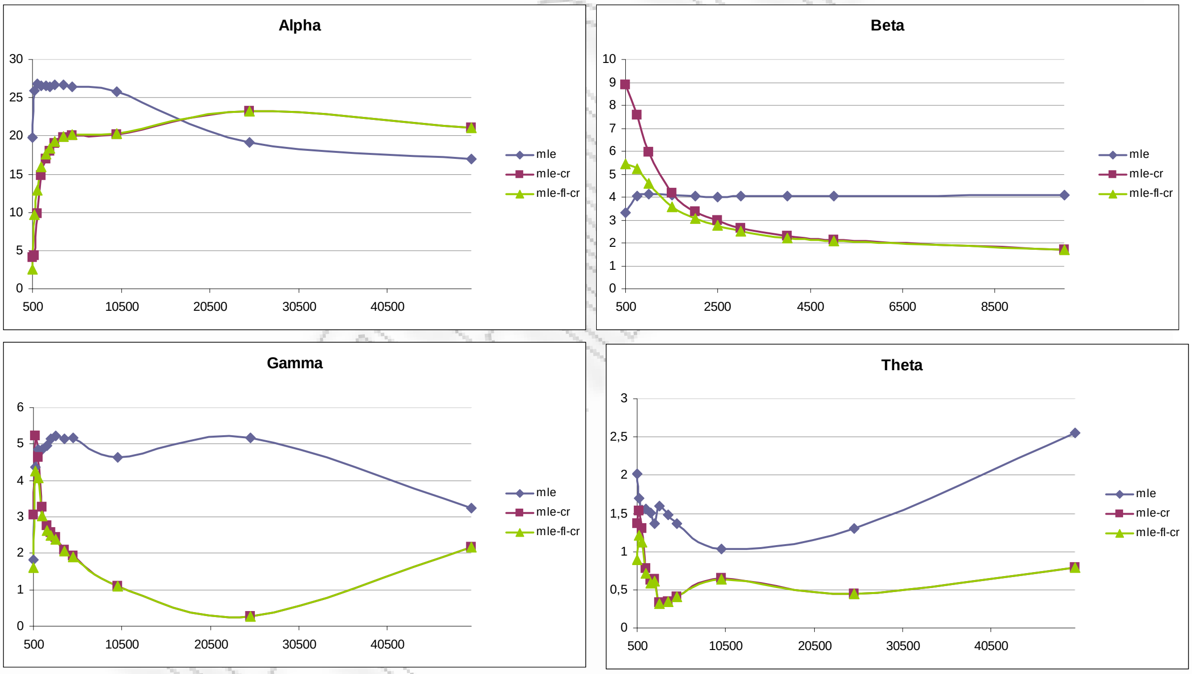
Figures 3.3 and 3.4 represent two sets of parameter values, in which we have selected different values of the beta coefficient, i.e. low (0.5) and high (0.9). Figure 3.1 (under normality) and Figure 3.4 (under mixture of normals) are constructed under the same set of parameter values and it is interesting to compare between the two cases. As in the case of normality, we see that in Figure 3.4 the bias correction of the estimators works in most cases and the results are satisfactory. In Figure 3.3, the corrected bias is again under the bias of the MLEs, indicating that the theoretical results correct the bias, under the assumptions made.

FIGURE 3.1: First- and full-step bias-correction



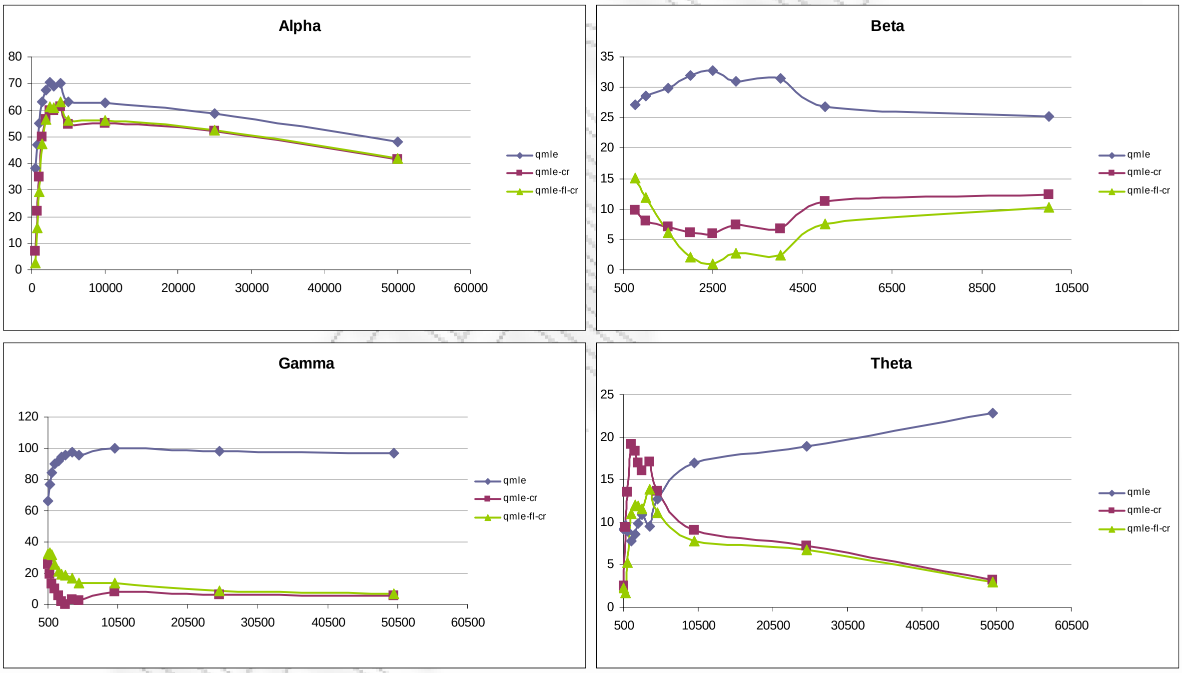
Note: $\alpha_0 = 0.1, \beta_0 = 0.9, \gamma_0 = 0.7, \theta_0 = -0.4$, under normality
1-step correction denoted by "mle-cr", full-step correction by "mle-fl-cr" (the same applies to all graphs)

FIGURE 3.2: First- and full-step bias-correction



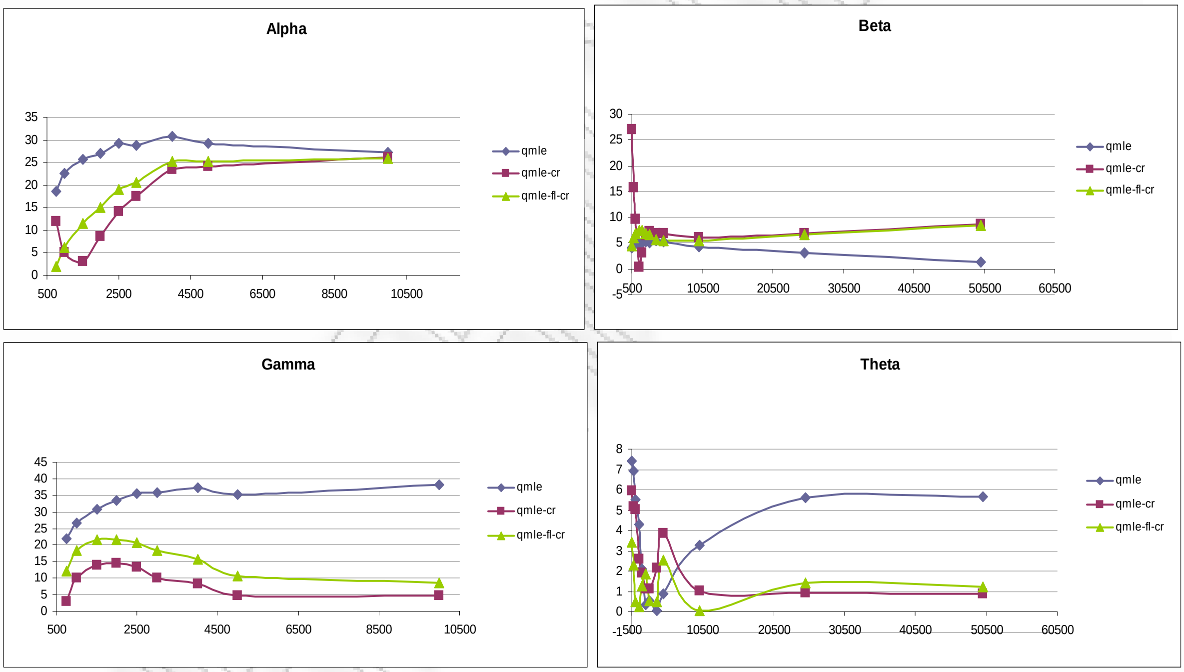
Note: $\alpha_0 = -0.1, \beta_0 = 0.9, \gamma_0 = 0.6, \theta_0 = -0.2,$ under normality

FIGURE 3.3: First- and full-step bias-correction



Note: $\alpha_0 = 0.5, \beta_0 = 0.5, \gamma_0 = 0.8, \theta_0 = -0.5,$ under mixture of normals with $p = 0.1$: $N(0.01, 9)$: $E\left(z^3\right) = 0.0266,\ E\left(z^4\right) = 24.334$

FIGURE 3.4: First- and full-step bias-correction



Note: $\alpha_0 = 0.1, \beta_0 = 0.9, \gamma_0 = 0.7, \theta_0 = -0.4,$ under mixture of normals with $p = 0.1$: $N(0.01, 9)$: $E\left(z^3\right) = 0.0266$, $E\left(z^4\right) = 24.334$

3.3.5 Theoretical Skewness and the Edgeworth Expansion

This section provides the theoretical skewness and the Edgeworth expansion of the estimators. After recentering the standardized estimator, we have:

$$P_i = \sqrt{T}\left(\widehat{\varphi}_i - \varphi_i\right) = A_i + \frac{B_i}{\sqrt{T}},$$

where

$$\begin{aligned} A_i &= -c^{ij} Z_j, \\ B_i &= C_i + D_i, \end{aligned}$$

with

$$\begin{aligned} C_i &= c^{ij} c^{kl} \left\{ Z_{jk} Z_l - E\left(Z_{jk} Z_l\right) \right\}, \quad \text{and} \\ D_i &= -c^{ij} c^{kl} c^{mn} c_{j\ln} \left\{ Z_k Z_m - E\left(Z_k Z_m\right) \right\}/2, \end{aligned}$$

where $Z_j = T^{-1/2}\mathcal{L}_j$ and $Z_{jk} = T^{-1/2}\left\{\mathcal{L}_{jk} - E\left(\mathcal{L}_{jk}\right)\right\}.$

Thus, the skewness is given by

$$skewness \approx E\left(A_{i_1} A_{i_2} A_{i_3}\right) + \frac{1}{\sqrt{T}} \left\{ E\left(A_{i_1} A_{i_2} B_{i_3}\right) + E\left(B_{i_1} A_{i_2} A_{i_3}\right) + E\left(A_{i_1} B_{i_2} A_{i_3}\right) \right\},$$

and generally we have

$$skewness \approx E\left(A_{i_1} A_{j_2} A_{k_3}\right) + \frac{1}{\sqrt{T}} \left\{ E\left(A_{i_1} A_{j_2} B_{k_3}\right) + E\left(B_{i_1} A_{j_2} A_{k_3}\right) + E\left(A_{i_1} B_{j_2} A_{k_3}\right) \right\},$$

where

$$\begin{aligned} E\left(A_{i_1} A_{i_2} A_{i_3}\right) &= -c^{i_1 j_1} c^{i_2 j_2} c^{i_3 j_3} E\left(Z_{j_1} Z_{j_2} Z_{j_3}\right), \\ E\left(A_{i_1} A_{i_2} C_{i_3}\right) &= c^{i_1 j_1} c^{i_2 j_2} c^{i_3 j_3} c^{k_3 l_3} E\left[Z_{j_1} Z_{j_2} \left\{ Z_{j_3 k_3} Z_{l_3} - E\left(Z_{j_3 k_3} Z_{l_3}\right) \right\}\right] \\ E\left(A_{i_1} A_{i_2} D_{i_3}\right) &= -\frac{1}{2} c^{i_1 j_1} c^{i_2 j_2} c^{i_3 j_3} c^{k_3 l_3} c^{m_3 n_3} c_{j_3 l_3 n_3} E\left[Z_{j_1} Z_{j_2} \left\{ Z_{k_3} Z_{m_3} - E\left(Z_{k_3} Z_{m_3}\right) \right\}\right]. \end{aligned}$$

Now

$$E\left(Z_{j_1} Z_{j_2} Z_{j_3}\right) = \frac{1}{\sqrt{T}} \kappa_{23} \tau_{j_1, j_2, j_3}/8,$$

where

$$\kappa_{23} = E\left\{\left(z_t^2 - 1\right)^3\right\}.$$

The following moments converge to finite limits as $T \to \infty$:

$$c_{i,j,k} = \frac{1}{T} E\left(\mathcal{L}_i \mathcal{L}_j \mathcal{L}_k\right) = \frac{1}{8}\left[\kappa_{23}\tau_{i,j,k} + (\kappa_4 + 2)\left(\tau^{zz}_{k;i,j} + \tau^{zz}_{i;k,j} + \tau^{zz}_{j;i,k}\right)\right]$$

Moreover,

$$E\left[Z_{j_1} Z_{j_2}\left\{Z_{j_3 k_3} Z_{l_3} - E\left(Z_{j_3 k_3} Z_{l_3}\right)\right\}\right] = c_{j_1,l_3} c_{j_3 k_3, j_2} + c_{j_1, j_3 k_3} c_{, j_2, l_3}$$

and

$$E\left[Z_{j_1} Z_{j_2}\left\{Z_{k_3} Z_{m_3} - E\left(Z_{k_3} Z_{m_3}\right)\right\}\right] = c_{j_1,m_3} c_{j_2,k_3} + c_{j_1,k_3} c_{j_2,m_3}.$$

For the Edgeworth expansion, we consider a vector $\mathbf{P} \in \mathbb{R}^4$ of standardized estimators that satisfies a joint Edgeworth Expansion. Suppose that the first three mixed cumulants of $\mathbf{P}$ satisfy

$$\begin{aligned} cum\left(\mathbf{P}_i\right) &= \frac{b_i}{\sqrt{T}} + o\left(T^{-1/2}\right), \quad \textit{where } b_i\textit{: bias} \\ cum\left(\mathbf{P}_i, \mathbf{P}_j\right) &= \frac{var_i}{\sqrt{T}} + o\left(T^{-1/2}\right), \quad \textit{where } var_i\textit{: variance} \\ cum\left(\mathbf{P}_i, \mathbf{P}_j, \mathbf{P}_k\right) &= \frac{skw_i}{\sqrt{T}} + o\left(T^{-1/2}\right), \quad \textit{where } skw_i\textit{: skewness.} \end{aligned}$$

Then for any Borel set B,

$$\Pr\left(\mathbf{P} \in B\right) = \int_B \phi_{o,\Omega}(y)\left\{1 + \sum_{i=1}^4 \frac{b_i}{\sqrt{T}} H_i(y) + \sum_{i=1}^4 \sum_{j=1}^4 \sum_{k=1}^4 \frac{skw_i}{6\sqrt{T}} H_{ijk}(y)\right\} dy + o\left(T^{-1/2}\right),$$

where $H_i(y)$ and $H_{ijk}(y)$ are the multivariate Hermite polynomials of first and third degree.

3.4 Conclusions

In this chapter we study the asymptotic properties of the MLEs and QMLEs in the EGARCH$(1,1)$ model of Nelson (1991) [71]. In the current context, we present analytic derivatives both of the log-likelihood and the log-variance functions and also their expected values. We further develop theoretical bias approximations for the estimators of the model parameters and we find conditions for the second-order stationarity of the log-variance derivatives. The theoretical results in this chapter can be used to bias-correct the QMLEs in practice directly. In small or moderate-sized samples, a bias correction could be appreciable and it is helpful to have a rough estimate of its size.

One might consider the case of the EGARCH-Mean model and employ the results presented here. It is well known that this model examines an important issue previously investigated in the economics and finance literature, namely the relation between the level of market risk and required return. To account for this relation, one might use the following model (instead of 3.1):

$$y_t = \lambda h_t + u_t, \tag{3.8}$$

where λ is the risk premium parameter. The justification for including λh_t is pragmatic: a number of researchers (for example, French, Schwert and Stambaugh 1987 [43] and Chou 1987 [25]) have found a statistically significant positive relation between conditional variance and excess returns on stock market indices. This idea might exist as an individual research project. The theoretical results under this new specification are available upon request by the author of this thesis.

Another interesting topic would be the investigation of necessary and sufficient conditions for the existence and validity of the Edgeworth approximations in this context[8].

This chapter is available as a Discussion Paper of Demos and Kyriakopoulou (2010a) [35].

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[8]The validity of the Edgeworth expansion could be examined along the lines of Corradi and Iglesias (2008) [27].

Rimini, the 10th International Vilnius Conference on Probability Theory and Mathematical Statistics, Vilnius, the 9th Conference on Research on Economic Theory & Econometrics (CRETE), Tinos, the Conference in Honour of Sir David F. Hendry, St. Andrews, Scotland, the European Meeting of Statisticians (EMS), Piraeus, the 4th International Conference on Computational and Financial Econometrics (CFE'10), London and the seminar participants at University of Piraeus and Athens University of Economics and Business. We also thank the participants at the Bank of Greece Workshop, Department of Economic Research. Financial support from the Basic Research Funding Program (PEVE 2) is gratefully acknowledged.

Chapter 4

Asymptotic Normality of the QML Estimators in the EGARCH(1,1) Model

4.1 Introduction

Over the last years, a lot of considerable attention has been given to the analysis of conditional heteroskedasticity and more specifically to the theoretical properties of the estimators in such models. One of the most popular models in applied financial work is the Exponential GARCH (EGARCH) model of Nelson (1991) [71], for which the investigation of the asymptotic properties of the estimators still remains unsolved. To model the returns of speculative assets, it is particularly important to derive the asymptotic theory and be able then to make statistical inference. While the asymptotic theory in the ARCH model of Engle (1982) [40] and the GARCH specification of Bollerslev (1986) [20] has been studied in the papers of Weiss (1986) [90] and, Lee and Hansen (1994) [61], Lumsdaine (1996) [66], respectively, in the case of the EGARCH model, only recently Zaffaroni (2009) [91] established the consistency and asymptotic normality of the Whittle estimates. The paper by Berkes, Horvath and Kokoszka (2003) [14] is a generalization of the above papers for the GARCH models, under minimal assumptions.

Recently, Hamadeh and Zakoian (2011) [51] established the asymptotic properties of LS and QML estimators for a class of nonlinear GARCH processes, under mild conditions[1].

The procedure most often used for estimating the conditional heteroskedastic models is the maximization of a likelihood function. The actual implementation of the maximization procedure requires an explicit assumption for the conditional density. The most commonly employed distribution in the literature is the normal, but the assumption of conditional normality for the standardized innovations is difficult to be justified in many empirical applications. For this reason, the method that we employ is the quasi-maximum likelihood estimation (QMLE), by maximizing a Gaussian quasi-log-likelihood function under the auxiliary assumption of an *iid* distribution for the standardized innovations.

In this chapter we study the asymptotic properties of the quasi-maximum likelihood estimators in the EGARCH$(1,1)$ process of Nelson (1991) [71]. The EGARCH$(1,1)$ model is then defined by the equations

$$X_t = Z_t \sigma_t \tag{4.1}$$

and

$$\log \sigma_t^2 = \alpha + \beta \log \sigma_{t-1}^2 + g\left(Z_{t-1}\right), \tag{4.2}$$

where $\alpha \in \mathbb{R}$ and $|\beta| < 1$. The real-valued function $g\left(Z_t\right)$ is given by

$$g\left(Z_t\right) = \gamma Z_t + \delta\left[\left|Z_t\right| - E\left|Z_t\right|\right], \tag{4.3}$$

where $\gamma, \delta \in \mathbb{R}$ are the parameters for the asymmetries that the model captures. We also assume throughout this chapter that

$$\left\{Z_i, -\infty < i < \infty\right\} \text{ are independent, identically distributed random variables,} \tag{4.4}$$

with mean zero and variance unity.

[1] Their proofs follow the same lines as in Francq and Zakoian (2004) [42].

Straumann and Mikosch (2006) [83] give an almost sure representation of $\log \sigma_t^2$ by recursive substitution, which is

$$\log \sigma_t^2 = \alpha (1-\beta)^{-1} + \sum_{k=0}^{\infty} \beta^k \left(\gamma Z_{t-1-k} + \delta \left|Z_{t-1-k}\right|\right). \tag{4.5}$$

The notion of invertibility plays an important role in the investigation of the asymptotic properties of the estimators in time series models. Invertibility is necessary for the observed likelihood function to be well-behaved asymptotically without exploding nor converging toward zero for any admissible parameter value. To this end, we provide below with the result that was obtained by Straumann and Mikosch (2006) [83], which gives the sufficient condition for the invertibility of the EGARCH(1, 1) model to hold. This is summarized in the next lemma, but before we give a useful definition, needed for this lemma:

Definition 4.1 (Straumann and Mikosch 2006 [83], section 2.5)**.** Let (E, d) be a Polish space equipped with its Borel σ-algebra $B(E)$. A map $\phi : E \to E$ is called Lipschitz if

$$\Lambda(\phi) := \sup_{x,y\in E, x\neq y} \left(\frac{d(\phi(x), \phi(y))}{d(x,y)}\right)$$

is finite.

Lemma 4.2 (Straumann and Mikosch 2006 [83], p. 2469)**.** *Assume* $0 \leq \beta < 1$ *and* $\gamma Z + \delta |Z| \geq 0$ *for all* $Z \in \mathbb{R}$ *(that is, the squared volatility should be nondecreasing on the positive real line and nonincreasing on the negative real line). The condition* $E[\Lambda(\phi_0)] < 0$*, which implies invertibility, is then given by*

$$E\left[\log\max\left\{\begin{array}{c}\beta, 2^{-1}\exp\left(2^{-1}\sum_{k=0}^{\infty}\beta^k\left(\gamma Z_{t-1-k}+\delta\left|Z_{t-1-k}\right|\right)\right)\\ \times\left(\gamma Z_0+\delta\left|Z_0\right|\right)-\beta\end{array}\right\}\right] < 0, \tag{4.6}$$

where $\Lambda(\phi_0)$ *is the Lipschitz coefficient.*

For the proof, see Straumann and Mikosch (2006) [83], p. 2468-2469.

Remark. It seems impossible to have an explicit representation for σ_t^2 in terms of past observations, as the above condition is difficult in practice to be verified. However, Straumann and Mikosch (2006) [83] end up with a simpler condition in the case of $\beta = 0$, which is practically feasible; this means that we can verify that there exist invertible EGARCH models. The last condition is given by the following summarizing result.

Lemma 4.3 (Straumann and Mikosch 2006 [83], p. 2469)**.** *In case of $\beta = 0$, the above condition (4.6) becomes*

$$-\log 2 + \left(\frac{\delta}{2}\right) E\left|Z_0\right| + E\left[\log\left(\left(\gamma Z_0 + \delta\left|Z_0\right|\right)\right)\right] < 0.$$

Remark. If we assume $\delta \leq 1$, the latter implies the above condition.

The invertibility in the EGARCH model has been an important matter in other papers, see e.g. Aue, Berkes and Horvath (2006) [8] in which they define as $\Lambda(x) = \log x$ and the invertibility implies that $\Lambda^{-1}(x)$ exists, in order to solve for σ_t^2. In fact, the EGARCH process is included in the general framework of the Augmented GARCH model, introduced by Duan (1997) [38], for which its asymptotic and dependence properties have been studied by Aue, Berkes and Horvath (2006) [8], Hormann (2008) [53] and Berkes, Hormann and Schauer (2010) [15], to state a few papers.

Straumann and Mikosch (2006) [83] showed that in the case of the EGARCH$(1,1)$ sequence, (4.1), (4.2) and (4.3) have a unique stationary solution if and only if

$$|\beta| < 1 \quad \text{and} \quad E\left[\log^{+}\left(\alpha + \gamma Z_0 + \delta\left|Z_0\right|\right)\right] < \infty, \tag{4.7}$$

where for instance, $log^{+}x = \log\left(\max\{x, 1\}\right)$[2]. This result can be summarized in the following Theorem, which is due to Aue, Berkes and Horvath (2006) [8]:

Theorem 4.4 (Theorems 2.1 and 2.3 of Aue, Berkes and Horvath 2006 [8])**.** *(i) Given the specification of (4.1)-(4.4) and that*

$$E\left[\log^{+}\left(\alpha + \gamma Z_0 + \delta\left|Z_0\right|\right)\right] < \infty,$$

if

$$\beta < 1, \tag{4.8}$$

then the infinite sum in eq. (4.5) is absolutely convergent with probability one.

(ii) We assume that (4.1)-(4.4) and (4.7) are satisfied. If (4.8) holds, then (4.5) is the only stationary solution of (4.1) and (4.2).

[2] See also Aue, Berkes and Horvath (2006) [8].

Remark. If $\beta > 0$ and $P\{(\alpha + \gamma Z_0 + \delta |Z_0|) \geq 0\} = 1$, then $\beta < 1$ is necessary and sufficient for the existence of a stationary solution of the EGARCH equations, see, for instance, Aue, Berkes and Horvath (2006) [8].

Remark. Aue, Berkes and Horvath (2006) [8] presented the general case of $\Lambda\left(\sigma_t^2\right)$, specifying by real-valued functions[3], where for example $\Lambda(x) = \log(x)$ in the case of the EGARCH process.

Remark. Straumann and Mikosch (2006) [83] obtained a stationary approximation to the log-variance process and its first and second derivatives, with the stochastic recurrence equation (SRE)[4] approach, in order to apply the Ergodic Theorem for sequences of continuous-valued random functions in a Banach space. This is really important if someone wants to tackle the limit properties of the estimators and this arises from the fact that the log-variance is generally nonstationary because it just represents an estimate. A stationary and ergodic sequence when is available, can be used in order to apply the Ergodic Theorem, which is one of the main devices in this chapter.

Remark. Another paper that deals with the existence of solutions in the general framework of the GARCH specification, is that of Carrasco and Chen (2002) [23]. In their paper the mixing properties of the sequences are also derived, which yield the weak convergence as well as the approximation of partial sums of the squares of the observed process. But the existing theory on dependence structures assumes restrictive moment and smoothness conditions. This is the main difference between the paper by Aue et al. (2006) [8] and that of Carrasco and Shen (2002) [23]. The former shows that these conditions can be weakened to logarithmic moment conditions.

In fact, (4.2) can be defined as a stochastic recurrence equation of the form treated in Straumann and Mikosch (2006) [83][5], i.e.

$$\begin{aligned} \log h_{t+1} &= g_\theta\left(X_t, \log h_t\right) \\ &= \alpha + \beta \log h_t + \left(\gamma X_t + \delta\left|X_t\right|\right) \exp\left(-2^{-1} \log h_t\right), \end{aligned} \tag{4.9}$$

[3] The specification of the volatility is given by equation (1.2) in their paper.

[4] Sufficient conditions for the existence of a stationary solution of this form can be found in Diaconis and Freedman (1999) [37].

[5] This appoach is a classical one as it was introduced by Bougerol (1993) [21], in which conditions of Lyapunov type for the existence of a stationary solution are given.

where the parametric family $\{g_\theta|\theta \in \Theta\}$ of nonnegative functions on $\mathbb{R} \times [0, \infty)$ fulfills certain regularity conditions. Here, $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^d$ is the parameter vector of interest, i.e. $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \delta)'$. The process h_t is the process defined by the filtered variance, i.e.

$$\log h_t = \alpha_0 + \gamma_0 Z^*_{t-1} + \delta_0 \left|Z^*_{t-1}\right| + \beta_0 \log h_{t-1},$$

where $Z^*_t = \frac{X_t}{\sqrt{h_t}}$ and

$$Z^*_0 = 0, \qquad \log h_0 = \frac{\alpha_0}{1-\beta_0}.$$

Thus, the filtered variance approximates σ_t^2, which is unobserved. Moreover, the initial values can be shown to be asymptotically irrelevant (see, for instance, Lumsdaine 1996 [66], Lemma 6, p. 587, as well as Dahl and Iglesias 2008 [29], Lemma 1, and Bardet and Wintenberger 2009 [11], p. 2731). The reader is referred to Straumann and Mikosch (2006) [83], Theorem 2.8 in p. 2458 and Theorem 2.10 in p. 2459, which are the key results due to Bougerol (1993) [21] about stationary solutions of SREs used throughout this chapter.

In the case of the EGARCH(1, 1) model, the classical estimation theory considers an approximate conditional log-likelihood function. Given some proper initial values, we obtain a good approximation to the conditional Gaussian log-likelihood, as follows:

$$\ell\left(\alpha, \beta, \gamma, \delta \middle| Z_0, \sigma_0^2\right) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\log\left(\sigma_t^2\right) - \frac{1}{2}\sum_{t=1}^{n}\frac{X_t^2}{\sigma_t^2}. \tag{4.10}$$

The first order conditions are recursive and consequently do not have explicit solutions. The likelihood function is derived as though the errors are conditionally normal and is still maximized at the true parameters. Having specified the log-likelihood function, the quasi maximum likelihood estimator is then defined as

$$\widehat{\theta_n} = \arg\max_{\theta\in\Theta} \frac{1}{n}\sum_{t=1}^{n}\ell(\theta). \tag{4.11}$$

The parameter space is of the form

$$\Theta = \mathbb{R} \times [0, 1) \times D,$$

where $D = \left\{(\gamma, \delta)' \in \mathbb{R}^2 \mid \gamma \in \mathbb{R}, \delta \geq |\gamma|\right\}$.

The EGARCH model has gained a lot of considerable attention through the last decade. More specifically, Surgailis and Viano (2002) [85] studied the covariance structure and dependence properties of the EGARCH model and they showed that normalized partial sums of powers of the observed process tend to fractional Brownian motion. Recently, Berkes, Hormann and Schauer (2010) [15] consider weakly $\mathcal{M}$-dependent processes and as an example they study the case of the Augmented GARCH sequences that include also the EGARCH model. Under some technical conditions stated in Hormann (2008) [53], one can show that Augmented GARCH sequences are weakly $\mathcal{M}$-dependent in L^p-norm with exponentially fast decaying rate. The following definition is due to Berkes, Hormann and Schauer (2010) [15]:

Definition 4.5 (Weakly $\mathcal{M}$-dependent process)**.** Let $\{Y_k, k \in \mathbb{Z}\}$ be a stochastic process, let $p \geq 1$ and let $\delta(m) \to 0$. We say that $\{Y_k, k \in \mathbb{Z}\}$ is weakly $\mathcal{M}$-dependent in L^p with rate $\delta(\cdot)$ if:

(A) For any $k \in \mathbb{Z}$, $m \in \mathbb{N}$ one can find a random variable $Y_k^{(m)}$ with finite p-th moment such that

$$\left\| Y_k - Y_k^{(m)} \right\|_p \leq \delta(m).$$

(B) For any disjoint intervals $I_1, ..., I_r$ $(r \in \mathbb{N})$ of integers and any positive integers $m_1, ..., m_r$, the vectors $\left\{ Y_j^{(m_1)}, j \in I_1 \right\}, ..., \left\{ Y_j^{(m_r)}, j \in I_r \right\}$ are independent provided $d(I_k, I_l) > \max\{m_k, m_l\}$ for $1 \leq k \leq l \leq r$.

Remark. In Hormann (2008) [53], an approximation of the original random variables is deduced by an m-dependent sequence (see Lemma 2 in Hormann 2008 [53], p. 548, for the L^2-approximation). Truncating an infinite series, the new sequence converges now very fast and hence considering the finite sums will only cause a small error. Consequently, their method yields sharp convergence rates to the normal law, using a Berry-Essen bound. It becomes clear that m-dependence, rather than mixing, is the crucial structural property required in order to study the asymptotics of augmented GARCH variables.

Notation. In the sequel, we assume that $K \subset \mathbb{R}^d$ is a compact set. Then $\mathbb{C}\left(K, \mathbb{R}^{d'}\right)$ denotes the space of continuous $\mathbb{R}^{d'}-$valued functions on K, which is endowed with the supremum norm, i.e.

$$\|\omega\|_K = \sup_{s \in K} |\omega(s)|, \qquad \omega \in \mathbb{C}\left(K, \mathbb{R}^{d'}\right),$$

where $|\cdot|$ denotes the Euclidean norm of the vector $\omega(s)$.

In this chapter we aim at establishing the asymptotic normality of the QMLEs in the EGARCH$(1,1)$ model. Previous research on this topic includes the paper by Straumann and Mikosch (2006) [83], in which they prove the strong consistency and asymptotic normality for some asymmetric models, such as the EGARCH model of Nelson (1991) [71] and the Asymmetric GARCH (AGARCH) model[6]. But they don't prove the asymptotic normality for the EGARCH$(1,1)$ model, only for the model of a lower order, i.e. for the case of $\beta = 0$. This is presented as a discussion only in the monograph by Straumann [84]. Moreover, in a recent paper, Zaffaroni (2009) [91] estimates the EGARCH parameters with Whittle methods and the asymptotic distribution theory of these estimators is established. Furthermore, Dahl and Iglesias (2008) [29] analyzed the limiting properties, in terms of consistency and asymptotic normality, of the estimated parameters in an exponential-type model, which is related but in many aspects different to the traditional EGARCH model of Nelson. The investigation of the asymptotic properties of Nelson's model still remains unsolved.

In this chapter for the first time we provide analytic results for the second-order stationarity in the EGARCH$(1,1)$ process and we give higher-order moment conditions resulting from this analysis. These results are competitive with previous research, as we are now able to establish the asymptotic theory for Nelson's model. We mainly obtain tractable sufficient conditions that guarantee the integrability of the supremum norms of the log-variance derivatives, in a neighborhood around the true parameter vector.

The chapter is organized as follows. First, we present the first and second order log-likelihood derivatives and the conditions for the second-order stationarity of the log-variance derivatives. In section 3, we proceed with the asymptotic theory in the EGARCH$(1,1)$ process and we present our main Theorem. The main analysis and the proofs of important lemmas are given in Section 4. Last, we conclude. The proofs of the Theorem and some auxiliary lemmas appear in the Appendix.

[6] for further information on the last model the reader is referred to the monograph by Straumann (2005) [84].

4.2 The first and second order log-likelihood derivatives

Employing the method of Straumann and Mikosch (2006) [83], it can be found that the SRE approach is also useful for the treatment of the first and second derivatives of the h_t sequence.

The first order derivatives of the log-likelihood function.

$$\begin{aligned}
\frac{\partial}{\partial \alpha} l_n(\theta) &\equiv \mathcal{L}_{1t}(\theta) = \frac{1}{2}\sum_{t=1}^{n}\left(\frac{X_t^2}{h_t}-1\right)\frac{\partial h_t/\partial\alpha}{h_t}, \\
\frac{\partial}{\partial \beta} l_n(\theta) &\equiv \mathcal{L}_{2t}(\theta) = \frac{1}{2}\sum_{t=1}^{n}\left(\frac{X_t^2}{h_t}-1\right)\frac{\partial h_t/\partial\beta}{h_t}, \\
\frac{\partial}{\partial \gamma} l_n(\theta) &\equiv \mathcal{L}_{3t}(\theta) = \frac{1}{2}\sum_{t=1}^{n}\left(\frac{X_t^2}{h_t}-1\right)\frac{\partial h_t/\partial\gamma}{h_t}, \\
\frac{\partial}{\partial \delta} l_n(\theta) &\equiv \mathcal{L}_{4t}(\theta) = \frac{1}{2}\sum_{t=1}^{n}\left(\frac{X_t^2}{h_t}-1\right)\frac{\partial h_t/\partial\delta}{h_t}.
\end{aligned}$$

and

Evaluated at the true parameter value. *Let* $\theta_0^* = \left(\beta_0 - \frac{1}{2}\gamma_0 Z_{t-1} - \frac{1}{2}\delta_0\left|Z_{t-1}\right|\right)$:

$$\begin{aligned}
\mathcal{L}_{1t}(\theta_0) &= \frac{1}{2}\sum_{t=1}^{n}\left(Z_t^2-1\right)\left[1+\theta_0^*\frac{\partial h_{t-1}/\partial\alpha}{h_{t-1}}|_{\theta_0}\right], \\
\mathcal{L}_{2t}(\theta_0) &= \frac{1}{2}\sum_{t=1}^{n}\left(Z_t^2-1\right)\left[\ln h_{t-1}+\theta_0^*\frac{\partial h_{t-1}/\partial\beta}{h_{t-1}}|_{\theta_0}\right], \\
\mathcal{L}_{3t}(\theta_0) &= \frac{1}{2}\sum_{t=1}^{n}\left(Z_t^2-1\right)\left[z_{t-1}+\theta_0^*\frac{\partial h_{t-1}/\partial\gamma}{h_{t-1}}|_{\theta_0}\right], \\
\mathcal{L}_{4t}(\theta_0) &= \frac{1}{2}\sum_{t=1}^{n}\left(Z_t^2-1\right)\left[g\left(z_{t-1}\right)+\theta_0^*\frac{\partial h_{t-1}/\partial\delta}{h_{t-1}}|_{\theta_0}\right].
\end{aligned}$$

The second order derivatives of the log-likelihood function. *For* $i,j \in \{\alpha,\beta,\gamma,\delta\}$:

$$\frac{\partial^2}{\partial i\partial j} l_T(\theta) = \frac{1}{2}\sum_{t=1}^{n}\left(\frac{X_t^2}{h_t}-1\right)h_{t;i,j} - \frac{1}{2}\sum_{t=1}^{n}\frac{X_t^2}{h_t}h_{t;i}h_{t;j},$$

where the log-variance derivatives are given by

$$
\begin{aligned}
h_{t;i} &= \frac{1}{h_t}\frac{\partial h_t}{\partial i}, \\
h_{t;i,j} &= -\left(\frac{1}{h_t}\frac{\partial h_t}{\partial i}\right)\left(\frac{1}{h_t}\frac{\partial h_t}{\partial j}\right) + \frac{1}{h_t}\frac{\partial^2 h_t}{\partial i \partial j}.
\end{aligned}
$$

4.2.1 Second-order stationarity

The existence, stationarity and ergodicity of the second order derivatives of the conditional variance are necessary so that the Taylor expansion of the first order derivatives of the log-likelihood is validated. Demos and Kyriakopoulou (2010a) [35] provide higher-order moment conditions for the second-order stationarity of the log-variance derivatives and products between them. We summarize these conditions in the following Proposition, which is due to Demos and Kyriakopoulou (2010a) [35] (see also Chapter 3 of this thesis):

Proposition 4.6. *Given*

a) $\left|\beta_0 - \frac{1}{2}\delta_0 E|Z|\right| < 1$

b) $\left|\beta_0^2 + \frac{1}{4}\gamma_0^2 + \frac{1}{4}\delta_0^2 - \delta_0\beta_0 E|Z| + \frac{1}{2}\delta_0\gamma_0 E(Z|Z|)\right| < 1$

and

c) $\left|\begin{array}{c} \beta_0^3 + \frac{3}{4}\beta_0\gamma_0^2 + \frac{3}{4}\beta_0\delta_0^2 - \frac{1}{8}\gamma_0\left(\gamma_0^2 + 3\delta_0^2\right)E\left(Z^3\right) - \frac{3}{2}\beta_0^2\delta_0 E|Z| \\ +\frac{3}{2}\beta_0\gamma_0\delta_0 E(Z|Z|) - \frac{1}{8}\delta_0\left(\delta_0^2 + 3\gamma_0^2\right)E|Z|^3 \end{array}\right| < 1,$

then

the second-order stationarity of all log-variance derivatives follows.

The proof is given analytically in the paper by Demos and Kyriakopoulou (2010a) [35].

4.3 Asymptotic Theory in the EGARCH(1,1)

In the related literature the EGARCH process has gained a lot of considerable attention regarding the asymptotic theory of its estimators. At this point, we recall the substantial work by Straumann and Mikosch (2006) [83], who proved the strong consistency of the QMLE in the EGARCH(1,1) model. The next Theorem has been proved in their paper.

Theorem 4.7 (Straumann and Mikosch 2006 [83], Theorem 5.1, p. 2477)**.** *Let* (X_t) *be a stationary EGARCH process with parameters* $\theta_0 = (\alpha_0, \beta_0, \gamma_0, \delta_0)'$ *such that* $(\gamma_0, \delta_0) \neq$

$(0,0)$. *Suppose the distribution of* Z_0 *is not concentrated in two points. Let* K *be a compact set with* $\theta_0 \in K$ *and such that*

$$E\left(\log \|\lambda_0\|_K\right) < 0,$$

where λ_0 *is given by*

$$\lambda_0 = \max\left(\beta, 2^{-1}\exp\left(-2^{-1}m\right)\left(\gamma X_0 + \delta\left|X_0\right|\right) - \beta\right),$$

as $\sup_K \lambda_0 = \Lambda\left(\Phi_0\right)$, *the Lipschitz coefficient, and* $m = \inf_{\theta\in K} \alpha\left(1-\beta\right)^{-1}$.
Then, the QMLE $\widehat{\theta}_n$ *is strongly consistent, i.e.*

$$\widehat{\theta}_n \overset{a.s.}{\rightarrow} \theta_0, \qquad n \rightarrow \infty.$$

Remark. With the previous Theorem, we define the set K in such an appropriate way so as to verify the condition $E\left(\log \|\lambda_0\|_K\right) < 0$, which is the sufficient condition for the invertibility of the model to hold. The invertibility assures that the nonstationary $\log h_t$ can be approximated by the unique stationary solution of the model, which is also ergodic.

Remark. Zaffaroni (2009) [91] proved the almost sure consistency of the Whittle estimator in the EGARCH process, see for instance Theorem 1 in his paper, p. 192.

Passing to the asymptotic normality of the estimators, which is the main task in this chapter, first of all, we recall some important assumptions and results from the paper by Straumann and Mikosch (2006) [83], from the section of the asymptotic normality of the Quasi Maximum Likelihood Estimator.

Assumption 4.8 (Straumann and Mikosch 2006 [83]). Let

(N1). The assumptions C1-C4 (see Straumann and Mikosch, 2006 [83], p. 2473) which imply consistency are satisfied and the true parameter θ_0 lies in the interior of the compact set K.

(N2). The assumptions D1-D3 of Proposition 6.2 in Straumann and Mikosch (2006) [83] are met so that h_t is twice continuously differentiable on K.

(N3). The following moment conditions hold:

$$(i) \qquad EZ_0^4 < \infty,$$

$$(ii) \qquad E\left(\left|\log\left(h_0\right)'\left(\theta_0\right)\right|^2\right) < \infty,$$

$$(iii) \qquad E\left\|l_0'\right\|_K < \infty,$$

$$(iv) \qquad E\left\|l_0''\right\|_K < \infty.$$

(N4). The components of the vector $\frac{\partial g}{\partial \theta}\left(X_0, \sigma_0^2\right)|_{\theta=\theta_0}$ are linearly independent random variables.

Remark. We also assume that the stationarity and ergodicity of $(\log h_t)'$ and $(\log h_t)''$ hold, as it has been proved by Straumann and Mikosch (see Propositions 6.1 and 6.2).

Remark. The condition (N3ii) above is given analytically in Lemma 4.13 in this chapter, in which we obtain analytic results for the existence of the squares of the log-variance derivatives, under their second-order stationarity.

Remark. Hamadeh and Zakoian (2011) [51] in their paper established the asymptotic normality of the QMLE for a class of nonlinear GARCH processes. Due to the fact that they employ the traditional method to prove the asymptotic theory, they bound the expected norm of the third derivative of the log-likelihood uniformly in a neighborhood of the parameter space (see, for instance, their proof of Theorem 2.2, point iii, in p. 499). In our paper, we omit that point, as we use the Ergodic Theorem for continuous-valued random functions. This is really useful as we avoid more technical proofs and we focus only on the uniform boundedness of the second, not the third, derivative of the log-likelihood function.

Next, we state our results which are shedding light on the asymptotic normality for the general EGARCH model of order 1. Our contributions are on the bounds and moment inequalities that must hold in order to establish our Theorem, which appears below. We are presenting these technical conditions in the following group of assumptions. Before, some notation that is used:

Notation. We define $\overline{\delta} = \max\left(\sup\left(\gamma+\delta\right), \sup\left(\delta-\gamma\right)\right) : \gamma x + \delta\left|x\right| \leq \overline{\delta}\left|x\right|, \quad \forall x \in \mathbb{R}$, as in the monograph by Straumann (2005) [84]. Also, let $\underline{m} = \inf\left\{\frac{\alpha}{1-\beta}\right\}$.

The following technical conditions are sufficient for the asymptotic normality to hold.

Assumption 4.9. Let the model parameters be such that the following conditions are satisfied:

Condition A.

$$2^{-1}\overline{\delta}\exp\left(2^{-1}\left[\left(\frac{\alpha}{(1-\beta)}-\underline{m}\right)\right]\right)E\left[\left|Z_0\right|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right]<1.$$

Condition B.

$$4^{-1}\overline{\delta}^2\exp\left(\alpha\frac{1}{1-\beta}-\underline{m}\right)E\left[Z_0^2\exp\left(\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right]<1.$$

Condition C.

$$8^{-1}\overline{\delta}^3\exp\left[\frac{3}{2}\left(\alpha\frac{1}{1-\beta}-\underline{m}\right)\right]E\left[\left|Z_0\right|^3\exp\left(\frac{3}{2}\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right]<1.$$

Condition A is verified by evaluating the expectation under some further distributional assumption about $\{Z_t\}$ and next redefining the parameter space in such a way in order the condition A to be satisfied. Nelson (1991) [71] proposed to use the Generalized Error Distribution (GED) for the errors normalized to have a mean of zero and a variance of one, which includes the normal distribution as a special case, some more fat tailed than the normal (e.g. the double exponential) and some more thin tailed (e.g. the uniform). Using some relations for the gamma function that appear in Davis (1965) [32] and employing the formula 3.462 #1 in Gradshteyn and Ryzhik (1980) [48], we have the next result for k positive:

$$E\left[\left|Z\right|\exp\left(k\left|Z\right|\right)\right]=\lambda 2^{1/v}\sum_{j=0}^{\infty}\left[k\lambda 2^{1/v}\right]^j\frac{\Gamma\left[\left(2+j\right)/v\right]}{\Gamma\left(1/v\right)\Gamma\left(j+1\right)},$$

where $\Gamma(\cdot)$ is the gamma function, $\lambda \equiv \left[2^{(-2/v)}\Gamma\left(1/v\right)/\Gamma\left(3/v\right)\right]^{1/2}$ is the dispersion of the distribution and v is a tail-thickness parameter. When $v=2$, Z has a standard normal distribution, for $v<2$, the distribution of Z has thicker tails than the normal (i.e. when $\upsilon = 1$, Z has a double exponential distribution) and for $\upsilon > 2$, the distribution of Z has thinner tails than the normal (i.e. for $\upsilon = \infty$, Z is uniformly distributed on the interval $\left[-3^{1/2}, 3^{1/2}\right]$). In our analysis we are interested in $k = \frac{1}{2}\frac{1}{1-\beta}\bar{\delta}$.

For a double exponential distribution of the errors (i.e. for $\upsilon = 1$), the condition A above is satisfied if

$$\frac{1}{1-\beta}\bar{\delta} < 2\sqrt{2} \quad and \quad \bar{\delta} \in (-\infty, 0.16889) \cup (0.47369, \infty),$$

for the EGARCH coefficient to be 0.9 and the α parameter close to 0.1. Under the normal distribution (i.e. by letting $\upsilon = 2$), we have

$$\frac{1}{1-\beta}\bar{\delta} < 2\sqrt{2} \quad and \quad \bar{\delta} \in (-\infty, 0.216\,45).$$

The remaining conditions are verified in a similar way.

Theorem 4.10 (Asymptotic Normality of the QMLEs)**.** *Under Assumptions 4.8, 4.9 and those of Lemma 4.13 below so that the first derivative of the likelihood function to have finite variance, the QMLE $\widehat{\theta}$ is asymptotically normal as $n \to \infty$, i.e.*

$$\sqrt{n}\left(\widehat{\theta} - \theta_0\right) \xrightarrow{d} \mathcal{N}\left(0, F(\theta_0)^{-1} G(\theta_0) F(\theta_0)^{-1}\right),$$

where $F(\theta_0)$ and $G(\theta_0)$ are defined as

$$\begin{aligned} F(\theta_0) &= -2^{-1} E\left[\left((\log h_0(\theta_0))'\right)^T (\log h_0(\theta_0))'\right], \\ G(\theta_0) &= 4^{-1} E\left(Z_0^4 - 1\right) E\left[\left((\log h_0(\theta_0))'\right)^T (\log h_0(\theta_0))'\right]. \end{aligned}$$

The QMLE has covariance matrix:

$\mathbf{V}_0 = F_0^{-1} G_0 F_0^{-1} = 4^{-1} E\left(Z_0^4 - 1\right) E\left[\left((\log h_0)'(\boldsymbol{\theta}_0)\right)^T (\log h_0)'(\boldsymbol{\theta}_0)\right]^{-1} =$
$= 4^{-1} E\left(Z_0^4 - 1\right)\left(1 - \beta_0^2 - \frac{1}{4}E\left(\gamma_0^2 + \delta_0^2 + 2\gamma_0\delta_0 E\left(Z_0\left|Z_0\right|\right)\right) - 2\beta_0\left(1, \frac{\alpha_0}{1-\beta_0}, 0, E\left|Z_0\right|\right)\right) \times$
$\left\{\mathbf{U}_0 - \mathbf{W}_0\right\}^{-1}$,

where

$\mathbf{U}_0 = E\left[(1, \log h_0, Z_0, |Z_0|)^T (1, \log h_0, Z_0, |Z_0|)\right]$,

$\mathbf{W}_0 = \left(\delta_0 E\left|Z_0\right|, \delta_0 \frac{\alpha_0}{1-\beta_0} E\left|Z_0\right|, \gamma_0 + \delta_0 E\left(Z_0\left|Z_0\right|\right), \gamma_0 E\left(Z_0\left|Z_0\right|\right) + \delta_0\right)^T \left[\left(1, \frac{\alpha_0}{1-\beta_0}, 0, E\left|Z_0\right|\right) + \beta_0\right] \times$
$\left(1 + \frac{1}{2}\delta_0 E\left|Z_0\right|\right)^{-1}$.

Proof. See Appendix C.3. □

Remark. Berkes, Horvath and Kokoszka (2003) [14] established consistency and asymptotic normality of the QMLE in the GARCH(p, q) process under weak assumptions on the parameters and the distribution of the underlying noise sequence (Theorems 4.1 and 4.2). Their paper is a generalization of the work by Lee and Hansen (1994) [61] and Lumsdaine (1996) [66] on the GARCH$(1, 1)$ process.

4.4 The Main Analysis

In this section we provide with all the proofs needed in order to establish the asymptotic normality of the QMLEs in the EGARCH$(1, 1)$ model, Theorem "Asymptotic Normality of the QMLEs". The concept of our proof is based on the method developed by Straumann and Mikosch (2006) [83].

To establish the asymptotic normality of the QMLEs, first we develop a Taylor expansion of the first derivative of the log-likelihood, say $\mathcal{L}'_n$, evaluated at the estimator, that is

$$\mathcal{L}'_n\left(\widehat{\boldsymbol{\theta}}_n\right) = \mathcal{L}'_n\left(\boldsymbol{\theta}_0\right) + \mathcal{L}''_n\left(\boldsymbol{\zeta}_n\right)\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right),$$

where $|\zeta_n - \boldsymbol{\theta}_0| < \left|\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right|$. The validity of the Taylor expansion is proved by the strong consistency of the estimator (see for instance Theorem 5.1 in Straumann and Mikosch 2006 [83]). The next step involves the application of the ergodic theorem for sequences of random functions in $\mathbb{C}\left(K, \mathbb{R}^{d'}\right)$, which allows to establish uniform convergence of the second derivative of the log-likelihood function. The last step is to apply an appropriate central limit theorem for a martingale difference sequence, which in our analysis is the normalized first order derivative of the log-likelihood, evaluated at the true parameter value.

4.4.1 CLT of the First Order Derivative

We first start with the central limit theorem for the score functions. This result follows from a CLT for finite variance stationary ergodic martingale difference sequences (see

Billingsley 1999 [18], Theorem 18.3). The following lemmas prove the asymptotic normal distribution for the standardized first order derivative of the log-likelihood function, see Proposition 4.4.1. The establishment of the central limit theorem for the first derivative of the log-likelihood function evaluated at the true parameter value represents a necessary step to prove the asymptotic normal distribution of the QMLE.

After a brief discussion on the classical methodology available to prove the asymptotic normality, we pass to the application of the Ergodic Theorem in order to obtain the uniform Strong Law of Large Numbers. We prove the boundedness of the first and second order derivative of the log-variance process, which implies that the supremum norm of the second order derivative of the log-likelihood function is finite. This is the key to prove the asymptotic normality of the QMLEs, see for instance Theorem 4.10.

Lemma 4.11 (Martingale Difference Sequence)**.** *Let*

a) $\left|\beta_0 - \frac{1}{2}\delta_0 E\,|Z|\right| < 1$ *and*

b) $E\left(\left(Z_t^2 - 1\right)^2\right) = \zeta < \infty$ *hold and define the sequence* $I_{t-1} = \{X_{t-1}, X_{t-2}, ...\}$ *to be sub-*σ*-algebras of* I*. Then*

$\{\mathcal{L}_{it}(\theta_0), I_{t-1}\}$ *for* $i = 1, 2, 3, 4,$ *are martingale difference sequences.*

Proof. For each t, $\mathcal{L}_{it}(\theta_0)$ is measurable I_t, and $I_{t-1} \subset I_t$. It is also quite trivial to see that $\Pr\left(E\left(\mathcal{L}_{it}(\theta_0)|\, I_{t-1}\right) = 0\right) = 1$. To complete the proof of the Lemma it is sufficient to verify that $E\left(|\mathcal{L}_{it}(\theta_0)|\right) < \infty$, for $i = 1, 2, 3, 4$ (we make use of the Cauchy-Schwarz Inequality, that is $E\,|XY| \leq \sqrt{EX^2EY^2}$ and end up with condition b). □

Lemma 4.12 (Bounded Moments)**.** *Let* $\theta_0^* = \left(\beta_0 - \frac{1}{2}\gamma_0 Z_{t-1} - \frac{1}{2}\delta_0\,|Z_{t-1}|\right)$*.*

Define the processes:

$$\begin{aligned}
u_{1t}(\theta_0) &= 1 + \theta_0^* \frac{\partial h_{t-1}/\partial\alpha}{h_{t-1}}|_{\theta_0}, \\
u_{2t}(\theta_0) &= \ln h_{t-1} + \theta_0^* \frac{\partial h_{t-1}/\partial\beta}{h_{t-1}}|_{\theta_0}, \\
u_{3t}(\theta_0) &= Z_{t-1} + \theta_0^* \frac{\partial h_{t-1}/\partial\gamma}{h_{t-1}}|_{\theta_0}, \\
u_{4t}(\theta_0) &= g(Z_{t-1}) + \theta_0^* \frac{\partial h_{t-1}/\partial\delta}{h_{t-1}}|_{\theta_0}.
\end{aligned}$$

Given

a) $\left|\beta_0 - \frac{1}{2}\delta_0 E\,|Z|\right| < 1$

b) $\left|\beta_0^2 + \frac{1}{4}\gamma_0^2 + \frac{1}{4}\delta_0^2 - \delta_0\beta_0 E\,|Z| + \frac{1}{2}\delta_0\gamma_0 E\left(Z\,|Z|\right)\right| < 1$

and

c) $\left| \begin{array}{c} \beta_0^3 + \frac{3}{4}\beta_0\gamma_0^2 + \frac{3}{4}\beta_0\delta_0^2 - \frac{1}{8}\gamma_0\left(\gamma_0^2 + 3\delta_0^2\right)E\left(Z^3\right) - \frac{3}{2}\beta_0^2\delta_0 E\left|Z\right| \\ +\frac{3}{2}\beta_0\gamma_0\delta_0 E\left(Z\left|Z\right|\right) - \frac{1}{8}\delta_0\left(\delta_0^2 + 3\gamma_0^2\right)E\left|Z\right|^3 \end{array} \right| < 1,$

then

$E\left(\left|u_{it}\left(\theta_0\right)\right|^p\right) \leq M_{i,p} < \infty$, *for* $p = 1, 2, 3$ *and* $i = 1, 2, 3, 4$.

Proof. Assume that $E\left|Z\right|$ is bounded. Higher order moments exist to the extent that the higher order moments of Z_t and $\left|Z_t\right|$ exist. □

Lemma 4.13 (Square Integrability of the First Order Derivatives)**.** *Let*

a) $\left|\beta_0^2 + \frac{1}{4}\gamma_0^2 + \frac{1}{4}\delta_0^2 - \delta_0\beta_0 E\left|Z\right| + \frac{1}{2}\delta_0\gamma_0 E\left(Z\left|Z\right|\right)\right| < 1$ *and*

b) $E\left(\left(Z_t^2 - 1\right)^2\right) = \zeta < \infty$ *hold. Then*

$$\frac{1}{n}\mathcal{L}_{it}^2 \overset{a.s.}{\to} \frac{\zeta}{4}\varpi_i^2 \quad \text{as } n \to \infty$$

where $i = 1, 2, 3, 4$.

Proof. Let

$$\frac{\partial h_t / \partial i}{h_t} \equiv h_{t;i},$$

$$\varpi_i^2 \triangleq M_{i,2} = E\left(\left|h_{t;i}\right|^2\right)\Big|_{\theta_0}, \qquad i \in \{\alpha, \beta, \gamma, \delta\}.$$

Let also $\theta_0^{**} = \beta_0^2 + \frac{1}{4}\gamma_0^2 + \frac{1}{4}\delta_0^2 - \delta_0\beta_0 E\left|Z\right| + \frac{1}{2}\delta_0\gamma_0 E\left(Z\left|Z\right|\right)$.

Hence:

$$\begin{aligned}
\varpi_1^2 &= E\left(\left|h_{t;\alpha}\right|^2\right)\Big|_{\theta_0} = \frac{1 + 2\left(\beta_0 - \frac{1}{2}\delta_0 E\left|Z\right|\right)E\left(h_{t;\alpha}\right)|_{\theta_0}}{1 - \theta_0^{**}}, \\
\varpi_2^2 &= E\left(\left|h_{t;\beta}\right|^2\right)\Big|_{\theta_0} = \frac{E\left(\ln^2\left(h_t\right)\right)|_{\theta_0} + 2\left(\beta_0 - \frac{1}{2}\delta_0 E\left|Z\right|\right)E\left(\ln\left(h_t\right)h_{t;\beta}\right)|_{\theta_0}}{1 - \theta_0^{**}}, \\
\varpi_3^2 &= E\left(\left|h_{t;\gamma}\right|^2\right)\Big|_{\theta_0} = \frac{1}{1 - \theta_0^{**}}, \\
\varpi_4^2 &= E\left(\left|h_{t;\delta}\right|^2\right)\Big|_{\theta_0} = \frac{1 - E^2\left|Z\right|}{1 - \theta_0^{**}}.
\end{aligned}$$

□

Remark. The conditions from Lemma 4.13 are equivalent with the need of condition (N3ii) in Straumann (2005) [84], p. 116.

Proposition 1 (Central Limit Theorem for the Score Functions) *Let the assumptions of the previous lemmas hold and let the scores be as defined in $\mathcal{L}_{it}(\theta_0)$ form. Having* $\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}E\left(\mathcal{L}_{it}^2(\theta_0)\, I\{|\mathcal{L}_{it}(\theta_0)|>\delta\sqrt{n}\}\right)\to 0$ and $\sup_{n\geq 1}\frac{1}{n}\sum_{t=1}^{n}E\left(\mathcal{L}_{it}^2(\theta_0)\right)<\infty$,

$$\frac{1}{\sqrt{n}}\mathcal{L}_{it}(\theta_0)\xrightarrow{d}N\left(0,\frac{\zeta}{4}\varpi_i^2\right)$$

for $n\to\infty$ and $i=1,2,3,4$, where ϖ_i^2 is defined as in Lemma 4.13.

Proof. By Lemma 4.11, $\mathcal{L}_{it}(\theta_0)$, for all i, is a martingale difference sequence. Furthermore, the results in Lemma 4.13 and the relations specified in the Proposition (the Lindeberg condition and the uniformity over n of the variance boundedness) correspond to the conditions of the proof of the CLT for the scores. Therefore, the result of Proposition 1 follows immediately. □

4.4.2 Uniform SLLN of the Second Order Derivative

In this section we provide with the lemmas that are required in order to establish the uniform convergence of the second order derivative of the log-likelihood.

Proposition 2 (Moments Convergence of the Second Order Derivatives) *Let*

a) $\left|\beta_0\left(\beta_0-\frac{1}{2}\delta_0 E|Z|\right)\right|<1$ *and*

b) $\left|\beta_0^2+\frac{1}{4}\gamma_0^2+\frac{1}{4}\delta_0^2-\delta_0\beta_0 E|Z|+\frac{1}{2}\delta_0\gamma_0 E(Z|Z|)\right|<1$ *hold. Then*

(a) $\frac{1}{n}\left(-\frac{\partial^2}{\partial\alpha^2}l_n(\theta)|_{\theta=\theta_0}\right)\xrightarrow{p}\frac{1}{2}\varpi_1^2>0$

(b) $\frac{1}{n}\left(-\frac{\partial^2}{\partial\beta^2}l_n(\theta)|_{\theta=\theta_0}\right)\xrightarrow{p}\frac{1}{2}\varpi_2^2>0$

(c) $\frac{1}{n}\left(-\frac{\partial^2}{\partial\gamma^2}l_n(\theta)|_{\theta=\theta_0}\right)\xrightarrow{p}\frac{1}{2}\varpi_3^2>0$

(d) $\frac{1}{n}\left(-\frac{\partial^2}{\partial\delta^2}l_n(\theta)|_{\theta=\theta_0}\right)\xrightarrow{p}\frac{1}{2}\varpi_4^2>0$

(e) $\frac{1}{n}\left(-\frac{\partial^2}{\partial\alpha\partial\beta}l_n(\theta)|_{\theta=\theta_0}\right)\xrightarrow{p}\frac{1}{2}\varpi_{12}$

(f) $\frac{1}{n}\left(-\frac{\partial^2}{\partial\alpha\partial\gamma}l_n(\theta)|_{\theta=\theta_0}\right)\xrightarrow{p}\frac{1}{2}\varpi_{13}$

(g) $\frac{1}{n}\left(-\frac{\partial^2}{\partial\alpha\partial\delta}l_n(\theta)|_{\theta=\theta_0}\right)\xrightarrow{p}\frac{1}{2}\varpi_{14}$

(h) $\frac{1}{n}\left(-\frac{\partial^2}{\partial\beta\partial\gamma}l_n(\theta)|_{\theta=\theta_0}\right)\xrightarrow{p}\frac{1}{2}\varpi_{23}$

(i) $\frac{1}{n}\left(-\frac{\partial^2}{\partial\beta\partial\delta}l_n(\theta)|_{\theta=\theta_0}\right)\xrightarrow{p}\frac{1}{2}\varpi_{24}$

(j) $\frac{1}{n}\left(-\frac{\partial^2}{\partial\gamma\partial\delta}l_n(\theta)|_{\theta=\theta_0}\right)\xrightarrow{p}\frac{1}{2}\varpi_{34}$, *as $n\to\infty$.*

Proof. By applying the ergodic theorem and the results on the existence of moments in Lemma 4.12, then the proof of Proposition 2 follows. □

Definition 1 *Denote* $\theta_0 = (\alpha_0, \beta_0, \gamma_0, \delta_0)'$. *Define the lower and upper values for each parameter in* θ_0 *as*

$$\begin{aligned} \alpha_L &< \alpha_0 < \alpha_U; \qquad \beta_L < \beta_0 < \beta_U, \\ \gamma_L &< \gamma_0 < \gamma_U; \qquad \delta_L < \delta_0 < \delta_U, \end{aligned}$$

and the neighborhood $N(\theta_0)$ *around* θ_0 *as*

$$N(\theta_0) = \left\{ \begin{array}{c} \alpha_L < \alpha_0 < \alpha_U, \beta_L < \beta_0 < \beta_U, \gamma_L < \gamma_0 < \gamma_U, \\ \text{and } \delta_L < \delta_0 < \delta_U \end{array} \right\},$$

for which we have that $N(\theta_0) \subseteq K$.

There are papers in the related literature of an alternative methodology[7], which has been the traditional one over the past decades. The reader is referred to the papers by Lee and Hansen (1994) [61] and Lumsdaine (1996) [66] for the QMLE in the GARCH(1,1) process. Using that method, they prove that the second order derivative of the log-likelihood has a unique limit function and they also prove that this convergence is uniform as the second derivative is stochastically equicontinuous which comes from the boundedness of the third derivatives. When deriving consistency and asymptotic normality, the classical sufficient condition regarding bounds of the third derivatives of the log-likelihood function is that

$$E \sup_{\tilde{\theta} \in N(\theta_0)} \left| \frac{1}{n} \frac{\partial^3}{\partial \theta^3} l\left(\tilde{\theta}\right) \right| < \infty.$$

These authors apply the SLLN for stationary and ergodic sequences (see the pointwise ergodic theorem, Theorem 3.5.7, in Stout 1974 [82]) and examine the above condition which implies that the second derivative satisfies the Lipschitz condition of Andrews (1992) [4] and hence establish uniform convergence. Jensen and Rahbek (2004a) [58] noted in p. 645 that the above condition has been reproduced in other papers (see, for instance, Lumsdaine 1996 [66]) with a misleading way, such that the proofs in those papers might not be complete. The last holds with an exception of the paper by Berkes, Horváth and Kokoszka (2003) [14].

[7] This method utilizes the classic Cramer type conditions, that is central limit theorem for the score, convergence of the Hessian and uniformly bounded third-order derivatives; see e.g. Lehmann (1999) [62].

In order to establish the almost sure equicontinuity of the second derivatives of the log-likelihood, Lumsdaine (1996) [66], Lee and Hansen (1994) [61] and Berkes, Horváth and Kokoszka (2003) [14] stochastically bound the third derivatives. Such computations however can be avoided when the Ergodic Theorem for random elements with values in a separable Banach space is applied.

Straumann and Mikosch (2006) [83] suggest a simpler method that is based on the ergodic theorem for $\mathbb{C}\left(K, \mathbb{R}^{d'}\right)$ −valued sequences of random variables and requires that the stationary sequence is ergodic and has a bounded expected norm[8]. This result is summarized in the following Theorem and we refer to Ranga Rao (1962) [76] for its proof.

Theorem 4.14 (Theorem 2.7 of Straumann and Mikosch 2006 [83]). *Let (v_t) be a stationary ergodic sequence of random elements with values in $C(K, \mathbb{R}^{d'})$. Then the uniform SLLN is implied by $E\left\|v_0\right\|_K < \infty$.*

To what follows, we denote by $\|\mathbf{A}\|$ the Frobenius norm of a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{d\times d}$, defined by

$$\|\mathbf{A}\| = \left(\sum_{i,j=1}^{d} a_{ij}^2\right)^{1/2}.$$

The following inequality, which is valid for the Frobenius norm, is useful in our analysis. If $\mathbf{x}$,$\mathbf{y} \in \mathbb{C}\left(K, \mathbb{R}^{d}\right)$, then the Frobenius norm of the matrix $\mathbf{x}\mathbf{y}^T$ is bounded by

$$\left\|\mathbf{x}\mathbf{y}^T\right\|_K \leq \left\|\mathbf{x}\right\|_K \left\|\mathbf{y}\right\|_K.$$

In the sequel, we first prove the existence of the expected sup-norm of the first order derivative of the log-variance function, see Lemma.4.18 For this scope, we provide with useful lemmas in which we consider the higher order dependence in the EGARCH process in such a way so as to find accurate moment estimates and verify the moment conditions needed in the proof of our main Theorem 4.10.

Notation. As already stated, $\overline{\delta} = \max\left(\sup\left(\gamma+\delta\right), \sup\left(\delta-\gamma\right)\right) : \gamma x + \delta\left|x\right| \leq \overline{\delta}\left|x\right|,$ $\forall x \in \mathbb{R}$ and $\underline{m} = \inf\left\{\frac{\alpha}{1-\beta}\right\}$ .Let also c be a constant that is equal to $\frac{1}{\beta}2^{-1}\overline{\delta}\exp\left(-2^{-1}\underline{m}\right)$ (otherwise, it will be defined properly).

[8] For further analysis, see the paper by Ranga Rao (1962).

Lemma 4.15. *If* $E\left[\left(\left|Z_0\right|\sigma_0\right)\right] < \infty$ *and* $2^{-1}\overline{\delta}\exp\left(2^{-1}\left[\left(\frac{\alpha}{(1-\beta)} - \underline{m}\right)\right]\right)\left(E\left|Z_0\right|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right) = q^a < 1$, *then*

$$\sum_{k=1}^{\infty} E\left[\beta^{k-1}\prod_{i=1}^{k-1}\left[1 + c\left|X_{t-i}\right|\right]\right] < \infty$$

Proof. See Appendix C.1. □

Remark. The previous result uses the important Lemmas C.1 and C.2 from the Appendix.

Lemma 4.16. *If* $E\left|Z_0\exp\left[\frac{1}{2}\overline{\delta}\left|Z_0\right|\right]\right| < \infty$ *and*
$2^{-1}\overline{\delta}\exp\left(2^{-1}\left[\left(\frac{\alpha}{(1-\beta)} - \underline{m}\right)\right]\right)\left(E\left|Z_0\right|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right) = q^a < 1$, *then*

$$\sum_{k=1}^{\infty} E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1 + c\left|X_{t-i}\right|\right]\right)\left|X_{t-k}\right|\right]$$

is finite.

Proof. See Appendix C.1. □

Remark. The previous result makes use of the following dependence property:

$$\sum_{i=1}^{k-1}\left|X_{t-i}\right|\left|X_{t-k}\right| = \left(\left|X_{t-1}\right| + \left|X_{t-2}\right| + ... + \left|X_{t-k+1}\right|\right)\left|X_{t-k}\right|$$

is bounded from the higher dependence, which is between $\left|X_{t-k+1}\right|$ and $\left|X_{t-k}\right|$. Hence:

$$\sum_{i=1}^{k-1}\left|X_{t-i}\right|\left|X_{t-k}\right| \le (k-1)\left|X_{t-k+1}\right|\left|X_{t-k}\right|.$$

Lemma 4.17. *If the conditions of Lemma 4.15 are satisfied and moreover*
$E\left|Z_0\exp\left(\frac{1}{2}\overline{\delta}\left|Z_0\right|\right)\right| < \infty$,
$\beta\exp\left(\frac{1}{2(1-\beta)}\alpha\right)E\left(\left|Z_0\right|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right) = q^b < 1$,
$\exp\left(\frac{1}{2(1-\beta)}\alpha\right)E\left(\left|\exp\left\{\frac{1}{2}\overline{\delta}\left|Z_0\right|\right\}Z_0\right|\right) = q^c < 1$ *and* $E\left|\exp\left\{\frac{1}{2}\left(\frac{1}{1-\beta}+1\right)\overline{\delta}\left|Z_0\right|\right\}Z_0^2\right| < \infty$,

then

$$\sum_{k=1}^{\infty} E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1 + c\left|X_{t-i}\right|\right]\right)\left|\log h_{t-k}\right|\right] < \infty$$

Proof. See Appendix C.1. □

Remark. The following result is important in order to prove the previous lemma:

$$\sum_{k=1}^{\infty} E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}[1+c\,|X_{t-i}|]\right)|\log h_{t-k}|\right] \le \sum_{k=1}^{\infty} E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}[1+c\,|X_{t-i}|]\right)|\log h_{t}|\right],$$

by backward substitution of $\log h_t$.

Remark. The Lemmas C.4, C.5, C.6, C.7 and C.8 from the Appendix are useful to prove the previous lemma.

Lemma 4.18 (Boundedness of the expected value of the sup-norm of the first order derivative)**.** *Suppose the conditions imposed in Straumann (2005) [84], section 5.7.2.*[9] *If the conditions imposed to the previous Lemmas 4.15, 4.16 and 4.17 hold, then* $E\left\|(\log h_t)'\right\|_K < \infty$, *where* K *is a compact set of the parameter space.*

Proof. Differentiation with respect to $\boldsymbol{\theta}$ of both sides of

$$\log h_{t+1} = \alpha + \beta \log h_t + (\gamma X_t + \delta\,|X_t|)\exp\left(-2^{-1}\log h_t\right),$$

leads to

$$(\log h_{t+1})' = A_t\,(\log h_t)' + B_t, \tag{4.12}$$

where

$$\begin{aligned} A_t &= \frac{\partial \log h_{t+1}}{\partial \log h_t} = \beta - 2^{-1}(\gamma X_t + \delta\,|X_t|)\exp\left(-2^{-1}\log h_t\right), \\ B_t &= \frac{\partial \log h_{t+1}}{\partial \boldsymbol{\theta}} = \left(1, \log h_t, X_t\exp\left(-2^{-1}\log h_t\right), |X_t|\exp\left(-2^{-1}\log h_t\right)\right). \end{aligned}$$

The eq. (4.12) is linear and due to this fact, its unique stationary ergodic solution has the representation

$$(\log h_t)' = \sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1} A_{t-i}\right) B_{t-k} \qquad a.s.$$

[9] These conditions refer to the proof of a contractive SRE needed for the application of the Theorem 2.6.4., in Straumann (2005) [84].

Noticing that $\|B_t\|_K \leq c_1 (1 + |X_t| + |\log h_t|)$ for some constant $c_1 > 0$ and applying the triangle inequality to the latter representation, we get

$$\left\|(\log h_t)'\right\|_K \leq c_1 \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k-1} \|A_{t-i}\|_K \right) (1 + |X_{t-k}| + |\log h_{t-k}|) .$$

We also have that

$$\begin{aligned} \|A_t\| &\leq \beta + 2^{-1}\overline{\delta}\,|X_t| \exp\left(-2^{-1}\underline{m}\right), \quad \underline{m} = \inf\left\{\frac{\alpha}{1-\beta}\right\} \\ &\leq \beta\left[1 + \frac{1}{\beta}2^{-1}\overline{\delta}\,|X_t| \exp\left(-2^{-1}\underline{m}\right)\right], \end{aligned}$$

and hence, that

$$\prod_{i=1}^{k-1} \|A_{t-i}\| \leq \beta^{(k-1)} \prod_{i=1}^{k-1} \left[1 + \frac{1}{\beta}2^{-1}\overline{\delta}\,|X_{t-i}| \exp\left(-2^{-1}\underline{m}\right)\right].$$

Using the above inequalities, we obtain

$$E\left\|(\log h_t)'\right\| \leq c_1 E\left[\sum_{k=1}^{\infty}\left(\beta^{(k-1)} \prod_{i=1}^{k-1} \left[1 + c\,|X_{t-i}|\right]\right)(1 + |X_{t-k}| + |\log h_{t-k}|)\right],$$

where

$$c = \frac{1}{\beta}2^{-1}\overline{\delta} \exp\left(-2^{-1}\underline{m}\right).$$

We need to bound the next three elements, i.e. find the appropriate conditions in order these terms to be finite and then apply the Minkowski inequality to the infinite sum $\sum_{k=1}^{\infty}$, that is:

$$E\left[\beta^{(k-1)} \prod_{i=1}^{k-1} \left[1 + c\,|X_{t-i}|\right]\right],$$

$$E\left[\sum_{k=1}^{\infty}\left(\beta^{(k-1)} \prod_{i=1}^{k-1} \left[1 + c\,|X_{t-i}|\right]\right)|X_{t-k}|\right] \text{ and}$$

$$E\left[\sum_{k=1}^{\infty}\left(\beta^{(k-1)} \prod_{i=1}^{k-1} \left[1 + c\,|X_{t-i}|\right]\right)|\log h_{t-k}|\right].$$

We then make use of Lemmas 4.15, 4.16 and 4.17 for each of the above terms and this completes the proof. □

Next, we proceed with the finiteness of the sup-norm of the second order derivative of the log-variance function. To do so, we first calculate the following bounds that are useful for the remaining analysis. Recall that

$$\left\|(\log h_t)'\right\| \leq c_1 \left[\sum_{j=1}^{\infty}\left(\beta^{j-1}\prod_{i=1}^{j-1}\left[1+c\left|X_{t-i}\right|\right]\right)\left(1+\left|X_{t-j}\right|+\left|\log h_{t-j}\right|\right)\right].$$

Hence,

$$E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)\left|Z_{t-k}\right|\sigma_{t-k}\left\|(\log h_{t-k})'\right\|\right] \leq c_1 E\left[\begin{array}{c}\sum_{k=1}^{\infty}\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^2\left|Z_{t-k}\right|\sigma_{t-k} \\ \times\left(1+\left|X_{t-k}\right|+\left|\log h_{t-k}\right|\right)\end{array}\right], \tag{4.13}$$

by backward substitution of $(\log h_t)'$. Moreover, we have that

$$\begin{aligned} E\left\|(\log h_t)'\right\|^{\eta} &\leq E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)\|B_{t-k}\|\right]^{\eta} \\ &\leq c_1 E\left[\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\|A_{t-i}\|^{\eta}\left(1+\left|X_{t-k}\right|+\left|\log h_{t-k}\right|\right)^{\eta}\right] \\ &\leq c_1 E\left[\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\|A_{t-i}\|^{\eta}\right]\left[2^{\eta}+2^{2\eta}\left(\left|X_{t-k}\right|^{\eta}+\left|\log h_{t-k}\right|^{\eta}\right)\right], \end{aligned}$$

making use of the following inequality:

$$(x+y)^{\eta} \leq 2^{\eta}\left(x^{\eta}+y^{\eta}\right).$$

Hence

$$\begin{aligned}
\left\|(\log h_t)'\right\|^2 &\leq \left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)\|B_{t-k}\|\right]^2 \\
&\leq c_1\left[\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\|A_{t-i}\|^2\left(1+|X_{t-k}|+|\log h_{t-k}|\right)^2\right] \\
&\leq c_1\left[\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\|A_{t-i}\|^2\right]\left[4+16\left(|X_{t-k}|^2+|\log h_{t-k}|^2\right)\right] \\
&\leq c_1^*\left[\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\|A_{t-i}\|^2\right]\left[1+4\left(|X_{t-k}|^2+|\log h_{t-k}|^2\right)\right],
\end{aligned}$$

$$\text{where } \sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\|A_{t-i}\|^2 = \sum_{k=1}^{\infty}(2\beta)^{2(k-1)}\prod_{i=1}^{k-1}\left(1+c\,|X_{t-i}|^2\right)$$

$$\text{and } c = \frac{1}{(2\beta)^2}\overline{\delta}^2\exp\left(-\underline{m}\right)\ \text{(see Lemma C.9).}$$

Thus

$$E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)|Z_{t-k}|\,\sigma_{t-k}\left\|(\log h_{t-k})'\right\|^2\right] \leq c_1^* E\left[\begin{array}{c}\displaystyle\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|^3\right)|Z_{t-k}|\,\sigma_{t-k} \\ \times\left[1+4\left(|X_{t-k}|^2+|\log h_{t-k}|^2\right)\right]\end{array}\right]. \tag{4.14}$$

We provide below with useful lemmas that are important to prove the second main result, Lemma 4.23. Let the following:

Lemma 4.19. *If* $E\left(|Z_0|\exp\overline{\delta}\,|Z_0|\right)<\infty$ *and* $\frac{1}{4}\overline{\delta}^2\exp\left(\alpha\frac{1}{1-\beta}-\underline{m}\right)E\left[Z_0^2\exp\left(\frac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)\right]=q^d<1$, *then*

$$\sum_{k=1}^{\infty}E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\,|X_{t-i}|\right]\right)^2|Z_{t-k}|\,\sigma_{t-k}\right]<\infty$$

Proof. See Appendix C.2. □

Lemma 4.20. *If* $E\left(Z_0^2\exp\overline{\delta}\,|Z_0|\right)<\infty$ *and* $q^d<1$ *(see Lemma 4.19), then*

$$\sum_{k=1}^{\infty}E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\,|X_{t-i}|\right]\right)^2|Z_{t-k}|\,\sigma_{t-k}\,|X_{t-k}|\right]$$

is finite.

Proof. See Appendix C.2. □

Lemma 4.21. *If the conditions of Lemma 4.19 are satisfied and moreover*
$E\left[|Z_0|\exp\left(\frac{3}{2}\overline{\delta}\,|Z_0|\right)\right]<\infty$, $E\left[Z_0^2\exp\left(\overline{\delta}\,|Z_0|\right)\right]<\infty$,
$\exp\left(\frac{1}{(1-\beta)}\alpha\right)E\left[Z_0^2\exp\left(\overline{\delta}\,|Z_0|\right)\right]=q^e<1$, $E\left[|Z_0|\exp\left[\left(\frac{1}{1-\beta}+\frac{1}{2}\right)\overline{\delta}\,|Z_0|\right]\right]<\infty$ *and*
$\exp\left(\frac{1}{1-\beta}\alpha\right)E\left[Z_0^2\exp\left(\overline{\delta}\,|Z_0|\right)\right]=q^f<1$,
then

$$E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\,|X_{t-i}|\right]\right)^2|Z_{t-k}|\,\sigma_{t-k}\,|\log h_{t-k}|\right]<\infty.$$

Proof. See Appendix C.2. □

Lemma 4.22. *Let* $E\left(|Z_0|\exp\overline{\delta}\,|Z_0|\right)<\infty$,
$\frac{1}{8}\overline{\delta}^3\exp\left[\frac{3}{2}\left(\alpha\frac{1}{1-\beta}-\underline{m}\right)\right]E\left[|Z_0|^3\exp\left(\frac{3}{2}\frac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)\right]=q^g<1$, $E\left[|Z_0|^3\exp\left(\frac{3}{2}\overline{\delta}\,|Z_0|\right)\right]<\infty$ *(see Appendix A2 for all conditions),*
then

$$E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)|Z_{t-k}|\,\sigma_{t-k}\left\|(\log h_{t-k})'\right\|^2\right]$$

is finite.

Proof. See Appendix C.2. □

Lemma 4.23 (Boundedness of the expected value of the sup-norm of the second order derivative)**.** *Suppose all the conditions of the previous Lemmas 4.19, 4.20, 4.21 and 4.22 hold. Moreover, if the conditions imposed to Lemmas 4.15, 4.16 and 4.17 are also satisfied,*
then $E\left\|(\log h_t)''\right\|_K<\infty$.

Proof. Differentiation of (4.12) with respect to $\boldsymbol{\theta}$ yields

$$\begin{aligned}(\log h_{t+1})'' &= \left(A_t\,(\log h_t)'+B_t\right)' \\ &= \frac{\partial A_t^T}{\partial\theta}(\log h_t)'+\frac{\partial A_t}{\partial s}\left((\log h_t)'\right)^T(\log h_t)' \\ &\quad +A_t\,(\log h_t)''+\frac{\partial B_t}{\partial\theta}+\frac{\partial B_t^T}{\partial s}(\log h_t)' \\ &= A_t\,(\log h_t)''+C_t,\end{aligned}$$

where

$$\begin{aligned}
C_t &= \frac{\partial A_t^T}{\partial \theta} (\log h_t)' + \frac{\partial A_t}{\partial s} \left((\log h_t)'\right)^T (\log h_t)' + \frac{\partial B_t}{\partial \theta} + \frac{\partial B_t^T}{\partial s} (\log h_t)' \\
&= \left(0, 2, -X_t \exp\left(-2^{-1}\log h_t\right), -|X_t| \exp\left(-2^{-1}\log h_t\right)\right)^T (\log h_t)' \\
&\quad +4^{-1}\left(\gamma X_t + \delta |X_t|\right) \exp\left(-2^{-1}\log h_t\right) \left((\log h_t)'\right)^T (\log h_t)' \\
&\quad + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \log h_t & 0 & 0 \\ 0 & X_t \exp\left(-2^{-1}\log h_t\right) & 0 & 0 \\ 0 & |X_t| \exp\left(-2^{-1}\log h_t\right) & 0 & 0 \end{bmatrix}.
\end{aligned}$$

We can write

$$\begin{aligned}
C_t &= -\exp\left(-2^{-1}\log h_t\right) \left(0, -2/\exp\left(-2^{-1}\log h_t\right), X_t, |X_t|\right)^T (\log h_t)' \\
&\quad +4^{-1}\left(\gamma X_t + \delta |X_t|\right) \exp\left(-2^{-1}\log h_t\right) \left((\log h_{t-1})'\right)^T (\log h_{t-1})' \\
&\quad + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \log h_t & 0 & 0 \\ 0 & X_t \exp\left(-2^{-1}\log h_t\right) & 0 & 0 \\ 0 & |X_t| \exp\left(-2^{-1}\log h_t\right) & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Starting point for establishing $E\left\|(\log h_t)''\right\|_K < \infty$ is the almost sure representation

$$(\log h_t)'' = \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k-1} A_{t-i}\right) C_{t-k}.$$

By means of the inequality $\left\|xy^T\right\| \leq \|x\| \|y\|$ and the Frobenius norm, i.e. $\|A\| = \left(\sum_{i,j=1}^{d} a_{ij}^2\right)^{1/2}$, we have that

$$\begin{aligned}
\|C_t\| &\leq c_1 \left(c_2 + 2|X_t|\right) \left|(\log h_t)'\right| + c_3 |X_t| \left|(\log h_t)'\right|^2 + c_3 \left(1 + |X_t| + |\log h_t|\right) \\
&\leq c_1^* + c_2^* |X_t| \left[\left|(\log h_t)'\right| + \left|(\log h_t)'\right|^2\right] + c_3 \left(1 + |X_t| + |\log h_t|\right),
\end{aligned}$$

where the constants $c_1, ...c_3, c_1^*, ...$ define generic positive constants that take many different values.

Hence

$$\left\|(\log h_t)''\right\| \leq \sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)\left(\begin{array}{c} c_1^* + c_2^*\,|X_{t-k}|\left[\left\|(\log h_{t-k})'\right\| + \left\|(\log h_{t-k})'\right\|^2\right] \\ +c_3\left(1+|X_{t-k}|+|\log h_{t-k}|\right)\end{array}\right).$$

We need to bound the next three elements, i.e. find the appropriate conditions in order these terms to be finite and then apply the Minkowski inequality to the infinite sum $\sum_{k=1}^{\infty}$, that is:

$c_1^* E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)\right] \leq c_1^* E\left[\sum_{k=1}^{\infty}\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\,|X_{t-i}|\right]\right)\right]$: as in the proof in the first order derivative, i.e. Lemma 4.15,

$c_2^* E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)|X_{t-k}|\left[\left\|(\log h_{t-k})'\right\| + \left\|(\log h_{t-k})'\right\|^2\right]\right]$

$= c_2^* E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)|Z_{t-k}|\,\sigma_{t-k}\left[\left\|(\log h_{t-k})'\right\| + \left\|(\log h_{t-k})'\right\|^2\right]\right]$, where we have, for instance, two components:

(1) $E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)|Z_{t-k}|\,\sigma_{t-k}\left\|(\log h_{t-k})'\right\|\right]$ (see Lemmas 4.19, 4.20 and 4.21)

(2) $E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)|Z_{t-k}|\,\sigma_{t-k}\left\|(\log h_{t-k})'\right\|^2\right]$ (see Lemma 4.22) and

$c_3 E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)\left(1+|X_{t-k}|+|\log h_{t-k}|\right)\right]$: as in the proof in the first order derivative, i.e. Lemmas 4.15, 4.16 and 4.17.

This concludes the proof of the Lemma. □

The last step in our analysis is the following Lemma which provides moment estimates that are necessary in order to prove the asymptotic normality of the estimators, i.e. Theorem 4.10.

Lemma 4.24. *Under the conditions of Lemma 4.23, we have that*

$$E\left\|X_t^2\frac{1}{h_t}\left[(\log h_t)'\right]^T\left[(\log h_t)'\right]\right\| < \infty \quad \text{and} \quad E\left\|X_t^2\frac{1}{h_t}(\log h_t)''\right\| < \infty. \tag{4.15}$$

Proof. The second order derivative of the log-likelihood is given by:

$$\begin{aligned}\frac{\partial^2}{\partial i \partial j} l_T(\theta) &= \frac{1}{2}\left(\frac{X_t^2}{h_t} - 1\right)(\log h_t)'' - \frac{1}{2}\frac{X_t^2}{h_t}\left[(\log h_t)'\right]^T (\log h_t)' = \\ &= \frac{1}{2}\frac{X_t^2}{h_t}(\log h_t)'' - \frac{1}{2}(\log h_t)'' - \frac{1}{2}\frac{X_t^2}{h_t}\left[(\log h_t)'\right]^T (\log h_t)' = \end{aligned} \quad (4.16)$$

$$= \tfrac{1}{2}X_t^2\left[-\tfrac{1}{h_t}\left(\tfrac{1}{h_t}\tfrac{\partial h_t}{\partial i}\right)\left(\tfrac{1}{h_t}\tfrac{\partial h_t}{\partial j}\right) + \left(\tfrac{1}{h_t}\right)^2 \tfrac{\partial^2 h_t}{\partial i \partial j}\right] - \tfrac{1}{2}\left(\tfrac{1}{h_t}\tfrac{\partial h_t}{\partial i}\right)\left(\tfrac{1}{h_t}\tfrac{\partial h_t}{\partial j}\right) + \tfrac{1}{2}\tfrac{1}{h_t}\tfrac{\partial^2 h_t}{\partial i \partial j} - \tfrac{1}{2}X_t^2\tfrac{1}{h_t}\left(\tfrac{1}{h_t}\tfrac{\partial h_t}{\partial i}\right)\left(\tfrac{1}{h_t}\tfrac{\partial h_t}{\partial j}\right),$$

where

$$(\log h_t)' = \frac{1}{h_t}\frac{\partial h_t}{\partial i}, \quad (\log h_t)'' = -\left(\frac{1}{h_t}\frac{\partial h_t}{\partial i}\right)\left(\frac{1}{h_t}\frac{\partial h_t}{\partial j}\right) + \frac{1}{h_t}\frac{\partial^2 h_t}{\partial i \partial j}.$$

Hence, applying the triangle inequality to eq. (4.16) and next the Holder inequality implies:

$\|l_0''\| \le \frac{1}{2}\left(\left(\left\|\frac{X_0^2}{h_0}\right\| + 1\right)\|(\log h_0)''\| + \left\|\frac{X_0^2}{h_0}\right\|\left\|\left[(\log h_0)'\right]^T\left[(\log h_0)'\right]\right\|\right).$

Thus, to prove $E\left\|X_t^2\frac{1}{h_t}(\log h_t)''\right\| < \infty$ it suffices to show that $E\left\|X_t^2\frac{1}{h_t}\right\| < \infty$ and then use the Cauchy-Schwarz inequality to obtain the desired result. This is also the case for $E\left\|X_t^2\frac{1}{h_t}\left[(\log h_0)'\right]^T\left[(\log h_0)'\right]\right\|$ if and only if $E\left\|\left[(\log h_0)'\right]^T\left[(\log h_0)'\right]\right\| < \infty$.

We have that

$$\left\|\frac{\sigma_t^2}{h_t}\right\| \le \frac{\sigma_t^2}{h_t^{\inf}},$$

where $h_t^{\inf} = \exp\left(\frac{\underline{\alpha}}{1-\overline{\beta}}\right)$, $\underline{\alpha}$ = min of the space of α and $\overline{\beta}$ = max of the space of β. Hence

$$E\left\|\frac{\sigma_t^2}{h_t}\right\| \le \frac{1}{h_t^{\inf}} E\left(\sigma_t^2\right) < \infty.$$

This completes the proof. □

4.5 Conclusions

This paper studies the asymptotic properties of the quasi-maximum likelihood estimators in the EGARCH(1,1) process. For the first time we give higher-order moment conditions and we obtain tractable sufficient conditions that guarantee the integrability of the supremum norms of the log-variance derivatives. Under the Generalized Error Distribution assumption for the errors, the conditions presented in this chapter are verified. Our proofs are based on the application of the Ergodic Theorem for continuous valued sequences of random functions and our results comprise an extension of the work by Straumann and Mikosch (2006) [83].

The next step in our analysis is to formulate also the necessary conditions needed for the asymptotic normality of the QMLE to hold, which might be weaker. We leave this issue for future research.

This chapter is available as a Discussion Paper of Demos and Kyriakopoulou (2010b) [36].

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Chapter 5

Conclusions

This thesis has studied the asymptotic expansions of the econometric estimators in two time series models, the Moving Average (MA) and the Exponential GARCH (EGARCH) models. To this end, it has addressed primarily two research questions. First, it has analyzed the finite sample properties of the MM and QML estimators in the MA(1) model, as well as the QMLEs in the EGARCH$(1, 1)$ process and derived the approximate distribution of Edgeworth type. Second, it has examined the conditions under which the QMLEs in the EGARCH$(1, 1)$ are asymptotically normal.

Regarding the first research question, an extension of Sargan's (1976) [80] results was necessary so that the second order Edgeworth and Nagar-type expansions of the MM estimators were derived. Moreover, the first order expansions in Tanaka (1984) [87] were extended to include terms of second order for the QMLEs. A comparison of the expansions, either in terms of approximate bias or MSE, reveals that there is not uniform superiority of neither of the estimators of the MA parameter and the first order autocorrelation. Next, to the best of our knowledge, analytic derivatives both of the log-likelihood and the log-variance functions and also their expected values in the EGARCH$(1, 1)$ model are presented for the first time. By developing the theoretical bias approximations of all estimators, we were then able to bias correct the QMLEs in practice, under the specification of two methods and compare the simulation results between them. For given sets of parameters values, the bias correction works satisfactory for all parameters.

With respect to the second research question, sufficient conditions for the existence of moments of the log-variance derivatives, evaluated in the true parameter value and also the integrability of their supremum norms in a neighborhood around the true parameter vector, in the EGARCH$(1,1)$ model, were investigated for the first time. In particular, we extended the work by Straumann and Mikosch (2006) [83] in order to study the case when the EGARCH coefficient is not zero. The application of the Ergodic Theorem for continuous valued sequences of random elements was an important tool in order to establish the asymptotic normality of the QMLEs, avoiding more complicated calculations.

Future research should investigate the necessary and sufficient conditions for the existence and validity of the formal Edgeworth expansions that were presented in Chapter 3. This idea might be interesting to be also applied to the case of the EGARCH-M model, that was also briefly discussed at the end of Chapter 3. The EGARCH model has an substantial impact on finance and such results would be highly appreciated by financial practitioners, due to the fact that they would approximate the distributions of certain assets, for example options, and therefore derive higher order independent moments.

Appendix A

Appendix for "Edgeworth and Moment Approximations: The Case of MM and QML Estimators for the MA(1) Models"

A.1 Proof of Theorem 1

As the validity of Theorem 1 is dealt in Sargan (1976) [80] or Bhattacharya and Ghosh (1978) [16], we proceed with the coefficient derivation. Let us denote by $cf_{\overline{\varphi}}(s)$ the characteristic function of $\overline{\varphi}$. The Taylor series expansion of $\overline{\varphi}$ is:

$$\overline{\varphi} = \sum_{i=0}^{l} f^{i} A_i + \frac{1}{2} \sum_{i,j=0}^{l} f^{ij} A_i A_j + \frac{1}{6} \sum_{i,j,k=0}^{l} f^{ijk} A_i A_j A_k + o_p\left(n^{-1}\right),$$

where $f^i = \frac{\partial f}{\partial A_i}$, $f^{ij} = \frac{\partial^2 f}{\partial A_i \partial A_j}$, and $f^{ijk} = \frac{\partial^3 f}{\partial A_i \partial A_j \partial A_k}$, all evaluated at 0.

Adapting the summation convention, i.e. $f^{ij} A_i A_j = \sum_{i,j=0}^{l} f^{ij} A_i A_j$, the characteristic function of $\overline{\varphi}$ is:

$$cf_{\overline{\varphi}}(s) = \int \begin{bmatrix} \exp\left(isf^{i} A_i\right) \exp\left(\frac{is}{2} f^{ij} A_i A_j\right) \\ \exp\left(\frac{is}{6} f^{ij} A_i A_j A_k\right) \end{bmatrix} dF\left(A\right) + o\left(n^{-1}\right),$$

where $A = (A_0, ..., A_l)^{/}$.

Now

$$\exp\left(\frac{is}{2} f^{ij} A_i A_j\right) = 1 + \frac{is}{2\sqrt{n}} h^{ij} A_i A_j - \frac{s^2}{8n}\left(h^{ij} A_i A_j\right)^2 + o_p\left(n^{-1}\right),$$

and

$$\exp\left(\frac{is}{6} f^{ijk} A_i A_j A_k\right) = 1 + \frac{is}{6n} h^{ijk} A_i A_j A_k + o_p\left(n^{-1}\right).$$

Expanding $\exp\left(\frac{is}{2} f^{ij} A_i A_j\right)$ and $\exp\left(\frac{is}{6} f^{ijk} A_i A_j A_k\right)$ around $(0, ..., 0)^{/}$ the characteristic function of $\overline{\varphi}$ becomes:

$$cf_{\overline{\varphi}}(s) = \int \left[\exp\left(isf^i A_i\right)\left(1 + \frac{is}{2\sqrt{n}} h^{ij} A_i A_j + \frac{is}{6n} h^{ijk} A_i A_j A_k - \frac{s^2}{8n}\left(h^{ij} A_i A_j\right)^2\right)\right] dF\left(A\right) + o\left(n^{-1}\right),$$

where $h^{ij} = \sqrt{n} f^{ij}$ and $h^{ijk} = n f^{ijk}$.

Setting $s\left(f^1, ..., f^l\right)^{/} = z$ and noticing that

$$\begin{aligned}
\frac{\partial cf_A(z)}{\partial z_i} &= \int iA_i \exp\left(iz^{/} A\right) dF(A), \\
\frac{\partial^2 cf_A(z)}{\partial z_i \partial z_j} &= -\int A_i A_j \exp\left(iz^{/} A\right) dF(A), \\
\frac{\partial^3 cf_A(z)}{\partial z_i \partial z_j \partial z_k} &= -\int iA_i A_j A_k \exp\left(iz^{/} A\right) dF(A) \quad \textit{and} \\
\frac{\partial^4 cf_A(z)}{\partial z_i \partial z_j \partial z_k \partial z_m} &= \int A_i A_j A_k A_m \exp\left(iz^{/} A\right) dF(A),
\end{aligned}$$

we get

$$\begin{aligned}
cf_{\overline{\varphi}}(s) &= cf_A(z) - \frac{is}{2\sqrt{n}} h^{ij} \frac{\partial^2 cf_A(z)}{\partial z_i \partial z_j} - \frac{s}{6n} h^{ijk} \frac{\partial^3 cf_A(z)}{\partial z_i \partial z_j \partial z_k} \qquad \text{(app-1)} \\
&\quad - \frac{s^2}{8n} h^{ij} h^{km} \frac{\partial^4 cf_A(z)}{\partial z_i \partial z_j \partial z_k \partial z_m} + o\left(n^{-1}\right).
\end{aligned}$$

By definition, the characteristic function of A is:

$$cf_A(z) = \exp\left(ic_i z_i - \frac{1}{2} c_{ij} z_i z_j - \frac{i}{6} c_{ijk} z_i z_j z_k + \frac{1}{24} c_{ijkm} z_i z_j z_k z_m\right) + o\left(n^{-1}\right)$$

and expanding $\exp(ic_i z_i)$, $\exp\left(-\frac{i}{6}c_{ijk}z_iz_jz_k\right)$, $\exp\left(\frac{1}{24}c_{ijkm}z_iz_jz_kz_m\right)$ up to $o\left(n^{-1}\right)$ we get

$$cf_A(z) = \exp\left(-\frac{1}{2}c_{ij}z_iz_j\right)\begin{pmatrix} 1 + ic_iz_i - \frac{1}{2}\left(c_iz_i\right)^2 + \frac{1}{24}c_{ijkm}z_iz_jz_kz_m \\ -\frac{i}{6}c_{ijk}z_iz_jz_k + \frac{1}{6}\left(c_{ijk}z_iz_jz_k\right)\left(c_iz_i\right) \\ -\frac{1}{72}\left(c_{ijk}z_iz_jz_k\right)^2 \end{pmatrix} + o\left(n^{-1}\right).$$

Employing the above formula we can find the, up to 4^{th} order, derivatives of the characteristic function. Substituting into (app-1) and setting for $z_i = sf^i$ we get:

$$\begin{aligned} cf_{\overline{\varphi}}(s) &= \exp\left(-\frac{s^2}{2}c_{ij}f^if^j\right) \\ &\times\begin{pmatrix} 1 - \frac{is^3}{6}c_{ijk}f^if^jf^k + \frac{s^4}{24}c_{ijkm}f^if^jf^kf^m - \frac{s^6}{72}\left(c_{ijk}f^if^jf^k\right)^2 \\ +\frac{s^4}{6}c_{ijk}f^if^jf^k\left(c_if^i\right) - \frac{s^2}{2}\left(c_if^i\right)^2 + isc_jf^j \\ -\frac{is}{2}h^{pq}\begin{pmatrix} \frac{s^2}{\sqrt{n}}\left(c_{qj}f^j\right)\left(c_{pj}f^j\right) + \frac{i}{2}\frac{s^3}{\sqrt{n}}\left(c_{qj}f^j\right)\left(c_{pjk}f^jf^k\right) - \frac{1}{\sqrt{n}}c_{pq} \\ +\frac{i}{2}\frac{s^3}{\sqrt{n}}\left(c_{pj}f^j\right)\left(c_{qjk}f^jf^k\right) - \frac{i}{6}\frac{s^5}{\sqrt{n}}\left(c_{qj}f^j\right)\left(c_{pj}f^j\right)\left(c_{ijk}f^if^jf^k\right) \\ -i\frac{s}{\sqrt{n}}\left(c_{pqk}f^k\right) + \frac{i}{6}\frac{s^3}{\sqrt{n}}c_{pq}\left(c_{ijk}f^if^jf^k\right) + \frac{s^3}{\sqrt{n}}i\left(c_{qk}f^k\right)\left(c_{pj}f^j\right)\left(c_if^i\right) \\ -\frac{s}{\sqrt{n}}ic_p\left(c_{iq}f^i\right) - \frac{s}{\sqrt{n}}ic_{pq}\left(c_if^i\right) - \frac{s}{\sqrt{n}}ic_q\left(c_{pj}f^j\right) \end{pmatrix} \\ -\frac{s}{6}h^{pqr}\left(-\frac{s^3}{n}\left(c_{rj}f^j\right)\left(c_{qj}f^j\right)\left(c_{pj}f^j\right) + \frac{s}{n}\left[c_{qr}\left(c_{pj}f^j\right) + c_{pr}\left(c_{qj}f^j\right) + c_{pq}\left(c_{rj}f^j\right)\right]\right) \\ -\frac{s^2}{8}h^{pq}h^{rs}\begin{pmatrix} \frac{s^4}{n}\left(c_{sj}f^j\right)\left(c_{rj}f^j\right)\left(c_{qj}f^j\right)\left(c_{pj}f^j\right) \\ -\frac{s^2}{n}c_{sj}f^j\left[c_{qr}\left(c_{pj}f^j\right) + c_{pr}\left(c_{qj}f^j\right) + c_{pq}\left(c_{rj}f^j\right)\right] \\ -\frac{s^2}{n}\left[c_{rs}\left(c_{qj}f^j\right)\left(c_{pj}f^j\right) + c_{qs}\left(c_{rj}f^j\right)\left(c_{pj}f^j\right) + c_{ps}\left(c_{rj}f^j\right)\left(c_{qj}f^j\right)\right] \\ +\frac{1}{n}\left(c_{qr}c_{ps} + c_{pr}c_{qs} + c_{pq}c_{rs}\right) \end{pmatrix} \end{pmatrix} \end{aligned}$$

with a remainder of $o\left(n^{-1}\right)$.

However as $c_{ij} = c_{ij}^{(1)} + n^{-1/2}c_{ij}^{(2)} + n^{-1}c_{ij}^{(3)} + o\left(n^{-1}\right)$ there are terms of $O\left(n^{-1/2}\right)$ and $O\left(n^{-1}\right)$, in the exponential. Consequently, we have that

$$\exp\left(-\frac{s^2}{2}c_{ij}f^if^j\right) = \exp\left(-\frac{s^2c_{ij}^{(1)}f^if^j}{2}\right)\left(1 - \frac{s^2c_{ij}^{(2)}f^if^j}{2\sqrt{n}} - \frac{s^2c_{ij}^{(3)}f^if^j}{2n} + \frac{s^4\left(c_{ij}^{(2)}f^if^j\right)^2}{8n}\right) + o\left(n^{-1}\right)$$

and it follows that, with the same order of remainder,

$$
\begin{aligned}
cf_{\overline{\varphi}}(s) &= \exp\left(-\frac{s^2}{2}\omega^2\right) \\
&\times \left(\begin{array}{c}
1+\frac{is}{2\sqrt{n}}a_4^{(1)}+\frac{is}{2n}a_4^{(2)}-\frac{s^2}{2\sqrt{n}}\omega^{(2)}-\frac{s^2}{2n}\omega^{(3)}-\frac{s^2}{2n}h^{pq}\gamma_{pq}^{(1)} \\
-\frac{s^2}{6n}h^{pqr}\left(c_{qr}^{(1)}\gamma_p^{(1)}+c_{pr}^{(1)}\gamma_q^{(1)}+c_{pq}^{(1)}\gamma_r^{(1)}\right) \\
-\frac{s^2}{8n}h^{pq}h^{rs}\left(c_{qr}^{(1)}c_{ps}^{(1)}+c_{pr}^{(1)}c_{qs}^{(1)}+c_{pq}^{(1)}c_{rs}^{(1)}\right) \\
-\frac{is^3}{6\sqrt{n}}a_1^{(1)}-\frac{is^3}{6n}a_1^{(2)}-i\frac{s^3}{2\sqrt{n}}h^{pq}\gamma_q^{(1)}\gamma_p^{(1)}-i\frac{s^3}{2n}h^{pq}\gamma_q^{(2)}\gamma_p^{(1)}-i\frac{s^3}{2n}h^{pq}\gamma_q^{(1)}\gamma_p^{(2)} \\
-\frac{is^3}{4n}\omega^{(2)}a_4^{(1)}+\frac{s^4}{24n}a_2^{(1)}+\frac{s^4}{2n}h^{pq}\gamma_q^{(1)}\beta_p^{(1)}+\frac{s^4}{12n}a_4^{(1)}a_1^{(1)}+\frac{s^4}{6n}h^{pqr}\gamma_r^{(1)}\gamma_q^{(1)}\gamma_p^{(1)} \\
+\frac{s^4}{8n}\left(\omega^{(2)}\right)^2-\frac{s^6}{72n}\left(a_1^{(1)}\right)^2-\frac{s^6}{12n}a_1^{(1)}h^{pq}\gamma_q^{(1)}\gamma_p^{(1)}-\frac{s^6}{8n}h^{pq}h^{rs}\gamma_s^{(1)}\gamma_r^{(1)}\gamma_q^{(1)}\gamma_p^{(1)} \\
+\frac{is^5}{12n}\omega^{(2)}a_1^{(1)}+\frac{is^5}{4n}\omega^{(2)}h^{pq}\gamma_q^{(1)}\gamma_p^{(1)} \\
-\frac{s^2}{n}h^{pq}c_p^{(1)}\gamma_q^{(1)}+\frac{s^4}{8n}h^{pq}h^{rs}\left[\begin{array}{c}\gamma_s^{(1)}\left(c_{qr}^{(1)}\gamma_p^{(1)}+c_{pr}^{(1)}\gamma_q^{(1)}+c_{pq}^{(1)}\gamma_r^{(1)}\right) \\ +c_{rs}^{(1)}\gamma_q^{(1)}\gamma_p^{(1)}+c_{qs}^{(1)}\gamma_r^{(1)}\gamma_p^{(1)}+c_{ps}^{(1)}\gamma_r^{(1)}\gamma_q^{(1)}\end{array}\right] \\
+\frac{s^4}{6n}a_1^{(1)}a_{11}^{(1)}-\frac{s^2}{2n}\left(a_{11}^{(1)}\right)^2+\frac{is}{\sqrt{n}}a_{11}^{(1)}+\frac{is}{n}a_{11}^{(2)}-i\frac{s^3}{2n}\omega^{(2)}a_{11}^{(1)} \\
-\frac{s^2}{2n}a_4^{(1)}a_{11}^{(1)}+\frac{s^4}{2n}h^{pq}\gamma_q^{(1)}\gamma_p^{(1)}a_{11}^{(1)}
\end{array}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\omega^2 &= c_{ij}^{(1)}f^if^j, \quad \omega^{(2)}=c_{ij}^{(2)}f^if^j, \quad \omega^{(3)}=c_{ij}^{(3)}f^if^j, \\
a_1^{(1)} &= c_{ijk}^{(1)}f^if^jf^k, \quad a_1^{(2)}=c_{ijk}^{(2)}f^if^jf^k, \quad a_2^{(1)}=c_{ijkm}^{(1)}f^if^jf^kf^m, \\
\beta_p^{(1)} &= c_{pjk}^{(1)}f^jf^k, \quad \gamma_p^{(1)}=c_{pj}^{(1)}f^j, \quad \gamma_{pq}^{(1)}=c_{pqk}^{(1)}f^k, \quad \gamma_{pq}^{(2)}=c_{pqk}^{(2)}f^k, \\
a_4^{(1)} &= h^{pq}c_{pq}^{(1)}, \quad a_4^{(2)}=h^{pq}c_{pq}^{(2)}, \quad a_{11}^{(1)}=c_i^{(1)}f^i \quad \textit{and} \quad a_{11}^{(2)}=c_i^{(2)}f^i.
\end{aligned}
$$

As now $h^{pq}\gamma_q^{(2)}\gamma_p^{(1)}=h^{pq}\gamma_q^{(1)}\gamma_p^{(2)}$, $h^{pq}h^{rs}c_{qr}^{(1)}c_{ps}^{(1)}=h^{pq}h^{rs}c_{pr}^{(1)}c_{qs}^{(1)}$, $h^{pq}h^{rs}c_{pq}^{(1)}c_{rs}^{(1)}=\left(h^{pq}c_{pq}^{(1)}\right)^2$, $h^{pq}h^{rs}\gamma_s^{(1)}c_{pq}^{(1)}\gamma_r^{(1)}=h^{pq}h^{rs}c_{rs}^{(1)}\gamma_q^{(1)}\gamma_p^{(1)}=\left(h^{rs}c_{rs}^{(1)}\right)\left(h^{pq}\gamma_q^{(1)}\gamma_p^{(1)}\right)$, $h^{pq}h^{rs}\gamma_s^{(1)}c_{qr}^{(1)}\gamma_p^{(1)}=h^{pq}h^{rs}\gamma_s^{(1)}c_{pr}^{(1)}\gamma_q^{(1)}=h^{pq}h^{rs}c_{qs}^{(1)}\gamma_r^{(1)}\gamma_p^{(1)}=h^{pq}h^{rs}c_{ps}^{(1)}\gamma_r^{(1)}\gamma_q^{(1)}$, $h^{pq}h^{rs}\gamma_s^{(1)}\gamma_r^{(1)}\gamma_q^{(1)}\gamma_p^{(1)}=\left(h^{pq}\gamma_q^{(1)}\gamma_p^{(1)}\right)^2$, it follows that

$$cf_{\overline{\varphi}}(s) = \exp\left(-\frac{s^2}{2}\omega^2\right)\left(\begin{array}{c} 1+\left(\frac{1}{2\sqrt{n}}a_4^{(1)}+\frac{1}{2n}a_4^{(2)}+\frac{1}{\sqrt{n}}a_{11}^{(1)}+\frac{1}{n}a_{11}^{(2)}\right)is \\ -s^2\left(\begin{array}{c} \frac{1}{2n}a_5^{(1)}+\frac{1}{2n}a_7^{(1)}+\frac{1}{2\sqrt{n}}\omega^{(2)}+\frac{1}{2n}\omega^{(3)}+\frac{1}{4n}a_9^{(1)} \\ +\frac{1}{8n}\left(a_4^{(1)}\right)^2+\frac{1}{n}a_{12}^{(1)}+\frac{1}{2n}\left(a_{11}^{(1)}\right)^2+\frac{1}{2n}a_4^{(1)}a_{11}^{(1)} \end{array}\right) \\ -is^3\left[\frac{1}{6\sqrt{n}}a_1^{(1)}+\frac{1}{6n}a_1^{(2)}+\frac{1}{2\sqrt{n}}a_3^{(1)}+\frac{1}{n}a_3^{(2)}+\frac{\omega^{(2)}}{4n}\left(a_4^{(1)}+2a_{11}^{(1)}\right)\right] \\ +s^4\left(\begin{array}{c} \frac{1}{24n}a_2^{(1)}+\frac{1}{2n}a_{10}^{(1)}+\frac{1}{12n}a_4^{(1)}a_1^{(1)}+\frac{1}{6n}a_6^{(1)}+\frac{1}{2n}a_8^{(1)} \\ +\frac{1}{4n}a_3^{(1)}a_4^{(1)}+\frac{1}{8n}\left(\omega^{(2)}\right)^2+\frac{1}{6n}a_1^{(1)}a_{11}^{(1)}+\frac{1}{2n}a_3^{(1)}a_{11}^{(1)} \end{array}\right) \\ +is^5\left(\frac{1}{4n}\omega^{(2)}a_3^{(1)}+\frac{1}{12n}\omega^{(2)}a_1^{(1)}\right) \\ -s^6\left(\frac{1}{72n}\left(a_1^{(1)}\right)^2+\frac{1}{12n}a_1^{(1)}a_3^{(1)}+\frac{1}{8n}\left(a_3^{(1)}\right)^2\right) \end{array}\right)$$

where

$$\begin{aligned} a_3^{(1)} &= \gamma_p^{(1)}h^{pq}\gamma_q^{(1)}, \quad a_3^{(2)}=\gamma_p^{(1)}h^{pq}\gamma_q^{(2)}, \quad a_5^{(1)}=h^{pq}\gamma_{pq}^{(1)}, \quad a_6^{(1)}=h^{pqr}\gamma_p^{(1)}\gamma_r^{(1)}\gamma_q^{(1)}, \\ a_7^{(1)} &= h^{pqr}c_{pq}^{(1)}\gamma_r^{(1)}, \quad a_8^{(1)}=h^{pq}h^{rs}\gamma_s^{(1)}c_{pr}^{(1)}\gamma_q^{(1)}, \quad a_9^{(1)}=h^{pq}h^{rs}c_{qr}^{(1)}c_{ps}^{(1)}, \\ a_{10}^{(1)} &= \beta_p^{(1)}h^{pq}\gamma_q^{(1)}, \quad \textit{and} \quad a_{12}^{(1)}=h^{pq}c_p^{(1)}\gamma_q^{(1)}. \end{aligned}$$

Inverting the characteristic function of $\overline{\varphi}$ term by term, we deduce the corresponding asymptotic expansion of the density, say $g(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(-isx)\,cf_{\overline{\theta}}(s)\,ds$, and the probability function $G(m)=\Pr\left[\sqrt{n}\left(\widehat{\varphi}-\varphi\right)\leq m\right]$ as $n\to\infty$. To do so, we use the next relations:

$$\begin{aligned} (-1)^n\phi^{(n)}(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(it)^n e^{-t^2/2}e^{-itz}dt \Rightarrow \\ H_n(z)\phi(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(it)^n e^{-t^2/2}e^{-itz}dt \end{aligned}$$

where

$$\phi(z)=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itz}e^{-t^2/2}dt$$

denotes the standard normal density function, and $H_n(z)$ are the Hermite polynomials, for which we have:

$$\begin{aligned} H_0(z) &= 1, \quad H_1(z)=z, \quad H_2(z)=z^2-1, \quad H_3(z)=z^3-3z, \qquad \text{(app-2)} \\ H_4(z) &= z^4-6z^2+3, \quad H_5(z)=z^5-10z^3+15z \quad \textit{etc.} \end{aligned}$$

Now the probability function $G(m)$ is given as $G(m)=\Pr\left[\sqrt{n}\left(\widehat{\varphi}-\varphi\right)\leq m\right]=\int_{-\infty}^{m}g(x)dx$. Employing again the connection between the derivatives of the standard normal and the

Hermite polynomials we get:

$$G(m) = \Phi\left(\frac{m}{\omega}\right) - \left(\frac{1}{2\sqrt{n}}a_4^{(1)} + \frac{1}{2n}a_4^{(2)} + \frac{1}{\sqrt{n}}a_{11}^{(1)} + \frac{1}{n}a_{11}^{(2)}\right)\frac{1}{\omega}\phi\left(\frac{m}{\omega}\right)$$
$$-\left(\begin{array}{c} \frac{1}{2n}a_5^{(1)} + \frac{1}{2n}a_7^{(1)} + \frac{1}{2\sqrt{n}}\omega^{(2)} + \frac{1}{2n}\omega^{(3)} + \frac{1}{4n}a_9^{(1)} \\ +\frac{1}{8n}\left(a_4^{(1)}\right)^2 + \frac{1}{n}a_{12}^{(1)} + \frac{1}{2n}\left(a_{11}^{(1)}\right)^2 + \frac{1}{2n}a_4^{(1)}a_{11}^{(1)} \end{array}\right)\frac{1}{\omega^2}H_1\left(\frac{m}{\omega}\right)\phi\left(\frac{m}{\omega}\right)$$
$$-\left(\begin{array}{c} \frac{1}{6\sqrt{n}}a_1^{(1)} + \frac{1}{6n}a_1^{(2)} + \frac{1}{2\sqrt{n}}a_3^{(1)} + \frac{1}{n}a_3^{(2)} \\ +\frac{1}{4n}\omega^{(2)}a_4^{(1)} + \frac{1}{2n}\omega^{(2)}a_{11}^{(1)} \end{array}\right)\frac{1}{\omega^3}H_2\left(\frac{m}{\omega}\right)\phi\left(\frac{m}{\omega}\right)$$
$$-\left(\begin{array}{c} \frac{1}{24n}a_2^{(1)} + \frac{1}{2n}a_{10}^{(1)} + \frac{1}{12n}a_4^{(1)}a_1^{(1)} + \frac{1}{6n}a_6^{(1)} + \frac{1}{2n}a_8^{(1)} + \frac{1}{4n}a_3^{(1)}a_4^{(1)} \\ +\frac{1}{8n}\left(\omega^{(2)}\right)^2 + \frac{1}{6n}a_1^{(1)}a_{11}^{(1)} + \frac{1}{2n}a_3^{(1)}a_{11}^{(1)} \end{array}\right)\frac{1}{\omega^4}H_3\left(\frac{m}{\omega}\right)\phi\left(\frac{m}{\omega}\right)$$
$$-\left(\frac{1}{4n}\omega^{(2)}a_3^{(1)} + \frac{1}{12n}\omega^{(2)}a_1^{(1)}\right)\frac{1}{\omega^5}H_4\left(\frac{m}{\omega}\right)\phi\left(\frac{m}{\omega}\right)$$
$$-\left(\frac{1}{72n}\left(a_1^{(1)}\right)^2 + \frac{1}{12n}a_1^{(1)}a_3^{(1)} + \frac{1}{8n}\left(a_3^{(1)}\right)^2\right)\frac{1}{\omega^6}H_5\left(\frac{m}{\omega}\right)\phi\left(\frac{m}{\omega}\right)dx + o\left(n^{-1}\right),$$

and employing equation (app-2) we finally get the Edgeworth approximation of the distribution function of $\sqrt{n}\left(\widehat{\varphi}-\varphi\right)$, written compactly, as:

$$G(m) = \Phi\left(\frac{m}{\omega}\right) - \phi\left(\frac{m}{\omega}\right)\left[\psi_0 + \psi_1\left(\frac{m}{\omega}\right) + \psi_2\left(\frac{m}{\omega}\right)^2 + \psi_3\left(\frac{m}{\omega}\right)^3 + \psi_4\left(\frac{m}{\omega}\right)^4 + \psi_5\left(\frac{m}{\omega}\right)^5\right],$$

where

$$\psi_0 = \frac{1}{\sqrt{n}}\psi_0^{(1)} + \frac{1}{n}\psi_0^{(2)}, \quad \psi_1 = \frac{1}{\sqrt{n}}\psi_1^{(1)} + \frac{1}{n}\psi_1^{(2)}, \quad \psi_2 = \frac{1}{\sqrt{n}}\psi_2^{(1)} + \frac{1}{n}\psi_2^{(2)},$$
$$\psi_3 = \frac{1}{n}\psi_3^{(2)}, \quad \psi_5 = \frac{1}{72n}\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)^2}{\omega^6}, \quad \psi_4 = \frac{1}{12n}\frac{\omega^{(2)}\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^5},$$
$$\psi_0^{(1)} = \frac{1}{6\omega}\left\{3\left(a_4^{(1)} + 2a_{11}^{(1)}\right) - \frac{\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^2}\right\},$$
$$\psi_0^{(2)} = \frac{1}{6\omega}\left\{3\left[a_4^{(2)} + 2a_{11}^{(2)}\right] - \frac{a_1^{(2)}+6a_3^{(2)}+\frac{3}{2}\omega^{(2)}a_4^{(1)}+3\omega^{(2)}a_{11}^{(1)}}{\omega^2} + \frac{3}{2}\frac{\omega^{(2)}\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^4}\right\},$$
$$\psi_1^{(1)} = \frac{\omega^{(2)}}{2\omega^2}, \quad \psi_1^{(2)} = \frac{1}{24\omega^2}\left\{\begin{array}{c} 3\left[4\left(a_5^{(1)} + a_7^{(1)} + \omega^{(3)} + 2a_{12}^{(1)}\right) + 2a_9^{(1)} + \left(2a_{11}^{(1)} + a_4^{(1)}\right)^2\right] \\ -3\varsigma + 5\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)^2}{\omega^4} \end{array}\right\},$$
$$\psi_2^{(1)} = \frac{\left(a_1^{(1)}+3a_3^{(1)}\right)}{6\omega^3}, \quad \psi_2^{(2)} = \frac{1}{6\omega^3}\left[\begin{array}{c} a_1^{(2)} + 6a_3^{(2)} + \frac{3}{2}\omega^{(2)}a_4^{(1)} + 3\omega^{(2)}a_{11}^{(1)} \\ -3\frac{\omega^{(2)}\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^2} \end{array}\right],$$
$$\psi_3^{(2)} = -\frac{1}{72\omega^2}\left[10\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)^2}{\omega^4} - 3\varsigma\right], \text{ and}$$
$$\varsigma = \frac{a_2^{(1)}+2\left(a_4^{(1)}+2a_{11}^{(1)}\right)\left(a_1^{(1)}+3a_3^{(1)}\right)+4a_6^{(1)}+12\left(a_8^{(1)}+a_{10}^{(1)}\right)+3\left(\omega^{(2)}\right)^2}{\omega^2}.$$

In Sargan (1976) [80] we have that $\omega^{(2)} = \omega^{(3)} = \gamma_{pq}^{(2)} = 0$, $a_i^{(2)} = 0$ for $i = 1,2,3,4$ and $a_{11}^{(1)} = a_{11}^{(2)} = a_{12}^{(1)} = 0$. Under these assumptions our coefficients become identical to the ones in Sargan (1977) [81] (the corrected version of the 1976 paper [80]).

A.2 Proof of Lemma 1

To easy the notation, let $w = \frac{m}{\omega}$. Then we would like to find $d_0^{(1)}$, $d_1^{(1)}$, $d_2^{(1)}$, and $d_0^{(2)}$, $d_1^{(2)}$, $d_2^{(2)}$, and $d_3^{(2)}$ such that

$$\begin{aligned}
&\Phi(w) - \phi(w)\left[\begin{array}{r} \left(\psi_0^{(1)} + \psi_1^{(1)} w + \psi_2^{(1)} w^2\right) y \\ + \left(\psi_0^{(2)} + \psi_1^{(2)} w + \psi_2^{(2)} w^2 + \psi_3 w^3 + \psi_4 w^4 + \psi_5 w^5\right) y^2 \end{array}\right] \\
= \;& \Phi\left[w + \left(d_0^{(1)} + d_1^{(1)} w + w^2 d_2^{(1)}\right) y + \left(d_0^{(2)} + d_1^{(2)} w + w^2 d_2^{(2)} + d_3^{(2)} w^3\right) y^2\right] + o\left(n^{-1}\right)
\end{aligned}$$

where $y = \frac{1}{\sqrt{n}}$. Employing a Taylor series expansion of the right-hand side around $y = 0$ and equating terms of the same order of y we get:

$$\begin{aligned}
d_0^{(1)} &= -\psi_0^{(1)}, \quad d_0^{(2)} = -\psi_0^{(2)}, \quad d_1^{(1)} = -\psi_1^{(1)}, \quad d_1^{(2)} = -\psi_1^{(2)} + \frac{1}{2}\left(\psi_0^{(1)}\right)^2 \\
d_2^{(1)} &= -\psi_2^{(1)}, \quad d_2^{(2)} = -\psi_2^{(2)} + \psi_0^{(1)} \psi_1^{(1)}, \\
d_3^{(2)} &= -\psi_3 + \frac{1}{2}\left(\psi_1^{(1)}\right)^2 + \psi_0^{(1)} \psi_2^{(1)},
\end{aligned}$$

and

$$\psi_4 = d_2^{(1)} d_1^{(1)}, \quad \textit{and} \quad \psi_5 = \frac{1}{2}\left(d_2^{(1)}\right)^2$$

which are always true.

As $\Phi\left[w + \left(d_0^{(1)} + d_1^{(1)} w + w^2 d_2^{(1)}\right) \frac{1}{\sqrt{n}} + \left(d_0^{(2)} + d_1^{(2)} w + w^2 d_2^{(2)} + d_3^{(2)} w^3\right) \frac{1}{n}\right] + o\left(n^{-1}\right)$ one can find a standard normal variate, say z, such that $z = w + \left(d_0^{(1)} + d_1^{(1)} w + w^2 d_2^{(1)}\right) \frac{1}{\sqrt{n}} + \left(d_0^{(2)} + d_1^{(2)} w + w^2 d_2^{(2)} + d_3^{(2)} w^3\right) \frac{1}{n} + o\left(n^{-1}\right)$.

Let $w = a + bz + cz^2 + ez^3 + o\left(n^{-1}\right)$ where the coefficients a, b, c, and e are to be determined. Then substituting out z, by employing the above formula, letting $a = a^{(0)} + \frac{1}{\sqrt{n}} a^{(1)} + \frac{1}{n} a^{(2)}$ and the same for b, c, and e, and equating coefficients we get a, b, c, and e as functions of the $d_i^{(j)} s$:

$$a^{(0)} = 0, \quad a^{(1)} = -d_0^{(1)}, \quad a^{(2)} = d_0^{(1)} d_1^{(1)} - d_0^{(2)}$$

$$b^{(0)} = 1, \quad b^{(1)} = -d_1^{(1)}, \quad b^{(2)} = 2 d_2^{(1)} d_0^{(1)} + \left(d_1^{(1)}\right)^2 - d_1^{(2)},$$

$$c^{(0)} = 0, \quad c^{(1)} = -d_2^{(1)}, \quad c^{(2)} = 3 d_1^{(1)} d_2^{(1)} - d_2^{(2)},$$

$$e^{(0)} = 0, \quad e^{(1)} = 0 \quad \textit{and} \quad e^{(2)} = 2\left(d_2^{(1)}\right)^2 - d_3.$$

Hence

$w = -\frac{1}{\sqrt{n}} d_0^{(1)} + \frac{1}{n}\left(d_0^{(1)} d_1^{(1)} - d_0^{(2)}\right) + \left(1 - \frac{1}{\sqrt{n}} d_1^{(1)} + \frac{1}{n}\left(2 d_2^{(1)} d_0^{(1)} + \left(d_1^{(1)}\right)^2 - d_1^{(2)}\right)\right) z$ $+ \left(-\frac{1}{\sqrt{n}} d_2^{(1)} + \frac{1}{n}\left(3 d_1^{(1)} d_2^{(1)} - d_2^{(2)}\right)\right) z^2 + \frac{1}{n}\left(2\left(d_2^{(1)}\right)^2 - d_3^{(2)}\right) z^3 + o\left(n^{-1}\right)$ or in terms of the $\psi_j^{(i)}s$

$$\begin{aligned} w \quad = \quad & \frac{1}{\sqrt{n}} \psi_0^{(1)} + \frac{1}{n}\left(\psi_0^{(1)} \psi_1^{(1)} + \psi_0^{(2)}\right) \\ & + \left(1 + \frac{1}{\sqrt{n}} \psi_1^{(1)} + \frac{1}{n}\left(2 \psi_2^{(1)} \psi_0^{(1)} + \left(\psi_1^{(1)}\right)^2 + \psi_1^{(2)} - \frac{1}{2}\left(\psi_0^{(1)}\right)^2\right)\right) z \\ & + \left(\frac{1}{\sqrt{n}} \psi_2^{(1)} + \frac{1}{n}\left(3 \psi_1^{(1)} \psi_2^{(1)} + \psi_2^{(2)} - \psi_0^{(1)} \psi_1^{(1)}\right)\right) z^2 \\ & + \frac{1}{n}\left(2\left(\psi_2^{(1)}\right)^2 + \psi_3^{(2)} - \frac{1}{2}\left(\psi_1^{(1)}\right)^2 - \psi_0^{(1)} \psi_2^{(1)}\right) z^3 + o\left(n^{-1}\right). \end{aligned} \tag{A.1}$$

Hence, employing the connection between the $\psi_j^{(i)}s$ and the Edgeworth coefficients, $a_l^{(k)}$, setting $w = \overline{\varphi}$ we get the results of Lemma 1:

The first cumulant of w is

$$k_1^w = E\left(w\right) = \frac{1}{\sqrt{n}} \frac{1}{2\omega}\left(a_4^{(1)} + 2a_{11}^{(1)}\right) + \frac{1}{n}\frac{1}{2\omega}\left[a_4^{(2)} + 2a_{11}^{(2)}\right] + o\left(n^{-1}\right).$$

Squaring w in (A.1) and taking the expectation we get

$$E\left(w^2\right) = 1 + 2\frac{1}{\sqrt{n}}\frac{\omega^{(2)}}{2\omega^2} + \frac{1}{n}\frac{1}{4\omega^2}\left[4\left(a_5^{(1)} + a_7^{(1)} + \omega^{(3)} + 2a_{12}^{(1)}\right) + 2a_9^{(1)} + \left(2a_{11}^{(1)} + a_4^{(1)}\right)^2\right] + o\left(n^{-1}\right)$$

and

$$k_2^w = 1 + \frac{1}{\sqrt{n}}\frac{\omega^{(2)}}{\omega^2} + \frac{1}{n}\frac{1}{2\omega^2}\left[a_9^{(1)} + 2\left(a_5^{(1)} + a_7^{(1)} + \omega^{(3)} + 2a_{12}^{(1)}\right)\right] + o\left(n^{-1}\right).$$

Finally,

$$k_3^w = \frac{1}{\sqrt{n}}\frac{\left(a_1^{(1)} + 3a_3^{(1)}\right)}{\omega^3} + \frac{1}{n}\frac{1}{\omega^3}\left(a_1^{(2)} + 6a_3^{(2)}\right) + o\left(n^{-1}\right)$$

and

$$k_4^w = \frac{1}{n}\frac{1}{\omega^4}\left[a_2^{(1)} + 4a_6^{(1)} + 12\left(a_8^{(1)} + a_{10}^{(1)}\right) + \frac{9}{4}\left(\omega^{(2)}\right)^2\right] + o\left(n^{-1}\right).$$

A.3 Cumulants needed for $\widehat{\rho}$

A_0, A_1, A_2, and A_3 can be expressed as

$$
\begin{aligned}
A_0 &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(u_{t-1}+\theta u_{t-2}\right), \quad A_1=\frac{\sum_{t=2}^{n}u_{t-1}u_{t-2}}{\sqrt{n}} \\
A_2 &= \frac{1}{\sqrt{n}}\left[\begin{array}{c} \theta u_1 u_{-1}+\theta^2 u_0 u_{-1}+\theta\sum_{t=2}^{n}u_t u_{t-2}+u_n u\varepsilon_{n-1}-\theta\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+2\theta u_0 u_{-1}+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)} \\ +\frac{1}{n}\left[\left(u_0+\theta u_{-1}\right)-u_n-\theta u_{n-1}\right]\left[\left(1+\theta\right)\sum_{t=1}^{n}u_{t-1}-\theta u_{n-1}+\theta u_{-1}\right] \end{array}\right], \\
A_3 &= \frac{\sum_{t=1}^{n}\left(u_{t-1}^2-\sigma^2\right)+\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+2\theta u_0 u_{-1}+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)}}{\sqrt{n}}.
\end{aligned}
$$

It is obvious that

$$E\left(A_0\right)=E\left(A_1\right)=E\left(A_3\right)=0, \quad E\left(A_2\right)=o\left(n^{-1}\right),$$

and consequently

$$c_0=c_1=c_3=0, \quad and \quad c_2=o\left(n^{-1}\right).$$

Hence

$$c_i^{(1)}=c_i^{(2)}=0 \quad for \quad i=0,1,2,3$$

In terms of second moments, notice that

$$
\begin{aligned}
E\left(A_0^2\right) &= \frac{1}{n}\left(n\left(1+\theta\right)^2\sigma^2+2\theta^2\sigma^2-2\theta\left(1+\theta\right)\sigma^2\right)=\left(1+\theta\right)^2\sigma^2-\frac{2}{n}\theta\sigma^2, \\
E\left(A_1^2\right) &= \sigma^4-\frac{\sigma^4}{n}, \\
E\left(A_2^2\right) &= \theta^2\sigma^4+\frac{1}{n}\left[2\frac{\theta^6}{\left(\theta^2+1\right)^2}\kappa_4+\sigma^4\left(\theta^4+1\right)\right]+o\left(n^{-1}\right), \\
E\left(A_3^2\right) &= V\left(u_0^2\right)-2\frac{1}{n}\frac{\theta^2}{\left(1+\theta^2\right)^2}\kappa_4+o\left(n^{-1}\right),
\end{aligned}
$$

where $\kappa_4=\frac{E\left(\varepsilon_0^4\right)-3\sigma^4}{\sigma^4}=\frac{V\left(\varepsilon_0^2\right)-2\sigma^4}{\sigma^4}$.

Now

$$
\begin{aligned}
E\left(A_0 A_1\right) &= 0, \quad E\left(A_0 A_2\right)=\frac{1}{n}\theta^3\frac{1-\theta}{\theta^2+1}\kappa_3\sigma^3+o\left(n^{-1}\right), \\
E\left(A_0 A_3\right) &= \left(1+\theta\right)\kappa_3\sigma^3-\frac{1}{n}\theta\frac{\theta+1}{\theta^2+1}\kappa_3\sigma^3+o\left(n^{-1}\right),
\end{aligned}
$$

where $\kappa_3 = \frac{E\left(\varepsilon_0^3\right)}{\sigma^3}$ and

$$\begin{aligned} E\left(A_1 A_2\right) &= o\left(n^{-1}\right), \quad E\left(A_1 A_3\right) = 0, \\ E\left(A_2 A_3\right) &= \frac{1}{n}\frac{\theta^3\left(1-\theta^2\right)}{\left(1+\theta^2\right)^2}\sigma^4\kappa_4 + o\left(n^{-1}\right). \end{aligned}$$

Hence

$$\begin{aligned} c_{00} &= (1+\theta)^2\sigma^2 - \frac{2}{n}\theta\sigma^2, \quad c_{01} = 0, \quad c_{02} = \frac{1}{n}\theta^3\frac{1-\theta}{\theta^2+1}\kappa_3\sigma^3 + o\left(n^{-1}\right), \\ c_{03} &= (1+\theta)\kappa_3\sigma^3 - \frac{1}{n}\theta\frac{\theta+1}{\theta^2+1}\kappa_3\sigma^3 + o\left(n^{-1}\right), \quad c_{11} = \sigma^4 - \frac{\sigma^4}{n}, \quad c_{12} = 0 + o\left(n^{-1}\right), \quad c_{13} = 0, \\ c_{22} &= \theta^2\sigma^4 + \frac{1}{n}\left[2\frac{\theta^6}{\left(\theta^2+1\right)^2}\kappa_4 + \left(\theta^4+1\right)\right]\sigma^4 + o\left(n^{-1}\right), \quad c_{23} = \frac{1}{n}\frac{\theta^3\left(1-\theta^2\right)}{\left(1+\theta^2\right)^2}\sigma^4\kappa_4 + o\left(n^{-1}\right), \\ c_{33} &= \sigma^4\left(\kappa_4+2\right) - 2\frac{1}{n}\frac{\theta^2}{\left(1+\theta^2\right)^2}\sigma^4\kappa_4 + o\left(n^{-1}\right). \end{aligned}$$

For the cubes,

$$E\left(A_1^3\right) = \frac{1}{\sqrt{n}}\sigma^6\kappa_3^2 + o\left(n^{-1}\right), \quad E\left(A_2^3\right) = \frac{1}{\sqrt{n}}\theta^3\sigma^6\kappa_3^2 + o\left(n^{-1}\right).$$

$$E\left(A_0^2 A_1\right) = \frac{2}{\sqrt{n}}(1+\theta)^2\sigma^4 + o\left(n^{-1}\right), \quad E\left(A_0^2 A_2\right) = \frac{1}{\sqrt{n}}2(1+\theta)^2\theta\sigma^4 + o\left(n^{-1}\right),$$

$$E\left(A_1^2 A_2\right) = 2\frac{1}{\sqrt{n}}\theta\sigma^6 + o\left(n^{-1}\right),$$

$$E\left(A_1^2 A_3\right) = \frac{1}{\sqrt{n}}2\sigma^2 V\left(u_0^2\right) + o\left(n^{-1}\right),$$

$$E\left(A_1 A_2^2\right) = o\left(n^{-1}\right),$$

$$E\left(A_2^2 A_3\right) = \frac{2}{\sqrt{n}}\theta^2\sigma^2 V\left(u_0^2\right) + o\left(n^{-1}\right),$$

$$E\left(A_1 A_2 A_3\right) = o\left(n^{-1}\right).$$

Finally,

$$E\left(A_0^3\right) = \frac{1}{\sqrt{n}}(1+\theta)^3\sigma^3\kappa_3 + o\left(n^{-1}\right).$$

Now, as $E\left(A_j\right)=0$ for all js, we have, up to $o\left(n^{-1}\right)$,

$$\begin{aligned} c_{000} &= \frac{1}{\sqrt{n}}\left(1+\theta\right)^3\sigma^3\kappa_3,\ c_{001}=\frac{2}{\sqrt{n}}\left(1+\theta\right)^2\sigma^4,\ c_{002}=\frac{1}{\sqrt{n}}2\left(1+\theta\right)^2\theta\sigma^4,\ c_{111}=\frac{1}{\sqrt{n}}\sigma^6\kappa_3^2,\\ c_{112} &= \frac{2\theta\sigma^6}{\sqrt{n}},\ c_{113}=\frac{2\sigma^2}{\sqrt{n}}\left(\kappa_4+2\right),\ c_{122}=c_{123}=0,\ c_{222}=\frac{\theta^3\sigma^6}{\sqrt{n}}\kappa_3^2,\ c_{223}=\frac{2\theta^2\sigma^6}{\sqrt{n}}\left(\kappa_4+2\right).\end{aligned}$$

With some tedious algebra, for the fourth order cumulants, first notice that

$$E\left(A_1^4\right)=3\sigma^8+\frac{\sigma^8\left(\kappa_4^2+12\kappa_4+12\right)}{n}+o\left(n^{-1}\right),$$

$E\left[\sum_{t=2}^n u_tu_{t-2}\right]^4=3n^2\sigma^8+n\left[\kappa_4^2+12\kappa_4\sigma^4+12\sigma^8\right]+O\left(1\right)$ and

$E\left[\sum_{t=2}^n u_tu_{t-2}\right]\left[\theta u_1u_{-1}+\theta^2u_0u_{-1}+u_nu_{n-1}-\theta\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+2\theta u_0u_{-1}+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)}\right]^3=O\left(1\right)$.

Further $E\left[\sum_{t=2}^n u_tu_{t-2}\right]^3\left[\theta u_1u_{-1}+\theta^2u_0u_{-1}+u_nu_{n-1}-\theta\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+2\theta u_0u_{-1}+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)}\right]=$ $O\left(1\right)$, and $E\left\{\frac{1}{n}\left[\sum_{t=2}^n u_tu_{t-2}\right]^3\left[u_0+\theta u_{-1}-u_n-\theta u_{n-1}\right]\left[\left(1+\theta\right)\sum_{t=1}^n u_{t-1}-\theta u_{n-1}+u\varepsilon_{-1}\right]\right\}=$ $O\left(1\right)$.

Now $E\left[\sum_{t=2}^n u_tu_{t-2}\right]^2\left[\theta u_1u_{-1}+\theta^2u_0u_{-1}+u_nu_{n-1}-\theta\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+2\theta u_0u_{-1}+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)}\right]^2=$

$=\left(\theta^2+\theta^4+1\right)n\sigma^8+n\theta^2\frac{2\theta^4\sigma^4V\left(\varepsilon_0^2\right)+4\theta^2\sigma^8}{\left(1+\theta^2\right)^2}-n\frac{4\theta^4}{\left(1+\theta^2\right)}\sigma^8+O\left(1\right).$

Also $E\left[\sum_{t=2}^n u_tu_{t-2}\right]^2\left\{\frac{1}{n}\left[\left(u_0+\theta u_{-1}\right)-u_n-\theta u_{n-1}\right]\left[\left(1+\theta\right)\sum_{t=1}^n u_{t-1}-\theta u_{n-1}+\theta u_{-1}\right]\right\}^2=$ $O\left(1\right).$

Hence

$$E\left(A_2^4\right)=3\theta^4\sigma^8+\frac{1}{n}\sigma^8\left[\theta^4\kappa_4^2+12\theta^4\kappa_4\frac{2\theta^2+2\theta^4+1}{\left(\theta^2+1\right)^2}+6\theta^2\left(3\theta^2+\theta^4+1\right)\right].$$

Now to find $E\left(A_1^3A_2\right)$ notice that $E\left(\sum_{t=2}^n u_{t-1}u_{t-2}\right)^3\sum_{t=2}^n u_tu_{t-2}=6n\sigma^2E^2\left(u_0^3\right)+$ $O\left(1\right)$ and $E\left(\sum_{t=2}^n u_{t-1}u_{t-2}\right)^3u_nu_{n-1}=0$.

Furthermore, $E\left(\sum_{t=2}^n u_{t-1}u_{t-2}\right)^3\left[\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)}\right]=O\left(1\right)$ and

$E\left(\sum_{t=2}^n u_{t-1}u_{t-2}\right)^3\frac{1}{n}\left[-\theta u_0u_{n-1}+\theta^2u_{-1}^2+\theta u_nu_{n-1}+\theta^2u_{n-1}^2\right]=O\left(1\right)$.

Finally, $\frac{1}{n}E\left(\sum_{t=2}^{n}u_{t-1}u_{t-2}\right)^3\left[\left[u_0+u\varepsilon_{-1}-u_n-\theta u_{n-1}\right]\sum_{t=1}^{n}u_{t-1}\right]=O(1)$. Hence

$$E\left(A_1^3A_2\right)=\frac{1}{n}6\theta\sigma^8\kappa_3^2+o\left(n^{-1}\right).$$

To find $E\left(A_1A_2^3\right)$, first notice that $E\sum_{t=2}^{n}u_{t-1}u_{t-2}\left[\theta\sum_{t=2}^{n}u_tu_{t-2}\right]^3=0$, and

$$E\left\{\begin{array}{c}\sum_{t=2}^{n}u_{t-1}u_{t-2}\left[\sum_{t=2}^{n}u_t^2u_{t-2}^2+2\sum_{t=2}^{n-1}u_tu_{t-2}\sum_{j=t+1}^{n}u_ju_{j-2}\right]\\ \left[\theta u_1u_{-1}+\theta^2u_0u_{-1}+u_nu_{n-1}-\theta\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+2\theta u_0u_{-1}+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)}\right]\end{array}\right\}=O(1).$$

Also $$E\left\{\begin{array}{c}\sum_{t=2}^{n}u_{t-1}u_{t-2}\left[\sum_{t=2}^{n}u_t^2u_{t-2}^2+2\sum_{t=2}^{n-1}u_tu_{t-2}\sum_{j=t+1}^{n}u_ju_{j-2}\right]\\ \left[\frac{1}{n}\left[\left(u_0+\theta u_{-1}\right)-u_n-\theta u_{n-1}\right]\left[\left(1+\theta\right)\sum_{t=1}^{n}u_{t-1}-\theta u_{n-1}+\theta u_{-1}\right]\right]\end{array}\right\}=O(1),$$

$$E\left\{\begin{array}{c}\sum_{t=2}^{n}u_{t-1}u_{t-2}\sum_{t=2}^{n}u_tu_{t-2}\\ \left[\theta u_1u_{-1}+\theta^2u_0u_{-1}+u_nu_{n-1}-\theta\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+2\theta u_0u_{-1}+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)}\right]^2\end{array}\right\}=O(1),$$

$$E\left\{\begin{array}{c}\sum_{t=2}^{n}u_{t-1}u_{t-2}\sum_{t=2}^{n}u_tu_{t-2}\\ \left[\theta u_1u_{-1}+\theta^2u_0u_{-1}+u_nu_{n-1}-\theta\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+2\theta u_0u_{-1}+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)}\right]\\ \frac{1}{n}\left[\left(u_0+\theta u_{-1}\right)-u_n-\theta u_{n-1}\right]\left[\left(1+\theta\right)\sum_{t=1}^{n}u_{t-1}-\theta u_{n-1}+\theta u_{-1}\right]\end{array}\right\}=O(1).$$ Hence

$$E\left(A_1A_2^3\right)=o\left(n^{-1}\right).$$

In the same way we find $E\left(\sum_{t=2}^{n}u_{t-1}u_{t-2}\right)^2\left(\theta\sum_{t=2}^{n}u_tu_{t-2}\right)^2=n^2\theta^2\sigma^8+4n\theta^2\sigma^4\kappa_4+2n\theta^2\sigma^2E^2\left(\varepsilon_0^3\right)+10n\theta^2\sigma^8+O(1)$ and

$$E\left(A_2^2A_1^2\right)=\theta^2\sigma^8+\frac{1}{n}\sigma^8\left[\frac{13\theta^2+24\theta^4+21\theta^6+\theta^8+1}{\left(\theta^2+1\right)^2}+2\theta^2\kappa_3^2+2\frac{\left(4\theta^2+3\theta^4+2\right)\theta^2}{\left(\theta^2+1\right)^2}\kappa_4\right]+o\left(n^{-1}\right).$$

Finally,

$$E\left(A_0^4\right)=3\left(1+\theta\right)^4\sigma^4+\frac{1}{n}\left(\left(1+\theta\right)^4E\left(\varepsilon_0^4\right)-3\left(1+\theta\right)^4\sigma^4+12\left(1+\theta\right)^2\sigma^4-12\left(1+\theta\right)^3\sigma^4\right)+o\left(n^{-1}\right).$$

Consequently, and due to zero mean we get

$$\begin{aligned}c_{0000}&=\frac{\left(1+\theta\right)^4\sigma^4}{n}\kappa_4,\quad c_{1111}=\frac{\sigma^8}{n}\left(\kappa_4^2+12\kappa_4+18\right),\quad c_{2222}=\frac{\sigma^8}{n}\left(\theta^4\kappa_4^2+12\theta^4\kappa_4+18\theta^4\right)\\ c_{1122}&=\frac{\sigma^8}{n}\left(4\theta^2\kappa_4+2\theta^2\kappa_3^2+12\theta^2\right),\quad c_{1112}=\frac{6\theta\sigma^8}{n}\kappa_3^2,\quad c_{1222}=0\end{aligned}$$

with an error of order $o\left(n^{-1}\right)$.

A.4 Expansion of $\widehat{\rho}$

As the validity of the approximation is established in Kakizawa (1999) [59], let us concentrate on deriving the Edgeworth coefficients. As $\sqrt{n}\left(\widehat{\rho}-\rho\right)=f\left(A_0,A_1,A_2,A_3\right)$ by (2.2), the first derivatives evaluated at 0 are

$$f^0=0,\quad f^1=\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^2\sigma^2},\quad f^2=\frac{1}{\left(1+\theta^2\right)\sigma^2},\quad f^3=0.$$

The non-zero second order derivatives, evaluated at 0, are

$$\begin{aligned}
f^{00} &= \frac{1}{\sqrt{n}}\frac{-2\left(1-\theta+\theta^2\right)}{\left(1+\theta^2\right)^2\sigma^2},\quad f^{11}=-\frac{1}{\sqrt{n}}4\theta\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^3\sigma^4},\\
f^{12} &= -\frac{1}{\sqrt{n}}\frac{2\theta}{\left(1+\theta^2\right)^2\sigma^4},\quad f^{13}=-\frac{1}{\sqrt{n}}\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^2\sigma^4},\quad f^{23}=-\frac{1}{\sqrt{n}}\frac{1}{\left(1+\theta^2\right)\sigma^4},
\end{aligned}$$

whereas $h^{ij}=\sqrt{n}f^{ij}$, e.g. $h^{00}=\frac{-2\left(1-\theta+\theta^2\right)}{\left(1+\theta^2\right)^2\sigma^2}$, $h^{11}=-4\theta\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^3\sigma^4}$ etc.

Finally, the non-zero third order derivatives, evaluated at 0, are

$$\begin{aligned}
f^{001} &= \frac{1}{n}2\frac{\left(2\theta-2\theta^2+2\theta^3+\theta^4+1\right)}{\left(\theta^2+1\right)^3\sigma^4},\quad f^{002}=\frac{1}{n}\frac{2}{\left(1+\theta^2\right)^2\sigma^4},\quad f^{003}=\frac{1}{n}2\frac{\left(1-\theta+\theta^2\right)}{\left(1+\theta^2\right)^2\sigma^4},\\
f^{111} &= \frac{1}{n}24\frac{\theta^2\left(1+\theta^4\right)}{\left(1+\theta^2\right)^4\sigma^6},\quad f^{112}=\frac{1}{n}\frac{8\theta^2}{\left(1+\theta^2\right)^3\sigma^6},\quad f^{113}=\frac{1}{n}\frac{8\theta\left(1+\theta^4\right)}{\left(1+\theta^2\right)^3\sigma^6},\\
f^{123} &= 4\frac{1}{n}\frac{\theta}{\left(1+\theta^2\right)^2\sigma^6},\quad f^{133}=2\frac{1}{n}\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^2\sigma^6},\quad f^{233}=\frac{1}{n}\frac{2}{\left(1+\theta^2\right)\sigma^6},
\end{aligned}$$

whereas $h^{ijk}=nf^{ijk}$, e.g. $h^{001}=2\frac{\left(2\theta-2\theta^2+2\theta^3+\theta^4+1\right)}{\left(\theta^2+1\right)^3\sigma^4}$ etc.

Now from Theorem 1 (Appendix A.1) we have

$$\begin{aligned}
\omega^2 &= c_{11}^{(1)}\left(\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^2\sigma^2}\right)^2+2c_{12}^{(1)}\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^2\sigma^2}\frac{1}{\left(1+\theta^2\right)\sigma^2}+c_{22}^{(1)}\left(\frac{1}{\left(1+\theta^2\right)\sigma^2}\right)^2\\
&= \frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^4}.
\end{aligned}$$

Notice that ω^2 is the asymptotic variance of $\sqrt{T}\left(\widehat{\rho}-\rho\right)$ under the assumption of normality of the errors.

Further

$$\omega^{(2)} = c_{11}^{(2)}\left(f^1\right)^2 + 2c_{12}^{(2)} f^1 f^2 + c_{22}^{(2)}\left(f^2\right)^2 = 0$$

and

$$\begin{aligned} \omega^{(3)} &= c_{11}^{(3)}\left(f^1\right)^2 + 2c_{12}^{(3)} f^1 f^2 + c_{22}^{(3)}\left(f^2\right)^2 \\ &= \frac{2\theta^2\left(\theta^4+1\right)}{\left(1+\theta^2\right)^4} + 2\frac{\theta^6}{\left(\theta^2+1\right)^4}\kappa_4. \end{aligned}$$

Next

$$\begin{aligned} \gamma_0^{(1)} &= c_{01}^{(1)} f^1 + c_{02}^{(1)} f^2 = 0, \quad \gamma_0^{(2)} = \theta^3 \frac{1-\theta}{\left(1+\theta^2\right)^2}\sigma\kappa_3, \quad \gamma_1^{(1)} = \sigma^2 \frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^2}, \\ \gamma_1^{(2)} &= -\sigma^2 \frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^2}, \quad \gamma_2^{(1)} = \frac{\theta^2\sigma^2}{\left(1+\theta^2\right)}, \quad \gamma_2^{(2)} = \left[2\frac{\theta^6}{\left(\theta^2+1\right)^2}\kappa_4 + \left(\theta^4+1\right)\right]\frac{\sigma^2}{\left(1+\theta^2\right)}, \\ \gamma_3^{(1)} &= 0, \quad \gamma_3^{(2)} = \frac{\theta^3\left(1-\theta^2\right)}{\left(1+\theta^2\right)^3}\sigma^2\kappa_4. \end{aligned}$$

Also

$$a_1^{(1)} = \frac{6\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^5} + \frac{\left(1+\theta^4\right)^3 + \theta^3\left(1+\theta^2\right)^3}{\left(1+\theta^2\right)^6}\kappa_3^2, \quad a_1^{(2)} = 0,$$

$$\begin{aligned} a_2^{(1)} &= \kappa_4^2 \frac{5\theta^4 + 4\theta^6 + 12\theta^8 + 4\theta^{10} + 5\theta^{12} + \theta^{16} + 1}{\left(\theta^2+1\right)^8} + 12\kappa_4 \frac{\left(\theta^2 + 4\theta^4 + \theta^6 + \theta^8 + 1\right)^2}{\left(\theta^2+1\right)^8} \\ &\quad + 12\theta\left(\theta + \theta^3 + 2\theta^4 + 2\right)\frac{\left(\theta^4+1\right)^2}{\left(\theta^2+1\right)^7}\kappa_3^2 \\ &\quad + 18\frac{4\theta^2 + 13\theta^4 + 16\theta^6 + 28\theta^8 + 16\theta^{10} + 13\theta^{12} + 4\theta^{14} + \theta^{16} + 1}{\left(\theta^2+1\right)^8}, \end{aligned}$$

$$a_3^{(1)} = -4\frac{\left(\theta^2 + 4\theta^4 + \theta^6 + \theta^8 + 1\right)\left(\theta^4+1\right)\theta}{\left(\theta^2+1\right)^7}, \quad a_3^{(2)} = 0,$$

$$a_4^{(1)} = -2\left(2\theta - 2\theta^2 + 2\theta^3 + \theta^4 + 1\right)\frac{\theta + \theta^2 + 1}{\left(\theta^2+1\right)^3}, \quad a_4^{(2)} = 0,$$

$$
\begin{aligned}
a_5^{(1)} &= -4\left(-\theta+\theta^2+1\right)\frac{5\theta+9\theta^2+6\theta^3+9\theta^4+5\theta^5+3\theta^6+3}{\left(\theta^2+1\right)^4}-4\frac{\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^5}\kappa_3^2 \\
&\quad -4\frac{\left(1+\theta^4\right)^2+\theta^2\left(1+\theta^2\right)^2}{\left(1+\theta^2\right)^4}\kappa_4, \\
a_5^{(2)} &= 0,
\end{aligned}
$$

as

$$
\begin{aligned}
\gamma_{00}^{(1)} &= \frac{2}{\sqrt{n}}\left(-\theta+\theta^2+1\right)\frac{(\theta+1)^4}{\left(\theta^2+1\right)^2}\sigma^2, \quad \gamma_{11}^{(1)}=\frac{1}{\sqrt{n}}\frac{1}{\left(1+\theta^2\right)}\sigma^4\left(\kappa_3^2E^2\left(\varepsilon_0^3\right)\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)}+2\theta\right), \\
\gamma_{12}^{(1)} &= 2\frac{1}{\sqrt{n}}\theta\sigma^4\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^2}, \quad \gamma_{13}^{(1)}=\frac{1}{\sqrt{n}}2V\left(\varepsilon_0^2\right)\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^2}, \quad \gamma_{23}^{(1)}=\frac{2}{\sqrt{n}}V\left(\varepsilon_0^2\right)\frac{\theta^2}{\left(1+\theta^2\right)}, \\
\gamma_{pq}^{(2)} &= 0, \quad for \;\; p,q=0,1,2,3.
\end{aligned}
$$

Next

$$
a_6^{(1)}=24\theta^2\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)\frac{\left(\theta^4+1\right)^2}{\left(\theta^2+1\right)^{10}},
$$

$$
\begin{aligned}
a_7^{(1)} &= 2\frac{4\theta+23\theta^2+6\theta^3+29\theta^4+14\theta^5+66\theta^6+14\theta^7+29\theta^8+6\theta^9+23\theta^{10}+4\theta^{11}+3\theta^{12}+3}{\left(\theta^2+1\right)^6} \\
&\quad +2\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(\theta^2+1\right)^4}\kappa_4,
\end{aligned}
$$

$$
\begin{aligned}
a_8^{(1)} &= 2\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)\frac{11\theta^2+9\theta^4+30\theta^6+9\theta^8+11\theta^{10}+\theta^{12}+1}{\left(\theta^2+1\right)^{10}} \\
&\quad +\frac{\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)^2}{\left(\theta^2+1\right)^8}\kappa_4,
\end{aligned}
$$

$$
\begin{aligned}
a_9^{(1)} &= 8\frac{\theta+5\theta^2+3\theta^3+8\theta^4+4\theta^5+16\theta^6+4\theta^7+8\theta^8+3\theta^9+5\theta^{10}+\theta^{11}+\theta^{12}+1}{\left(\theta^2+1\right)^6} \\
&\quad +2\kappa_4\frac{\left(1+\theta^4\right)^2+\theta^2\left(1+\theta^2\right)^2}{\left(1+\theta^2\right)^4}.
\end{aligned}
$$

Further,

$$\beta_1^{(1)} = \left(\frac{\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^4}\right)\sigma^2\kappa_3^2 + 4\left(\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^3}\right)\theta\sigma^2,$$

$$\beta_2^{(1)} = 2\left(\frac{\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^4}\right)\theta\sigma^2 + \frac{1}{\left(1+\theta^2\right)^2}\theta^3\sigma^2\kappa_3^2,$$

$$\beta_3^{(1)} = \frac{\left(1+\theta^4\right)^2 + \theta^2\left(1+\theta^2\right)^2}{\left(1+\theta^2\right)^4\sigma^2}V\left(\varepsilon_0^2\right)$$

and

$$\begin{aligned} a_{10}^{(1)} = & -2\frac{\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)^2}{\left(\theta^2+1\right)^8}\kappa_4 \\ & -2\theta\left(\theta^2+\theta^3+8\theta^4+3\theta^5+2\theta^6+3\theta^7+8\theta^8+\theta^9+\theta^{10}+2\theta^{12}+2\right)\frac{\theta^4+1}{\left(\theta^2+1\right)^9}\kappa_3^2 \\ & -4\frac{7\theta^2+11\theta^4+29\theta^6+24\theta^8+29\theta^{10}+11\theta^{12}+7\theta^{14}+\theta^{16}+1}{\left(\theta^2+1\right)^8}. \end{aligned}$$

Finally,

$$a_{11}^{(1)} = a_{11}^{(2)} = a_{12}^{(1)} = 0.$$

For the **zero-mean** case, all Edgeworth coefficients are the same as in the non-zero mean one, apart from $a_4^{(1)}$, $a_5^{(1)}$, $a_7^{(1)}$, and $a_9^{(1)}$, which now stand as:

$$a_4^{(1)} = -4\theta\frac{\theta^4+1}{\left(\theta^2+1\right)^3} + \frac{1}{n}4\theta\frac{\theta^4+1}{\left(\theta^2+1\right)^3} - 2\frac{1}{n}\frac{\theta^3\left(1-\theta^2\right)}{\left(1+\theta^2\right)^3}\kappa_4,$$

$$a_5^{(1)} = -8\frac{\left(1-\theta+\theta^2\right)\left(\theta+\theta^2+1\right)}{\left(1+\theta^2\right)^2} - 4\frac{\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^5}\kappa_3^2 - 4\frac{\left(1+\theta^4\right)^2+\theta^2\left(1+\theta^2\right)^2}{\left(1+\theta^2\right)^4}\kappa_4,$$

$$a_7^{(1)} = 4\frac{9\theta^2+9\theta^4+26\theta^6+9\theta^8+9\theta^{10}+\theta^{12}+1}{\left(\theta^2+1\right)^6} + 2\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(\theta^2+1\right)^4}\kappa_4$$

and

$$a_9^{(1)} = 4\frac{7\theta^2+9\theta^4+22\theta^6+9\theta^8+7\theta^{10}+\theta^{12}+1}{\left(\theta^2+1\right)^6} + 2\kappa_4\frac{\left(1+\theta^4\right)^2+\theta^2\left(1+\theta^2\right)^2}{\left(1+\theta^2\right)^4}.$$

To find the expansion of $\sqrt{n}\,(\widehat{\mu}-\mu)$ we can apply Theorem 1 with

$$f(A_0)=A_0,\quad f^0=1\quad and\quad f^{00}=f^{000}=h^{00}=h^{000}=0.$$

Hence the non-zero Edgeworth coefficients are:

$$\begin{aligned}\omega^2 &= (1+\theta)^2\sigma^2,\quad \omega^{(3)}=-2\theta\sigma^2,\\ a_1^{(1)} &= (1+\theta)^3\sigma^3\kappa_3\quad and\quad a_2^{(1)}=(1+\theta)^4\sigma^4\kappa_4.\end{aligned}$$

Consequently, the polynomials in the expansion are:

$$\begin{aligned}\psi_0 &= -\frac{\kappa_3}{6\sqrt{n}},\quad \psi_1=\frac{1}{24n}\left[-24\frac{\theta}{(1+\theta)^2}-3\kappa_4+5\kappa_3^2\right],\\ \psi_2 &= \frac{1}{6\sqrt{n}}\kappa_3,\quad \psi_3=-\frac{1}{72n}\left[10\kappa_3^2-3\kappa_4\right],\quad \psi_4=0,\quad \psi_5=\frac{1}{72n}\kappa_3^2.\end{aligned}$$

A.5 Expansion of $\widehat{\theta}$

For $|\widehat{\rho}|<0.5$ the solution for $\widehat{\theta}$ is given in equation (2.6). Hence

$$\begin{aligned}f(\rho) &= 0,\quad \frac{\partial f(\rho)}{\partial\widehat{\rho}}=\frac{1}{2}\frac{1-\sqrt{1-4\rho^2}}{\rho^2\sqrt{1-4\rho^2}}=\frac{(1+\theta^2)^2}{(1-\theta^2)}>0,\\ \frac{\partial^2 f(\rho)}{\partial\widehat{\rho}^2} &= \frac{\sqrt{(1-4\rho^2)^3}-1+6\rho^2}{\rho^3\sqrt{(1-4\rho^2)^3}}=\frac{2\theta(3-\theta^2)(1+\theta^2)^3}{(1-\theta^2)^3},\\ \frac{\partial^3 f(\rho)}{\partial\widehat{\rho}^3} &= 3\frac{(1-4\rho^2)(1-2\rho^2)-4\rho^2(1-6\rho^2)-(1-4\rho^2)^{\frac{5}{2}}}{(1-4\rho^2)^{\frac{5}{2}}}=6\theta^4\frac{11\theta^2-5\theta^4+\theta^6+1}{(1-\theta^2)^5}.\end{aligned}$$

It follows that for $\overline{\theta}=\sqrt{n}\left(\widehat{\theta}-\theta\right)$ we have

$$\overline{\theta}=\frac{\partial f(\rho)}{\partial\widehat{\rho}}\sqrt{n}\,(\widehat{\rho}-\rho)+\frac{1}{2\sqrt{n}}\frac{\partial^2 f(\rho)}{\partial\widehat{\rho}^2}\left[\sqrt{n}\,(\widehat{\rho}-\rho)\right]^2+\frac{1}{6n}\frac{\partial^3 f(\rho)}{\partial\widehat{\rho}^3}\left[\sqrt{n}\,(\widehat{\rho}-\rho)\right]^3+o\left(n^{-1}\right),$$

where the cumulants of $\sqrt{n}\,(\widehat{\rho}-\rho)$, $k_1^{\widehat{\rho}}$, $k_2^{\widehat{\rho}}$, $k_3^{\widehat{\rho}}$ and $k_4^{\widehat{\rho}}$, are presented in section 2.3.1.

Hence Theorem 1 can be applied with $f^1=\frac{(1+\theta^2)^2}{(1-\theta^2)}$, $h^{11}=\frac{2\theta(3-\theta^2)(1+\theta^2)^3}{(1-\theta^2)^3}$, $h^{111}=6\theta^4\frac{11\theta^2-5\theta^4+\theta^6+1}{(1-\theta^2)^5}$, and $c_1^{(1)}=-\frac{(\theta+\theta^2+1)(2\theta-2\theta^2+2\theta^3+\theta^4+1)}{(\theta^2+1)^3}$, $c_{11}^{(1)}=\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{(1+\theta^2)^4}$, $c_{11}^{(3)}=$

$-\frac{2\left(-4\theta-\theta^2+6\theta^3-12\theta^5+6\theta^7-\theta^8-4\theta^9+\theta^{10}+1\right)(\theta+1)^2}{\left(\theta^2+1\right)^6} - \frac{4\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^5}\kappa_3^2 - \frac{\theta^2+4\theta^4-\theta^6+\theta^8+1}{\left(\theta^2+1\right)^4}\kappa_4$, $c_{111}^{(1)} = \sqrt{n}k_3^{\widehat{\rho}}$, $c_{1111}^{(1)} = nk_4^{\widehat{\rho}}$, and $c_1^{(2)} = c_2^{(2)} = c_{111}^{(2)} = 0$.

$$
\begin{aligned}
\omega^2 &= \frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1-\theta^2\right)^2}, \quad \omega^{(2)} = 0, \\
\omega^{(3)} &= -\frac{2\left(-4\theta-\theta^2+6\theta^3-12\theta^5+6\theta^7-\theta^8-4\theta^9+\theta^{10}+1\right)(\theta+1)^2}{\left(1-\theta^4\right)^2} \\
&\quad -\frac{4\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\kappa_3^2 - \frac{\theta^2+4\theta^4-\theta^6+\theta^8+1}{\left(1-\theta^2\right)^2}\kappa_4,
\end{aligned}
$$

$$
a_1^{(1)} = -6\theta\left(\theta^4+1\right)\frac{6\theta^4+\theta^8+1}{\left(\theta^2+1\right)\left(1-\theta^2\right)^3} + \frac{\left(1+\theta^4\right)^3+\theta^3\left(1+\theta^2\right)^3}{\left(1-\theta^2\right)^3}\kappa_3^2, \quad a_1^{(2)} = 0,
$$

$$
\begin{aligned}
a_2^{(1)} &= -6\frac{\begin{pmatrix} 1-10\theta^2+30\theta^4-106\theta^6+129\theta^8-216\theta^{10} \\ +129\theta^{12}-106\theta^{14}+30\theta^{16}-10\theta^{18}+\theta^{20} \end{pmatrix}}{\left(\theta^2+1\right)^2\left(1-\theta^2\right)^4} \\
&\quad -12\theta\left(\theta^4+1\right)\frac{\begin{pmatrix} -\theta-2\theta^2-\theta^3+10\theta^4+2\theta^5-4\theta^6+2\theta^7 \\ +10\theta^8-\theta^9-2\theta^{10}-\theta^{11}+2\theta^{12}+2 \end{pmatrix}}{\left(\theta^2+1\right)\left(1-\theta^2\right)^4}\kappa_3^2 \\
&\quad +\frac{5\theta^4+4\theta^6+12\theta^8+4\theta^{10}+5\theta^{12}+\theta^{16}+1}{\left(1-\theta^2\right)^4}\kappa_4^2,
\end{aligned}
$$

$$
\begin{aligned}
\gamma_1^{(1)} &= \frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^2\left(1-\theta^2\right)}, \quad \gamma_1^{(2)} = 0 \\
a_3^{(1)} &= 2\theta\left(3-\theta^2\right)\frac{\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^5}, \quad a_3^{(2)} = 0,
\end{aligned}
$$

$$
a_4^{(1)} = 2\theta\left(3-\theta^2\right)\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)\left(1-\theta^2\right)^3}, \quad a_4^{(2)} = 0,
$$

$$
\begin{aligned}
\gamma_{11}^{(1)} &= -6\theta\left(\theta^4+1\right)\frac{6\theta^4+\theta^8+1}{\left(\theta^2+1\right)^5\left(1-\theta^2\right)}+\frac{\left(1+\theta^4\right)^3+\theta^3\left(1+\theta^2\right)^3}{\left(1+\theta^2\right)^4\left(1-\theta^2\right)}\kappa_3^2,\\
a_5^{(1)} &= -12\theta^2\left(\theta^4+1\right)\left(3-\theta^2\right)\frac{6\theta^4+\theta^8+1}{\left(\theta^2+1\right)^2\left(1-\theta^2\right)^4}+2\theta\left(3-\theta^2\right)\frac{\left(1+\theta^4\right)^3+\theta^3\left(1+\theta^2\right)^3}{\left(1+\theta^2\right)\left(1-\theta^2\right)^4}\kappa_3^2,
\end{aligned}
$$

$$
a_6^{(1)} = 6\theta^4\frac{\left(11\theta^2-5\theta^4+\theta^6+1\right)\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)^3}{\left(1+\theta^2\right)^6\left(1-\theta^2\right)^8},
$$

$$
a_7^{(1)} = 6\theta^4\frac{11\theta^2-5\theta^4+\theta^6+1}{\left(1-\theta^2\right)^6}\frac{\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)^2}{\left(1+\theta^2\right)^6},
$$

$$
a_8^{(1)} = \left(h^{11}\right)^2 c_{11}^{(1)}\left(\gamma_1^{(1)}\right)^2 = 4\theta^2\left(3-\theta^2\right)^2\frac{\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)^3}{\left(1+\theta^2\right)^2\left(1-\theta^2\right)^8},
$$

$$
a_9^{(1)} = \left(h^{11}\right)^2\left(c_{11}^{(1)}\right)^2 = \frac{4\theta^2\left(3-\theta^2\right)^2}{\left(1-\theta^2\right)^6}\frac{\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)^2}{\left(1+\theta^2\right)^2},
$$

$$
\begin{aligned}
\beta_1^{(1)} &= 6\theta\left(\theta^4+1\right)\frac{6\theta^4+\theta^8+1}{\left(\theta^2+1\right)^3\left(1-\theta^2\right)^2}+\frac{\left(1+\theta^4\right)^3+\theta^3\left(1+\theta^2\right)^3}{\left(1+\theta^2\right)^2\left(1-\theta^2\right)^2}\kappa_3^2,\\
a_{10}^{(1)} &= -12\theta^2\left(\theta^4+1\right)\left(3-\theta^2\right)\frac{6\theta^4+\theta^8+1}{\left(1-\theta^2\right)^6}\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^2}\\
&\quad +2\theta\left(3-\theta^2\right)\left(1+\theta^2\right)\frac{\left(1+\theta^4\right)^3+\theta^3\left(1+\theta^2\right)^3}{\left(1-\theta^2\right)^6}\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^2}\kappa_3^2,
\end{aligned}
$$

$$
a_{11}^{(1)} = -\frac{\left(\theta+\theta^2+1\right)\left(2\theta-2\theta^2+2\theta^3+\theta^4+1\right)}{\left(1-\theta^4\right)}, \quad a_{11}^{(2)} = 0
$$

and

$$
a_{12}^{(1)} = -\left(\theta+\theta^2+1\right)\left(2\theta-2\theta^2+2\theta^3+\theta^4+1\right)\frac{2\theta\left(3-\theta^2\right)}{\left(1-\theta^2\right)^4}\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^2}.
$$

For the **zero mean** case, all Edgeworth coefficients which are different from the coefficients given above are:

$$
\omega^{(3)} = -2\frac{\left(1-\theta^2\right)^4}{\left(\theta^2+1\right)^2}-4\frac{\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\kappa_3^2-\frac{\theta^2+4\theta^4-\theta^6+\theta^8+1}{\left(1-\theta^2\right)^2}\kappa_4,
$$

$$a_{11}^{(1)} = -2\theta\frac{\theta^4+1}{(\theta^2+1)(1-\theta^2)}, \quad and$$
$$a_{12}^{(1)} = -4\theta^2(3-\theta^2)\frac{\theta^4+1}{(1-\theta^2)^4}\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{(1+\theta^2)^2}.$$

A.6 Expansion of QMLEs

Consider the first order conditions in equation (2.8). Now let $\overline{\varphi} = (\overline{\theta}_1, \overline{\theta}_2)^{/} = = \left(\sqrt{n}\left(\widetilde{\theta}-\theta\right), \sqrt{n}(\widetilde{\mu}-\mu)\right)^{/}$. The Taylor expansion of $\frac{1}{\sqrt{n}}\frac{\partial\ell(\widetilde{\varphi})}{\partial\varphi}$, where $\widetilde{\varphi} = \left(\widetilde{\theta}_1, \widetilde{\theta}_2\right)^{/} = \left(\widetilde{\theta}, \widetilde{\mu}\right)^{/}$ around the true value $\varphi = (\theta_1, \theta_2)^{/} = (\theta, \mu)^{/}$ can be written as:

$$\begin{aligned} 0 &= \frac{1}{\sqrt{n}}\frac{\partial\ell}{\partial\theta_j} + \sum_{i=1}^{2}\left(M_{ji}+\frac{w_{ji}}{\sqrt{n}}\right)\overline{\theta}_i + \frac{1}{2\sqrt{n}}\sum_{k,i=1}^{2}\left(M_{jik}+\frac{q_{jik}}{\sqrt{n}}\right)\overline{\theta}_i\overline{\theta}_k \\ &+\frac{1}{6n}\sum_{l,k,i=1}^{2}M_{jikl}\overline{\theta}_l\overline{\theta}_i\overline{\theta}_k + O_p\left(n^{-\frac{3}{2}}\right), \\ &\equiv g_j(\overline{\varphi}, v) + O_p\left(n^{-\frac{3}{2}}\right), \quad j=1,2, \end{aligned}$$

where $j=1,2$, $A_{ij} = \frac{1}{n}E\left(\frac{\partial^2\ell(\varphi)}{\partial\theta_j\partial\theta_i}\right)$, $K_{jik} = \frac{1}{n}E\left(\frac{\partial^3\ell(\varphi)}{\partial\theta_j\partial\theta_i\partial\theta_k}\right)$, $M_{jikl} = \frac{1}{n}E\left(\frac{\partial^4\ell(\varphi)}{\partial\theta_j\partial\theta_i\partial\theta_k\partial\theta_l}\right)$, $w_{ij} = \frac{1}{\sqrt{n}}\left(\frac{\partial^2\ell(\varphi)}{\partial\theta_j\partial\theta_i} - nA_{ij}\right)$, $q_{ijk} = \frac{1}{\sqrt{n}}\left(\frac{\partial^3\ell(\varphi)}{\partial\theta_j\partial\theta_i\partial\theta_k} - nK_{jik}\right)$, for $i,j,k=1,2$ and all derivatives are evaluated at the true values.

Let us define a vector A containing the non-zero elements of $\frac{1}{\sqrt{n}}\frac{\partial\ell}{\partial\theta_i}$, w_{ij}, q_{ijk}, for $i,j,k=1,2$. As however $w_{22}=q_{122}=q_{222}=0$ (see below) we define A as $A = (A_1,A_2,A_3,A_4,A_5,A_6)^{/} = \left(\frac{1}{\sqrt{n}}\frac{\partial\ell}{\partial\theta_1}, \frac{1}{\sqrt{n}}\frac{\partial\ell}{\partial\theta_2}, w_{11}, w_{12}, q_{111}, q_{112}\right)^{/}$. Solving for $\overline{\theta}_j$, and $j=1,2$, as continuously differentiable functions of A, gives:

$$\begin{aligned} \overline{\theta}_j(A) &= \sum_{a=1}^{6}\frac{\partial\overline{\theta}_j(0)}{\partial A_a}A_a + \frac{1}{2}\sum_{a,b=1}^{6}\frac{\partial^2\overline{\theta}_j(0)}{\partial A_a\partial A_b}A_aA_b + \frac{1}{6}\sum_{a,b,c=1}^{6}\frac{\partial^3\overline{\theta}_j(0)}{\partial A_a\partial A_b\partial A_c}A_aA_bA_c + O_p\left(n^{-\frac{3}{2}}\right) \\ &\equiv \sum_{a=1}^{6}f_j^aA_a + \frac{1}{2\sqrt{n}}\sum_{a,b=1}^{6}h_j^{ab}A_aA_b + \frac{1}{6n}\sum_{a,b,c=1}^{6}h_j^{abc}A_aA_bA_c + O_p\left(n^{-\frac{3}{2}}\right), \end{aligned}$$

where $f_j^a = \frac{\partial\overline{\theta}_j(0)}{\partial A_a}$, $h_j^{ab} = \sqrt{n}\frac{\partial^2\overline{\theta}_j(0)}{\partial A_a\partial A_b}$ and $h_j^{abc} = n\frac{\partial^3\overline{\theta}_j(0)}{\partial A_a\partial A_b\partial A_c}$ (employing the notation of Theorem 1).

Now the derivatives can be found by solving the following system of equations, for $j,k=1,2$ and $a,b,c=1,...,6$:

$0 = \sum_{k=1}^{2} M_{jk} f_k^a + \frac{\partial g_j(0,0)}{\partial A_a}$,

$0 = \sum_{k=1}^{2} \left(\frac{1}{\sqrt{n}} \sum_{l=1}^{2} M_{jkl} f_l^b + \frac{\partial^2 g_j(0,0)}{\partial A_b \partial \bar{\theta}_k} \right) f_k^a + \sum_{k=1}^{2} \frac{\partial^2 g_j(0,0)}{\partial A_a \partial \bar{\theta}_k} f_k^b + \frac{1}{\sqrt{n}} \sum_{k=1}^{2} M_{jk} h_k^{ab}$, and

$0 = \sum_{k=1}^{2} \left(\frac{1}{n} \sum_{p,l=1}^{2} M_{jlkp} f_b^l f_c^p + \frac{1}{n} \sum_{l=1}^{2} M_{jkl} h_l^{bc} + \sum_{l=1}^{2} \frac{\partial^3 g_j(0,0)}{\partial A_c \partial \widetilde{\theta}_l \partial \bar{\theta}_k} f_l^b + \sum_{p=1}^{2} \frac{\partial^3 g_j(0,0)}{\partial \widetilde{\theta}_p \partial A_b \partial \bar{\theta}_k} f_c^p \right) f_k^a$
$+ \sum_{k=1}^{2} \left(\frac{1}{n} \sum_{l=1}^{2} M_{jkl} h_l^{ac} + \sum_{p=1}^{2} \frac{\partial^3 g_j(0,0)}{\partial \widetilde{\theta}_p \partial \bar{\theta}_k \partial A_a} f_c^p \right) f_k^b + \sum_{k=1}^{2} \left(\frac{1}{n} \sum_{p=1}^{2} M_{jkp} f_c^p + \frac{1}{\sqrt{n}} \frac{\partial^2 g_j(0,0)}{\partial A_c \partial \bar{\theta}_k} \right) h_k^{ab}$
$+ \frac{1}{\sqrt{n}} \sum_{k=1}^{2} \frac{\partial^2 f_j(0,0)}{\partial \bar{\theta}_k \partial A_a} h_k^{bc} + \frac{1}{\sqrt{n}} \sum_{k=1}^{2} \frac{\partial^2 f_j(0,0)}{\partial A_b \partial \bar{\theta}_k} h_k^{ac} + \frac{1}{n} \sum_{k=1}^{2} M_{jk} h_k^{abc}$. Notice that the first two equations are as in Tanaka (1984) [87]. However, the third is completely new (Tanaka 1984 [87] is developing a 1^{st} order expansion).

Hence, first consider $j = 1$ and observe that $\frac{\partial g_1(0,0)}{\partial A_1} = 1$, and $\frac{\partial g_1(0,0)}{\partial A_a} = 0$ for $a = 2, ..., 6$. It follows that

$$f_1^1 = 1 - \theta^2, \quad \textit{and} \quad f_1^2 = f_1^3 = f_1^4 = f_1^5 = f_1^6 = 0.$$

For $j = 2$, observe that $\frac{\partial g_2(0,0)}{\partial A_2} = 1$, and $\frac{\partial g_2(0,0)}{\partial A_1} = ... = \frac{\partial g_2(0,0)}{\partial A_6} = 0$ and it follows that

$$f_2^2 = \sigma^2 (1 + \theta)^2, \quad f_1^2 = f_3^2 = f_4^2 = f_5^2 = f_6^2 = 0.$$

Applying the same logic and by the notation of Theorem 1 we find that the non-zero second derivatives for $j = 1$ are:

$$h_1^{11} = -6\theta \left(1 - \theta^2\right), \; h_1^{13} = \left(1 - \theta^2\right)^2,$$

$$h_1^{22} = 2\sigma^2 (1 + \theta) \left(1 - \theta^2\right), \; \textit{and} \; h_1^{24} = \sigma^2 (1 + \theta)^2 \left(1 - \theta^2\right).$$

Finally we have

$$h_1^{111} = \left(-12 + 72\theta^2\right) \left(1 - \theta^2\right), \; h_1^{113} = -18\theta \left(1 - \theta^2\right)^2, \; h_1^{115} = \left(1 - \theta^2\right)^3,$$

$$h_1^{122} = 2\sigma^2 (1 - 7\theta) (1 + \theta) \left(1 - \theta^2\right), \; h_1^{124} = 2\sigma^2 \left(2 - 3\theta - 5\theta^2\right) (1 + \theta) \left(1 - \theta^2\right),$$

$$h_1^{126} = \sigma^2 (1 + \theta)^2 \left(1 - \theta^2\right)^2, \; h_1^{133} = 2 \left(1 - \theta^2\right)^3, \; h_1^{144} = 2\sigma^2 (1 + \theta)^2 \left(1 - \theta^2\right)^2,$$

$$h_1^{234} = \sigma^2 (1 + \theta)^2 \left(1 - \theta^2\right)^2, \; h_1^{223} = 2\sigma^2 (1 + \theta) \left(1 - \theta^2\right)^2,$$

and

$$
\begin{aligned}
h_2^{112} &= -2\left(1-\theta^2\right)(7\theta-1)(\theta+1)\sigma^2, \quad h_2^{114} = 2(5\theta-2)(\theta-1)(\theta+1)^3\sigma^2, \\
e_2^{116} &= \sigma^2(1+\theta)^2\left(1-\theta^2\right)^2, \quad e_2^{123} = 2\sigma^2\left(1-\theta^2\right)^2(1+\theta), \quad e_2^{134} = \sigma^2(1+\theta)^2\left(1-\theta^2\right)^2, \\
e_2^{222} &= 12\sigma^4\left(1-\theta^2\right)(1+\theta)^2, \quad e_2^{224} = 6\sigma^4\left(1-\theta^2\right)(1+\theta)^3 \quad e_2^{244} = 2\sigma^4(1+\theta)^4\left(1-\theta^2\right),
\end{aligned}
$$

whereas all the other derivatives are 0.

A.7 Cumulants needed for $\widetilde{\theta}$

The derivatives of $\ell(\theta,\mu)$ w.r.t. θ are:

$$
\begin{aligned}
\frac{\partial \ell}{\partial \theta} &= -\frac{1}{\sigma^2}\sum_{t=1}^{n} u_t \frac{\partial u_t}{\partial \theta}, \quad \frac{\partial^2 \ell}{\partial \theta^2} = -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(u_t\frac{\partial^2 u_t}{\partial \theta^2} + \left(\frac{\partial u_t}{\partial \theta}\right)^2\right) \\
\frac{\partial^3 \ell}{\partial \theta^3} &= -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(u_t\frac{\partial^3 u_t}{\partial \theta^3} + 3\left(\frac{\partial u_t}{\partial \theta}\right)\frac{\partial^2 u_t}{\partial \theta^2}\right), \\
\frac{\partial^4 \ell}{\partial \theta^4} &= -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(u_t\frac{\partial^4 u_t}{\partial \theta^4} + 3\left(\frac{\partial^2 u_t}{\partial \theta^2}\right)^2 + 4\frac{\partial u_t}{\partial \theta}\frac{\partial^3 u_t}{\partial \theta^3}\right).
\end{aligned}
$$

Noting now that

$$
\begin{aligned}
\frac{\partial u_t}{\partial \theta} &= -u_{t-1} - \theta\frac{\partial u_{t-1}}{\partial \theta} = \ldots = -\sum_{i=0}^{\infty}(-\theta)^i u_{t-1-i}, \\
\frac{\partial^2 u_t}{\partial \theta^2} &= -2\frac{\partial u_{t-1}}{\partial \theta} - \theta\frac{\partial^2 u_{t-1}}{\partial \theta^2} = \ldots = 2\sum_{k=0}^{\infty}(-\theta)^k\left(-\frac{\partial u_{t-1-k}}{\partial \theta}\right) = 2\sum_{i=0}^{\infty}(i+1)(-\theta)^i u_{t-2-i}, \\
\frac{\partial^3 u_t}{\partial \theta^3} &= -3\frac{\partial^2 u_{t-1}}{\partial \theta^2} - \theta\frac{\partial^3 u_{t-1}}{\partial \theta^3} = \ldots = -6\sum_{i=0}^{\infty}\frac{(i+1)(i+2)}{2}(-\theta)^i u_{t-3-i}, \\
\frac{\partial^4 u_t}{\partial \theta^4} &= -4\frac{\partial^3 u_{t-1}}{\partial \theta^3} - \theta\frac{\partial^4 u_{t-1}}{\partial \theta^4} = \ldots = 4\sum_{i=0}^{\infty}(i+1)(i+2)(i+3)(-\theta)^i u_{t-4-i},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
E\left(\frac{\partial u_t}{\partial \theta}\right) &= 0, \quad E\left(\frac{\partial u_t}{\partial \theta}\right)^2 = \frac{\sigma^2}{1-\theta^2}, \quad E\left(\frac{\partial^2 u_t}{\partial \theta^2}\right)^2 = 4\sigma^2\frac{1+\theta^2}{\left(1-\theta^2\right)^3}, \\
E\left(\frac{\partial^2 u_t}{\partial \theta^2}\frac{\partial u_t}{\partial \theta}\right) &= \frac{2\sigma^2}{\theta}\sum_{i=0}^{\infty}(i+1)\left(\theta^2\right)^{i+1} = \sigma^2\frac{2\theta}{\left(1-\theta^2\right)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(\frac{\partial^3 u_t}{\partial \theta^3}\frac{\partial u_t}{\partial \theta}\right) &= 6\theta^2 E\left[\left(\sum_{i=0}^{\infty}\frac{(i+1)(i+2)}{2}(-\theta)^i u_{t-3-i}\right)\left(\sum_{k=0}^{\infty}(-\theta)^k u_{t-3-k}\right)\right] \\
&= 6\theta^2 E\left(\sum_{i=0}^{\infty}\frac{(i+1)(i+2)}{2}\left(\theta^2\right)^i u_{t-3-i}^2\right) = \frac{6\theta^2\sigma^2}{\left(1-\theta^2\right)^3}.
\end{aligned}
$$

Hence the expectations of the derivatives evaluated at the true $\varphi = (\theta, \mu)^{/}$ are:

$$
\begin{aligned}
E\left(\frac{\partial \ell(\varphi)}{\partial \theta}\right) &= 0, \quad E\left(\frac{\partial^2 l(\varphi)}{\partial \theta^2}\right) = -\frac{1}{\sigma^2}\sum_{t=1}^{n} E\left(\frac{\partial u_t}{\partial \theta}\right)^2 = -\frac{n}{1-\theta^2}, \\
E\left(\frac{\partial^3 \ell(\varphi)}{\partial \theta^3}\right) &= -\frac{3}{\sigma^2}\sum_{t=1}^{n} E\left(\left(\frac{\partial u_t}{\partial \theta}\right)\frac{\partial^2 u_t}{\partial \theta^2}\right) = -6\frac{n\theta}{\left(1-\theta^2\right)^2}, \\
E\left(\frac{\partial^4 \ell(\varphi)}{\partial \theta^4}\right) &= -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(3E\left(\frac{\partial^2 u_t}{\partial \theta^2}\right)^2 + 4E\left(\frac{\partial u_t}{\partial \theta}\frac{\partial^3 u_t}{\partial \theta^3}\right)\right) = -12n\frac{1+3\theta^2}{\left(1-\theta^2\right)^3}
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
M_{11} &= \frac{1}{n}E\left(\frac{\partial^2 \ell(\theta)}{\partial \theta^2}\right) = -\frac{1}{1-\theta^2}, \quad M_{111} = \frac{1}{n}E\left(\frac{\partial^3 \ell(\varphi)}{\partial \theta^3}\right) = -6\frac{\theta}{\left(1-\theta^2\right)^2}, \\
M_{1111} &= \frac{1}{n}E\left(\frac{\partial^4 \ell(\varphi)}{\partial \theta^4}\right) = -12\frac{1+3\theta^2}{\left(1-\theta^2\right)^3}.
\end{aligned}
$$

Now, let us calculate the derivatives of $\ell(\theta, \mu)$ with respect to the parameter μ:

$$
\begin{aligned}
\frac{\partial \ell}{\partial \mu} &= -\frac{1}{\sigma^2}\sum_{t=1}^{n} u_t\frac{\partial u_t}{\partial \mu}, \quad \frac{\partial^2 \ell}{\partial \mu^2} = -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(\frac{\partial u_t}{\partial \mu}\right)^2 - \frac{1}{\sigma^2}\sum_{t=1}^{n} u_t\frac{\partial^2 u_t}{\partial \mu^2} \\
\frac{\partial^3 \ell}{\partial \mu^3} &= -\frac{3}{\sigma^2}\sum_{t=1}^{n}\frac{\partial u_t}{\partial \mu}\frac{\partial^2 u_t}{\partial \mu^2} - \frac{1}{\sigma^2}\sum_{t=1}^{n} u_t\frac{\partial^3 u_t}{\partial \mu^3} \\
\frac{\partial^4 \ell}{\partial \mu^4} &= -\frac{3}{\sigma^2}\sum_{t=1}^{n}\left(\frac{\partial^2 u_t}{\partial \mu^2}\right)^2 - \frac{4}{\sigma^2}\sum_{t=1}^{n}\frac{\partial u_t}{\partial \mu}\frac{\partial^3 u_t}{\partial \mu^3} - \frac{1}{\sigma^2}\sum_{t=1}^{n} u_t\frac{\partial^4 u_t}{\partial \mu^4}
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial u_t}{\partial \mu} &= -1 - \theta\frac{\partial u_{t-1}}{\partial \mu} = \ldots = -\sum_{i=0}^{\infty}(-\theta)^i = -\frac{1}{1+\theta}, \quad \textit{and} \\
\frac{\partial^2 u_t}{\partial \mu^2} &= \frac{\partial^3 u_t}{\partial \mu^3} = \frac{\partial^4 u_t}{\partial \mu^4} = 0.
\end{aligned}
$$

Hence their expectations at the true θ are:

$$E\left(\frac{\partial \ell(\theta)}{\partial \mu}\right) = 0, \quad E\left(\frac{\partial^2 \ell(\theta)}{\partial \mu^2}\right) = -\frac{n}{(1+\theta)^2\sigma^2}, \quad E\left(\frac{\partial^3 \ell(\theta)}{\partial \mu^3}\right) = E\left(\frac{\partial^4 \ell(\theta)}{\partial \mu^4}\right) = 0.$$

It follows that

$$M_{22} = \frac{1}{n}E\left(\frac{\partial^2 \ell(\varphi)}{\partial \theta_2^2}\right) = -\frac{1}{(1+\theta)^2\sigma^2}, \; M_{222} = \frac{1}{n}E\left(\frac{\partial^3 \ell(\varphi)}{\partial \theta_2^3}\right) = 0, \; M_{2222} = \frac{1}{n}E\left(\frac{\partial^4 \ell(\varphi)}{\partial \theta_2^4}\right) = 0.$$

Furthermore, the cross derivatives are

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \mu \partial \theta} &= -\frac{1}{\sigma^2}\left(\sum_{t=1}^{n}\frac{\partial u_t}{\partial \mu}\frac{\partial u_t}{\partial \theta} + \sum_{t=1}^{n} u_t \frac{\partial^2 u_t}{\partial \mu \partial \theta}\right), \quad \textit{and} \\
E\left(\frac{\partial u_t}{\partial \mu}\right) &= -\frac{1}{1+\theta}, \quad E\left(\frac{\partial u_t}{\partial \mu}\right)^2 = \frac{1}{(1+\theta)^2}, \quad \textit{and} \quad E\left(\frac{\partial^2 \ell}{\partial \mu \partial \theta}\right) = A_{12} = 0,
\end{aligned}$$

as

$$\begin{aligned}
\frac{\partial^2 u_t}{\partial \mu \partial \theta} &= -\frac{\partial u_{t-1}}{\partial \mu} - \theta \frac{\partial^2 u_{t-1}}{\partial \mu \partial \theta} = \ldots = -\sum_{i=0}^{\infty}(-\theta)^i \frac{\partial u_{t-1-i}}{\partial \mu} = \frac{1}{(1+\theta)^2}, \\
\left(\frac{\partial^2 u_t}{\partial \mu \partial \theta}\right)^2 &= \frac{1}{(1+\theta)^4}, \\
\frac{\partial^3 u_t}{\partial \mu \partial \theta^2} &= -2\frac{\partial^2 u_{t-1}}{\partial \theta \partial \mu} - \theta \frac{\partial^3 u_{t-1}}{\partial \theta^2 \partial \mu} = -2\frac{1}{(1+\theta)^3}, \\
\frac{\partial^3 u_t}{\partial \theta \partial \mu^2} &= -\frac{\partial^2 u_{t-1}}{\partial \mu^2} - \theta \frac{\partial^3 u_{t-1}}{\partial \theta \partial \mu^2} = 0 - \theta \frac{\partial^3 u_{t-1}}{\partial \theta \partial \mu^2} = \ldots = 0.
\end{aligned}$$

We have also

$$\begin{aligned}
\frac{\partial^3 \ell}{\partial \mu^2 \partial \theta} &= -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(\frac{\partial^2 u_t}{\partial \mu^2}\frac{\partial u_t}{\partial \theta} + 2\frac{\partial u_t}{\partial \mu}\frac{\partial^2 u_t}{\partial \mu \partial \theta} + u_t \frac{\partial^3 u_t}{\partial \mu^2 \partial \theta}\right), \\
\textit{where} \; \frac{\partial^2 u_t}{\partial \mu^2} &= 0, \quad \frac{\partial^2 u_t}{\partial \mu \partial \theta} = \frac{1}{(1+\theta)^2}.
\end{aligned}$$

So, its expected value is

$$E\left(\frac{\partial^3 \ell}{\partial \mu^2 \partial \theta}\right) = \frac{2n}{(1+\theta)^3\sigma^2} \quad \textit{and} \quad M_{122} = \frac{1}{n}E\left(\frac{\partial^3 \ell(\varphi)}{\partial \theta_1 \partial \theta_2 \partial \theta_2}\right) = \frac{2}{(1+\theta)^3\sigma^2},$$

as we have

$$E\left(\frac{\partial^2 u_t}{\partial \mu^2}\frac{\partial u_t}{\partial \theta}\right) = 0 \quad \textit{and} \quad E\left(\frac{\partial u_t}{\partial \mu}\frac{\partial^2 u_t}{\partial \mu \partial \theta}\right) = -\frac{1}{(1+\theta)^3}.$$

Next

$$\begin{aligned}
\frac{\partial^3 \ell}{\partial\mu\partial\theta^2} &= -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(2\frac{\partial^2 u_t}{\partial\mu\partial\theta}\frac{\partial u_t}{\partial\theta}+\frac{\partial u_t}{\partial\mu}\frac{\partial^2 u_t}{\partial\theta^2}+u_t\frac{\partial^3 u_t}{\partial\mu\partial\theta^2}\right),\\
E\left(\frac{\partial^3 \ell}{\partial\mu\partial\theta^2}\right) &= 0 \quad and \quad M_{112}=\frac{1}{n}E\left(\frac{\partial^3 \ell(\varphi)}{\partial\theta_1\partial\theta_1\partial\theta_2}\right)=0.
\end{aligned}$$

Hence,

$$M_{12}=E\left(\frac{\partial^2\ell}{\partial\mu\partial\theta}\right)=M_{112}=\frac{1}{n}E\left(\frac{\partial^3\ell(\varphi)}{\partial\theta^2\partial\mu}\right)=0,\quad M_{122}=\frac{1}{n}E\left(\frac{\partial^3\ell(\varphi)}{\partial\theta\partial\mu^2}\right)=\frac{2}{(1+\theta)^3\sigma^2}.$$

Moreover, we calculate the next derivatives

$$\begin{aligned}
\frac{\partial^4 \ell}{\partial\mu\partial\theta^3} &= -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(3\frac{\partial^3 u_t}{\partial\mu\partial\theta^2}\frac{\partial u_t}{\partial\theta}+3\frac{\partial^2 u_t}{\partial\mu\partial\theta}\frac{\partial^2 u_t}{\partial\theta^2}+\frac{\partial u_t}{\partial\mu}\frac{\partial^3 u_t}{\partial\theta^3}+u_t\frac{\partial^4 u_t}{\partial\mu\partial\theta^3}\right),\\
\frac{\partial^4 \ell}{\partial\mu^2\partial\theta^2} &= -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(2\frac{\partial^3 u_t}{\partial\mu^2\partial\theta}\frac{\partial u_t}{\partial\theta}+\frac{\partial^2 u_t}{\partial\mu^2}\frac{\partial^2 u_t}{\partial\theta^2}+2\left(\frac{\partial^2 u_t}{\partial\mu\partial\theta}\right)^2+2\frac{\partial u_t}{\partial\mu}\frac{\partial^3 u_t}{\partial\mu\partial\theta^2}+u_t\frac{\partial^4 u_t}{\partial\mu^2\partial\theta^2}\right),\\
\frac{\partial^4 \ell}{\partial\mu^3\partial\theta} &= -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(\frac{\partial^3 u_t}{\partial\mu^3}\frac{\partial u_t}{\partial\theta}+3\frac{\partial^2 u_t}{\partial\mu^2}\frac{\partial^2 u_t}{\partial\mu\partial\theta}+3\frac{\partial u_t}{\partial\mu}\frac{\partial^3 u_t}{\partial\mu^2\partial\theta}+u_t\frac{\partial^4 u_t}{\partial\mu^3\partial\theta}\right),
\end{aligned}$$

and their expected values

$$\begin{aligned}
E\left(\frac{\partial^4 \ell}{\partial\mu\partial\theta^3}\right) &= 0 \quad as \quad \frac{\partial^4 u_t}{\partial\mu\partial\theta^3}=6\frac{1}{(1+\theta)^4},\\
E\left(\frac{\partial^4 \ell}{\partial\mu^2\partial\theta^2}\right) &= -6\frac{T}{(1+\theta)^4\sigma^2} \quad as \quad \frac{\partial^4 u_t}{\partial\mu^2\partial\theta^2}=0 \quad and\\
E\left(\frac{\partial^4 \ell}{\partial\mu^3\partial\theta}\right) &= 0 \quad as \quad \frac{\partial^4 u_t}{\partial\mu^3\partial\theta}=0.
\end{aligned}$$

Hence,

$$\begin{aligned}
M_{1112} &= \frac{1}{n}E\left(\frac{\partial^4\ell(\varphi)}{\partial\theta^3\partial\mu}\right)=0,\quad M_{1122}=\frac{1}{n}E\left(\frac{\partial^4\ell(\varphi)}{\partial\theta^2\partial\mu^2}\right)=-6\frac{1}{(1+\theta)^4\sigma^2},\\
M_{1222} &= \frac{1}{n}E\left(\frac{\partial^4\ell(\varphi)}{\partial\theta\partial\mu^3}\right)=0.
\end{aligned}$$

For the cumulants of $v_i s$, the $A_i s$ in terms of Theorem 1, notice that in the maximization of the likelihood we have that for any admissible θ and μ we have that $u_t = y_t - \mu - \theta u_{t-1}$, with u_0 drawn from the stationary distribution. Hence we have that the derivatives of the $u_t\prime s$ with respect to the parameters θ and μ are:

$\frac{\partial u_t}{\partial \theta} = -u_{t-1} - \theta \frac{\partial u_{t-1}}{\partial \theta} = ... = -\sum_{i=0}^{t-1} (-\theta)^i u_{t-1-i}$, for $t > 1$ and $\frac{\partial u_0}{\partial \theta} = 0$. $\frac{\partial^2 u_t}{\partial \theta^2} = -2\frac{\partial u_{t-1}}{\partial \theta} - \theta \frac{\partial^2 u_{t-1}}{\partial \theta^2} = 2\sum_{i=0}^{t-2} (i+1)(-\theta)^i u_{t-2-i}$, for $t > 2$ and $\frac{\partial^2 u_0}{\partial \theta^2} = \frac{\partial^2 u_1}{\partial \theta^2} = 0$. $\frac{\partial^3 u_t}{\partial \theta^3} = -3\frac{\partial^2 u_{t-1}}{\partial \theta^2} - \theta \frac{\partial^3 u_{t-1}}{\partial \theta^3} = -6\sum_{i=0}^{t-3} \frac{(i+1)(i+2)}{2} (-\theta)^i u_{t-3-i}$, for $t > 3$ and $\frac{\partial u_0}{\partial \theta} = \frac{\partial^2 u_1}{\partial \theta^2} = \frac{\partial^2 u_2}{\partial \theta^2} = 0$. $\frac{\partial u_t}{\partial \mu} = -1 - \theta \frac{\partial u_{t-1}}{\partial \mu} - \sum_{i=0}^{t-1} (-\theta)^i = -\frac{1-(-\theta)^t}{1+\theta}$, $\frac{\partial^2 u_t}{\partial \mu^2} = \frac{\partial^3 u_t}{\partial \mu^3} = \frac{\partial^4 u_t}{\partial \mu^4} = 0$, and $\frac{\partial^2 u_t}{\partial \mu \partial \theta} = -\frac{[t+(t+1)(-\theta)](-\theta)^{t-1}-1}{(1+\theta)^2}$, $\frac{\partial^3 u_t}{\partial \mu \partial \theta^2} = \frac{t(t-1)(-\theta)^{t-2}+2t(t+1)(-\theta)^{t-1}+(t+1)(t+2)(-\theta)^t-2}{(1+\theta)^3}$, and $\frac{\partial^3 u_t}{\partial \theta \partial \mu^2} = 0$.

Hence, adapting the notation of Theorem 1, and as all first order cumulants of the $A_i s$ are 0, we have that.$c_i^{(1)} = c_i^{(2)} = 0$ for $i = 1, .., 6$. The second order cumulants are:

$$\begin{aligned}
c_{11}^{(1)} &= cum(v_1, v_1) = \frac{1}{n}\frac{1}{\sigma^4}\sum_{t=1}^{n} E\left(u_t^2\left(\sum_{i=0}^{\infty}(-\theta)^i u_{t-1-i}\right)^2\right) - 2\frac{1}{n}\frac{1}{\sigma^4}\sum_{t=1}^{n-1} E\left(u_t\frac{\partial u_t}{\partial \theta}\sum_{j=t+1}^{n} u_j \frac{\partial u_j}{\partial \theta}\right) \\
&= \frac{1}{1-\theta^2}
\end{aligned}$$

$$c_{12}^{(1)} = cum(v_1, v_2) = \frac{1}{n}E\left(\frac{\partial \ell}{\partial \theta}\frac{\partial \ell}{\partial \mu}\right) = 0$$

$$\begin{aligned}
c_{13}^{(1)} &= cum(v_1, v_3) = \frac{1}{n}E\left(\frac{\partial \ell}{\partial \theta}\frac{\partial^2 \ell(\varphi)}{\partial \theta^2}\right) = \\
&= \frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n} u_t\frac{\partial u_t}{\partial \theta}\sum_{t=1}^{n} u_t\frac{\partial^2 u_t}{\partial \theta^2}\right) + \frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n} u_t\frac{\partial u_t}{\partial \theta}\sum_{t=1}^{n}\left(\frac{\partial u_t}{\partial \theta}\right)^2\right) = \\
&= 2\theta\frac{1}{(1-\theta^2)^2} + 2\frac{1}{n}\frac{\theta}{1-\theta^2}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}\theta^{2(k-1)} = 4\frac{\theta}{(1-\theta^2)^2} - 2\frac{1}{n}\frac{\theta}{(1-\theta^2)^3} + O\left(\frac{1}{n}\theta^{2n}\right)
\end{aligned}$$

$$c_{14}^{(1)} = cum(v_1, v_4) = \frac{1}{n}E\left(\frac{\partial \ell}{\partial \theta}\frac{\partial^2 \ell(\varphi)}{\partial \theta \partial \mu}\right) = \frac{1}{n}\frac{1}{(1+\theta)^2}E\left(\frac{\partial \ell}{\partial \theta}\right) = 0$$

$$\begin{aligned}
c_{15}^{(1)} &= \frac{1}{n}E\left(\frac{\partial \ell}{\partial \theta}\frac{\partial^3 \ell(\varphi)}{\partial \theta^3}\right) \\
&= \frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n} u_t \frac{\partial u_t}{\partial \theta}\sum_{t=1}^{n} u_t \frac{\partial^3 u_t}{\partial \theta^3}\right) + 3\frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n} u_t \frac{\partial u_t}{\partial \theta}\sum_{t=1}^{n} \frac{\partial u_t}{\partial \theta}\frac{\partial^2 u_t}{\partial \theta^2}\right) \\
&= \frac{6\theta^2}{\left(1-\theta^2\right)^3} + \frac{1}{n}\frac{1}{\sigma^4}\sum_{t=1}^{n-1} E\left(u_t \frac{\partial u_t}{\partial \theta}\sum_{k=1}^{T-t}\frac{\partial u_{t+k}}{\partial \theta}\frac{\partial^2 u_{t+k}}{\partial \theta^2}\right) \\
&= \frac{6\theta^2}{\left(1-\theta^2\right)^3} + 12\frac{1}{n}\frac{1}{1-\theta^2}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}(k-1)\,\theta^{2(k-1)} + 6\frac{1}{n}\frac{1}{\left(1-\theta^2\right)^2}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}\theta^{2(k-1)} \\
&= \frac{18\theta^2}{\left(1-\theta^2\right)^3} + 6\frac{1}{\left(1-\theta^2\right)^3} + 12\frac{1}{n}\frac{1}{\left(1-\theta^2\right)^2}\frac{\theta^{2n-2}-1}{\left(1-\left(\theta^{-2}\right)\right)^2} - 18\frac{1}{n}\frac{1}{\left(1-\theta^2\right)^3}\frac{\theta^2-\theta^{2n}}{1-\theta^2} \\
&= 6\frac{3\theta^2+1}{\left(1-\theta^2\right)^3} + \frac{6}{n}\left(3-2\theta^2\right)\frac{\theta^2}{\left(1-\theta^2\right)^4} + O\left(\frac{1}{n}\theta^{2n}\right)
\end{aligned}$$

as

$$\sum_{i=0}^{n-t-1} i\left(\theta^2\right)^i = \frac{-(n-t-1)\,\theta^{2(n-t)}}{1-\theta^2} + \frac{\theta^2-\theta^{2(n-t)}}{\left(1-\theta^2\right)^2}, \quad \sum_{t=1}^{n-1} t\theta^{2(n-t)} = \frac{\theta^2 n}{1-\theta^2} - \frac{\theta^2}{1-\theta^2} + \frac{\theta^{2n-2}-1}{\left(1-\left(\theta^{-2}\right)\right)^2}$$

$$\begin{aligned}
c_{16}^{(1)} &= \frac{1}{n}E\left(\frac{\partial \ell}{\partial \theta}\frac{\partial^3 \ell(\varphi)}{\partial \theta^2 \partial \mu}\right) = \frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n} u_t \frac{\partial u_t}{\partial \theta}\sum_{t=1}^{n}\left(2\frac{\partial^2 u_t}{\partial \mu \partial \theta}\frac{\partial u_t}{\partial \theta} + \frac{\partial u_t}{\partial \mu}\frac{\partial^2 u_t}{\partial \theta^2} + u_t\frac{\partial^3 u_t}{\partial \mu \partial \theta^2}\right)\right) \\
&= 2\frac{1}{\left(1+\theta\right)^2}\frac{1}{n}\frac{1}{\sigma^4}\sum_{t=1}^{n-1} E\left(u_t \frac{\partial u_t}{\partial \theta}\sum_{k=1}^{n-t}\frac{\partial u_{t+k}}{\partial \theta}\right) - \frac{1}{n}\frac{1}{\sigma^4}\frac{1}{1+\theta}\sum_{t=1}^{n-1} E\left(u_t \frac{\partial u_t}{\partial \theta}\sum_{k=1}^{n-t}\frac{\partial^2 u_{t+k}}{\partial \theta^2}\right) = 0
\end{aligned}$$

Now

$$c_{22}^{(1)} = \frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n} u_t \frac{\partial u_t}{\partial \mu}\sum_{t=1}^{n} u_t \frac{\partial u_t}{\partial \mu}\right) = \frac{1}{n}\frac{1}{\sigma^4}\frac{1}{\left(1+\theta\right)^2}E\left(\sum_{t=1}^{n} u_t^2\right) = \frac{1}{\sigma^2}\frac{1}{\left(1+\theta\right)^2}$$

$$\begin{aligned}
c_{23}^{(1)} &= \frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n}u_t\frac{\partial u_t}{\partial\mu}\sum_{t=1}^{n}\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}+\left(\frac{\partial u_t}{\partial\theta}\right)^2\right)\right)\\
&= -\frac{1}{n}\frac{1}{\sigma^4}\frac{1}{(1+\theta)}\sum_{t=1}^{n}E\left(u_t^2\frac{\partial^2 u_t}{\partial\theta^2}\right)-\frac{1}{n}\frac{1}{\sigma^4}\frac{1}{(1+\theta)}\sum_{t=1}^{n}E\left(u_t\sum_{t=1}^{n}\left(\frac{\partial u_t}{\partial\theta}\right)^2\right)\\
&= -\frac{1}{n}\frac{1}{\sigma^4}\frac{E\left(u_0^3\right)}{(1+\theta)}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}\theta^{2(k-1)}=-\frac{1}{n}\frac{1}{\sigma^4}\frac{E\left(u_0^3\right)}{(1+\theta)}\sum_{t=1}^{n-1}\frac{1-\theta^{2(n-t)}}{1-\theta^2}\\
&= -\frac{1}{\sigma^4}\frac{E\left(u_0^3\right)}{(1+\theta)\left(1-\theta^2\right)}+\frac{1}{n}\frac{1}{\sigma^4}\frac{E\left(u_0^3\right)}{(1+\theta)\left(1-\theta^2\right)^2}+O\left(\frac{1}{n}\theta^{2n}\right)
\end{aligned}$$

$$\begin{aligned}
c_{24}^{(1)} &= \frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n}u_t\frac{\partial u_t}{\partial\mu}\left(\sum_{t=1}^{n}\frac{\partial u_t}{\partial\mu}\frac{\partial u_t}{\partial\theta}+\sum_{t=1}^{n}u_t\frac{\partial^2 u_t}{\partial\mu\partial\theta}\right)\right)\\
&= \frac{1}{n}\frac{1}{\sigma^4}\frac{1}{(1+\theta)^2}E\left(\sum_{t=1}^{n}u_t\sum_{t=1}^{n}\frac{\partial u_t}{\partial\theta}\right)-\frac{1}{(1+\theta)^3}\frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n}u_t\sum_{t=1}^{n}u_t\right)\\
&= \frac{1}{n}\frac{1}{\sigma^4}\frac{1}{(1+\theta)^2}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}E\left(u_t\frac{\partial u_{t+k}}{\partial\theta}\right)-\frac{1}{(1+\theta)^3}\frac{1}{\sigma^2}\\
&= -\frac{1}{n}\frac{1}{\sigma^2}\frac{1}{(1+\theta)^2}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}(-\theta)^{k-1}-\frac{1}{(1+\theta)^3}\frac{1}{\sigma^2}\\
&= -\frac{2}{\sigma^2}\frac{1}{(1+\theta)^3}+\frac{1}{n}\frac{1}{\sigma^2}\frac{1}{(1+\theta)^4}+O\left(-\frac{1}{n}\theta^n\right)
\end{aligned}$$

$$\begin{aligned}
c_{25}^{(1)} &= \frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n}u_t\frac{\partial u_t}{\partial\mu}\sum_{t=1}^{n}\left(u_t\frac{\partial^3 u_t}{\partial\theta^3}+3\left(\frac{\partial u_t}{\partial\theta}\right)\frac{\partial^2 u_t}{\partial\theta^2}\right)\right)\\
&= -3\frac{1}{1+\theta}\frac{1}{n}\frac{1}{\sigma^4}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}E\left(u_t\frac{\partial u_{t+k}}{\partial\theta}\frac{\partial^2 u_{t+k}}{\partial\theta^2}\right)=-6\frac{1}{1+\theta}\frac{1}{n}\frac{E\left(u_0^3\right)}{\sigma^4}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}(k-1)\,\theta^{2k-3}\\
&= -6\frac{1}{1+\theta}\frac{1}{n}\frac{E\left(u_0^3\right)}{\sigma^4}\frac{1}{\theta}\sum_{t=1}^{n-1}\left(\frac{-(n-t-1)\,\theta^{2(n-t)}}{1-\theta^2}+\frac{\theta^2-\theta^{2(n-t)}}{\left(1-\theta^2\right)^2}\right)\\
&= -6\frac{\theta}{(1+\theta)\left(1-\theta^2\right)^2}\frac{E\left(u_0^3\right)}{\sigma^4}+6\frac{\theta^3+\theta}{(1+\theta)\left(1-\theta^2\right)^3}\frac{1}{n}\frac{E\left(u_0^3\right)}{\sigma^4}+O\left(\theta^{2n}\right)
\end{aligned}$$

$$
\begin{aligned}
c_{26}^{(1)} &= \frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n}u_t\frac{\partial u_t}{\partial\mu}\sum_{t=1}^{n}\left(2\frac{\partial^2 u_t}{\partial\mu\partial\theta}\frac{\partial u_t}{\partial\theta}+\frac{\partial u_t}{\partial\mu}\frac{\partial^2 u_t}{\partial\theta^2}+u_t\frac{\partial^3 u_t}{\partial\mu\partial\theta^2}\right)\right)\\
&= -\frac{1}{T}\frac{1}{\sigma^4}2\frac{1}{(1+\theta)^3}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}E\left(u_t\frac{\partial u_{t+k}}{\partial\theta}\right)+\frac{1}{(1+\theta)^2}\frac{1}{n}\frac{1}{\sigma^4}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}E\left(u_t\frac{\partial^2 u_{t+k}}{\partial\theta^2}\right)\\
&\quad +2\frac{1}{\sigma^2}\frac{1}{(1+\theta)^4}\\
&= 2\frac{1}{n}\frac{1}{\sigma^2}\frac{1}{(1+\theta)^3}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}(-\theta)^{k-1}+2\frac{1}{(1+\theta)^2}\frac{1}{n}\frac{1}{\sigma^2}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}(k-1)(-\theta)^{k-2}+2\frac{1}{\sigma^2}\frac{1}{(1+\theta)^4}\\
&= 6\frac{1}{\sigma^2}\frac{1}{(1+\theta)^4}+\frac{2}{n}\frac{1}{\sigma^2}\frac{(2\theta-1)}{(1+\theta)^5}+O\left(\frac{1}{n}\theta^{n-1}\right).
\end{aligned}
$$

$$
\begin{aligned}
c_{33}^{(1)} &= \frac{1}{n}E\left(\frac{\partial^2\ell(\varphi)}{\partial\theta^2}\right)^2-n\left(\frac{1}{1-\theta^2}\right)^2=\frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n}\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}+\left(\frac{\partial u_t}{\partial\theta}\right)^2\right)\right)^2-\frac{n}{(1-\theta^2)^2}\\
&= \frac{1}{n}\frac{1}{\sigma^4}\sum_{t=1}^{n}E\left(\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\right)^2\right)+2\frac{1}{n}\frac{1}{\sigma^4}\sum_{t=1}^{n}E\left(\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\right)\left(\frac{\partial u_t}{\partial\theta}\right)^2\right)+\frac{1}{n}\frac{1}{\sigma^4}\sum_{t=1}^{n}E\left(\left(\frac{\partial u_t}{\partial\theta}\right)^4\right)\\
&\quad +2\frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}u_t\frac{\partial^2 u_t}{\partial\theta^2}u_{t+k}\frac{\partial^2 u_{t+k}}{\partial\theta^2}\right)+2\frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}u_t\frac{\partial^2 u_t}{\partial\theta^2}\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\right)\\
&\quad +2\frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}\left(\frac{\partial u_t}{\partial\theta}\right)^2u_{t+k}\frac{\partial^2 u_{t+k}}{\partial\theta^2}\right)+2\frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}\left(\frac{\partial u_t}{\partial\theta}\right)^2\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\right)\\
&\quad -\frac{n}{(1-\theta^2)^2}\\
&= 2\frac{7\theta^2+3}{(1-\theta^2)^3}+\frac{1}{(1-\theta^2)^2}\kappa_4-\frac{4}{n}\left(-\theta+\theta^2+1\right)\frac{\theta+\theta^2+1}{(1-\theta^2)^4}-2\frac{1}{n}\frac{1}{1-\theta^4}\frac{1-\theta^2+\theta^4}{(1-\theta^2)^2}\kappa_4\\
&\quad +O\left(\frac{1}{n}\theta^{2n}\right)
\end{aligned}
$$

where $\kappa_4=\frac{E\left(u_0^4\right)}{\sigma^4}-3$.

Finally,

$$\begin{aligned}
c_{44}^{(1)} &= \frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n}\frac{\partial u_t}{\partial\mu}\frac{\partial u_t}{\partial\theta}+\sum_{t=1}^{n}u_t\frac{\partial^2 u_t}{\partial\mu\partial\theta}\right)^2 \\
&= \frac{1}{n}\frac{1}{\sigma^4}\frac{1}{(1+\theta)^2}E\left(\sum_{t=1}^{n}\frac{\partial u_t}{\partial\theta}\right)^2-2\frac{1}{(1+\theta)^3}\frac{1}{n}\frac{1}{\sigma^4}E\left(\sum_{t=1}^{n}\frac{\partial u_t}{\partial\theta}\sum_{t=1}^{n}u_t\right) \\
&\quad+\frac{1}{n}\frac{1}{\sigma^4}\frac{1}{(1+\theta)^4}E\left(\sum_{t=1}^{n}u_t\right)^2 \\
&= 4\frac{1}{(1+\theta)^4}\frac{1}{\sigma^2}+\frac{2}{n}\frac{2\theta-1}{\sigma^2(1-\theta)(\theta+1)^5}+O\left(\frac{1}{n}\theta^n\right)
\end{aligned}$$

Hence, the needed cumulants, and employing the notation of Theorem 1, are:

$$c_{11}^{(1)}=\frac{1}{1-\theta^2},\quad c_{12}^{(1)}=c_{14}^{(1)}=c_{16}^{(1)}=0,\quad c_{13}^{(1)}=4\frac{\theta}{(1-\theta^2)^2},\quad c_{15}^{(1)}=6\frac{3\theta^2+1}{(1-\theta^2)^3},$$

$$\begin{aligned}
c_{22}^{(1)} &= \frac{1}{\sigma^2}\frac{1}{(1+\theta)^2},\ c_{23}^{(1)}=-\frac{1}{\sigma^4}\frac{E\left(u_0^3\right)}{(1+\theta)\left(1-\theta^2\right)},\ c_{24}^{(1)}=-\frac{2}{\sigma^2}\frac{1}{(1+\theta)^3}, \\
c_{25}^{(1)} &= -6\frac{\theta}{(1+\theta)\left(1-\theta^2\right)^2}\frac{E\left(u_0^3\right)}{\sigma^4},\ c_{26}^{(1)}=6\frac{1}{\sigma^2}\frac{1}{(1+\theta)^4},\ c_{33}^{(1)}=2\frac{7\theta^2+3}{(1-\theta^2)^3}+\frac{1}{(1-\theta^2)^2}\kappa_4, \\
c_{44}^{(1)} &= 4\frac{1}{(1+\theta)^4}\frac{1}{\sigma^2},
\end{aligned}$$

$$c_{ij}^{(2)}=0\quad for\quad i,j=1,...,6,$$

$$c_{11}^{(3)}=c_{12}^{(3)}=c_{14}^{(3)}=c_{16}^{(3)}=c_{22}^{(3)}=0,\quad c_{13}^{(3)}=-2\frac{\theta}{(1-\theta^2)^3},\quad c_{15}^{(3)}=6\left(3-2\theta^2\right)\frac{\theta^2}{(1-\theta^2)^4},$$

$$\begin{aligned}
c_{23}^{(3)} &= \frac{1}{\sigma^4}\frac{E\left(u_0^3\right)}{(1+\theta)\left(1-\theta^2\right)^2},\quad c_{24}^{(3)}=\frac{1}{\sigma^2}\frac{1}{(1+\theta)^4},\quad c_{25}^{(3)}=6\frac{\theta^3+\theta}{(1+\theta)\left(1-\theta^2\right)^3}\frac{E\left(u_0^3\right)}{\sigma^4}, \\
c_{26}^{(3)} &= 2\frac{1}{\sigma^2}\frac{(2\theta-1)}{(1+\theta)^5},\quad c_{33}^{(3)}=-4\left(-\theta+\theta^2+1\right)\frac{\theta+\theta^2+1}{(1-\theta^2)^4}-2\frac{1}{1-\theta^4}\frac{1-\theta^2+\theta^4}{(1-\theta^2)^2}\kappa_4, \\
c_{44}^{(3)} &= 2\frac{2\theta-1}{\sigma^2(1-\theta)(\theta+1)^5}.
\end{aligned}$$

From the $c_{ij}^{(3)}s$ we need only $c_{11}^{(3)}$, mainly due to fact that $f_1^2=f_1^3=f_1^4=f_1^5=f_1^6=0$. Hence, $c_{11}^{(3)}=-\frac{\theta^2}{(1-\theta^2)^2}$.

Out of all 3$^{\text{rd}}$ order cumulants, we only need $c_{111}, c_{113}, c_{122}$ and c_{124}.

$$
\begin{aligned}
c_{111} &= E\left(v_1\right)^3 = E\left(\frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \theta}\right)^3 = -\frac{1}{\sqrt{n^3}}\frac{1}{\sigma^6}E\left(\sum_{t=1}^{n} u_t \frac{\partial u_t}{\partial \theta}\right)^3 \\
&= \frac{1}{\sqrt{n}}\frac{1}{1+\theta^3}\frac{E^2\left(u_0^3\right)}{\sigma^6} - 3\frac{1}{\sqrt{n^3}}\frac{1}{\sigma^4}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}E\left(u_t\frac{\partial u_t}{\partial \theta}\left(\frac{\partial u_{t+k}}{\partial \theta}\right)^2\right) \\
&= \frac{1}{\sqrt{n}}\frac{1}{1+\theta^3}\frac{E^2\left(u_0^3\right)}{\sigma^6} - 6\frac{1}{\sqrt{n^3}}\frac{1}{1-\theta^2}\sum_{t=1}^{n-1}\sum_{k=1}^{n-t}\theta^{2k-1} = -\frac{6}{\sqrt{n}}\frac{\theta}{\left(1-\theta^2\right)^2} + \frac{1}{\sqrt{n}}\frac{\kappa_3^2}{1+\theta^3} + o\left(n^{-1}\right)
\end{aligned}
$$

where $\kappa_3 = \frac{E\left(u_0^3\right)}{\sigma^3}$.

Now $c_{113} = -\frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^4}E\left[\left(\sum_{t=1}^{n} u_t\frac{\partial u_t}{\partial \theta}\right)^2\left(\frac{1}{\sigma^2}\sum_{t=1}^{n}\left(u_t\frac{\partial^2 u_t}{\partial \theta^2} + \left(\frac{\partial u_t}{\partial \theta}\right)^2\right) + nA_{11}\right)\right]$

$= -\frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^6}E\left[\sum_{t=1}^{n} u_t^2\left(\frac{\partial u_t}{\partial \theta}\right)^2\sum_{t=1}^{n} u_t\frac{\partial^2 u_t}{\partial \theta^2}\right] - 2\frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^6}E\left[\sum_{t=1}^{n-1} u_t\frac{\partial u_t}{\partial \theta}\sum_{k=1}^{n-t} u_{t+k}\frac{\partial u_{t+k}}{\partial \theta}\sum_{t=1}^{n} u_t\frac{\partial^2 u_t}{\partial \theta^2}\right]$

$-\frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^6}E\left[\sum_{t=1}^{n} u_t^2\left(\frac{\partial u_t}{\partial \theta}\right)^2\sum_{t=1}^{n}\left(\frac{\partial u_t}{\partial \theta}\right)^2\right] - 2\frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^6}E\left[\sum_{t=1}^{n-1} u_t\frac{\partial u_t}{\partial \theta}\sum_{k=1}^{n-t} u_{t+k}\frac{\partial u_{t+k}}{\partial \theta}\sum_{t=1}^{n}\left(\frac{\partial u_t}{\partial \theta}\right)^2\right]$

$+\sqrt{n}\frac{1}{\left(1-\theta^2\right)^2} =$

$= -\frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^6}E\left[\sum_{t=1}^{n} u_t^3\left(\frac{\partial u_t}{\partial \theta}\right)^2\frac{\partial^2 u_t}{\partial \theta^2}\right] - \frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^4}\sum_{t=1}^{n-1}E\left[u_t\frac{\partial^2 u_t}{\partial \theta^2}\sum_{k=1}^{n-t}\left(\frac{\partial u_{t+k}}{\partial \theta}\right)^2\right]$

$-\frac{2}{\sqrt{n^3}}\frac{1}{\sigma^6}E\left[\sum_{t=2}^{n} u_t\frac{\partial^2 u_t}{\partial \theta^2}\sum_{i=1}^{t-1} u_i\frac{\partial u_i}{\partial \theta}\sum_{k=1}^{n-i} u_{i+k}\frac{\partial u_{i+k}}{\partial \theta}\right] - \frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^6}E\left[\sum_{t=2}^{n}\left(\frac{\partial u_t}{\partial \theta}\right)^2\sum_{i=1}^{t-1} u_i^2\left(\frac{\partial u_i}{\partial \theta}\right)^2\right]$

$-\frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^4}\sum_{t=1}^{n}E\left[\left(\frac{\partial u_t}{\partial \theta}\right)^4\right] - \frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^4}\sum_{t=1}^{n-1}E\left[\left(\frac{\partial u_t}{\partial \theta}\right)^2\sum_{k=1}^{n-t}\left(\frac{\partial u_{t+k}}{\partial \theta}\right)^2\right]$

$-2\frac{1}{n}\frac{1}{\sqrt{n}}\frac{1}{\sigma^6}E\left[\sum_{t=2}^{n}\left(\frac{\partial u_t}{\partial \theta}\right)^2\sum_{i=1}^{t-1} u_i\frac{\partial u_i}{\partial \theta}\sum_{k=1}^{n-i} u_{i+k}\frac{\partial u_{i+k}}{\partial \theta}\right] + \sqrt{n}\frac{1}{\left(1-\theta^2\right)^2}$

Now $E\left[\sum_{t=2}^{n} u_t\frac{\partial^2 u_t}{\partial \theta^2}\sum_{i=1}^{t-1} u_i\frac{\partial u_i}{\partial \theta}\sum_{k=1}^{n-i} u_{i+k}\frac{\partial u_{i+k}}{\partial \theta}\right] = \sigma^2\sum_{t=2}^{n}\sum_{i=1}^{t-1}E\left[\frac{\partial^2 u_t}{\partial \theta^2}\frac{\partial u_t}{\partial \theta}u_i\frac{\partial u_i}{\partial \theta}\right]$

$= \sigma^2\sum_{t=2}^{n}\sum_{k=1}^{t-1}E\left[\frac{\partial^2 u_t}{\partial \theta^2}\frac{\partial u_t}{\partial \theta}u_{t-k}\frac{\partial u_{t-k}}{\partial \theta}\right]$

$= \sigma^6\frac{4}{1-\theta^2}\sum_{t=2}^{n}\sum_{k=1}^{t-1}(k-1)\,\theta^{2k-2} + \sigma^6\frac{2}{\left(1-\theta^2\right)^2}\sum_{t=2}^{n}\sum_{k=1}^{t-1}\theta^{2k-2}$

$$= -\sigma^6 \frac{4}{\left(1-\theta^2\right)^2} \sum_{k=1}^{n-1} k \left(\theta^2\right)^k + \sigma^6 n \frac{4\theta^2}{\left(1-\theta^2\right)^3} + \sigma^6 n \frac{2}{\left(1-\theta^2\right)^3} + O\left(1\right)$$

$$= \sigma^6 n \frac{4\theta^{2n}}{\left(1-\theta^2\right)^3} + \sigma^6 n \frac{4\theta^2}{\left(1-\theta^2\right)^3} + \sigma^6 n \frac{2}{\left(1-\theta^2\right)^3} + O\left(1\right)$$

Also $E\left[\sum_{t=2}^{n} \left(\frac{\partial u_t}{\partial \theta}\right)^2 \sum_{i=1}^{t-1} u_i \frac{\partial u_i}{\partial \theta} \sum_{k=1}^{n-i} u_{i+k} \frac{\partial u_{i+k}}{\partial \theta}\right] = \sum_{t=3}^{n} E\left[\left(\frac{\partial u_t}{\partial \theta}\right)^2 \sum_{j=2}^{t-1} u_{t-j} \frac{\partial u_{t-j}}{\partial \theta} \sum_{k=1}^{j-1} u_{t-j+k} \frac{\partial u_{t-j+k}}{\partial \theta}\right]$

$$= \sum_{t=3}^{n} E\left[\left(\frac{\partial u_t}{\partial \theta}\right)^2 \sum_{k=1}^{t-2} u_{t-k} \frac{\partial u_{t-k}}{\partial \theta} \sum_{i=1}^{t-k-1} u_{t-k-i} \frac{\partial u_{t-k-i}}{\partial \theta}\right]$$

$$= \sum_{t=3}^{n} \sum_{k=1}^{t-2} \sum_{i=1}^{t-k-1} E\left[\left(\frac{\partial u_t}{\partial \theta}\right)^2 u_{t-k} \frac{\partial u_{t-k}}{\partial \theta} u_{t-k-i} \frac{\partial u_{t-k-i}}{\partial \theta}\right]$$

$$= \sigma^6 \frac{4}{1-\theta^2} \sum_{t=3}^{n} \sum_{k=1}^{t-2} \sum_{i=1}^{t-k-1} \left(-\theta\right)^{2k-1} \left(-\theta\right)^{2i-1}$$

$$= \sigma^6 \frac{4(-\theta)}{\left(1-\theta^2\right)^2} \sum_{t=3}^{T} \sum_{k=1}^{t-2} \left(-\theta\right)^{2k-1} - \sigma^6 \frac{4\theta^2}{\left(1-\theta^2\right)^2} \sum_{t=3}^{n} \sum_{k=1}^{t-2} \theta^{2(k-1)} \theta^{2(t-k-1)} =$$

$$= n\sigma^6 \frac{4\theta^2}{\left(1-\theta^2\right)^3} - \sigma^6 \frac{4}{\left(1-\theta^2\right)^2} \sum_{t=3}^{n} \left(t-1\right) \theta^{2(t-1)} + O\left(1\right)$$

$$= n\sigma^6 \frac{4\theta^2}{\left(1-\theta^2\right)^3} + n\sigma^6 \frac{4}{\left(1-\theta^2\right)^3} \theta^{2n} + O\left(1\right).$$

Hence $c_{113} = -1 \overline{\sqrt{n^3}\frac{1}{\sigma^6} E\left[\sum_{t=1}^{n} u_t^3 \left(\frac{\partial u_t}{\partial \theta}\right)^2 \frac{\partial^2 u_t}{\partial \theta^2}\right] - \frac{1}{\sqrt{n^3}} \frac{1}{\sigma^4} \sum_{t=1}^{n-1} E\left[u_t \frac{\partial^2 u_t}{\partial \theta^2} \sum_{k=1}^{T-t} \left(\frac{\partial u_{t+k}}{\partial \theta}\right)^2\right] - 2\frac{1}{\sqrt{n}}\left[\frac{4\theta^2}{\left(1-\theta^2\right)^3} + \frac{2}{\left(1-\theta^2\right)^3}\right]}$

$$-\frac{1}{\sqrt{n^3}} \frac{1}{\sigma^6} E\left[\sum_{t=2}^{n} \left(\frac{\partial u_t}{\partial \theta}\right)^2 \sum_{i=1}^{t-1} u_i^2 \left(\frac{\partial u_i}{\partial \theta}\right)^2\right] - \frac{1}{\sqrt{n^3}} \frac{1}{\sigma^4} \sum_{t=1}^{n} E\left[\left(\frac{\partial u_t}{\partial \theta}\right)^4\right]$$

$$-\frac{1}{\sqrt{n^3}} \frac{1}{\sigma^4} \sum_{t=1}^{T-1} E\left[\left(\frac{\partial u_t}{\partial \theta}\right)^2 \sum_{k=1}^{T-t} \left(\frac{\partial u_{t+k}}{\partial \theta}\right)^2\right] - 2\frac{1}{\sqrt{n}} \frac{4\theta^2}{\left(1-\theta^2\right)^3} + \sqrt{n} \frac{1}{\left(1-\theta^2\right)^2} + O\left(n^{-\frac{1}{2}} \theta^{2n}\right)$$

$$= -\frac{1}{\sqrt{n}} \frac{2\theta^2}{\left(1+\theta^3\right)^2} \frac{E^2\left(u_0^3\right)}{\sigma^6} - \frac{1}{\sqrt{n^3}} \frac{4}{\left(1-\theta^2\right)^2} \sum_{t=1}^{n-1} \sum_{k=1}^{n-t} \theta^{2k} - 2\frac{1}{\sqrt{n}} \left[\frac{4\theta^2}{\left(1-\theta^2\right)^3} + \frac{2}{\left(1-\theta^2\right)^3}\right]$$

$$-\frac{1}{\sqrt{n^3}} \frac{1}{\sigma^6} \sum_{t=2}^{n} \sum_{k=1}^{t-1} \left[\begin{array}{c} \frac{1-\theta^{2(k-1)}}{\left(1-\theta^2\right)^2} \sigma^6 + \theta^{2(k-1)} E\left(u_0^4\right) \sigma^2 \frac{1}{1-\theta^2} - 2\theta^{2k-1} \frac{1}{1+\theta^3} E^2\left(u_0^3\right) \\ +\theta^{2k} \sigma^2 \left(\frac{1}{1-\theta^4} E\left(u_0^4\right) + 6\sigma^4 \frac{\theta^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\right) \end{array}\right]$$

$$-\frac{1}{\sqrt{n}} \frac{1}{\sigma^4} \left(\frac{1}{1-\theta^4} E\left(u_0^4\right) + 6\sigma^4 \frac{\theta^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\right)$$

$$-\frac{1}{\sqrt{n^3}} \frac{1}{\sigma^4} \sum_{t=1}^{n-1} \sum_{k=1}^{n-t} \left[\sigma^4 \frac{1}{\left(1-\theta^2\right)^2} + \theta^{2k} \frac{1}{1-\theta^4} E\left(u_0^4\right) + \sigma^4 \frac{5\theta^2-1}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2} \theta^{2k}\right]$$

$$-2\frac{1}{\sqrt{n}}\frac{4\theta^2}{\left(1-\theta^2\right)^3}+\sqrt{n}\frac{1}{\left(1-\theta^2\right)^2}+O\left(n^{-\frac{1}{2}}\theta^{2n}\right)$$

$$=-\frac{1}{\sqrt{n}}\frac{2\theta^2}{\left(1+\theta^3\right)^2}\frac{E^2\left(u_0^3\right)}{\sigma^6}-\frac{1}{\sqrt{n}}\frac{1}{1-\theta^4}\frac{E\left(u_0^4\right)}{\sigma^4}-\frac{1}{\sqrt{n^3}}\frac{4}{\left(1-\theta^2\right)^2}\sum_{t=1}^{n-1}\frac{\theta^2-\theta^{2(n-t)}\theta^2}{1-\theta^2}$$

$$-\frac{1}{\sqrt{n}}\frac{16\theta^2}{\left(1-\theta^2\right)^3}-\frac{1}{\sqrt{n}}\frac{4}{\left(1-\theta^2\right)^3}-\frac{6}{\sqrt{n}}\frac{\theta^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}$$

$$+\frac{1}{\sqrt{n}}\frac{1}{\left(1-\theta^2\right)^2}-\frac{1}{\sqrt{n^3}}\left[-\frac{1}{\left(1-\theta^2\right)^2}+\frac{E\left(u_0^4\right)}{\sigma^4}\frac{1}{1-\theta^2}-2\frac{\theta}{1+\theta^3}\frac{E^2\left(u_0^3\right)}{\sigma^6}\right]\sum_{t=2}^{n}\frac{1-\theta^{2(t-1)}}{1-\theta^2}$$

$$-\frac{1}{\sqrt{n^3}}\left(\frac{1}{1-\theta^4}\frac{E\left(u_0^4\right)}{\sigma^4}+6\frac{\theta^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\right)\sum_{t=2}^{n}\frac{\theta^2-\theta^{2(t-1)}\theta^2}{1-\theta^2}$$

$$-\frac{1}{\sqrt{n^3}}\left[\frac{1}{1-\theta^4}\frac{1}{\sigma^4}E\left(u_0^4\right)+\frac{5\theta^2-1}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\right]\sum_{t=1}^{n-1}\frac{\theta^2-\theta^{2(n-t)}\theta^2}{1-\theta^2}+O\left(n^{-\frac{1}{2}}\theta^{2n}\right)$$

$$=2\frac{1}{\sqrt{n}}\left(\frac{1}{1-\theta^2}-\frac{\theta}{1+\theta^3}\right)\frac{\theta}{1+\theta^3}\frac{E^2\left(u_0^3\right)}{\sigma^6}-\frac{1}{\sqrt{n}}\frac{1}{1-\theta^4}\frac{E\left(u_0^4\right)}{\sigma^4}-\frac{1}{\sqrt{n}}\frac{1+3\theta^2}{1+\theta^2}\frac{1}{\left(1-\theta^2\right)^2}\frac{E\left(u_0^4\right)}{\sigma^4}$$

$$-\frac{1}{\sqrt{n}}\frac{4\theta^2}{\left(1-\theta^2\right)^3}-\frac{1}{\sqrt{n}}\frac{16\theta^2}{\left(1-\theta^2\right)^3}-\frac{1}{\sqrt{n}}\frac{4}{\left(1-\theta^2\right)^3}-\frac{1}{\sqrt{n}}6\frac{\theta^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}+\frac{1}{\sqrt{n}}\frac{1}{\left(1-\theta^2\right)^3}$$

$$-\frac{1}{\sqrt{n}}6\frac{\theta^4}{\left(1+\theta^2\right)\left(1-\theta^2\right)^3}+\frac{1}{\sqrt{n}}\frac{1}{\left(1-\theta^2\right)^2}-\frac{1}{\sqrt{n}}\frac{5\theta^2-1}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\frac{\theta^2}{1-\theta^2}+O\left(n^{-\frac{1}{2}}\theta^{2n}\right).$$

or for $\kappa_4=\frac{E\left(u_0^4\right)}{\sigma^4}-3$ and $\kappa_3=\frac{E\left(u_0^3\right)}{\sigma^3}$

$$c_{113}=-\frac{4}{\sqrt{n}}\frac{5\theta^2+2}{\left(1-\theta^2\right)^3}+2\frac{1}{\sqrt{n}}\left(\frac{1}{1-\theta^2}-\frac{\theta}{1+\theta^3}\right)\frac{\theta}{1+\theta^3}\kappa_3^2-\frac{1}{\sqrt{n}}\frac{2}{\left(1-\theta^2\right)^2}\kappa_4+O\left(n^{-\frac{1}{2}}\theta^{2n}\right)$$

In the same logic we get

$$\begin{aligned}
c_{122} &= -\frac{1}{\sqrt{n^3}}\frac{1}{\sigma^6}E\left[\sum_{t=1}^{n}u_t\frac{\partial u_t}{\partial\theta}\left(\sum_{t=1}^{n}u_t\frac{\partial u_t}{\partial\mu}\right)^2\right]\\
&= -\frac{1}{\sqrt{n^3}}\frac{1}{(1+\theta)^2}\frac{1}{\sigma^6}\sum_{t=1}^{n}E\left[u_t\frac{\partial u_t}{\partial\theta}\sum_{t=1}^{n}u_t^2\right]-\frac{2}{\sqrt{n^3}}\frac{1}{(1+\theta)^2}\frac{1}{\sigma^6}\sum_{t=1}^{n}E\left[u_t\frac{\partial u_t}{\partial\theta}\sum_{t=1}^{n-1}u_t\sum_{k=1}^{n-t}u_{t+k}\right]\\
&= -\frac{2}{\sqrt{n^3}}\frac{1}{(1+\theta)^2}\frac{1}{\sigma^6}\sum_{t=2}^{n}E\left[u_t^2\frac{\partial u_t}{\partial\theta}\sum_{t=1}^{t-1}u_i\right]=-\frac{2}{\sqrt{n^3}}\frac{1}{(1+\theta)^2}\frac{1}{\sigma^4}\sum_{t=2}^{n}\sum_{k=1}^{t-1}E\left[\frac{\partial u_t}{\partial\theta}u_{t-k}\right]\\
&= \frac{2}{\sigma^2\sqrt{n}}\frac{1}{(1+\theta)^3}+o\left(n^{-1}\right),
\end{aligned}$$

$$c_{124}=-\frac{4}{\sqrt{n}}\frac{1}{\sigma^2}\frac{1}{(1+\theta)^4}+O\left(n^{-\frac{1}{2}}\theta^{n}\right)+o\left(n^{-1}\right).$$

Hence, the needed 3^{rd} order cumulants, employing the notation of Theorem 1, are:$c_{111}^{(1)} = -6\frac{\theta}{\left(1-\theta^2\right)^2}+\frac{\kappa_3^2}{1+\theta^3}$, $c_{113}^{(1)} = -\frac{19\theta^2+8}{\left(1-\theta^2\right)^3}+\frac{2\theta}{1+\theta^3}\left(\frac{1}{1-\theta^2}-\frac{\theta}{1+\theta^3}\right)\kappa_3^2-\frac{2}{\left(1-\theta^2\right)^2}\kappa_4$, $c_{122}^{(1)} = \frac{2}{\sigma^2}\frac{1}{(1+\theta)^3}$, $c_{124}^{(1)} = -\frac{4}{\sigma^2}\frac{1}{(1+\theta)^4}$, and $c_{113}^{(2)} = c_{122}^{(2)} = c_{124}^{(2)} = 0$.and

$$c_{ijk}^{(2)} = 0 \quad for \ the \ relevant \ values \ of \ i, j \ and \ k.$$

From all the 4^{th} order cumulants we only need $c_{1111} = E\left((v_1)^4\right) - 3\left(c_{11}\right)^2$ with $c_{11} = \frac{1}{1-\theta^2}$, and $c_{2222} = E\left((v_2)^4\right) - 3\left(c_{22}\right)^2$, with $c_{22} = \frac{1}{\sigma^2}\frac{1}{(1+\theta)^2}$.

Now $E\left((v_1)^4\right) = \frac{1}{n}\frac{1}{1-\theta^4}\frac{E^2\left(u_0^4\right)}{\sigma^8}+\frac{6}{n}\frac{\theta^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\frac{E\left(u_0^4\right)}{\sigma^4}+3\frac{1}{\left(1-\theta^2\right)^2}-3\frac{1}{n}\frac{1}{\left(1-\theta^2\right)^2}$

$+6\frac{1}{n}\left(-\frac{1}{\left(1-\theta^2\right)^2}+\frac{E\left(u_0^4\right)}{\sigma^4}\frac{1}{1-\theta^2}-2\frac{\theta}{1+\theta^3}\frac{E^2\left(u_0^3\right)}{\sigma^6}\right)\frac{1}{1-\theta^2}$

$+6\frac{\theta^2}{n}\left(\frac{1}{1-\theta^4}\frac{E\left(u_0^4\right)}{\sigma^4}+6\frac{\theta^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\right)\frac{1}{1-\theta^2}+4\frac{1}{n}\frac{1}{\sigma^6}E^2\left(u_0^3\right)\left(-3\frac{\theta}{1-\theta^2}+3\frac{\theta^2}{1+\theta^3}\right)\frac{1}{1+\theta^3}+96\frac{1}{n}\frac{\theta^2}{\left(1-\theta^2\right)^3}$

$-48\frac{1}{n}\frac{\theta^4}{\left(1-\theta^2\right)^3}-48\frac{1}{n}\frac{\theta^2}{\left(1-\theta^2\right)^2}+O\left(n^{-1}\theta^{2n}\right)+O\left(n^{-2}\right).$

Hence employing the notation of Theorem 1

$$c_{1111}^{(1)} = \kappa_4^2\frac{1}{1-\theta^4}+12\frac{1}{\left(1-\theta^2\right)^2}\kappa_4-12\frac{\theta}{1+\theta^3}\kappa_3^2\left(\frac{2}{1-\theta^2}-\frac{\theta}{1+\theta^3}\right)+6\frac{7\theta^2+3}{\left(1-\theta^2\right)^3}.$$

Now

$$\begin{aligned} E\left(v_2\right)^4 &= \frac{1}{\sigma^8}\frac{1}{n^2}E\left(-\frac{1}{1+\theta}\sum_{t=1}^{n}u_t\right)^4 = \frac{1}{\sigma^8}\frac{1}{n}\frac{1}{(1+\theta)^4}E\left(u_0^4\right)+6\frac{1}{\sigma^4}\frac{1}{n^2}\frac{1}{(1+\theta)^4}\sum_{t=1}^{n-1}(T-t) \\ &= \frac{1}{\sigma^8}\frac{1}{n}\frac{1}{(1+\theta)^4}E\left(u_0^4\right)-3\frac{1}{\sigma^4}\frac{1}{n}\frac{1}{(1+\theta)^4}+3\frac{1}{\sigma^4}\frac{1}{(1+\theta)^4} \end{aligned}$$

and

$$c_{2222}^{(1)} = \frac{1}{\sigma^4}\frac{1}{(1+\theta)^4}\kappa_4.$$

A.8 Expansion of $\widetilde{\theta}$

For the validity of the expansion we have that under the assumptions of Lemma 2, $A = (A_1, A_2, A_3, A_4, A_5, A_6)^{/}$ is a martingale satisfying all the assumptions of Götze and Hipp (1983 [45], 1994 [46]) and Hall and Horowitz (1996) [49] (see also Corradi and Iglesias 2008 [27]).

Now applying the results of Theorem 1 (see Appendix A.1) we get: $\omega^2 = \left(1-\theta^2\right)$, $\omega^{(3)} = -\theta^2$, $a_1^{(1)} = -6\theta\left(1-\theta^2\right) + \frac{\kappa_3^2\left(1-\theta^2\right)^3}{1+\theta^3}$, $a_2^{(1)} = \kappa_4^2\frac{\left(1-\theta^2\right)^3}{1+\theta^2} + 12\left(1-\theta^2\right)^2\kappa_4 - 12\frac{\theta\left(1-\theta^2\right)^4}{1+\theta^3}\kappa_3^2\left(\frac{2}{1-\theta^2} - \frac{\theta}{1+\theta^3}\right) + 6\left(7\theta^2+3\right)\left(1-\theta^2\right)$, $a_3^{(1)} = 2\theta\left(1-\theta^2\right)$, $a_4^{(1)} = 2\left(2\theta-1\right)$, $a_5^{(1)} = 2\left(4\theta-3\theta^2-10\right) - \frac{\kappa_3^2}{1+\theta^3}6\theta\left(1-\theta^2\right)^2 + 4\left(\frac{1}{1-\theta^2} - \frac{\theta}{1+\theta^3}\right)\frac{\theta\left(1-\theta^2\right)^3}{\left(1+\theta^3\right)}\kappa_3^2 - 4\left(1-\theta^2\right)\kappa_4$, $a_6^{(1)} = 6\left(1-\theta^4\right)$, $a_7^{(1)} = 4\left(-2\theta+\theta^2+6\right) + 2\left(1-\theta^2\right)\kappa_4$, $a_8^{(1)} = 2\left(1-\theta^2\right)\left(\theta^2+3\right) + \left(1-\theta^2\right)^2\kappa_4$, $a_9^{(1)} = 4\left(-2\theta+\theta^2+4\right) + 2\left(1-\theta^2\right)\kappa_4$, $a_{10}^{(1)} = -\left(1-\theta^2\right)\left(7\theta^2+8\right) - 2\frac{(1-\theta)\theta^2}{\left(\theta^2-\theta+1\right)}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2 - 2\kappa_4\left(1-\theta^2\right)^2$, and $\omega^{(2)} = a_1^{(2)} = a_3^{(2)} = a_4^{(2)} = a_{11}^{(1)} = a_{11}^{(2)} = a_{12}^{(1)} = 0$.

Now from Lemma 1 we get that

$$k_1^{\widetilde{\theta}} = \frac{2\theta-1}{\sqrt{n}} + o\left(n^{-1}\right), \quad k_2^{\widetilde{\theta}} = \omega^2 + \frac{1}{n}\left(\theta+6\right)\left(2-\theta\right) + \frac{1}{n}\xi_2^{\widetilde{\theta}} + o\left(n^{-1}\right),$$

where $\omega^2 = 1-\theta^2$, and $\xi_2^{\widetilde{\theta}} = 2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2 - \left(1-\theta^2\right)\kappa_4$.

Also

$$E\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^2 = \left(1-\theta^2\right) + \frac{1}{n}\left[-8\theta+3\theta^2+13+2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2 - \left(1-\theta^2\right)\kappa_4\right],$$

$$k_3^{\widetilde{\theta}} = \frac{1}{\sqrt{n}}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2 + o\left(n^{-1}\right), \quad k_4^{\widetilde{\theta}} = \frac{1}{n}6\left(1-\theta^2\right)\left(\theta^2+3\right) + \frac{1}{n}\xi_4^{\widetilde{\theta}} + o\left(n^{-1}\right),$$

where $\xi_4^{\widetilde{\theta}} = 12\theta\frac{\theta-\theta^2-2}{1-\theta+\theta^2}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2 + \frac{\left(1-\theta^2\right)^3}{1+\theta^2}\kappa_4^2$.

To find the expansion of $\sqrt{n}\left(\widetilde{\mu}-\mu\right)$ we can apply Theorem 1 with

$$f^2 = \sigma^2\left(1+\theta\right)^2, \quad f^1 = f^3 = f^4 = f^5 = f^6 = 0,$$

$$h^{12} = 2\sigma^2\left(1+\theta\right)\left(1-\theta^2\right), \quad h^{14} = \sigma^2\left(1+\theta\right)^2\left(1-\theta^2\right).$$

and all other $h^{ij}\prime s = 0$ for $i, j = 1, ..., 6$, and the non-zero $h^{ijk}\prime s$ given by

$$\begin{aligned}
h^{112} &= -2\left(1-\theta^2\right)(7\theta-1)(\theta+1)\sigma^2, \quad h^{114} = 2(5\theta-2)(\theta-1)(\theta+1)^3\sigma^2, \\
h^{116} &= \sigma^2(1+\theta)^2\left(1-\theta^2\right)^2, \quad h^{123} = 2\sigma^2\left(1-\theta^2\right)^2(1+\theta), \\
h^{134} &= \sigma^2(1+\theta)^2\left(1-\theta^2\right)^2, \quad h^{222} = 12\sigma^4\left(1-\theta^2\right)(1+\theta)^2, \\
h^{224} &= 6\sigma^4\left(1-\theta^2\right)(1+\theta)^3 \quad h^{244} = 2\sigma^4(1+\theta)^4\left(1-\theta^2\right).
\end{aligned}$$

(see Appendix A.6) Employing the cumulants from Appendix A.7, the non-zero Edgeworth coefficients are:

$$\omega^2 = \sigma^2(1+\theta)^2, \quad a_1^{(1)} = E\left(u_0^3\right)(1+\theta)^3, \quad a_2^{(1)} = \sigma^4(1+\theta)^4\kappa_4,$$

and it follows that

$$\begin{aligned}
\psi_0 &= -\frac{1}{6\sqrt{n}}\kappa_3, \quad \psi_1 = \frac{1}{24n}\left(5\kappa_3^2 - 3\kappa_4\right), \quad \psi_2 = \frac{1}{6\sqrt{n}}\kappa_3, \\
\psi_3 &= -\frac{1}{72n}\left(10\kappa_3^2 - 3\kappa_4\right), \quad \psi_4 = 0, \quad \psi_5 = \frac{1}{72n}\kappa_3^2.
\end{aligned}$$

For $\mu = 0$, we play the above procedure with the difference that now the vector A is $A = (A_1, A_2, A_3)^{/} = (g_1, w_{11}, q_{111})^{/}$. The coefficients which are different from the above ones are: $a_4^{(1)} = 2\theta$, $a_5^{(1)} = -2\left(\theta^2+8\right) + 2\frac{-\theta+\theta^2-1}{-\theta+\theta^2+1}\theta\frac{\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2 - 4\left(1-\theta^2\right)\kappa_4$, $a_7^{(1)} = 2\left(\theta^2+9\right) + 2\left(1-\theta^2\right)\kappa_4$, and $a_9^{(1)} = 12 + 2\left(1-\theta^2\right)\kappa_4$.

Hence

$$k_1^{\widetilde{\theta}_0} = \frac{1}{\sqrt{n}}\theta + o\left(n^{-1}\right),$$

$$E\left[\sqrt{n}\left(\widetilde{\theta}_0 - \theta\right)\right]^2 = 1 - \theta^2 + \frac{1}{n}\left[8 + 2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2 - \left(1-\theta^2\right)\kappa_4\right],$$

$$k_2^{\widetilde{\theta}_0} = \omega^2 + \frac{1}{n}\left(8-\theta^2\right) + \frac{1}{n}\xi_2^{\widetilde{\theta}} + o\left(n^{-1}\right),$$

where ω^2 and $\xi_2^{\widetilde{\theta}}$ given above. Finally, $k_3^{\widetilde{\theta}_0} = k_3^{\widetilde{\theta}}$ and $k_4^{\widetilde{\theta}_0} = k_4^{\widetilde{\theta}}$ as neither of these cumulants are functions of the Edgeworth coefficients which are different in the non-zero mean case, i.e. $a_4^{(1)}$, $a_5^{(1)}$, $a_7^{(1)}$ and $a_9^{(1)}$.

A.9 Expansion of $\widetilde{\rho}$

With the definition of $\widetilde{\rho}$ let us call $m\left(\widetilde{\theta}\right) = \frac{\widetilde{\theta}}{1+\widetilde{\theta}^2} - \rho$, where ρ is the true value of the parameter. Then we have that

$$\sqrt{n}\left(\widetilde{\rho}-\rho\right) = \frac{\partial m(\theta)}{\partial \widetilde{\theta}}\sqrt{n}\left(\widetilde{\theta}-\theta\right) + \frac{1}{2\sqrt{n}}\frac{\partial^2 m(\theta)}{\partial \widetilde{\theta}^2}\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^2 + \frac{1}{6n}\frac{\partial^3 m(\theta)}{\partial \widetilde{\theta}^3}\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^3,$$

with a $o\left(n^{-1}\right)$ error. Consequently, we can apply Theorem 1 with $f^1 = \frac{(1-\theta^2)}{(1+\theta^2)^2}$, $h^{11} = -2\frac{\theta(3-\theta^2)}{(1+\theta^2)^3}$, $h^{111} = -6\frac{(1-\theta^2)^2-4\theta^2}{(1+\theta^2)^4}$, and $c_1^{(1)} = 2\theta - 1$, $c_{11}^{(1)} = 1-\theta^2$, $c_{11}^{(3)} = (\theta+6)(2-\theta) + \xi_2^{\widetilde{\theta}}$, $c_{111}^{(1)} = \frac{(1-\theta^2)^3}{1+\theta^3}\kappa_3^2$, $c_{1111}^{(1)} = 6\left(1-\theta^2\right)\left(\theta^2+3\right) + 120\theta\frac{\theta-\theta^2-2}{1-\theta+\theta^2}\frac{(1-\theta^2)^3}{1+\theta^3}\kappa_3^2 + \frac{(1-\theta^2)^3}{1+\theta^2}\kappa_4^2$, and all other cumulants being zero. Hence applying the formulae of Appendix A.1 we get that the non-zero Edgeworth coefficients are: $\omega^2 = \frac{(1-\theta^2)^3}{(1+\theta^2)^4}$, $\omega^{(3)} = \left[(\theta+6)(2-\theta) + 2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta(1-\theta^2)^2}{1+\theta^3}\kappa_3^2 - \left(1-\theta^2\right)\kappa_4\right]\frac{(1-\theta^2)^2}{(1+\theta^2)^4}$, $a_1^{(1)} = \frac{(1-\theta^2)^6}{(1+\theta^3)(1+\theta^2)^6}\kappa_3^2$, $a_2^{(1)} = \left(6\left(1-\theta^2\right)\left(\theta^2+3\right) + 120\theta\frac{\theta-\theta^2-2}{1-\theta+\theta^2}\frac{(1-\theta^2)^3}{1+\theta^3}\kappa_3^2 + \frac{(1-\theta^2)^3}{1+\theta^2}\kappa_4^2\right)\frac{(1-\theta^2)^4}{(1+\theta^2)^8}$, $a_3^{(1)} = -2\frac{\theta(3-\theta^2)(1-\theta^2)^4}{(1+\theta^2)^7}$, $a_4^{(1)} = -2\frac{\theta(3-\theta^2)}{(1+\theta^2)^3}\left(1-\theta^2\right)$, $a_5^{(1)} = -2\frac{\theta(3-\theta^2)}{1+\theta^3}\frac{(1-\theta^2)^4}{(1+\theta^2)^5}\kappa_3^2$, $a_6^{(1)} = -6\frac{(1-\theta^2)^2-4\theta^2}{(1+\theta^2)^4}\frac{(1-\theta^2)^6}{(1+\theta^2)^6}$, $a_7^{(1)} = -6\frac{(1-\theta^2)^2-4\theta^2}{(1+\theta^2)^4}\frac{(1-\theta^2)^3}{(1+\theta^2)^2}$, $a_8^{(1)} = 4\frac{\theta^2(3-\theta^2)^2}{(1+\theta^2)^6}\frac{(1-\theta^2)^4}{(1+\theta^2)^4}\left(1-\theta^2\right)$, $a_9^{(1)} = 4\frac{\theta^2(3-\theta^2)^2}{(1+\theta^2)^6}\left(1-\theta^2\right)^2$, $a_{10}^{(1)} = -2\frac{\theta(3-\theta^2)}{1+\theta^3}\frac{(1-\theta^2)^7}{(1+\theta^2)^9}\kappa_3^2$, $a_{11}^{(1)} = (2\theta-1)\frac{(1-\theta^2)}{(1+\theta^2)^2}$, and $a_{12}^{(1)} = -2\frac{\theta(3-\theta^2)}{(1+\theta^2)^3}(2\theta-1)\frac{(1-\theta^2)^2}{(1+\theta^2)^2}$.

Hence By Lemma 1

$$k_1^{\widetilde{\rho}} = -\frac{(1-\theta)\left(1+2\theta+3\theta^2\right)}{\sqrt{n}}\frac{\left(1-\theta^2\right)}{\left(1+\theta^2\right)^3},$$

and

$$\begin{aligned} E\left[\sqrt{n}\left(\widetilde{\rho}-\rho\right)\right]^2 &= \frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4} + \frac{1}{4n}\frac{\left(34\theta+239\theta^2-4\theta^3-245\theta^4-38\theta^5+69\theta^6+25\right)\left(1-\theta^2\right)^2}{\left(\theta^2+1\right)^6} \\ &+ \frac{1}{n}\left[-\theta(1-\theta)\frac{\left(1-\theta^2\right)^3\left(2-\theta+\theta^2+\theta^3-\theta^4\right)}{\left(1+\theta^2\right)^5\left(\theta^2-\theta+1\right)^2}\kappa_3^2 - \frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4}\kappa_4\right]. \end{aligned}$$

For the **zero-mean** case, notice that the Edgeworth coefficients that are different from the ones given above are: $\omega^{(3)} = \left(8-\theta^2 + 2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta(1-\theta^2)^2}{1+\theta^3}\kappa_3^2 - \left(1-\theta^2\right)\kappa_4\right)\frac{(1-\theta^2)^2}{(1+\theta^2)^4}$,

$a_{11}^{(1)} = \theta \frac{(1-\theta^2)}{(1+\theta^2)^2}$, and $a_{12}^{(1)} = -2\frac{\theta^2(3-\theta^2)}{(1+\theta^2)^3}\frac{(1-\theta^2)^2}{(1+\theta^2)^2}$. It follows that

$$k_1^{\widetilde{\rho_0}} = -\frac{2}{\sqrt{n}}\frac{\theta\left(1-\theta^2\right)^2}{\left(1+\theta^2\right)^3},$$

and

$$E\left[\sqrt{n}\left(\widetilde{\rho}_0-\rho\right)\right]^2 = \frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4}+\frac{1}{n}\left(\begin{array}{c} 2\left(32\theta^2-29\theta^4+6\theta^6+1\right)\frac{(1-\theta^2)^2}{(\theta^2+1)^6} \\ -\theta\left(1-\theta\right)\frac{(1-\theta^2)^3(2-\theta+\theta^2+\theta^3-\theta^4)}{(1+\theta^2)^5(\theta^2-\theta+1)^2}\kappa_3^2-\frac{(1-\theta^2)^3}{(1+\theta^2)^4}\kappa_4 \end{array}\right).$$

A.10 Useful Formulae for QMLE

$$\begin{aligned}
E\left(\frac{\partial u_t}{\partial\theta}\right)^2 &= \sigma^2\frac{1}{1-\theta^2} \\
E\left(\frac{\partial u_t}{\partial\theta}\right)^3 &= E\left(-\sum_{i=0}^{\infty}(-\theta)^i u_{t-1-i}\right)^3 = -E\left(\sum_{i=0}^{\infty}(-\theta)^{3i}u_{t-1-i}^3\right) = -\frac{E\left(u_0^3\right)}{1+\theta^3} \\
E\left(\frac{\partial u_t}{\partial\theta}\right)^4 &= E\left(-\sum_{i=0}^{\infty}(-\theta)^i u_{t-1-i}\right)^4 = \frac{1}{1-\theta^4}E\left(u_0^4\right)+6\sigma^4\frac{\theta^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}
\end{aligned}$$

$$\begin{aligned}
E\left(\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k}}{\partial\theta}\right) &= -E\left(\frac{\partial u_t}{\partial\theta}u_{t+k-1}\right)-\theta E\left(\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k-1}}{\partial\theta}\right) = -\theta E\left(\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k-1}}{\partial\theta}\right) \\
&= (-\theta)^k E\left(\frac{\partial u_t}{\partial\theta}\right)^2 = \frac{(-\theta)^k\sigma^2}{1-\theta^2}
\end{aligned}$$

$$E\left(\left(\frac{\partial u_t}{\partial\theta}\right)^2\frac{\partial u_{t+k}}{\partial\theta}\right) = (-\theta)E\left(\left(\frac{\partial u_t}{\partial\theta}\right)^2\left(\frac{\partial u_{t+k-1}}{\partial\theta}\right)\right) = (-\theta)^k E\left(\frac{\partial u_t}{\partial\theta}\right)^3 = -(-\theta)^k\frac{E\left(u_0^3\right)}{1+\theta^3}$$

$$E\left(\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\frac{\partial u_t}{\partial\theta}\right) = \theta^2 E\left(\frac{\partial u_t}{\partial\theta}\left(\frac{\partial u_{t+k-1}}{\partial\theta}\right)^2\right) = \theta^{2k}E\left(\frac{\partial u_t}{\partial\theta}\right)^3 = -\theta^{2k}\frac{E\left(u_0^3\right)}{1+\theta^3}$$

$$E\left(\left(\frac{\partial u_t}{\partial\theta}\right)^2\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\right) = \sigma^4\frac{1}{1-\theta^2}+\theta^2E\left(\left(\frac{\partial u_t}{\partial\theta}\right)^2\left(\frac{\partial u_{t+k-1}}{\partial\theta}\right)^2\right)=...=$$

$$= \sigma^4\frac{1}{1-\theta^2}\left(1+\theta^2+...+\theta^{2k-2}\right)+\theta^{2k}E\left(\left(\frac{\partial u_t}{\partial\theta}\right)^4\right)$$

$$= \sigma^4\frac{1}{\left(1-\theta^2\right)^2}+\theta^{2k}\frac{1}{1-\theta^4}E\left(u_0^4\right)+\sigma^4\frac{5\theta^2-1}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\theta^{2k}$$

$$E\left(u_t\frac{\partial u_{t+k}}{\partial\theta}\right)=-E\left(u_t\left(\sum_{i=1}^{k}(-\theta)^{i-1}u_{t+k-i}\right)\right)+(-\theta)^kE\left(u_t\frac{\partial u_t}{\partial\theta}\right)=-(-\theta)^{k-1}\sigma^2$$

$$E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k}}{\partial\theta}\right)=0$$

$$E\left(u_t\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\right)=\theta^{2(k-1)}E\left(u_0^3\right)$$

$$E\left(u_t\frac{\partial u_t}{\partial\theta}\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\right) = \theta^2E\left(u_t\frac{\partial u_t}{\partial\theta}\left(\frac{\partial u_{t+k-1}}{\partial\theta}\right)^2\right)=\theta^{2(k-1)}E\left(u_t\frac{\partial u_t}{\partial\theta}\left(\frac{\partial u_{t+1}}{\partial\theta}\right)^2\right)$$

$$= 2\theta^{2k-1}\sigma^4\frac{1}{1-\theta^2}$$

$$E\left(u_t^2\left(\frac{\partial u_t}{\partial\theta}\right)^2\frac{\partial u_{t+k}}{\partial\theta}\right) = (-\theta)E\left(u_t^2\left(\frac{\partial u_t}{\partial\theta}\right)^2\frac{\partial u_{t+k-1}}{\partial\theta}\right)=(-\theta)^{k-1}E\left(u_t^2\left(\frac{\partial u_t}{\partial\theta}\right)^2\frac{\partial u_{t+1}}{\partial\theta}\right)$$

$$= -(-\theta)^{k-1}E\left(u_t^3\left(\frac{\partial u_t}{\partial\theta}\right)^2\right)+(-\theta)^kE\left(u_t^2\left(\frac{\partial u_t}{\partial\theta}\right)^3\right)$$

$$= -(-\theta)^{k-1}E\left(u_0^3\right)E\left(\frac{\partial u_t}{\partial\theta}\right)^2+(-\theta)^k\sigma^2E\left(\frac{\partial u_t}{\partial\theta}\right)^3$$

$$= -(-\theta)^{k-1}E\left(u_0^3\right)\sigma^2\frac{1}{1-\theta^2}-(-\theta)^k\sigma^2\frac{1}{1+\theta^3}E\left(u_0^3\right)$$

$$E\left(u_t\frac{\partial u_t}{\partial\theta}\right)^3=E\left(u_0^3\right)E\left(\frac{\partial u_t}{\partial\theta}\right)^3=-\frac{E^2\left(u_0^3\right)}{1+\theta^3}$$

$$
\begin{aligned}
E\left(u_t^2\left(\frac{\partial u_t}{\partial\theta}\right)^2\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\right) &= E\left(u_t^2\left(\frac{\partial u_t}{\partial\theta}\right)^2\left(\sum_{i=0}^{k-1}(-\theta)^i u_{t+k-1-i}\right)^2\right)\\
&\quad -2(-\theta)^{2k-1}E\left(u_t^3\left(\frac{\partial u_t}{\partial\theta}\right)^3\right)+\theta^{2k}E\left(u_t^2\left(\frac{\partial u_t}{\partial\theta}\right)^4\right)\\
&= \sigma^6\frac{1-\theta^{2(k-1)}}{\left(1-\theta^2\right)^2}+\theta^{2(k-1)}\frac{E\left(u_0^4\right)\sigma^2}{1-\theta^2}-2\theta^{2(k-1)}\frac{\theta E^2\left(u_0^3\right)}{1+\theta^3}\\
&\quad +\theta^{2k}\sigma^2\left(\frac{E\left(u_0^4\right)}{1-\theta^4}+\frac{6\sigma^4\theta^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
E\left(u_t^2\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k}}{\partial\theta}\right) &= E\left(u_t^2\frac{\partial u_t}{\partial\theta}\left(-u_{t+k-1}-(-\theta)u_{t+k-2}-\ldots-(-\theta)^{k-1}u_t+(-\theta)^k\frac{\partial u_t}{\partial\theta}\right)\right)\\
&= (-\theta)^k E\left(u_t^2\left(\frac{\partial u_t}{\partial\theta}\right)^2\right)=(-\theta)^k\sigma^4\frac{1}{1-\theta^2}
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(\frac{\partial u_t}{\partial\theta}\right)^2 u_{t+k}^2\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\right] &= \sigma^2\left[\sigma^4\frac{1+3\theta^{2k}}{\left(1-\theta^2\right)^2}+\frac{\theta^{2k}}{1-\theta^4}E\left(u_0^4\right)\right]\\
&= \sigma^6\frac{1+3\theta^{2k}}{\left(1-\theta^2\right)^2}+\sigma^2\frac{\theta^{2k}}{1-\theta^4}E\left(u_0^4\right).
\end{aligned}
$$

$$
\begin{aligned}
E\left(u_t\frac{\partial u_t}{\partial\theta}\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^3\right) &= E\left(u_t\frac{\partial u_t}{\partial\theta}\left(-\sum_{i=0}^{k-1}(-\theta)^i u_{t+k-1-i}+(-\theta)^k\frac{\partial u_t}{\partial\theta}\right)^3\right)\\
&= -E\left(u_t\frac{\partial u_t}{\partial\theta}\left(\sum_{i=0}^{k-1}(-\theta)^i u_{t+k-1-i}\right)^3\right)\\
&\quad +3(-\theta)^k E\left(u_t\left(\frac{\partial u_t}{\partial\theta}\right)^2\left(\sum_{i=0}^{k-1}(-\theta)^i u_{t+k-1-i}\right)^2\right)\\
&\quad -3\theta^{2k}E\left(u_t\left(\frac{\partial u_t}{\partial\theta}\right)^3\sum_{i=0}^{k-1}(-\theta)^i u_{t+k-1-i}\right)+(-\theta)^{3k}E\left(u_t\left(\frac{\partial u_t}{\partial\theta}\right)^4\right)\\
&= 3(-\theta)^{3k-2}\sigma^2\frac{1}{1-\theta^2}E\left(u_0^3\right)+3(-\theta)^{3k-1}\sigma^2\frac{1}{1+\theta^3}E\left(u_0^3\right)
\end{aligned}
$$

$$E\left[u_t^2\left(\frac{\partial u_t}{\partial\theta}\right)^4\right]=\sigma^2E\left[\left(\frac{\partial u_t}{\partial\theta}\right)^4\right]=\frac{1}{1-\theta^4}\sigma^2E\left(u_0^4\right)+6\sigma^6\frac{1}{\left(1-\theta^2\right)^2}.$$

$$E\left(u_t\frac{\partial u_t}{\partial\theta}u_{t+k}\frac{\partial u_{t+k+i}}{\partial\theta}\right)=0$$

$$\begin{aligned}E\left[u_{t-i}u_t\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k}}{\partial\theta}\right] &= E\left[u_{t-i}u_t\frac{\partial u_t}{\partial\theta}\left(-u_{t+k-1}-(-\theta)\,u_{t+k-2}-..-(-\theta)^{k-1}u_t+(-\theta)^k\frac{\partial u_t}{\partial\theta}\right)\right]\\ &= -(-\theta)^{k-1}\sigma^2E\left[u_{t-i}\frac{\partial u_t}{\partial\theta}\right]=(-\theta)^{k+i-2}\sigma^4\end{aligned}$$

$$\begin{aligned}E\left(u_t\frac{\partial u_t}{\partial\theta}u_{t+k}\frac{\partial u_{t+k}}{\partial\theta}\frac{\partial u_{t+k+i}}{\partial\theta}\right) &= -(-\theta)^{i-1}E\left(u_t\frac{\partial u_t}{\partial\theta}u_{t+k}^2\frac{\partial u_{t+k}}{\partial\theta}\right)\\ &\quad+(-\theta)^iE\left(u_t\frac{\partial u_t}{\partial\theta}u_{t+k}\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\right)\\ &= -(-\theta)^{i-1}\sigma^2E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k}}{\partial\theta}\right)=0\end{aligned}$$

$$\begin{aligned}E\left(u_t\frac{\partial u_t}{\partial\theta}u_{t+k}\frac{\partial u_{t+k}}{\partial\theta}\left(\frac{\partial u_{t+k+i}}{\partial\theta}\right)^2\right) &= (-\theta)^{2(i-1)}E\left(u_0^3\right)E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k}}{\partial\theta}\right)\\ &\quad-2(-\theta)^{2i-1}\sigma^2E\left(u_t\frac{\partial u_t}{\partial\theta}\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\right)\\ &= 4(-\theta)^{2i-1}(-\theta)^{2k-1}\sigma^6\frac{1}{1-\theta^2}\end{aligned}$$

$$E\left[\frac{\partial^2u_t}{\partial\theta^2}\left(\frac{\partial u_t}{\partial\theta}\right)^2\right]=-2\theta^2E\left(\frac{\partial u_{t-1}}{\partial\theta}\right)^3+(-\theta)^3E\left[\frac{\partial^2u_{t-1}}{\partial\theta^2}\left(\frac{\partial u_{t-1}}{\partial\theta}\right)^2\right]=2\theta^2\frac{1}{\left(1+\theta^3\right)^2}E\left(u_0^3\right),$$

$$\begin{aligned}
E\left(u_t \frac{\partial^2 u_{t+k}}{\partial\theta^2}\right) &= E\left(u_t\left(-2\frac{\partial u_{t+k-1}}{\partial\theta} - \theta\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right)\right) = -2E\left(u_t\frac{\partial u_{t+k-1}}{\partial\theta}\right) - \theta E\left(u_t\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right) \\
&= 2(-\theta)^{k-2}\sigma^2 - \theta E\left(u_t\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right) = 2(-\theta)^{k-2}\sigma^2 + 2(-\theta)^{k-2}\sigma^2 \\
&\quad + (-\theta)^2 E\left(u_t\frac{\partial^2 u_{t+k-2}}{\partial\theta^2}\right) \\
&= 2(k-1)(-\theta)^{k-2}\sigma^2
\end{aligned}$$

$$E\left(u_t^2\frac{\partial^2 u_t}{\partial\theta^2}\right) = \sigma^2 E\left(\frac{\partial^2 u_t}{\partial\theta^2}\right) = 0$$

$$E\left(u_t^2\left(\frac{\partial^2 u_t}{\partial\theta^2}\right)^2\right) = \sigma^2 E\left(\frac{\partial^2 u_t}{\partial\theta^2}\right)^2 = 4\sigma^4\frac{1+\theta^2}{(1-\theta^2)^3}$$

$$\begin{aligned}
E\left(\left(\frac{\partial^2 u_t}{\partial\theta^2}\right)^2\right) &= 4E\left(\frac{\partial u_{t-1}}{\partial\theta}\right)^2 + 4\theta E\left(\frac{\partial u_{t-1}}{\partial\theta}\frac{\partial^2 u_{t-1}}{\partial\theta^2}\right) + \theta^2 E\left(\left(\frac{\partial^2 u_{t-1}}{\partial\theta^2}\right)^2\right) \\
&= 4\sigma^2\frac{1}{1-\theta^2} + 8\theta^2\sigma^2\frac{1}{(1-\theta^2)^2} + \theta^2 E\left(\left(\frac{\partial^2 u_{t-1}}{\partial\theta^2}\right)^2\right) \\
&= 4\sigma^2\frac{1}{(1-\theta^2)^2} + 8\theta^2\sigma^2\frac{1}{(1-\theta^2)^3} = 4\sigma^2\frac{1+\theta^2}{(1-\theta^2)^3}
\end{aligned}$$

as

$$E\left(\frac{\partial u_t}{\partial\theta}\frac{\partial^2 u_t}{\partial\theta^2}\right) = 2\theta\sigma^2\frac{1}{1-\theta^2} + \theta^2 E\left(\frac{\partial u_{t-1}}{\partial\theta}\frac{\partial^2 u_{t-1}}{\partial\theta^2}\right) = \frac{2\theta\sigma^2}{(1-\theta^2)^2}$$

$$E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial^2 u_{t+k}}{\partial\theta^2}\right) = -2E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k-1}}{\partial\theta}\right) - \theta E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right) = -\theta E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right) = 0$$

$$E\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\frac{\partial u_{t+k}}{\partial\theta}\right) = (-\theta)E\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\frac{\partial u_{t+k-1}}{\partial\theta}\right) = \ldots = (-\theta)^{k-1}E\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\frac{\partial u_{t+1}}{\partial\theta}\right) = 0$$

$$E\left(u_t\frac{\partial u_{t+k}}{\partial\theta}\frac{\partial^2 u_{t+k}}{\partial\theta^2}\right) = E\left(u_t\left(u_{t+k-1}+\theta\frac{\partial u_{t+k-1}}{\partial\theta}\right)\left(2\frac{\partial u_{t+k-1}}{\partial\theta}+\theta\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right)\right)$$

$$= 2\theta E\left(u_t\left(\frac{\partial u_{t+k-1}}{\partial\theta}\right)^2\right)+\theta^2 E\left(u_t\frac{\partial u_{t+k-1}}{\partial\theta}\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right)$$

$$= 2\theta^{2k-3}E\left(u_t^3\right)+\theta^2 E\left(u_t\frac{\partial u_{t+k-1}}{\partial\theta}\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right) = 2\left(k-1\right)\theta^{2k-3}E\left(u_t^3\right)$$

$$E\left(u_t^2\frac{\partial u_t}{\partial\theta}\frac{\partial^2 u_t}{\partial\theta^2}\right) = \sigma^2 E\left(\frac{\partial u_t}{\partial\theta}\frac{\partial^2 u_t}{\partial\theta^2}\right) = \sigma^2 E\left(\left(-u_{t-1}+(-\theta)\frac{\partial u_{t-1}}{\partial\theta}\right)\left(-2\frac{\partial u_{t-1}}{\partial\theta}-\theta\frac{\partial^2 u_{t-1}}{\partial\theta^2}\right)\right)$$

$$= (-2)\,\sigma^2\,(-\theta)\,E\left(\frac{\partial u_{t-1}}{\partial\theta}\right)^2+\sigma^2\,(-\theta)^2\,E\left(\frac{\partial u_{t-1}}{\partial\theta}\frac{\partial^2 u_{t-1}}{\partial\theta^2}\right) =$$

$$= (-2)\,\sigma^4\,(-\theta)\,\frac{1}{1-\theta^2}+\sigma^2\,(-\theta)^2\,E\left(\frac{\partial u_{t-1}}{\partial\theta}\frac{\partial^2 u_{t-1}}{\partial\theta^2}\right) = 2\theta\sigma^4\frac{1}{\left(1-\theta^2\right)^2}$$

$$E\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\left(\frac{\partial u_{t+k}}{\partial\theta}\right)^2\right) = E\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\left(u_{t+k-1}^2+2u_{t+k-1}\theta\frac{\partial u_{t+k-1}}{\partial\theta}+\theta^2\left(\frac{\partial u_{t+k-1}}{\partial\theta}\right)^2\right)\right)$$

$$= \theta^2 E\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\left(\frac{\partial u_{t+k-1}}{\partial\theta}\right)^2\right) = \ldots = \theta^{2k-2}E\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\left(\frac{\partial u_{t+1}}{\partial\theta}\right)^2\right)$$

$$= \theta^{2k-2}E\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\left(u_t^2+2u_t\theta\frac{\partial u_t}{\partial\theta}+\theta^2\left(\frac{\partial u_t}{\partial\theta}\right)^2\right)\right)$$

$$= 2\theta^{2k-1}\sigma^2 E\left(\frac{\partial^2 u_t}{\partial\theta^2}\frac{\partial u_t}{\partial\theta}\right)+\theta^{2k}E\left(u_t\frac{\partial^2 u_t}{\partial\theta^2}\left(\frac{\partial u_t}{\partial\theta}\right)^2\right)$$

$$= 4\theta^{2k}\sigma^4\frac{1}{\left(1-\theta^2\right)^2}$$

$$
\begin{aligned}
E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k}}{\partial\theta}\frac{\partial^2 u_{t+k}}{\partial\theta^2}\right) &= E\left(u_t\frac{\partial u_t}{\partial\theta}\left(-u_{t+k-1}+(-\theta)\frac{\partial u_{t+k-1}}{\partial\theta}\right)\left(-2\frac{\partial u_{t+k-1}}{\partial\theta}-\theta\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right)\right)\\
&= -E\left(u_t\frac{\partial u_t}{\partial\theta}u_{t+k-1}\left(-2\frac{\partial u_{t+k-1}}{\partial\theta}-\theta\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right)\right)\\
&\quad -\theta E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k-1}}{\partial\theta}\left(-2\frac{\partial u_{t+k-1}}{\partial\theta}-\theta\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right)\right)\\
&= -2\left(-\theta\right)E\left(u_t\frac{\partial u_t}{\partial\theta}\left(\frac{\partial u_{t+k-1}}{\partial\theta}\right)^2\right)+\theta^2 E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k-1}}{\partial\theta}\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right)\\
&= 4\left(-\theta\right)^{2k-2}\sigma^4\frac{1}{1-\theta^2}+\theta^2 E\left(u_t\frac{\partial u_t}{\partial\theta}\frac{\partial u_{t+k-1}}{\partial\theta}\frac{\partial^2 u_{t+k-1}}{\partial\theta^2}\right)\\
&= 4\left(k-1\right)\theta^{2k-2}\sigma^4\frac{1}{1-\theta^2}+2\theta^{2k-2}\sigma^4\frac{1}{\left(1-\theta^2\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
E\left(u_t^2\frac{\partial u_t}{\partial\theta}\frac{\partial^3 u_t}{\partial\theta^3}\right) &= \sigma^2 E\left(\frac{\partial u_t}{\partial\theta}\left(-3\frac{\partial^2 u_{t-1}}{\partial\theta^2}-\theta\frac{\partial^3 u_{t-1}}{\partial\theta^3}\right)\right)\\
&= -3\sigma^2 E\left(\frac{\partial u_t}{\partial\theta}\frac{\partial^2 u_{t-1}}{\partial\theta^2}\right)-\theta\sigma^2 E\left(\frac{\partial u_t}{\partial\theta}\frac{\partial^3 u_{t-1}}{\partial\theta^3}\right)=\\
&= 3\sigma^2 E\left(\left(u_{t-1}+\theta\frac{\partial u_{t-1}}{\partial\theta}\right)\frac{\partial^2 u_{t-1}}{\partial\theta^2}\right)+\theta\sigma^2 E\left(\left(u_{t-1}+\theta\frac{\partial u_{t-1}}{\partial\theta}\right)\frac{\partial^3 u_{t-1}}{\partial\theta^3}\right)\\
&= 6\theta^2\sigma^4\frac{1}{\left(1-\theta^2\right)^2}+\left(-\theta\right)^2\sigma^2 E\left(\frac{\partial u_{t-1}}{\partial\theta}\frac{\partial^3 u_{t-1}}{\partial\theta^3}\right)=6\theta^2\sigma^4\frac{1}{\left(1-\theta^2\right)^3}
\end{aligned}
$$

Appendix B

Appendix for "Bias Correction of ML and QML Estimators in the EGARCH(1,1) Model"

B.1 Proof of the unconditional variance

We write the variance equation as follows:

$$\ln\left(h_t\right) = \alpha^* + \theta \sum_{i=0}^{\infty} \beta^i z_{t-1-i} + \gamma \sum_{i=0}^{\infty} \beta^i \left(\left|z_{t-1-i}\right| - E\left|z_{t-1-i}\right|\right),$$

where $\alpha^* = \frac{\alpha}{1-\beta}$. Taking the expectation of the exponential of $\ln\left(h_t\right)$ we have:

$E\exp\left(\ln h_t\right) = \exp\left(\alpha^*\right) E\exp\left[\sum_{i=0}^{\infty}\left(\theta\beta^i z_{t-1-i} + \gamma\beta^i\left(\left|z_{t-1-i}\right| - E\left|z_{t-1-i}\right|\right)\right)\right] =$

$= \exp\left(\alpha^*\right) E \prod_{i=0}^{\infty} \exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\left(\left|z_{t-1-i}\right| - E\left|z_{t-1-i}\right|\right)\right] =$

$= \exp\left(\alpha^*\right) \prod_{i=0}^{\infty} E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\left(\left|z_{t-1-i}\right| - E\left|z_{t-1-i}\right|\right)\right]$

Now,

$\prod_{i=0}^{\infty} E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\left(\left|z_{t-1-i}\right| - E\left|z_{t-1-i}\right|\right)\right] =$

$$= \prod_{i=0}^{\infty} \exp\left(-\gamma E\left|z_{t-1-i}\right|\beta^i\right) E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\left|z_{t-1-i}\right|\right] =$$

$$= \exp\left(-\frac{\gamma E|z|}{1-\beta}\right)\prod_{i=0}^{\infty} E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\left|z_{t-1-i}\right|\right]$$

$$E\exp\left[\theta\beta^i z + \gamma\beta^i|z|\right] \overset{\substack{\kappa_1=\theta\beta^i \\ \kappa_2=\gamma\beta^i}}{=} E\exp\left[\kappa_1 z + \kappa_2|z|\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(\kappa_1 z + \kappa_2|z| - \frac{1}{2}z^2\right)dz =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\frac{1}{2}\left(-2(\kappa_1-\kappa_2)z + z^2 \pm (\kappa_1-\kappa_2)^2\right)\right)dz$$

$$+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\exp\left(-\frac{1}{2}\left(-2(\kappa_1+\kappa_2)z + z^2 \pm (\kappa_1+\kappa_2)^2\right)\right)dz =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\frac{1}{2}\left(-2(\kappa_1-\kappa_2)z + z^2 \pm (\kappa_1-\kappa_2)^2\right)\right)dz$$

$$+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\exp\left(-\frac{1}{2}\left(-2(\kappa_1+\kappa_2)z + z^2 \pm (\kappa_1+\kappa_2)^2\right)\right)dz =$$

$$= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\frac{1}{2}\left(z-(\kappa_1-\kappa_2)\right)^2\right)dz$$

$$+\exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\frac{1}{2}\left(z-(\kappa_1+\kappa_2)\right)^2\right)dz =$$

$$= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-(\kappa_1-\kappa_2)}\exp\left(-\frac{1}{2}u^2\right)du + \exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-(\kappa_1+\kappa_2)}\exp\left(-\frac{1}{2}u^2\right)du =$$

$$= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right)\Phi\left(-(\kappa_1-\kappa_2)\right) + \exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right)\left(1-\Phi\left(-(\kappa_1+\kappa_2)\right)\right) =$$

$$= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right)\Phi\left(-(\kappa_1-\kappa_2)\right) + \exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right)\Phi\left(\kappa_1+\kappa_2\right) =$$

$$= \exp\left(\frac{\beta^{2i}(\gamma-\theta)^2}{2}\right)\Phi\left(\beta^i(\gamma-\theta)\right) + \exp\left(\frac{\beta^{2i}(\gamma+\theta)^2}{2}\right)\Phi\left(\beta^i(\gamma+\theta)\right)$$

$$= \exp(\Delta)\Phi(-B) + \exp(\Gamma)\Phi(A),$$

where $\Gamma = \frac{\beta^{2i}(\gamma+\theta)^2}{2}$, $\Delta = \frac{\beta^{2i}(\gamma-\theta)^2}{2}$, $A = \beta^i(\gamma+\theta)$ and $B = \beta^i(\gamma-\theta)$, and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Therefore,

$$\begin{aligned} E\exp(q\ln h_t) &= \exp\left(\frac{\alpha-\gamma E|z|}{1-\beta}\right)\prod_{i=0}^{\infty}\begin{pmatrix} \exp\left(\frac{\beta^{2i}(\gamma-\theta)^2}{2}\right)\Phi\left(\beta^i(\gamma-\theta)\right) \\ +\exp\left(\frac{\beta^{2i}(\gamma+\theta)^2}{2}\right)\Phi\left(\beta^i(\gamma+\theta)\right)\end{pmatrix} \\ &= \exp(\Psi)\prod_{i=0}^{\infty}\left(\exp(\Delta)\Phi(-B) + \exp(\Gamma)\Phi(A)\right) \\ &= b^*, \end{aligned}$$

where $\Psi = \frac{\alpha - \gamma E|z|}{1-\beta}$.■

B.2 Expected values of the log-likelihood derivatives

The expected values of all first order derivatives are equal to zero.

Second order derivatives:

For $i, j \in \{\alpha, \theta, \gamma, \beta\}$,

$$\begin{aligned} E(\mathcal{L}_{ij}) &= -\frac{T}{2} E(h_{t;i} h_{t;j}), \\ E(\mathcal{L}_{\mu j}) &= -\frac{T}{2} E(h_{t;\mu} h_{t;j}), \\ E(\mathcal{L}_{\mu\mu}) &= -TE\left(\frac{1}{h_t}\right) - \frac{T}{2} E\left(h_{t;\mu}^2\right). \end{aligned}$$

Third order derivatives:

For $i \in \{\alpha, \theta, \gamma, \beta\}$,

$$E(\mathcal{L}_{iii}) = -\frac{T}{2} E\left(3h_{t;i} h_{t;i,i} - h_{t;i}^3\right),$$

for $i \in \{\alpha, \theta, \gamma, \beta\}, j \in \{\alpha, \theta, \gamma, \beta, \mu\}$,

$$E(\mathcal{L}_{iij}) = -\frac{T}{2} E\left(h_{t;j} h_{t;i,i} - h_{t;i}^2 h_{t;j} + 2h_{t;i} h_{t;i,j}\right),$$

for $i, j \in \{\alpha, \theta, \gamma, \beta\}, k \in \{\alpha, \theta, \gamma, \beta, \mu\}$,

$$E(\mathcal{L}_{ijk}) = -\frac{T}{2} E(h_{t;j} h_{t;i,k} + h_{t;k} h_{t;i,j} + h_{t;i} h_{t;j,k} - h_{t;j} h_{t;i} h_{t;k},),$$

for $i \in \{\alpha, \theta, \gamma, \beta\}, j \in \{\mu\}$,

$$E(\mathcal{L}_{ijj}) = -\frac{T}{2} E\left(h_{t;i} h_{t;j,j} + 2h_{t;j} h_{t;i,j} - h_{t;i} (h_{t;j})^2\right) + TE\left(\frac{1}{h_t} h_{t;i}\right),$$

for $j \in \{\mu\}$,

$$E(\mathcal{L}_{jjj}) = -\frac{T}{2} E\left(3h_{t;j} h_{t;j,j} - h_{t;j}^3\right) + TE\left(3\frac{1}{h_t} h_{t;j}\right).$$

In this Appendix we make a list of the results that are needed for the bias approximations. Please note that the last Appendix should be studied first in order to be familiarized with the symbols used.

First, provided that $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)\right| < 1$, the expected values of all second order derivatives are:

1. $E\left(\mathcal{L}_{\alpha\alpha}\right) = -\frac{T}{2}\frac{1+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\theta\gamma E(z|z|)\right)}$

2. $E\left(\mathcal{L}_{\alpha\beta}\right) = -\frac{T}{2}\frac{E(\ln(h_{t-1}))+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\theta\gamma E(z|z|)\right)}$

3. $E\left(\mathcal{L}_{\alpha\gamma}\right) = -\frac{T}{2}\frac{-\frac{1}{2}\left[E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

4. $E\left(\mathcal{L}_{\alpha\theta}\right) = -\frac{T}{2}\frac{-\frac{1}{2}[\theta+\gamma E(z|z|)]E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

5. $E\left(\mathcal{L}_{\mu\alpha}\right) = -\frac{T}{2}\frac{-(\theta+\gamma EI)E\left(\frac{1}{\sqrt{h}}\right)+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\mu}+[\theta(\beta-\gamma E|z|)+\gamma\beta EI]E_{-\frac{1}{2}}E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

6. $E\left(\mathcal{L}_{\beta\beta}\right) = -\frac{T}{2}\frac{E\left(\ln^2(h_{t-1})\right)+2\left(\beta-\frac{1}{2}\gamma|z_{t-1}|\right)LE_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

7. $E\left(\mathcal{L}_{\beta\gamma}\right) = -\frac{T}{2}\frac{\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\gamma}-\frac{1}{2}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

8. $E\left(\mathcal{L}_{\beta\theta}\right) = -\frac{T}{2}\frac{-\frac{1}{2}[\theta+\gamma E(z|z|)]E_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

9. $E\left(\mathcal{L}_{\beta\mu}\right) = -\frac{T}{2}\frac{-(\theta+\gamma EI)LE_{-\frac{1}{2}}+[\theta(\gamma E|z|-\beta)-\beta\gamma EI]E_{-\frac{1}{2}}E_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\mu}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

10. $E\left(\mathcal{L}_{\gamma\gamma}\right) = -\frac{T}{2}\frac{1-E^2|z|}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

11. $E\left(\mathcal{L}_{\theta\gamma}\right) = -\frac{T}{2}\frac{E(z|z|)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

12. $E\left(\mathcal{L}_{\mu\gamma}\right) = -\frac{T}{2}\frac{-\gamma EI_{g(z)}E\left(\frac{1}{\sqrt{h}}\right)-\frac{1}{2}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\mu}-(\theta(\beta-\gamma E|z|)+\gamma\beta EI)E_{-\frac{1}{2}}E_{;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

13. $E\left(\mathcal{L}_{\theta\theta}\right) = -\frac{T}{2}\frac{1}{1-\left(\beta_2^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

14. $E\left(\mathcal{L}_{\mu\theta}\right) = -\frac{T}{2}\frac{-\frac{1}{2}(\theta+\gamma E(z|z|))E(h_{t;\mu})-\gamma E|z|E\left(\frac{1}{\sqrt{h}}\right)+[\theta(\gamma E|z|-\beta)-\beta\gamma EI]E_{-\frac{1}{2}}E_{;\theta}}{1-\left(\beta_2^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

15. $E\left(\mathcal{L}_{\mu\mu}\right) = -TE\left(\frac{1}{h_t}\right) - \frac{T}{2}\frac{\left(\theta^2+\gamma^2+2\gamma\theta EI\right)E\left(\frac{1}{\sqrt{h_{t-1}}}\right)-2(\theta(\beta-\gamma E|z|)+\gamma\beta EI)E_{-\frac{1}{2}}E_{;\mu}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}.$

Second, the expected values of the third order derivatives are:

1. $E\left(\mathcal{L}_{\alpha\alpha\alpha}\right) = -\frac{T}{2}E\left(3\left(h_{t;\alpha}h_{t;\alpha,\alpha}\right) - h^3_{t;\alpha}\right)$

2. $E\left(\mathcal{L}_{\alpha\alpha\beta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\alpha} - h^2_{t;\alpha}h_{t;\beta} + 2h_{t;\alpha}h_{t;\alpha,\beta}\right)$

3. $E\left(\mathcal{L}_{\alpha\alpha\gamma}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\alpha,\alpha} - h^2_{t;\alpha}h_{t;\gamma} + 2h_{t;\alpha}h_{t;\alpha,\gamma}\right)$

4. $E\left(\mathcal{L}_{\alpha\alpha\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\alpha,\alpha} - h^2_{t;\alpha}h_{t;\theta} + 2h_{t;\alpha}h_{t;\alpha,\theta}\right)$

5. $E\left(\mathcal{L}_{\mu\alpha\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha,\alpha}h_{t;\mu} + 2\left(h_{t;\alpha}h_{t;\mu,\alpha}\right) - h^2_{t;\alpha}h_{t;\mu}\right)$

6. $E\left(\mathcal{L}_{\beta\beta\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\beta,\beta} + 2h_{t;\beta}h_{t;\beta,\alpha} - h_{t;\alpha}h^2_{t;\beta}\right)$

7. $E\left(\mathcal{L}_{\alpha\beta\gamma}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\gamma} + h_{t;\gamma}h_{t;\alpha,\beta} + h_{t;\alpha}h_{t;\beta,\gamma} - h_{t;\beta}h_{t;\alpha}h_{t;\gamma}\right)$

8. $E\left(\mathcal{L}_{\alpha\beta\theta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\theta} + h_{t;\alpha}h_{t;\beta,\theta} + h_{t;\theta}h_{t;\alpha,\beta} - h_{t;\alpha}h_{t;\beta}h_{t;\theta}\right)$

9. $E\left(\mathcal{L}_{\mu\beta\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\beta,\mu} + h_{t;\beta,\alpha}h_{t;\mu} + h_{t;\beta}h_{t;\mu,\alpha} - h_{t;\alpha}h_{t;\beta}h_{t;\mu}\right)$

10. $E\left(\mathcal{L}_{\mu\beta\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\beta,\mu} + h_{t;\beta,\alpha}h_{t;\mu} + h_{t;\beta}h_{t;\mu,\alpha} - h_{t;\alpha}h_{t;\beta}h_{t;\mu}\right)$

11. $E\left(\mathcal{L}_{\alpha\gamma\gamma}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\gamma,\gamma} + 2h_{t;\gamma}h_{t;\alpha,\gamma} - h_{t;\alpha}h^2_{t;\gamma}\right)$

12. $E\left(\mathcal{L}_{\alpha\gamma\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\alpha,\gamma} + h_{t;\gamma}h_{t;\alpha,\theta} + h_{t;\alpha}h_{t;\gamma,\theta} - h_{t;\alpha}h_{t;\gamma}h_{t;\theta}\right)$

13. $E\left(\mathcal{L}_{\alpha\gamma\mu}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\gamma,\mu} - h_{t;\alpha}h_{t;\gamma}h_{t;\mu} + h_{t;\gamma,\alpha}h_{t;\mu} + h_{t;\gamma}h_{t;\alpha,\mu}\right)$

14. $E\left(\mathcal{L}_{\alpha\theta\theta}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\theta,\theta} + 2h_{t;\theta}h_{t;\alpha,\theta} - h_{t;\alpha}h^2_{t;\theta}\right)$

15. $E\left(\mathcal{L}_{\alpha\theta\mu}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\theta,\mu} - h_{t;\alpha}h_{t;\theta}h_{t;\mu} + h_{t;\theta,\alpha}h_{t;\mu} + h_{t;\theta}h_{t;\alpha,\mu}\right)$

16. $E\left(\mathcal{L}_{\alpha\mu\mu}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\alpha,\mu} - h_{t;\alpha}\left(h_{t;\mu}\right)^2\right) + TE\left(\frac{1}{h_t}h_{t;\alpha}\right)$

17. $E\left(\mathcal{L}_{\beta\beta\beta}\right) = -\frac{T}{2}E\left(3h_{t;\beta}h_{t;\beta,\beta} - Eh^3_{t;\beta}\right)$

18. $E\left(\mathcal{L}_{\beta\beta\gamma}\right) = -\frac{T}{2}E\left(2h_{t;\beta}h_{t;\gamma,\beta} + h_{t;\gamma}h_{t;\beta,\beta} - h^2_{t;\beta}h_{t;\gamma}\right)$

19. $E\left(\mathcal{L}_{\beta\beta\theta}\right) = -\frac{T}{2}E\left(2h_{t;\beta}h_{t;\theta,\beta} + h_{t;\theta}h_{t;\beta,\beta} - h^2_{t;\beta}h_{t;\theta}\right)$

20. $E\left(\mathcal{L}_{\beta\beta\mu}\right) = -\frac{T}{2}\left(2E\left(h_{t;\beta}h_{t;\beta,\mu}\right) + E\left(h_{t;\beta;\beta}h_{t;\mu}\right) - E\left(h^2_{t;\beta}h_{t;\mu}\right)\right)$

21. $E\left(\mathcal{L}_{\beta\gamma\gamma}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\gamma,\gamma} + 2h_{t;\gamma}h_{t;\beta,\gamma} - h_{t;\beta}h^2_{t;\gamma}\right)$

22. $E\left(\mathcal{L}_{\beta\gamma\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\beta,\gamma} + h_{t;\gamma}h_{t;\beta,\theta} + h_{t;\beta}h_{t;\gamma,\theta} - h_{t;\beta}h_{t;\gamma}h_{t;\theta}\right)$

23. $E\left(\mathcal{L}_{\beta\gamma\mu}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\gamma,\mu} - h_{t;\beta}h_{t;\gamma}h_{t;\mu} + h_{t;\gamma,\beta}h_{t;\mu} + h_{t;\gamma}h_{t;\beta,\mu}\right)$

24. $E\left(\mathcal{L}_{\beta\theta\theta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\theta,\theta} + 2h_{t;\theta}h_{t;\beta,\theta} - h_{t;\beta}h^2_{t;\theta}\right)$

25. $E\left(\mathcal{L}_{\beta\theta\mu}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\theta,\mu} - h_{t;\beta}h_{t;\theta}h_{t;\mu} + h_{t;\theta,\beta}h_{t;\mu} + h_{t;\theta}h_{t;\beta,\mu}\right)$

26. $E\left(\mathcal{L}_{\mu\mu_\beta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\mu,\beta} - h_{t;\beta}\left(h_{t;\mu}\right)^2\right) + TE\left(\frac{1}{h_t}h_{t;\beta}\right)$

27. $E\left(\mathcal{L}_{\gamma\gamma\gamma}\right) = -\frac{T}{2}E\left(3\left(h_{t;\gamma}h_{t;\gamma,\gamma}\right) - \left(h^3_{t;\gamma}\right)\right)$

28. $E\left(\mathcal{L}_{\gamma\gamma\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\gamma,\gamma} + 2h_{t;\gamma}h_{t;\gamma,\theta} - h^2_{t;\gamma}h_{t;\theta}\right)$

29. $E\left(\mathcal{L}_{\gamma\gamma\mu}\right) = -\frac{T}{2}E\left(2h_{t;\gamma}h_{t;\gamma,\mu} - h^2_{t;\gamma}h_{t;\mu} + h_{t;\gamma,\gamma}h_{t;\mu}\right)$

30. $E\left(\mathcal{L}_{\beta\theta\theta}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\theta,\theta} + 2h_{t;\theta}h_{t;\gamma,\theta} - h_{t;\gamma}h^2_{t;\theta}\right)$

31. $E\left(\mathcal{L}_{\gamma\theta\mu}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\theta,\mu} - h_{t;\gamma}h_{t;\theta}h_{t;\mu} + h_{t;\theta,\gamma}h_{t;\mu} + h_{t;\theta}h_{t;\gamma,\mu}\right)$

32. $E\left(\mathcal{L}_{\gamma\mu\mu}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\gamma,\mu} - h_{t;\gamma}h^2_{t;\mu}\right) + TE\left(\frac{1}{h_t}h_{t;\gamma}\right)$

33. $E\left(\mathcal{L}_{\theta\theta\mu}\right) = -\frac{T}{2}E\left(2h_{t;\theta}h_{t;\theta,\mu} - h^2_{t;\theta}h_{t;\mu} + h_{t;\theta,\theta}h_{t;\mu}\right)$

34. $E\left(\mathcal{L}_{\theta\mu\mu}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\theta,\mu} - h_{t;\theta}h^2_{t;\mu}\right) + TE\left(\frac{1}{h_t}h_{t;\theta}\right)$

35. $E\left(\mathcal{L}_{\mu\mu\mu}\right) = -\frac{T}{2}E\left(3h_{t;\mu}h_{t;\mu,\mu} - h^3_{t;\mu}\right) + TE\left(3\frac{1}{h_t}h_{t;\mu}\right).$

B.3 Expected values of the log-variance derivatives

In the current Appendix, we present some of the results for the expected values of the log-variance derivatives and more specifically those that are needed for the evaluation of some of the expected values of the third order log-likelihood derivatives of the previous Appendix, that is:

Assuming first $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$ and $\left|\beta^3 + \frac{3}{4}\beta\theta^2 - \frac{1}{8}\theta^3 Ez^3 - \frac{3}{2}\gamma\left(\beta^2 E\left|z\right| - \beta\theta E\left(z\left|z\right|\right) + \frac{1}{4}\theta^2 E\left|z\right|^3\right) + \frac{3}{4}\gamma^2\left(\beta - \frac{1}{2}\theta Ez^3\right) - \frac{1}{8}\gamma^3 E\left|z\right|^3\right| < 1$, we have:

1. $E\left(h_{t;\alpha}h_{t;\alpha,\alpha}\right) = \frac{\frac{1}{4}\gamma E|z|E_{(;\alpha)^2} + \left(\frac{1}{4}\beta\gamma E|z| - \frac{1}{8}\gamma^2 - \frac{1}{8}\theta^2 - \frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^3} + \left(\beta - \frac{1}{2}\gamma E|z|\right)E_{;\alpha,\alpha}}{1 - \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)}$

2. $E\left(h^3_{t;\alpha}\right) = \frac{1 + 3\left(\beta - \frac{1}{2}\gamma E|z|\right)E_{;\alpha} + 3\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)E_{(;\alpha)^2}}{1 - \left[\beta^3 + \frac{3}{4}\beta\theta^2 - \frac{1}{8}\theta^3 Ez^3 - \frac{3}{2}\gamma\left(\beta^2 E|z| - \beta\theta E(z|z|) + \frac{1}{4}\theta^2 E|z|^3\right) + \frac{3}{4}\gamma^2\left(\beta - \frac{1}{2}\theta Ez^3\right) - \frac{1}{8}\gamma^3 E|z|^3\right]}$

3. $E\left(h_{t;\beta}h_{t;\alpha,\alpha}\right) = \frac{\frac{1}{4}\gamma E|z|LE_{(;\alpha)^2} + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)E_{(;\alpha)^2;\beta} + \left(\beta - \frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\alpha}}{1 - \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)}$

4. $E\left(h_{t;\alpha}h_{t;\alpha,\beta}\right) = \frac{E_{;\alpha} + \left(\beta - \frac{1}{2}\gamma E|z|\right)E_{(;\alpha)^2} + \frac{1}{4}\gamma E|z|E_{;\alpha;\beta} + \left(\frac{1}{4}\beta\gamma E|z| - \frac{1}{8}\gamma^2 - \frac{1}{8}\theta^2 - \frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\beta} + \left(\beta - \frac{1}{2}\gamma E|z|\right)E_{;\alpha,\beta}}{1 - \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)}$

5. $E\left(h_{t;\gamma}h_{t;\alpha,\alpha}\right)=\frac{\frac{1}{4}[\theta E(z|z|)+\gamma(1-E|z|)]E_{(;\alpha)^2}-\frac{1}{2}[\theta E(z|z|)+\gamma(1-E|z|)]E_{;\alpha,\alpha}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

6. $E\left(h_{t;\theta}h_{t;\alpha,\alpha}\right)=\frac{\frac{1}{4}[\theta+\gamma E(z|z|)]E_{(;\alpha)^2}-\frac{1}{2}[\theta+\gamma E(z|z|)]E_{;\alpha,\alpha}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

7. $E\left(h^2_{t;\alpha}h_{t;\theta}\right)=\frac{-[\theta+\gamma E(z|z|)]E_{;\alpha}+\left(\frac{1}{2}\gamma\theta E|z|^3-\theta\beta+\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3-\beta\gamma E(z|z|)\right)E_{(;\alpha)^2}+2\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)E_{;\alpha;\theta}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$

8. $E\left(h_{t;\alpha}h_{t;\alpha,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{;\alpha;\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha,\theta}+\frac{1}{4}[\theta+\gamma E(z|z|)]E_{(;\alpha)^2}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

9. $E\left(h_{t;\alpha}h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\beta)^2}+2E_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta,\beta}+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\beta}+\left(\frac{1}{4}\beta\gamma|z_{t-1}|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{;\alpha(;\beta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

10. $E\left(h_{t;\beta}h_{t;\alpha,\beta}\right)=\frac{LE_{;\alpha}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\beta}+\frac{1}{4}\gamma E|z|LE_{;\alpha;\beta}+\left(\frac{1}{4}\beta\gamma|z_{t-1}|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{;\alpha(;\beta)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

11. $E\left(h_{t;\beta}h_{t;\alpha,\gamma}\right)=\frac{-\frac{1}{2}E|z|LE_{;\alpha}+\frac{1}{4}\gamma E|z|LE_{;\alpha;\gamma}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\gamma}-\frac{1}{2}\left(\beta E|z|-\frac{1}{2}\theta E(z|z|)-\frac{1}{2}\gamma\right)E_{;\alpha}E_{;\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\theta\gamma E(z|z|)\right.}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

12. $E\left(h_{t;\beta}h_{t;\alpha,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{;\alpha;\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

13. $E\left(h_{t;\alpha}h_{t;\beta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{;\beta;\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta,\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

14. $E\left(h_{t;\theta}h_{t;\alpha,\beta}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\beta}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\alpha,\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\theta}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\theta^2-\frac{1}{8}\gamma^2-\frac{1}{4}\gamma\theta E(z|z|)\right)E_{;\alpha;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

15. $E\left(h_{t;\alpha}h_{t;\gamma,\gamma}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\gamma)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\gamma,\gamma}-\left(\beta E|z|-\frac{1}{2}\gamma-\frac{1}{2}\theta E(z|z|)\right)E_{;\alpha;\gamma}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha(;\gamma)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

16. $E\left(h_{t;\alpha}h_{t;\gamma,\theta}\right)=\frac{-\frac{1}{2}\left(\beta E|z|-\frac{1}{2}\theta E(z|z|)-\frac{1}{2}\gamma\right)E_{;\alpha;\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\gamma}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha;\gamma;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

17. $E\left(h_{t;\alpha}h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\theta)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\theta,\theta}+\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\alpha;\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

18. $E\left(h_{t;\theta}h_{t;\alpha,\theta}\right)=\frac{-\frac{1}{2}E_{;\alpha}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\theta}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\alpha,\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

19. $E\left(h_{t;\beta}h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{(;\beta)^2}+2LE_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\beta,\beta}+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{(;\beta)^2}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\beta)^3}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

20. $E\left(h^3_{t;\beta}\right)=\frac{L^3+3\left(\beta-\frac{1}{2}\gamma E|z|\right)L^2E_{;\beta}+3\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)LE_{(;\beta)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$

21. $E\left(h_{t;\theta}h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{(;\beta)^2}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\beta,\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\beta)^2;\theta}+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

22. $E\left(h_{t;\beta}h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{(;\theta)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\theta,\theta}+\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\beta;\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\beta(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

23. $E\left(h^3_{t;\gamma}\right)=\frac{\left(E|z|^3-3E|z|+2E^3|z|\right)+3\left[\left(\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2\right)\left(E|z|^3-E|z|\right)-\gamma\beta\left(1-E^2|z|\right)-\beta\theta E(z|z|)+\frac{1}{2}\gamma\theta\left(Ez^3-E|z|E(z|z|)\right)\right]E_{(;\gamma)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$

24. $E\left(h_{t;\theta}h_{t;\gamma,\gamma}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{(;\gamma)^2}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\gamma,\gamma}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\gamma)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

25. $E\left(h_{t;\gamma}h_{t;\gamma,\theta}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{(;\gamma)^2}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\gamma)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

26. $E\left(h^2_{t;\gamma}h_{t;\theta}\right)=\frac{Ez^3-2E|z|E(z|z|)+\left[\theta\left(\frac{1}{2}\gamma E|z|^3-\beta\right)+\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3-\beta\gamma E(z|z|)\right]E_{(;\gamma)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$

27. $E\left(h_{t;\gamma}h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{(;\theta)^2}-\frac{1}{2}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\theta,\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\gamma(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

28. $E\left(h_{t;\theta}h_{t;\gamma,\theta}\right)=\frac{-\frac{1}{2}\left(\beta E|z|-\frac{1}{2}\theta E(z|z|)-\frac{1}{2}\gamma\right)E_{(;\theta)^2}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\gamma(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

29. $E\left(h_{t;\gamma}h^2_{t;\theta}\right)=\frac{E|z|^3-E|z|+\left[\left(\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2\right)\left(E|z|^3-E|z|\right)-\gamma\beta\left(1-E^2|z|\right)-\beta\theta E(z|z|)+\frac{1}{2}\gamma\theta\left(Ez^3-E|z|E(z|z|)\right)\right]E_{(;\theta)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}.$

The whole results are available on demand from the corresponding author.

B.4 Expected values of cross products of the log-likelihood derivatives

In this Appendix, we present the expected values of cross-products of the log-likelihood derivatives. To conserve space, we present only some indicative. That is,

1. $\frac{1}{T}E\left(\mathcal{L}_\alpha\mathcal{L}_{\alpha\alpha}\right)=-\frac{1}{4}\left[\sum_{s<t}\sum E\left[\left(z_s^2-1\right)h_{s;\alpha}h_{t;\alpha}h_{t;\alpha}\right]-(\kappa_4+2)\sum_{t=1}^T E\left(h_{t;\alpha}h_{t;\alpha,\alpha}-h^3_{t;\alpha}\right)\right]$

2. $\frac{1}{T}E\left(\mathcal{L}_\alpha\mathcal{L}_{\alpha\mu}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum_{s<t}\sum E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\alpha}h_{t;\mu}\right]-(\kappa_4+2)\sum_{t=1}^T E\left(h_{t;\alpha}h_{t;\alpha,\mu}-h_{t;\mu}h^2_{t;\alpha}\right)\\+2\kappa_3\sum_{t=1}^T E\left(\frac{1}{\sqrt{h_t}}h^2_{t;\alpha}\right)\end{array}\right]$

3. $\frac{1}{T}E\left(\mathcal{L}_\alpha\mathcal{L}_{\mu\mu}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum_{s<t}\sum E\left[\left(z_s^2-1\right)h_{s;\alpha}h^2_{t;\mu}\right]-(\kappa_4+2)\sum_{t=1}^T E\left(h_{t;\alpha}h_{t;\mu,\mu}-h_{t;\alpha}h^2_{t;\mu}\right)\\+4\kappa_3\sum_{t=1}^T E\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\mu}\right)\end{array}\right]$

4. $\frac{1}{T}E\left(\mathcal{L}_\mu\mathcal{L}_{\alpha\alpha}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum_{s<t}\sum E\left[\left(z_s^2-1\right)h_{s;\mu}h^2_{t;\alpha}\right]-(\kappa_4+2)\sum_{t=1}^T E\left(h_{t;\mu}h_{t;\alpha,\alpha}-h_{t;\mu}h^2_{t;\alpha}\right)\\+2\sum_{s<t}\sum E\left(z_s\frac{1}{\sqrt{h_t}}h^2_{t;\alpha}\right)+2\kappa_3\sum_{t=1}^T E\left(\frac{1}{\sqrt{h_t}}h^2_{t;\alpha}-\frac{1}{\sqrt{h_t}}h_{t;\alpha,\alpha}\right)\end{array}\right]$

5. $\frac{1}{T}E\left(\mathcal{L}_\mu\mathcal{L}_{\alpha\mu}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum_{s<t}\sum E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\alpha}h_{t;\mu}\right]-(\kappa_4+2)\sum_{t=1}^T E\left(h_{t;\mu}h_{t;\alpha,\mu}-h_{t;\alpha}h^2_{t;\mu}\right)\\+2\sum_{s<t}\sum E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\mu}\right)+2\kappa_3\sum_{t=1}^T E\left(\frac{2}{\sqrt{h_t}}h_{t;\alpha}h_{t;\mu}-\frac{1}{\sqrt{h_t}}h_{t;\alpha,\mu}\right)\\+4\sum_{t=1}^T E\left(\frac{1}{h_t}h_{t;\alpha}\right)\end{array}\right]$

6. $\frac{1}{T}E\left(\mathcal{L}_{\mu}\mathcal{L}_{\mu\mu}\right) = -\frac{1}{4}\left[\begin{array}{c} \sum\sum_{s<t} E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\mu}^2\right] - \left(\kappa_4+2\right)\sum_{t=1}^{T}E\left(h_{t;\mu}h_{t;\mu,\mu} - h_{t;\mu}^3\right) \\ +2\sum\sum_{s<t} E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;\mu}^2\right) + 2\kappa_3\sum_{t=1}^{T}E\left(\frac{3}{\sqrt{h_t}}h_{t;\mu}^2 - \frac{1}{\sqrt{h_t}}h_{t;\mu,\mu}\right) \\ +8\sum_{t=1}^{T}E\left(\frac{1}{h_t}h_{t;\mu}\right) \end{array}\right]$

At this point, we should note that these results differ from those in the paper of Linton (1997), due to the fact that we assume non-symmetric distribution of the errors and also none of these expressions are zero, since the block-diagonality of the information matrix in our case that we study the EGARCH(1, 1) model does not hold.

Analytic proof of the first result is given as follows:

$$\begin{aligned}
\mathcal{L}_{\alpha}\mathcal{L}_{\alpha\alpha} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha,\alpha} - \frac{1}{2}\sum_{t=1}^{T}z_t^2\left(h_{t;\alpha}\right)^2\right) \\
&= \frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha,\alpha} - \frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{t=1}^{T}z_t^2h_{t;\alpha}^2 \\
&= \frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)^2 h_{t;\alpha}h_{t;\alpha,\alpha} + \frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=t+1}^{T}\left(z_s^2-1\right)h_{s;\alpha,\alpha} \\
&\quad +\frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=1}^{t-1}\left(z_s^2-1\right)h_{s;\alpha,\alpha} \\
&\quad -\frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)z_t^2h_{t;\alpha}^3 - \frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=t+1}^{T}z_s^2h_{s;\alpha}^2 - \frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=1}^{t-1}z_s^2h_{s;\alpha}^2
\end{aligned}$$

Hence

$$E\left(\mathcal{L}_{\alpha}\mathcal{L}_{\alpha\alpha}\right) = \frac{T\left(\kappa_4+2\right)}{4}\left[E\left(h_{t;\alpha}h_{t;\alpha,\alpha}\right) - E\left(h_{t;\alpha}^3\right)\right] - \frac{1}{4}E\sum\sum_{s<t}\left(z_s^2-1\right)z_t^2h_{t;\alpha}^2h_{s;\alpha},$$

where

$h_{t;\alpha} = 1+\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}$ and $h_{t;\alpha}^2 = 1+2\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}+$ $\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha}^2$.

Let

$h_{t+k;\alpha} = 1+\left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right)h_{t+k-1;\alpha}$ and $h_{t+k;\alpha}^2 = 1+2\left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right)h_{t+k-1;\alpha}+$ $\left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right)^2 h_{t+k-1;\alpha}^2$.

Hence,

$E\left[\left(z_t^2-1\right)h_{t+k;\alpha}^2h_{t;\alpha}\right] =$

$= E\left[\left(z_t^2-1\right)\left[h_{t;\alpha}+2\left(\beta-\frac{1}{2}\theta z_{t+k-1}-\frac{1}{2}\gamma\left|z_{t+k-1}\right|\right)h_{t+k-1;\alpha}h_{t;\alpha}+\left(\beta-\frac{1}{2}\theta z_{t+k-1}-\frac{1}{2}\gamma\left|z_{t+k-1}\right|\right)^2 h_{t+k-1;\alpha}^2 h_{t;\alpha}\right]\right]=$

$\overset{k\geq 1}{=} 2\left(\beta-\frac{1}{2}\gamma E|z|\right)E\left(z_t^2-1\right)h_{t+k-1;\alpha}h_{t;\alpha}+\left[\beta^2+\frac{1}{4}\left(\theta^2+\gamma^2\right)-\beta\gamma E|z|+\frac{1}{2}\theta\gamma E\left(z|z|\right)\right]E\left(z_t^2-1\right)h_{t+k-1;\alpha}^2 h_{t;\alpha}.$

$k=1: E\left[\left(z_t^2-1\right)h_{t+1;\alpha}^2 h_{t;\alpha}\right]=E\left[\left(z_t^2-1\right)\left[h_{t;\alpha}+2\left(\beta-\frac{1}{2}\theta z_t-\frac{1}{2}\gamma|z_t|\right)h_{t;\alpha}^2+\left(\beta-\frac{1}{2}\theta z_t-\frac{1}{2}\gamma|z_t|\right)^2 h_{t;\alpha}^3\right]\right]=$

$=2E\left[\left(z_t^2-1\right)\left(\beta-\frac{1}{2}\theta z_t-\frac{1}{2}\gamma|z_t|\right)\right]Eh_{t;\alpha}^2+E\left[\left(z_t^2-1\right)\left(\beta-\frac{1}{2}\theta z_t-\frac{1}{2}\gamma|z_t|\right)^2\right]Eh_{t;\alpha}^3.$

Hence,

$$E\left[\left(z_t^2-1\right)h_{t+k;\alpha}^2 h_{t;\alpha}\right]\overset{k\geq 1}{=}-\left(\beta-\frac{1}{2}\gamma E|z|\right)\left[\theta E z^3+\gamma\left(E|z|^3-E|z|\right)\right]\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-2}Eh_{t;\alpha}^2$$
$$+\left[\beta^2+\frac{1}{4}\left(\theta^2+\gamma^2\right)-\beta\gamma E|z|+\frac{1}{2}\theta\gamma E\left(z|z|\right)\right]E\left(z_t^2-1\right)h_{t+k-1;\alpha}^2 h_{t;\alpha}.$$

Set: $A=-\left(\beta-\frac{1}{2}\gamma E|z|\right)\left[\theta E z^3+\gamma\left(E|z|^3-E|z|\right)\right]Eh_{t;\alpha}^2$ and $C=\beta^2+\frac{1}{4}\left(\theta^2+\gamma^2\right)-\beta\gamma E|z|+\frac{1}{2}\theta\gamma E\left(z|z|\right).$

We have that: $E\left[\left(z_t^2-1\right)h_{t+k;\alpha}^2 h_{t;\alpha}\right]\overset{k\geq 1}{=}A\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-2}+CE\left(z_t^2-1\right)h_{t+k-1;\alpha}^2 h_{t;\alpha}.$

By repeating substitution, $E\left[\left(z_t^2-1\right)h_{t+k;\alpha}^2 h_{t;\alpha}\right]\overset{k\geq 1}{=}A\left[\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-2}+C\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-3}+\ldots+C^{k-2}\right]$

$+C^{k-1}E\left(z_t^2-1\right)h_{t+1;\alpha}^2 h_{t;\alpha}.$

This formula can be written as:

$E\left[\left(z_t^2-1\right)h_{t+k;\alpha}^2 h_{t;\alpha}\right]\overset{k\geq 1}{=}A\frac{C^{k-1}-\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-1}}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}+C^{k-1}E\left(z_t^2-1\right)h_{t+1;\alpha}^2 h_{t;\alpha}.$

Consequently,

$E\left[\left(z_t^2-1\right)h_{t+k;\alpha}^2 h_{t;\alpha}\right]\overset{k\geq 1}{=}A\frac{C^{k-1}-\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-1}}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

$+C^{k-1}\left[2E\left[\left(z_t^2-1\right)\left(\beta-\frac{1}{2}\theta z_t-\frac{1}{2}\gamma|z_t|\right)\right]Eh_{t;\alpha}^2+E\left[\left(z_t^2-1\right)\left(\beta-\frac{1}{2}\theta z_t-\frac{1}{2}\gamma|z_t|\right)^2\right]Eh_{t;\alpha}^3\right],$

where

$2E\left[\left(z_t^2-1\right)\left(\beta-\frac{1}{2}\theta z_t-\frac{1}{2}\gamma|z_t|\right)\right]Eh_{t;\alpha}^2+E\left[\left(z_t^2-1\right)\left(\beta-\frac{1}{2}\theta z_t-\frac{1}{2}\gamma|z_t|\right)^2\right]Eh_{t;\alpha}^3=$

$$=-\left(\theta E z^3+\gamma\left(E\left(|z|^3\right)-E|z|\right)\right)Eh_{t;\alpha}^2+\left[\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta E z^3\\+\beta\gamma\left(E|z|-E|z|^3\right)+\frac{1}{2}\theta\gamma\left(E\left(z^3|z|\right)-E\left(z|z|\right)\right)\end{array}\right]Eh_{t;\alpha}^3$$

Hence we have $E\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(z_s^2-1\right)z_t^2 h_{t;\alpha}^2 h_{s;\alpha}=E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right)h_{t+k;\alpha}^2 h_{t;\alpha}=$

$=\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}A\frac{C^{k-1}-\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-1}}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}+C^{k-1}\Lambda$

where $\Lambda=-\left(\theta E z^3+\gamma\left(E\left(|z|^3\right)-E|z|\right)\right)Eh_{t;\alpha}^2+\left[\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta E z^3\\+\beta\gamma\left(E|z|-E|z|^3\right)+\frac{1}{2}\theta\gamma\left(E\left(z^3|z|\right)-E\left(z|z|\right)\right)\end{array}\right]Eh_{t;\alpha}^3.$

Hence, $E\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(z_s^2-1\right)z_t^2h_{t;\alpha}^2h_{s;\alpha}=\left(\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}+\Lambda\right)\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}C^{k-1}$

$-\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-1}=$

...(keeping only terms of order T)= $\left(\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}+\Lambda\right)\frac{T}{1-C}-\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\frac{T}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$,

provided that $|C|<1$ and $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$. Hence

$$\begin{aligned}E\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(z_s^2-1\right)z_t^2h_{t;\alpha}^2h_{s;\alpha} &= T\left(\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\frac{1}{1-C}+\Lambda\frac{1}{1-C}-\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\frac{1}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\right)\\ &= T\frac{A}{\left(1-C\right)\left(1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right)}+T\frac{\Lambda}{1-C}+O\left(1\right)\end{aligned}$$

where $A=-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\left[\theta Ez^3+\gamma\left(E\left|z\right|^3-E\left|z\right|\right)\right]Eh_{t;\alpha}^2$ and $C=\beta^2+\frac{1}{4}\left(\theta^2+\gamma^2\right)-\beta\gamma E\left|z\right|+\frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)$, $\Lambda=-\left(\theta Ez^3+\gamma\left(E\left(\left|z\right|^3\right)-E\left|z\right|\right)\right)Eh_{t;\alpha}^2$

$$+\left[\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta Ez^3\\ +\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right)+\frac{1}{2}\theta\gamma\left(E\left(z^3\left|z\right|\right)-E\left(z\left|z\right|\right)\right)\end{array}\right]Eh_{t;\alpha}^3.$$

B.5 Proof of the Main Theorem

The proof comes immediately from the results of *Appendix B.2* and *Appendix B.4*.

B.6 The log-variance derivatives

In this Appendix we present the expressions of the log-variance derivatives, in a form useful to explore their properties.

$$h_{t;\alpha}=1+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}$$

$$h_{t;\alpha,\alpha}=\left(\frac{1}{4}\theta z_{t-1}+\frac{1}{4}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^2+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\alpha}$$

$$h_{t;\alpha,\beta}=h_{t-1;\alpha}+\left(\frac{1}{4}\theta z_{t-1}+\frac{1}{4}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\beta}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\beta}$$

$$h_{t;\alpha,\gamma}=-\frac{1}{2}\left|z_{t-1}\right|h_{t-1;\alpha}+\left(\frac{1}{4}\theta z_{t-1}+\frac{1}{4}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\gamma}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\gamma}$$

$$h_{t;\alpha,\theta} = -\frac{1}{2}z_{t-1}h_{t-1;\alpha}+\left(\frac{1}{4}\theta z_{t-1} + \frac{1}{4}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha}h_{t-1;\theta}+\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha,\theta}$$

$$\begin{aligned} h_{t;\alpha,\mu} &= \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\alpha} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha}h_{t-1;\mu} \\ &+ \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha,\mu} \end{aligned}$$

$$h_{t;\beta} = \ln\left(h_{t-1}\right) + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\beta}$$

$$h_{t;\beta,\beta} = \left(\frac{1}{4}\theta z_{t-1} + \frac{1}{4}\gamma\left|z_{t-1}\right|\right) h^2_{t-1;\beta} + 2h_{t-1;\beta} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\beta,\beta}$$

$$h_{t;\beta,\gamma} = h_{t-1;\gamma}-\frac{1}{2}\left|z_{t-1}\right| h_{t-1;\beta}+\frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\beta}h_{t-1;\gamma}+\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\beta,\gamma}$$

$$h_{t;\beta,\theta} = h_{t-1;\theta}-\frac{1}{2}z_{t-1}h_{t-1;\beta}+\frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\beta}h_{t-1;\theta}+\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\beta,\theta}$$

$$\begin{aligned} h_{t;\beta,\mu} &= h_{t-1;\mu} + \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\beta} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\beta}h_{t-1;\mu} \\ &+ \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\beta,\mu} \end{aligned}$$

$$h_{t;\gamma} = g\left(z_{t-1}\right) + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma}$$

$$h_{t;\gamma,\gamma} = -\left|z_{t-1}\right| h_{t-1;\gamma} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h^2_{t-1;\gamma} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma,\gamma}$$

$$h_{t;\gamma,\theta} = -\frac{1}{2}\left|z_{t-1}\right| h_{t-1;\theta}-\frac{1}{2}z_{t-1}h_{t-1;\gamma}+\frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma}h_{t-1;\theta}+\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma,\theta}$$

$$\begin{aligned} h_{t;\gamma,\mu} &= -\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\frac{1}{\sqrt{h_{t-1}}} + \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\gamma} \\ &-\frac{1}{2}\left|z_{t-1}\right| h_{t-1;\mu} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma}h_{t-1;\mu} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma,\mu} \end{aligned}$$

$$h_{t;\theta} = z_{t-1} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\theta}$$

$$h_{t;\theta,\theta} = -z_{t-1}h_{t-1;\theta} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\theta}^{2} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta,\theta}$$

$$\begin{aligned} h_{t;\theta,\mu} \quad &= \quad -\frac{1}{\sqrt{h_{t-1}}} + \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\theta} - \frac{1}{2}z_{t-1}h_{t-1;\mu} \\ &\quad + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\theta}h_{t-1;\mu} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta,\mu} \end{aligned}$$

$$h_{t;\mu} = -\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\mu}$$

$$\begin{aligned} h_{t;\mu,\mu} \quad &= \quad \left(\theta + \gamma\left[I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\mu} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\mu}^{2} \\ &\quad + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\mu,\mu} \end{aligned}$$

B.7 Expected values of the first & second order log-variance derivatives

We assume $\left|\beta - \frac{1}{2}\gamma E\left|z\right|\right| < 1$.

First order derivatives:

1. $E\left(h_{t;\alpha}\right) = \frac{1}{1-\beta+\frac{1}{2}\gamma E(|z|)}$
2. $E\left(h_{t;\beta}\right) = \frac{\alpha}{\left(1-\beta+\frac{\gamma}{2}E|z|\right)(1-\beta)}$
3. $E\left(h_{t;\gamma}\right) = 0$
4. $E\left(h_{t;\theta}\right) = 0$
5. $E\left(h_{t;\mu}\right) = -\frac{\theta E_{-\frac{1}{2}}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$.

Second order derivatives:

1. $E\left(h_{t;\alpha,\alpha}\right) = \frac{\frac{1}{4}\gamma E|z| E_{(;\alpha)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

2. $E\left(h_{t;\alpha,\beta}\right)=\frac{E_{;\alpha}+\frac{1}{4}\gamma E|z|E_{;\alpha;\beta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

3. $E\left(h_{t;\alpha,\gamma}\right)=\frac{-\frac{1}{2}E|z|E_{;\alpha}+\frac{1}{4}\gamma E|z|E_{;\alpha;\gamma}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

4. $E\left(h_{t;\alpha,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{;\alpha;\theta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

5. $E\left(h_{t;\alpha,\mu}\right)=\frac{\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\alpha}+\frac{1}{4}\gamma E|z|E_{;\alpha;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

6. $E\left(h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\beta)^2}+2E_{;\beta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

7. $E\left(h_{t;\beta,\gamma}\right)=\frac{-\frac{1}{2}E|z|E_{;\beta}+\frac{1}{4}\gamma E|z|E_{;\beta;\gamma}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

8. $E\left(h_{t;\beta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{;\beta;\theta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

9. $E\left(h_{t;\beta,\mu}\right)=\frac{E_{;\mu}+\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\beta}+\frac{1}{4}\gamma E|z|E_{;\beta;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

10. $E\left(h_{t;\gamma,\gamma}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\gamma)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

11. $E\left(h_{t;\gamma,\theta}\right)=0$

12. $E\left(h_{t;\gamma,\mu}\right)=\frac{-EIE_{-\frac{1}{2}}+\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\gamma}-\frac{1}{2}E|z|E_{;\mu}+\frac{1}{4}\gamma E|z|E_{;\gamma;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

13. $E\left(h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\theta)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

14. $E\left(h_{t;\theta,\mu}\right)=\frac{-E_{-\frac{1}{2}}+\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\theta}+\frac{1}{4}\gamma E|z|E_{;\theta;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

15. $E\left(h_{t;\mu,\mu}\right)=\frac{(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\mu}+\frac{1}{4}\gamma E|z|E_{(;\mu)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}.$

B.8 Symbols

The next symbols are used in the paper and more specifically in the expressions of the expected values of all the derivatives.

$$E\left(\ln^2\left(h_t\right)\right)=L^2 \quad E\left(\ln\left(h_t\right)\right)^3=L^3 \quad etc.$$

$$E\left(h_{t;\beta}\right)=E_{;\beta} \quad E\left(h_{t;\alpha}\right)=E_{;\alpha} \quad E\left(h_{t;\beta}\right)^2=E_{(;\beta)^2} \quad E\left(h_{t;\mu}\right)^3=E_{(;\mu)^3} \quad etc.$$

$$\begin{aligned} E\left(\ln\left(h_t\right)h_{t;\mu}\right) &= LE_{;\mu} \quad E\left(\ln\left(h_t\right)h_{t;\alpha}\right) = LE_{;\alpha} \quad E\left(h_{t;\beta}\ln\left(h_t\right)\right) = LE_{;\beta} \quad etc \\ E\left(h_{t;\beta}\left(\ln\left(h_t\right)\right)^2\right) &= L^2E_{;\beta} \quad E\left(\ln\left(h_t\right)h_{t;\beta}^2\right) = LE_{(;\beta)^2} \end{aligned}$$

$$\begin{aligned} E\left(\exp\left(\kappa\ln h_t\right)h_{t;\beta}\right) &= E_kE_{;\beta} \quad E\left(\exp\left(k\ln h_t\right)h_{t;\alpha}\right) = E_kE_{;\alpha} \\ E\left(\exp\left(k\ln h_t\right)h_{t;\mu}\right) &= E_kE_{;\mu} \quad etc. \\ E\left(\exp\left(\kappa\ln h_t\right)h_{t;\mu}^2\right) &= E_\kappa E_{(;\mu)^2} \quad E\left(\exp\left(\kappa\ln h_t\right)h_{t;\mu}^3\right) = E_\kappa E_{(;\mu)^3} \quad etc. \end{aligned}$$

$$E\left(h_{t;\beta}\left(h_{t;\mu}\right)^2\right) = E_{;\beta(;\mu)^2}$$

$$\begin{aligned} E\left(h_{t-1;\beta}h_{t-1;\mu}\right) &= E_{;\beta;\mu} \quad E\left(h_{t;\beta}h_{t;\alpha}\right) = E_{;\beta;\alpha} \quad E\left(h_{t;\beta}h_{t;\gamma}\right) = E_{;\beta;\gamma} \quad etc. \\ E\left(\exp\left[\kappa\ln\left(h_t\right)\right]h_{t;\beta}h_{t;\mu}\right) &= E_\kappa E_{;\beta;\mu} \quad etc \end{aligned}$$

$$E\left(h_{t;\beta,\mu}\right) = E_{;\beta,\mu} \quad E\left(h_{t;\mu,\mu}\right) = E_{;\mu,\mu} \quad etc.$$

$$\begin{aligned} E\left(h_{t;\mu}h_{t;\mu,\mu}\right) &= E_{;\mu;\mu,\mu} \\ E\left(\ln\left(h_t\right)h_{t;\beta,\beta}\right) &= LE_{;\beta,\beta} \end{aligned}$$

$$E\left(\exp\left(\kappa\ln h_t\right)h_{t;\mu,\mu}\right) = E_\kappa E_{;\mu,\mu}$$

$$E\left(h_{t;\beta}h_{t;\beta,\beta}\right) = E_{;\beta;\beta,\beta}$$

$$E\left(\exp\left(\kappa\ln h_t\right)\ln\left(h_t\right)h_{t;\mu}\right) = E_\kappa LE_{;\mu}$$

$$\begin{aligned} E\left(h_{t;\beta}h_{t;\mu,\mu}\right) &= E_{;\beta;\mu,\mu} \quad E\left(h_{t;\mu}h_{t;\mu,\beta}\right) = E_{;\mu;\mu,\beta} \\ E\left(\exp\left(\kappa\ln h_t\right)h_{t;\mu,\beta}\right) &= E_\kappa E_{;\mu,\beta} \end{aligned}$$

Appendix C

Appendix for "Asymptotic Normality of the QMLEs in the EGARCH(1,1) Model"

C.1 Proofs of the Main Lemmas 4.15, 4.16 and 4.17

Recall that $c = \frac{1}{\beta} 2^{-1} \overline{\delta} \exp\left(-2^{-1} \underline{m}\right)$, where
$\overline{\delta} = \max\left(\sup\left(\gamma + \delta\right), \sup\left(\delta - \gamma\right)\right) : \gamma x + \delta\left|x\right| \leq \overline{\delta}\left|x\right|, \ \ \forall x \in \mathbb{R}$ and $\underline{m} = \inf\left\{\frac{\alpha}{1-\beta}\right\}$.

Proof of Lemma 4.15. Taking the first and the last term of the product $\prod_{i=1}^{k-1}\left[1 + c\left|X_{t-i}\right|\right]$, see Lemma C.2 in p. 176, we have first:

$$E\left[\beta^{k-1} c \sum_{i=1}^{n-1}\left|X_{t-i}\right|\right] = \left(k-1\right) 2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right) E\left[\left(\left|Z_0\right|\sigma_0\right)\right].$$

Hence, $\sum_{k=1}^{\infty}\left(k-1\right) 2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right) E\left[\left(\left|Z_0\right|\sigma_0\right)\right]$ is bounded if and only if: $E\left[\left(\left|Z_0\right|\sigma_0\right)\right] < \infty$.

Second, we have (see Lemma C.1 in p. 176):

$$E\left[\beta^{k-1} c^{k-1} \prod_{i=1}^{k-1}\left|X_{t-i}\right|\right] = 2^{-(k-1)}\overline{\delta}^{(k-1)}\exp\left(-2^{-1}\underline{m}\left(k-1\right)\right)\exp\left(\frac{1}{2(1-\beta)}\alpha\left(k-2-\frac{\beta-\beta^{k-1}}{1-\beta}\right)\right)\times$$
$$\times\left[\left(\begin{array}{c} E\exp\left(\frac{1}{2}\frac{1-\beta^{k-1}}{1-\beta}\log\sigma_{t-k+1}^2\right)\times \\ E\left|Z_{t-k+1}\right|\exp\left(\frac{1}{2}\frac{1-\beta^{k-2}}{1-\beta}\left(\gamma Z_{t-k+1} + \delta\left|Z_{t-k+1}\right|\right)\right)\times \ldots \times E\left|Z_{t-2}\right|\exp\left(\frac{1}{2}\left(\gamma Z_{t-2} + \delta\left|Z_{t-2}\right|\right)\right)\end{array}\right)\right]$$

$$\leq 2^{-(k-1)}\overline{\delta}^{(k-1)}\exp\left(-2^{-1}\underline{m}\,(k-1)\right)\exp\left(\tfrac{1}{2(1-\beta)}\alpha\left(k-2-\tfrac{\beta-\beta^{k-1}}{1-\beta}\right)\right)\times$$
$$\times\left[\left(\begin{array}{c} E\exp\left(\frac{1}{2}\frac{1-\beta^{k-1}}{1-\beta}\log\sigma^2_{t-k+1}\right)\times \\ E\left|Z_{t-k+1}\right|\exp\left(\frac{1}{2}\frac{1-\beta^{k-2}}{1-\beta}\overline{\delta}\left|Z_{t-k+1}\right|\right)\times\ldots\times E\left|Z_{t-2}\right|\exp\left(\frac{1}{2}\overline{\delta}\left|Z_{t-2}\right|\right)\end{array}\right)\right]$$
$$\leq 2^{-(k-1)}\overline{\delta}^{(k-1)}\exp\left(-2^{-1}\underline{m}\,(k-1)\right)\exp\left(\tfrac{1}{2(1-\beta)}\alpha\left(k-2-\tfrac{\beta-\beta^{k-1}}{1-\beta}\right)\right)\times$$
$$\times\left[\left(\begin{array}{l} E\exp\left(\frac{1}{2}\frac{1-\beta^{k-1}}{1-\beta}\log\sigma^2_{t-k+1}\right)\times \\ \left(E\left|Z_0\right|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)^{k-2}\end{array}\right)\right]$$
$$\leq 2^{-(k-1)}\overline{\delta}^{(k-1)}\exp\left(2^{-1}\left[\left(\tfrac{\alpha}{(1-\beta)}-\underline{m}\right)(k-1)\right]\right)\exp\left(-\tfrac{1}{2(1-\beta)}\alpha\left[1+\tfrac{\beta-\beta^{k-1}}{1-\beta}\right]\right)\times$$
$$\times\left(E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)^{k-1}\times\left(E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)^{-1}E\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\log\sigma_0^2\right)=$$
$$=\left[2^{-1}\overline{\delta}\exp\left(2^{-1}\left[\left(\tfrac{\alpha}{(1-\beta)}-\underline{m}\right)\right]\right)\left(E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)\right]^{k-1}E\exp\left(-2^{-1}\tfrac{\alpha}{(1-\beta)^2}\right)\times$$
$$\times\left(E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)^{-1}E\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\log\sigma_0^2\right),$$

where $2^{-1}\overline{\delta}\exp\left(2^{-1}\left[\left(\tfrac{\alpha}{(1-\beta)}-\underline{m}\right)\right]\right)\left(E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)=q^a<1$.

Hence: $\sum_{k=1}^{\infty}E\left[\beta^{k-1}c^{k-1}\prod_{i=1}^{k-1}\left|X_{t-i}\right|\right]=$

$$=\exp\left(-2^{-1}\tfrac{\alpha}{(1-\beta)^2}\right)\left(E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)^{-1}E\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\log\sigma_0^2\right)\sum_{k=1}^{\infty}q^{k-1}=$$
$$=\tfrac{1}{1-q}\exp\left(-2^{-1}\tfrac{\alpha}{(1-\beta)^2}\right)\left(E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)^{-1}E\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\log\sigma_0^2\right).\qquad\square$$

Proof of Lemma 4.16. Again, applying Lemma C.2 in p. 176 we have first:

$$E\left[\beta^{k-1}c\sum_{i=1}^{k-1}\left|X_{t-i}\right|\left|X_{t-k}\right|\right]=2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right)E\left(\sum_{i=1}^{k-1}\left|X_{t-i}\right|\left|X_{t-k}\right|\right).$$

Examining the higher dependence, which is $E\left(\left|X_{t-k+1}\right|\left|X_{t-k}\right|\right)$, we have that:

$$E\left[\beta^{k-1}c\sum_{i=1}^{k-1}\left|X_{t-i}\right|\left|X_{t-k}\right|\right]\leq(k-1)\,2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right)E\left(\left|X_{t-1}\right|\left|X_{t-2}\right|\right)$$
$$\leq(k-1)\,2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right)E\left|Z_0\right|E\exp\left(\tfrac{1}{2}\alpha\right)E\exp\left[\tfrac{1}{2}(\beta+1)\log\sigma^2_{t-2}\right]\times$$
$$\times E\left|Z_{t-2}\exp\left[\tfrac{1}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right]\right|$$
$$\leq(k-1)\,2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right)E\left|Z_0\right|E\exp\left(\tfrac{1}{2}\alpha\right)E\exp\left[\tfrac{1}{2}(\beta+1)\log\sigma_0^2\right]\times E\left|Z_0\exp\left[\tfrac{1}{2}\overline{\delta}\left|Z_0\right|\right]\right|.$$

Hence, $\sum_{k=1}^{\infty}(k-1)\,2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right)E\left|Z_0\right|E\exp\left(\tfrac{1}{2}\alpha\right)E\exp\left[\tfrac{1}{2}(\beta+1)\log\sigma_0^2\right]\times$
$E\left|Z_0\exp\left[\tfrac{1}{2}\overline{\delta}\left|Z_0\right|\right]\right|$

is bounded if and only if: $E\left|Z_0\exp\left[\tfrac{1}{2}\overline{\delta}\left|Z_0\right|\right]\right|<\infty$.

Through the analytic way, we have:

$$E\left[\beta^{k-1}c\sum_{i=1}^{k-1}\left|X_{t-i}\right|\left|X_{t-k}\right|\right]=2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right)E\left(\sum_{i=1}^{k-1}\left|X_{t-i}\right|\left|X_{t-k}\right|\right)$$

$$\leq 2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right)\sum_{i=1}^{k-1}E\left|Z_0\right|\exp\left(\tfrac{1}{2}\alpha\tfrac{1-\beta^{k-i}}{1-\beta}\right)\left(\prod_{j=1}^{k-1-i}E\left|\exp\left(\tfrac{1}{2}\beta^{k-1-i-j}\left(\gamma Z_0+\delta\left|Z_0\right|\right)\right)\right|\right)\times$$
$$\times E\left[\left|Z_0\exp\left(\tfrac{1}{2}\beta^{k-1-i}\left(\gamma Z_0+\delta\left|Z_0\right|\right)\right)\right|\right]E\left[\exp\left(\tfrac{1}{2}\left(\beta^{k-i}+1\right)\log\sigma_0^2\right)\right]$$
$$\leq 2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right)E\left|Z_0\right|\exp\left(\tfrac{1}{2}\alpha\tfrac{(k-1)(1-\beta)-\beta\left(1-\beta^{k-1}\right)}{(1-\beta)^2}\right)\times$$
$$\times\sum_{i=1}^{k-1}\left(\prod_{j=1}^{k-1-i}E\left|\exp\left(\tfrac{1}{2}\beta^{k-1-i-j}\left(\gamma Z_0+\delta\left|Z_0\right|\right)\right)\right|\right)\times$$
$$\times E\left[\left|Z_0\exp\left(\tfrac{1}{2}\beta^{k-1-i}\left(\gamma Z_0+\delta\left|Z_0\right|\right)\right)\right|\right]E\left[\exp\left(\tfrac{1}{2}\left(\beta^{k-i}+1\right)\log\sigma_0^2\right)\right]$$
$$\leq 2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right)E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{\alpha}{1-\beta}\left(k-1-\tfrac{\beta-\beta^k}{1-\beta}\right)\right)\times$$
$$\times E\left[\left|Z_0\exp\left(\tfrac{1}{2}\left(\gamma Z_0+\delta\left|Z_0\right|\right)\right)\right|\right]E\left[\exp\left(\tfrac{1}{2}\left(\beta+1\right)\log\sigma_0^2\right)\right]\times\sum_{i=1}^{k-1}\left(E\left|\exp\left(\tfrac{1}{2}\overline{\delta}\left|Z_0\right|\right)\right|\right)^{k-1-i}$$
$$=2^{-1}\overline{\delta}\beta^{k-2}\exp\left(-2^{-1}\underline{m}\right)E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{\alpha}{1-\beta}\left(k-1-\tfrac{\beta-\beta^k}{1-\beta}\right)\right)E\left[\left|Z_0\exp\left(\tfrac{1}{2}\overline{\delta}\left|Z_0\right|\right)\right|\right]\times$$
$$\times E\left[\exp\left(\tfrac{1}{2}\left(\beta+1\right)\log\sigma_0^2\right)\right]\times\frac{1-\left(E\left|\exp\left(\frac{1}{2}\overline{\delta}\left|Z_0\right|\right)\right|\right)^{k-1}}{1-\left(E\left|\exp\left(\frac{1}{2}\overline{\delta}\left|Z_0\right|\right)\right|\right)}$$
$$\leq 2^{-1}\overline{\delta}\tfrac{1}{\beta}\exp\left(-2^{-1}\underline{m}\right)E\left|z_0\right|\left[\beta\exp\left(\tfrac{1}{2}\tfrac{\alpha}{1-\beta}\right)\right]^{k-1}\exp\left(-\tfrac{1}{2}\tfrac{\alpha\beta}{(1-\beta)^2}\right)E\left[\left|Z_0\exp\left(\tfrac{1}{2}\overline{\delta}\left|Z_0\right|\right)\right|\right]\times$$
$$\times E\left[\exp\left(\tfrac{1}{2}\left(\beta+1\right)\log\sigma_0^2\right)\right]\times\frac{1-\left(E\left|\exp\left(\frac{1}{2}\overline{\delta}\left|z_0\right|\right)\right|\right)^{k-1}}{1-\left(E\left|\exp\left(\frac{1}{2}\overline{\delta}\left|z_0\right|\right)\right|\right)}.$$

Hence, $\sum_{k=0}^{\infty}E\left[\beta^{k-1}c\sum_{i=1}^{k-1}\left|X_{t-i}\right|\left|X_{t-k}\right|\right]$ will be finite if and only if:

$E\left[\left|Z_0\exp\left(\frac{1}{2}\overline{\delta}\left|Z_0\right|\right)\right|\right]<\infty$ and $E\exp\left(\frac{1}{2}\overline{\delta}\left|Z_0\right|\right)<1$, in order the $\sum_{i=0}^{k-2}\left(E\left|\exp\left(\frac{1}{2}\overline{\delta}\left|Z_0\right|\right)\right|\right)^{k-1}$

as $k\to\infty$ to converge, as 1) $\sum_{i=1}^{k-1}E\left[\left|Z_0\exp\left(\frac{1}{2}\beta^{k-1-i}\left(\gamma Z_0+\delta\left|Z_0\right|\right)\right)\right|\right]\leq E\left[\left|Z_0\exp\left(\frac{1}{2}\left(\gamma Z_0+\delta\left|Z_0\right|\right)\right)\right|\right]$ and

2) $\sum_{i=1}^{k-1}E\left[\exp\left(\frac{1}{2}\left(\beta^{k-i}+1\right)\log\sigma_0^2\right)\right]\leq E\left[\exp\left(\frac{1}{2}\left(\beta+1\right)\log\sigma_0^2\right)\right]$.

Second, we have:

$$E\left[\beta^{k-1}c^{k-1}\prod_{i=1}^{k}\left|X_{t-i}\right|\right]=2^{-(k-1)}\overline{\delta}^{(k-1)}\exp\left(-2^{-1}\underline{m}\left(k-1\right)\right)\exp\left(\tfrac{1}{2(1-\beta)}\alpha\left(k-1-\tfrac{\beta-\beta^k}{1-\beta}\right)\right)\times$$
$$\times\left[\left(\begin{array}{c}E\exp\left(\frac{1}{2}\frac{1-\beta^k}{1-\beta}\log\sigma_{t-k}^2\right)\times\\ E\left|Z_{t-k}\exp\left(\frac{1}{2}\frac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k}+\delta\left|Z_{t-k}\right|\right)\right)\right|\times\ldots\times E\left|Z_{t-2}\exp\left(\frac{1}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right)\right|\end{array}\right)\right]$$
$$\leq 2^{-(k-1)}\overline{\delta}^{(k-1)}\exp\left(2^{-1}\left[\left(\tfrac{\alpha}{(1-\beta)}-\underline{m}\right)(k-1)\right]\right)\exp\left(2^{-1}\left[-\tfrac{\alpha\beta}{(1-\beta)}\tfrac{1-\beta^{k-1}}{1-\beta}\right]\right)\times$$
$$\times\left(E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)^{k-1}E\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\log\sigma_0^2\right)=$$
$$=\left[2^{-1}\overline{\delta}\exp\left(2^{-1}\left[\left(\tfrac{\alpha}{(1-\beta)}-\underline{m}\right)\right]\right)\left(E\left|Z_0\right|\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)\right]^{k-1}\exp\left(2^{-1}\left[-\tfrac{\alpha\beta}{(1-\beta)^2}\right]\right)\times$$
$$\times E\exp\left(\tfrac{1}{2}\tfrac{1}{1-\beta}\log\sigma_0^2\right),$$

where $2^{-1}\overline{\delta}\exp\left(2^{-1}\left[\left(\frac{\alpha}{(1-\beta)}-\underline{m}\right)\right]\right)\left(E\left|Z_0\right|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)=q^a<1$.

Hence: $\sum_{k=1}^{\infty} E\left[\beta^{k-1}c^{k-1}\prod_{i=1}^{k}|X_{t-i}|\right] = \exp\left(2^{-1}\left[-\frac{\alpha\beta}{(1-\beta)^2}\right]\right) E\exp\left(\frac{1}{2}\frac{1}{1-\beta}\log\sigma_0^2\right)\sum_{k=1}^{\infty} q^{k-1} =$
$= \frac{1}{1-q}\exp\left(2^{-1}\left[-\frac{\alpha\beta}{(1-\beta)^2}\right]\right) E\exp\left(\frac{1}{2}\frac{1}{1-\beta}\log\sigma_0^2\right).$ □

Proof of Lemma 4.17. We can write $E\left[\sum_{k=1}^{\infty}\left(\beta^{k-1}\prod_{i=1}^{k-1}[1+c|X_{t-i}|]\right)|\log h_{t-k}|\right] =$

$$= E\left[\sum_{k=1}^{\infty}\left(\beta^{k-1}\prod_{i=1}^{k}[1+c|X_{t-i}|]\right)\left(\left|\frac{\alpha}{1-\beta}\right| + \exp\left(-2^{-1}\underline{m}\right)\sum_{n=0}^{\infty}\beta^n\overline{\delta}|X_{t-n-1}|\right)\right] =$$

$$= \left|\frac{\alpha}{1-\beta}\right| E\left[\sum_{k=1}^{\infty}\left(\beta^{k-1}\prod_{i=1}^{k}[1+c|X_{t-i}|]\right)\right]$$

$$+\overline{\delta}\exp\left(-2^{-1}\underline{m}\right)E\left[\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left(\beta^{k-1}\beta^n\left(|X_{t-n-1}|\right)\prod_{i=1}^{k}[1+c|X_{t-i}|]\right)\right].$$

For the first term above, see Lemma 4.15 and its conditions.

Again, by using Lemma C.2 in p. 176, we can replace the product in the second term by considering only the first and the last term from its expansion, that is:

$$\overline{\delta}\exp\left(-2^{-1}\underline{m}\right)E\left[\sum_{n=0}^{\infty}\left(\beta^{k-1}\beta^n\left(|X_{t-n-1}|\right)\left(c\sum_{i=1}^{k-1}|X_{t-i}|\right)\right)\right] =$$

$$= 2^{-1}\overline{\delta}^2\exp\left(-\underline{m}\right)\beta^{k-2}E\left[\sum_{n=0}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)\right] \text{(see Lemma C.4 in p. 177)}=$$

$$= 2^{-1}\overline{\delta}^2\exp\left(-\underline{m}\right)\beta^{k-2}\left(\begin{array}{c} E\left[\sum_{i=0}^{k-2}\beta^i|X_{t-1-i}|^2 + \sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}|\right] \\ +E\left[\sum_{n=k-1}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)\right]\end{array}\right)$$

$$\le 2^{-1}\overline{\delta}^2\exp\left(-\underline{m}\right)\beta^{k-2}\left(\begin{array}{c} \frac{1-\beta^{k-1}}{1-\beta}\sigma_0^2 + E\sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}| \\ +E\left[\sum_{n=k-1}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)\right]\end{array}\right)$$

$$\le 2^{-1}\overline{\delta}^2\exp\left(-\underline{m}\right)\beta^{k-2}E\left(\begin{array}{c} \frac{1}{1-\beta}\sigma_0^2 + \sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}| \\ +\left[\sum_{n=k-1}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)\right]\end{array}\right),$$

where 1) (see Lemma C.5 in p. 178) $\sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}| =$
$= \frac{1-\beta^{k-2}}{1-\beta}\left(1+\beta\right)|X_{t-1}||X_{t-2}| + \frac{1-\beta^{k-3}}{1-\beta}\left(1+\beta^2\right)|X_{t-1}||X_{t-3}| + \ldots + \left(1+\beta^{k-2}\right)|X_{t-1}||X_{t-k+1}|,$

and we have that:

$\beta^{k-2}E\sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}| \le \beta^{k-2}\left(k-2\right)\frac{1-\beta^{k-2}}{1-\beta}\left(1+\beta\right)E|X_{t-1}||X_{t-2}|$
$\le \beta^{k-2}\left(k-2\right)\frac{1-\beta^{k-2}}{1-\beta}\left(1+\beta\right)E|Z_0|\,E\exp\left(\frac{1}{2}\alpha\right)E\exp\left(\frac{1}{2}\left(\beta+1\right)\log\sigma_0^2\right)E\left|\exp\left(\frac{1}{2}\overline{\delta}|Z_0|\right)Z_0\right|,$

and 2) (see Lemma C.6 in p. 179) $\sum_{n=k-1}^{\infty} \beta^n \left(|X_{t-n-1}|\right) \left(\sum_{i=1}^{k-1} |X_{t-i}|\right) =$

$= \beta^{k-1} |X_{t-k}| \left(|X_{t-1}| + |X_{t-2}| + ... + |X_{t-k+1}|\right) + \beta^k |X_{t-k-1}| \left(|X_{t-1}| + |X_{t-2}| + ... + |X_{t-k+1}|\right) +$
$\beta^{k+1} |X_{t-k-2}| \left(|X_{t-1}| + |X_{t-2}| + ... + |X_{t-k+1}|\right) + ...$

Hence,

$$E\left[\beta^{k-2} \sum_{n=k-1}^{\infty} \beta^n \left(|X_{t-n-1}|\right) \left(\sum_{i=1}^{k-1} |X_{t-i}|\right)\right] = \beta^{k-2} (k-1) \sum_{n=k-1}^{\infty} \beta^n E |X_t| |X_{t-n+k}|$$
$$\leq \beta^{k-2} (k-1) \tfrac{\beta^{k-1}}{1-\beta} E |Z_0| E \exp\left(\tfrac{1}{2}\alpha\right) E \exp\left(\tfrac{1}{2}(\beta+1)\log\sigma_0^2\right) E \left|\exp\left(\tfrac{1}{2}\overline{\delta}|Z_0|\right) Z_0\right|.$$

Hence,

$$\overline{\delta} \exp\left(-2^{-1}\underline{m}\right) E\left[\sum_{n=0}^{\infty} \left(\beta^{k-1}\beta^n \left(|X_{t-n-1}|\right) \left(c \sum_{i=1}^{k-1} |X_{t-i}|\right)\right)\right]$$
$$\leq 2^{-1}\overline{\delta}^2 \exp\left(-\underline{m}\right) \beta^{k-2} \begin{pmatrix} E\frac{1}{1-\beta}\sigma_0^2 + E\sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j} \beta^j \left(1+\beta^i\right) |X_{t-1-j}| |X_{t-1-j-i}| \\ +E\left[\sum_{n=k-1}^{\infty} \beta^n \left(|X_{t-n-1}|\right) \left(\sum_{i=1}^{k-1} |X_{t-i}|\right)\right] \end{pmatrix}$$
$$\leq 2^{-1}\overline{\delta}^2 \exp\left(-\underline{m}\right) \beta^{k-2} \begin{pmatrix} E\frac{1}{1-\beta}\sigma_0^2 + (k-2)\frac{1-\beta^{k-2}}{1-\beta}(1+\beta) E|Z_0| E\exp\left(\frac{1}{2}\alpha\right) \times \\ E\exp\left(\frac{1}{2}(\beta+1)\log\sigma_0^2\right) E\left|\exp\left(\frac{1}{2}\overline{\delta}|Z_0|\right) Z_0\right| \\ +(k-1)\frac{\beta^{k-1}}{1-\beta} E|Z_0| E\exp\left(\frac{1}{2}\alpha\right) E\exp\left(\frac{1}{2}(\beta+1)\log\sigma_{t-1}^2\right) \times \\ E\left|\exp\left(\frac{1}{2}\left(\gamma Z_{t-1} + \delta|Z_{t-1}|\right)\right) Z_{t-1}\right| \end{pmatrix}$$
$$\leq 2^{-1}\overline{\delta}^2 \exp\left(-\underline{m}\right) \beta^{k-2} \begin{pmatrix} E\frac{1}{1-\beta}\sigma_0^2 + (k-2)\frac{1}{1-\beta}(1+\beta) E|Z_0| E\exp\left(\frac{1}{2}\alpha\right) \times \\ E\exp\left(\frac{1}{2}(\beta+1)\log\sigma_0^2\right) E\left|Z_0\exp\left(\frac{1}{2}\overline{\delta}|Z_0|\right)\right| \\ +(k-1)\frac{\beta^{k-1}}{1-\beta} E|Z_0| E\exp\left(\frac{1}{2}\alpha\right) E\exp\left(\frac{1}{2}(\beta+1)\log\sigma_0^2\right) \times \\ E\left|Z_0\exp\left(\frac{1}{2}\overline{\delta}|Z_0|\right)\right| \end{pmatrix}$$
$$\leq 2^{-1}\overline{\delta}^2 \exp\left(-\underline{m}\right) \beta^{k-2} \left[\begin{array}{c} E\frac{1}{1-\beta}\sigma_0^2 + E|Z_0| E\exp\left(\frac{1}{2}\alpha\right) E\exp\left(\frac{1}{2}(\beta+1)\log\sigma_0^2\right) E\left|\exp\left(\frac{1}{2}\overline{\delta}|Z_0|\right) Z_0\right| \\ \times\left((k-2)\frac{1}{1-\beta}(1+\beta) + (k-1)\frac{\beta^{k-1}}{1-\beta}\right) \end{array}\right].$$

And $\sum_{k=1}^{\infty} (\cdot)$ is bounded if and only if $E\left|Z_0 \exp\left(\frac{1}{2}\overline{\delta}|Z_0|\right)\right| < \infty$.

In the sequel, we have that:

$$\overline{\delta} \exp\left(-2^{-1}\underline{m}\right) E\left[\sum_{n=0}^{\infty} \left(\beta^{k-1}\beta^n \left(|X_{t-n-1}|\right) \left(c^{k-1} \prod_{i=1}^{k-1} |X_{t-i}|\right)\right)\right] =$$
$$= 2^{-(k-1)}\overline{\delta}^k \exp\left(-2^{-1}\underline{m}k\right) E\left[\sum_{n=0}^{\infty} \beta^n \left(|X_{t-n-1}|\right) \left(\prod_{i=1}^{k-1} |X_{t-i}|\right)\right] =$$
$$= 2^{-(k-1)}\overline{\delta}^k \exp\left(-2^{-1}\underline{m}k\right) \left(E\left[\sum_{j=0}^{k-2} \beta^j |X_{t-j-1}| \left(\prod_{i=1}^{k-1} |X_{t-i}|\right) + E\sum_{n=k-1}^{\infty} \beta^n \left(|X_{t-n-1}|\right) \left(\prod_{i=1}^{k-1} |X_{t-i}|\right)\right]\right.$$

We need the following:

1) (see Lemma C.7 in p. 179) $E\left[\sum_{j=0}^{k-2} \beta^j |X_{t-j-1}| \left(\prod_{i=1}^{k-1} |X_{t-i}|\right)\right] =$

$= E\left[\left(|X_{t-1}| + \beta|X_{t-2}| + ... + \beta^{k-2}|X_{t-k+1}|\right) \times \left(|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|\right)\right]$
$\leq \sum_{j=0}^{k-2}\beta^j cov\left(|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|, |X_{t-j}|\right), \qquad j = 1, ..., k-1,$

where

$$cov\left(|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|, |X_{t-j}|\right) \leq \exp\left[\frac{1}{2(1-\beta)}\alpha\left[k-1-\beta^{k-2}-\beta\frac{1-\beta^{k-2}}{1-\beta}\right]\right] \times$$
$$\times\left[E\left(\left|\exp\left\{\frac{1}{2}\overline{\delta}|Z_0|\right\}Z_0\right|\right)\right]^{k-3}$$
$$\times E\left|\exp\left\{\frac{1}{2}\left(\frac{1}{1-\beta}+1\right)\overline{\delta}|Z_0|\right\}Z_0^2\right|$$
$$\times E\exp\left\{\frac{1}{2}\left(\frac{1-\beta^{k-1}}{1-\beta}+1\right)\log\sigma_0^2\right\}.$$

Hence,

$E\left[\sum_{j=0}^{k-2}\beta^j|X_{t-j-1}|\left(\prod_{i=1}^{k-1}|X_{t-i}|\right)\right]$
$\leq \sum_{j=0}^{k-2}\beta^j\exp\left[\frac{1}{2(1-\beta)}\alpha\left[k-1-\beta^{k-2}-\beta\frac{1-\beta^{k-2}}{1-\beta}\right]\right]\left[E\left(\left|\exp\left\{\frac{1}{2}\overline{\delta}|Z_0|\right\}Z_0\right|\right)\right]^{k-3}\times$
$\times E\left|\exp\left\{\frac{1}{2}\left(\frac{1}{1-\beta}+1\right)\overline{\delta}|Z_0|\right\}Z_0^2\right| E\exp\left\{\frac{1}{2}\left(\frac{1-\beta^{k-1}}{1-\beta}+1\right)\log\sigma_0^2\right\}$
$\leq \sum_{j=0}^{k-2}\beta^j\exp\left[\frac{1}{2(1-\beta)}\alpha(k-3)\right]\exp\left[-\frac{1}{2(1-\beta)}\alpha\left[\beta^{k-2}+\beta\frac{1-\beta^{k-2}}{1-\beta}-2\right]\right]\left[E\left(\left|\exp\left\{\frac{1}{2}\overline{\delta}|Z_0|\right\}Z_0\right|\right)\right]^{k-3}$
$\times E\left|\exp\left\{\frac{1}{2}\left(\frac{1}{1-\beta}+1\right)\overline{\delta}|Z_0|\right\}Z_0^2\right| E\exp\left\{\frac{1}{2}\left(\frac{1-\beta^{k-1}}{1-\beta}+1\right)\log\sigma_0^2\right\}$
$\leq \sum_{j=0}^{k-2}\beta^j\exp\left[-\frac{1}{2(1-\beta)}\alpha\left[\beta^{k-2}+\beta\frac{1-\beta^{k-2}}{1-\beta}-2\right]\right]\left[\exp\left(\frac{1}{2(1-\beta)}\alpha\right)E\left(\left|\exp\left\{\frac{1}{2}\overline{\delta}|Z_0|\right\}Z_0\right|\right)\right]^{k-3}$
$\times E\left|\exp\left\{\frac{1}{2}\left(\frac{1}{1-\beta}+1\right)\overline{\delta}|Z_0|\right\}Z_0^2\right| E\exp\left\{\frac{1}{2}\left(\frac{1-\beta^{k-1}}{1-\beta}+1\right)\log\sigma_0^2\right\},$

which is bounded if and only if:

$\exp\left(\frac{1}{2(1-\beta)}\alpha\right)E\left(\left|\exp\left\{\frac{1}{2}\overline{\delta}|Z_0|\right\}Z_0\right|\right) < 1$ and $E\left|\exp\left\{\frac{1}{2}\left(\frac{1}{1-\beta}+1\right)\overline{\delta}|Z_0|\right\}Z_0^2\right| < \infty,$

and 2) (see Lemma C.8 in p. 183) $\sum_{n=k-1}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|\right) =$
$= \beta^{k-1}|X_{t-k}|\left(|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|\right)+\beta^k|X_{t-k-1}|\left(|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|\right)+$
$\beta^{k+1}|X_{t-k-2}|\left(|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|\right)+...$

Hence,

$E\left[\sum_{n=k-1}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|\right)\right]$
$\leq \left(\beta\exp\left(\frac{1}{2(1-\beta)}\alpha\right)E|z_0|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\overline{\delta}|z_0|\right)\right)^{k-1}\frac{1}{1-\beta}\exp\left(-2^{-1}\frac{\alpha\beta}{(1-\beta)^2}\right)E\exp\left(\frac{1}{2}\frac{1}{1-\beta}\log\sigma_0^2\right),$

where $\beta\exp\left(\frac{1}{2(1-\beta)}\alpha\right)E|Z_0|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\overline{\delta}|Z_0|\right) = q < 1$. Hence:

$\sum_{k=1}^{\infty}E\left[\sum_{n=k-1}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|\right)\right] =$
$= \frac{1}{1-\beta}\exp\left(-2^{-1}\frac{\alpha\beta}{(1-\beta)^2}\right)E\exp\left(\frac{1}{2}\frac{1}{1-\beta}\log\sigma_0^2\right)\sum_{k=1}^{\infty}q^{k-1} =$
$= \frac{1}{1-q}\frac{1}{1-\beta}\exp\left(-2^{-1}\left(\frac{\alpha\beta}{(1-\beta)^2}\right)\right)E\exp\left(\frac{1}{2}\frac{1}{1-\beta}\log\sigma_0^2\right).$ □

C.2 Proofs of the Main Lemmas 4.19, 4.20, 4.21 and 4.22

The constant c now takes the following form:

$$c = \frac{1}{(2\beta)^2}\overline{\delta}^2 \exp\left(-\underline{m}\right),$$

in the proof of the Lemmas *4.19, 4.20 and 4.21.* For a new definition of c, that is used in the proof of Lemma *4.22*, see at that point below.

Proof of Lemma 4.19. $\prod_{i=1}^{k-1} (1 + c\,|X_{t-i}|)^2$ is expanded as follows:

$$\prod_{i=1}^{k-1} (1 + c\,|X_{t-i}|)^2 = 1 + 2c\sum |X_{t-i}| + c^2 \sum |X_{t-i}|^2 + ... + c^{2(k-1)} \sum\prod |X_{t-i}|^2 .$$

If we take the first and the last term of the product, which are squared, we have first:

$$E\left[\left(\beta^{2(k-1)} c \sum_{i=1}^{n-1} |X_{t-i}|^2\right) |Z_{t-k}|\,\sigma_{t-k}\right] = \tfrac{1}{4}\beta^{2(k-2)}\overline{\delta}^2 \exp\left(-\underline{m}\right) E\left[\left(\sum_{i=1}^{n-1} |X_{t-i}|^2\right) |Z_{t-k}|\,\sigma_{t-k}\right].$$

Examining the higher dependence, which is $E\left[\left(|X_{t-k+1}|^2\right) |Z_{t-k}|\,\sigma_{t-k}\right]$, we have that:

$$\begin{aligned} |X_{t-1}|^2 |Z_{t-2}|\,\sigma_{t-2} &= Z_{t-1}^2 \left|\sigma_{t-1}^2 \sigma_{t-2} Z_{t-2}\right| \\ &= Z_{t-1}^2 \exp\left(\alpha\right) \exp\left[\left(\beta + \frac{1}{2}\right) \log \sigma_{t-2}^2\right] \left[|Z_{t-2}| \exp\left(\gamma Z_{t-2} + \delta\,|Z_{t-2}|\right)\right]. \end{aligned}$$

Hence,

$$E\left[\left(\beta^{2(k-1)} c \sum_{i=1}^{n-1} |X_{t-i}|^2\right) |Z_{t-k}|\,\sigma_{t-k}\right]$$

$$\leq (k-1)\,\tfrac{1}{4}\beta^{2(k-2)}\overline{\delta}^2 \exp\left(-\underline{m}\right) E\exp\left(\alpha\right) E\exp\left[\left(\beta + \tfrac{1}{2}\right)\log\sigma_0^2\right] \times E\left[|Z_0| \exp\left(\overline{\delta}\,|Z_0|\right)\right].$$

Hence,

$$\sum_{k=1}^{\infty} (k-1)\,\tfrac{1}{4}\beta^{2(k-2)}\overline{\delta}^2 \exp\left(-\underline{m}\right) E\exp\left(\alpha\right) E\exp\left[\left(\beta + \tfrac{1}{2}\right)\log\sigma_0^2\right] \times E\left[|Z_0| \exp\left(\overline{\delta}\,|Z_0|\right)\right]$$

is bounded if and only if:

$E\left[|Z_0| \exp\left(\overline{\delta}\,|Z_0|\right)\right] < \infty.$

Second, we have

$$E\left[\left(\beta^{2(k-1)} c^{k-1} \prod_{i=1}^{k-1} |X_{t-i}|^2\right) |Z_{t-k}|\,\sigma_{t-k}\right] =$$

$$= \tfrac{1}{4^{k-1}}\overline{\delta}^{2(k-1)} \exp\left(-\underline{m}\,(k-1)\right) E\left[\left(\prod_{i=1}^{k-1} |X_{t-i}|^2\right) |Z_{t-k}|\,\sigma_{t-k}\right].$$

We need the following (see Lemma C.10 in p. 186):

$$E\left[\left(\prod_{i=1}^{k-1}|X_{t-i}|^2\right)|Z_{t-k}|\,\sigma_{t-k}\right]=$$
$$=\exp(\alpha)\exp[(\beta+1)\alpha]\exp\left[\left(\beta^2+\beta+1\right)\alpha\right]\times...\times\exp\left[\left(\beta^{k-2}+...+1\right)\alpha\right]\times$$
$$\times E\exp\left((\gamma Z_{t-2}+\delta|Z_{t-2}|)\right)Z_{t-2}^2 E\exp\left((\beta+1)(\gamma Z_{t-3}+\delta|Z_{t-3}|)\right)Z_{t-3}^2\times...$$
$$\times E\exp\left(\left(\beta^{k-3}+...+1\right)(\gamma Z_{t-k+1}+\delta|Z_{t-k+1}|)\right)Z_{t-k+1}^2\times$$
$$\times E\left|\exp\left(\left(\beta^{k-2}+...+1\right)(\gamma Z_{t-k}+\delta|Z_{t-k}|)\right)Z_{t-k}\right|\times E\exp\left(\left(\beta^{k-2}+...+1+\tfrac{1}{2}\right)\log\sigma_{t-k}^2\right).$$

Hence,

$$E\left[\left(\beta^{2(k-1)}c^{k-1}\prod_{i=1}^{k-1}|X_{t-i}|^2\right)|Z_{t-k}|\,\sigma_{t-k}\right]\leq\frac{1}{4^{k-1}}\overline{\delta}^{2(k-1)}\exp\left(-\underline{m}(k-1)\right)\times$$
$$\times\exp\left[\alpha\frac{1}{1-\beta}\left(k-1-\frac{\beta-\beta^k}{1-\beta}\right)\right]E\left[\exp\left(\frac{1-\beta^{k-2}}{1-\beta}\overline{\delta}|Z_0|\right)Z_0^2\right]^{k-2}\times$$
$$\times E\left[|Z_0|\exp\left(\frac{1-\beta^{k-1}}{1-\beta}\overline{\delta}|Z_0|\right)\right]E\exp\left(\left(\frac{1-\beta^{k-1}}{1-\beta}+\frac{1}{2}\right)\log\sigma_0^2\right)$$
$$\leq 4^{-(k-1)}\overline{\delta}^{2(k-1)}\exp\left(-\underline{m}(k-1)\right)\exp\left[\alpha\frac{1}{1-\beta}(k-1)\right]E\left[Z_0^2\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right]^{k-1}\times$$
$$\times E\left[Z_0^2\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right]^{-1}\exp\left(-\alpha\frac{\beta}{(1-\beta)^2}\right)\times$$
$$\times E\left[|Z_0|\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right]E\exp\left(\left(\frac{1}{1-\beta}+\frac{1}{2}\right)\log\sigma_0^2\right)$$
$$\leq\left[\frac{1}{4}\overline{\delta}^2\exp\left(-\underline{m}\right)\exp\left(\alpha\frac{1}{1-\beta}\right)E\left[Z_0^2\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right]\right]^{k-1}\times$$
$$\times E\left[Z_0^2\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right]^{-1}\exp\left(-\alpha\frac{\beta}{(1-\beta)^2}\right)\times$$
$$\times E\left[|Z_0|\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right]E\exp\left(\left(\frac{1}{1-\beta}+\frac{1}{2}\right)\log\sigma_0^2\right),$$

where $\frac{1}{4}\overline{\delta}^2\exp\left(\alpha\frac{1}{1-\beta}-\underline{m}\right)E\left[Z_0^2\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right]=q^d<1$.

Hence,

$$\sum_k E\left[\left(\beta^{2(k-1)}c^{k-1}\prod_{i=1}^{k-1}|X_{t-i}|^2\right)|Z_{t-k}|\,\sigma_{t-k}\right]=E\left[\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)Z_0^2\right]^{-1}\exp\left(-\alpha\frac{\beta}{(1-\beta)^2}\right)\times$$
$$\times E\left[|Z_0|\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right]E\exp\left(\left(\frac{1}{1-\beta}+\frac{1}{2}\right)\log\sigma_0^2\right)\sum_k q^{k-1}=$$
$$=\frac{1}{1=q}E\left[Z_0^2\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right]^{-1}\exp\left(-\alpha\frac{\beta}{(1-\beta)^2}\right)E\left[|Z_0|\exp\left(\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right]E\exp\left(\left(\frac{1}{1-\beta}+\frac{1}{2}\right)\log\sigma_0^2\right).$$

□

Proof of Lemma 4.20. Again, by using the first and the last term of the product, given in the proof of Lemma 4.19, we have:

$$E\left[\left(\beta^{2(k-1)}c\sum_{i=1}^{n-1}|X_{t-i}|^2\right)X_{t-k}^2\right]=\frac{1}{4}\beta^{2(k-2)}\overline{\delta}^2\exp\left(-\underline{m}\right)E\left[\left(\sum_{i=1}^{n-1}|X_{t-i}|^2\right)X_{t-k}^2\right].$$

Examining the higher dependence, which is $E\left[\left(|X_{t-k+1}|^2\right)X_{t-k}^2\right]$, we have that:

$$\begin{aligned}X_{t-1}^2X_{t-2}^2 &= Z_{t-1}^2\sigma_{t-1}^2\sigma_{t-2}^2Z_{t-2}^2\\ &= Z_{t-1}^2\exp(\alpha)\exp\left[(\beta+1)\log\sigma_{t-2}^2\right]\exp\left(\gamma Z_{t-2}+\delta|Z_{t-2}|\right)Z_{t-2}^2.\end{aligned}$$

Hence,

$$E\left[\left(\beta^{2(k-1)}c\sum_{i=1}^{n-1}|X_{t-i}|^2\right)|Z_{t-k}|\,\sigma_{t-k}\right]=$$

$$\leq (k-1)\,\tfrac{1}{4}\beta^{2(k-2)}\overline{\delta}^2\exp\left(-\underline{m}\right)\exp\left(\alpha\right)E\exp\left[\left(\beta+1\right)\log\sigma_0^2\right]E\left[Z_0^2\exp\left(\overline{\delta}\,|Z_0|\right)\right].$$

Hence,

$\sum_{k=1}^{\infty}(k-1)\,\frac{1}{4}\beta^{2(k-2)}\overline{\delta}^2\exp\left(-\underline{m}\right)\exp\left(\alpha\right)E\exp\left[\left(\beta+1\right)\log\sigma_0^2\right]E\left[Z_0^2\exp\left(\overline{\delta}\,|Z_0|\right)\right]$ is bounded if and only if $E\left[Z_0^2\exp\left(\overline{\delta}\,|Z_0|\right)\right]<\infty$.

Second we have,

$$E\left[\left(\beta^{2(k-1)}c^{k-1}\prod_{i=1}^{k-1}|X_{t-i}|^2\right)X_{t-k}^2\right]=\tfrac{1}{4^{k-1}}\overline{\delta}^{2(k-1)}\exp\left(-\underline{m}\,(k-1)\right)E\left[\left(\prod_{i=1}^{k-1}|X_{t-i}|^2\right)X_{t-k}^2\right].$$

We need the following:

$$E\left[\left(\prod_{i=1}^{k-1}|X_{t-i}|^2\right)X_{t-k}^2\right]=E\left(Z_{t-1}^2\sigma_{t-1}^2Z_{t-2}^2\sigma_{t-2}^2Z_{t-3}^2\sigma_{t-3}^2...Z_{t-k+1}^2\sigma_{t-k+1}^2Z_{t-k}^2\sigma_{t-k}^2\right)=$$ (see Lemma C.1 in p. 176)

$$=E\exp\left(\tfrac{1}{(1-\beta)}\alpha\left(k-1-\tfrac{\beta-\beta^k}{1-\beta}\right)\right)\times E\exp\left(\tfrac{1-\beta^k}{1-\beta}\log\sigma_{t-k}^2\right)\times E\exp\left(\tfrac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k}+\delta\,|Z_{t-k}|\right)\right)Z_{t-k}^2$$

$$\times...\times E\exp\left(\gamma Z_{t-2}+\delta\,|Z_{t-2}|\right)Z_{t-2}^2.$$

Hence,

$$E\left[\left(\beta^{2(k-1)}c^{k-1}\prod_{i=1}^{k-1}|X_{t-i}|^2\right)X_{t-k}^2\right]=\tfrac{1}{4^{k-1}}\overline{\delta}^{2(k-1)}\exp\left(-\underline{m}\,(k-1)\right)\times$$

$$\times E\exp\left(\tfrac{1}{(1-\beta)}\alpha\left(k-1-\tfrac{\beta-\beta^k}{1-\beta}\right)\right)\times E\exp\left(\tfrac{1-\beta^k}{1-\beta}\log\sigma_{t-k}^2\right)\times E\exp\left(\tfrac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k}+\delta\,|Z_{t-k}|\right)\right)Z_{t-k}^2$$

$$\times...\times E\exp\left(\gamma Z_{t-2}+\delta\,|Z_{t-2}|\right)Z_{t-2}^2$$

$$\leq\tfrac{1}{4^{k-1}}\overline{\delta}^{2(k-1)}\exp\left(-\underline{m}\,(k-1)\right)\times E\exp\left(\tfrac{1}{(1-\beta)}\alpha\left(k-1-\tfrac{\beta-\beta^k}{1-\beta}\right)\right)\times E\exp\left(\tfrac{1}{1-\beta}\log\sigma_0^2\right)$$

$$\times\left(EZ_0^2\exp\left(\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)\right)^{k-1}$$

$$\leq 4^{-(k-1)}\overline{\delta}^{2(k-1)}\exp\left(\left(\tfrac{\alpha}{(1-\beta)}-\underline{m}\right)(k-1)\right)\exp\left(-\tfrac{\alpha\beta}{(1-\beta)^2}\right)\times E\exp\left(\tfrac{1}{1-\beta}\log\sigma_0^2\right)$$

$$\times\left(EZ_0^2\exp\left(\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)\right)^{k-1}$$

$$\leq\left[4^{-1}\overline{\delta}^2\exp\left(\tfrac{\alpha}{(1-\beta)}-\underline{m}\right)EZ_0^2\exp\left(\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)\right]^{k-1}\exp\left(-\tfrac{\alpha\beta}{(1-\beta)^2}\right)E\exp\left(\tfrac{1}{1-\beta}\log\sigma_0^2\right),$$

where $4^{-1}\overline{\delta}^2\exp\left(\frac{\alpha}{(1-\beta)}-\underline{m}\right)E\left[Z_0^2\exp\left(\frac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)\right]=q^d<1$.

Hence:

$$\sum_k E\left[\left(\beta^{2(k-1)}c^{k-1}\prod_{i=1}^{k-1}|X_{t-i}|^2\right)X_{t-k}^2\right]=\exp\left(-\tfrac{\alpha\beta}{(1-\beta)^2}\right)E\exp\left(\tfrac{1}{1-\beta}\log\sigma_0^2\right)\sum_k q^{k-1}=$$

$$=\tfrac{1}{1-q}\exp\left(-\tfrac{\alpha\beta}{(1-\beta)^2}\right)E\exp\left(\tfrac{1}{1-\beta}\log\sigma_0^2\right).$$

□

Proof of Lemma 4.21. We can write $E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^2\left|Z_{t-k}\right|\sigma_{t-k}\left|\log h_{t-k}\right|\right]=$

$$=E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^2\left|Z_{t-k}\right|\sigma_{t-k}\left(\left|\frac{\alpha}{1-\beta}\right|+\exp\left(-2^{-1}\underline{m}\right)\sum_{n=0}^{\infty}\beta^n\overline{\delta}\left|X_{t-n-1}\right|\right)\right]=$$

$$=\left|\frac{\alpha}{1-\beta}\right|E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^2\left|Z_{t-k}\right|\sigma_{t-k}\right]$$

$$+\overline{\delta}\exp\left(-2^{-1}\underline{m}\right)E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^2\right)\left|Z_{t-k}\right|\sigma_{t-k}\right].$$

For the first term, see Lemma 4.19 and its conditions.

Again, we can replace the product in the second term by considering only the first and the last term from its expansion, that is:

$$\overline{\delta}\exp\left(-2^{-1}\underline{m}\right)E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\beta^{2(k-1)}c\sum_{i=1}^{k-1}\left|X_{t-i}\right|^2\right)\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]=$$

$$=2^{-2}\overline{\delta}^3\exp\left(-\tfrac{3}{2}\underline{m}\right)\beta^{2(k-2)}E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\sum_{i=1}^{k-1}\left|X_{t-i}\right|^2\right)\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]$$ (see Lemma C.11 in p. 186)

$$\leq 2^{-2}\overline{\delta}^3\exp\left(-\tfrac{3}{2}\underline{m}\right)\beta^{2(k-2)}(k-1)\left[\begin{array}{c}\frac{1-\beta^{k-1}}{1-\beta}\left|Z_0\right|^3\exp\left(\frac{3}{2}\alpha\right)E\exp\left(\left(\frac{3}{2}\beta+\frac{1}{2}\right)\log\sigma_{t-2}^2\right)\times\\ E\left[\left|z_0\right|\exp\left(\frac{3}{2}\overline{\delta}\left|z_0\right|\right)\right]\\ +\frac{\beta^{k-1}}{1-\beta}\exp\left(\alpha\right)E\exp\left(\left(\beta+1\right)\log\sigma_{t-2}^2\right)E\left[Z_0^2\exp\left(\overline{\delta}\left|Z_0\right|\right)\right]\end{array}\right].$$

Hence, $\sum_{k=1}^{\infty}(\cdot)$ is bounded if and only if

$E\left[\left|Z_0\right|\exp\left(\frac{3}{2}\overline{\delta}\left|Z_0\right|\right)\right]<\infty$ and $E\left[Z_0^2\exp\left(\overline{\delta}\left|Z_0\right|\right)\right]<\infty$.

In the sequel, we have that:

$$\overline{\delta}\exp\left(-2^{-1}\underline{m}\right)E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\beta^{2(k-1)}c^{k-1}\prod_{i=1}^{k-1}\left|X_{t-i}\right|^2\right)\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]=$$

$$=2^{-2(k-1)}\overline{\delta}^{2k-1}\exp\left(-\underline{m}\left(k-2^{-1}\right)\right)E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\prod_{i=1}^{k-1}\left|X_{t-i}\right|^2\right)\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]=$$

$$=2^{-2(k-1)}\overline{\delta}^{2k-1}\exp\left(-\underline{m}\left(k-2^{-1}\right)\right)E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\prod_{i=1}^{k-1}\left|X_{t-i}\right|^2\right)\right)\left|X_{t-k}\right|\\ +\sum_{n=k-1}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\prod_{i=1}^{k-1}\left|X_{t-i}\right|^2\right)\right)\left|X_{t-k}\right|\end{array}\right].$$

We need the following:

1) (see Lemma C.12 in p. 187) $E\left[\sum_{j=0}^{k-2}\left(\beta^j\left(\left|X_{t-j-1}\right|\right)\left(\prod_{i=1}^{k-1}\left|X_{t-i}\right|^2\right)\right)\left|X_{t-k}\right|\right]$

$$\leq\sum_{j=0}^{k-2}cov\left(\left|X_{t-1}\right|^2\left|X_{t-2}\right|^2\left|X_{t-3}\right|^2\ldots\left|X_{t-k+1}\right|^2\left|X_{t-k}\right|,\left|X_{t-j}\right|\right),\qquad j=1,\ldots,k-1,$$

where

$$cov\left(|X_{t-1}|^2|X_{t-2}|^2|X_{t-3}|^2\dots|X_{t-k+1}|^2|X_{t-k}|,|X_{t-j}|\right) \leq E|Z_0|^3 \times \times\exp\left[\frac{1}{(1-\beta)}\alpha\left[\begin{array}{c}\frac{1}{2}\left(1-\beta^{k-1}\right)\\+k-1-\beta\frac{1-\beta^{k-1}}{1-\beta}\end{array}\right]\right] \times\left[E\left(\left|\exp\left\{\overline{\delta}|Z_0|\right\}Z_0^2\right|\right)\right]^{k-2} \times E\left[|Z_0|\exp\left(\left(\frac{1-\beta^{k-1}}{1-\beta}+\frac{1}{2}\right)\overline{\delta}|Z_0|\right)\right] \times E\exp\left\{\left[\frac{1-\beta^k}{1-\beta}+\frac{1}{2}(\beta-1)\right]\log\sigma^2_{t-k}\right\}$$

Hence,

$$E\left[\sum_{j=0}^{k-2}\left(\beta^j\left(|X_{t-j-1}|\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|^2\right)\right)|X_{t-k}|\right]$$
$$\leq\sum_{j=0}^{k-2}\beta^jE|Z_{t-1}|^3\exp\left[\tfrac{1}{(1-\beta)}\alpha\left[\tfrac{1}{2}\left(1-\beta^{k-1}\right)+k-1-\beta\tfrac{1-\beta^{k-1}}{1-\beta}\right]\right]\left[E\left[Z_0^2\left(\exp\left\{\overline{\delta}|Z_0|\right\}\right)\right]\right]^{k-2}$$
$$\times E\left[|Z_0|\exp\left(\left(\tfrac{1-\beta^{k-1}}{1-\beta}+\tfrac{1}{2}\right)\overline{\delta}|Z_0|\right)\right]E\exp\left\{\left[\tfrac{1-\beta^k}{1-\beta}+\tfrac{1}{2}(\beta-1)\right]\log\sigma^2_{t-k}\right\}$$
$$\leq\sum_{j=0}^{k-2}\beta^jE|Z_0|^3\exp\left[\tfrac{1}{(1-\beta)}\alpha(k-2)\right]\exp\left[\tfrac{1}{(1-\beta)}\alpha\left[\tfrac{1}{2}\left(1-\beta^{k-1}\right)+\tfrac{1}{(1-\beta)}\alpha-\beta\tfrac{1-\beta^{k-1}}{1-\beta}\right]\right]\times$$
$$\times\left[E\left[Z_0^2\left(\exp\left\{\overline{\delta}|Z_0|\right\}\right)\right]\right]^{k-2}$$
$$\times E\left[|Z_0|\exp\left(\left(\tfrac{1-\beta^{k-1}}{1-\beta}+\tfrac{1}{2}\right)\overline{\delta}|Z_0|\right)\right]E\exp\left\{\left[\tfrac{1-\beta^k}{1-\beta}+\tfrac{1}{2}(\beta-1)\right]\log\sigma^2_{t-k}\right\}$$
$$\leq\sum_{j=0}^{k-2}\beta^jE|Z_0|^3\exp\left[\tfrac{1}{(1-\beta)}\alpha\left[\tfrac{1}{2}\left(1-\beta^{k-1}\right)+\tfrac{1}{(1-\beta)}\alpha-\beta\tfrac{1-\beta^{k-1}}{1-\beta}\right]\right]\times$$
$$\times\left[\exp\left(\tfrac{1}{(1-\beta)}\alpha\right)E\left[Z_0^2\left(\exp\left\{\overline{\delta}|Z_0|\right\}\right)\right]\right]^{k-2}$$
$$\times E\left[|Z_0|\exp\left(\left(\tfrac{1}{1-\beta}+\tfrac{1}{2}\right)\overline{\delta}|Z_0|\right)\right]E\exp\left\{\left[\tfrac{1-\beta^k}{1-\beta}+\tfrac{1}{2}(\beta-1)\right]\log\sigma^2_0\right\},$$

where it is bounded if and only if:

$\exp\left(\frac{1}{(1-\beta)}\alpha\right)E\left[Z_0^2\exp\left(\overline{\delta}|Z_0|\right)\right]=q^e<1$ and $E\left[|Z_0|\exp\left(\left(\frac{1}{1-\beta}+\frac{1}{2}\right)\overline{\delta}|Z_0|\right)\right]<\infty$.

And 2) (see Lemma C.13 in p. 190) $E\left[\sum_{n=k-1}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|^2\right)\right)|X_{t-k}|\right]=$

$$=E\left[\sum_{n=k-1}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(|X_{t-1}|^2|X_{t-2}|^2\dots|X_{t-k+1}|^2\right)\right)|X_{t-k}|\right]$$
$$\leq\tfrac{\beta^{k-1}}{1-\beta}\exp\left(\tfrac{1}{1-\beta}\alpha(k-1)\right)\exp\left(-\tfrac{1}{1-\beta}\alpha\left(\tfrac{\beta-\beta^k}{1-\beta}\right)\right)\times E\exp\left(\tfrac{1-\beta^k}{1-\beta}\log\sigma^2_{t-k}\right)$$
$$\times\left[E\exp\left(\gamma Z_{t-2}+\delta|Z_{t-2}|\right)Z^2_{t-2}\right]^{k-1}$$
$$\leq\tfrac{\beta^{k-1}}{1-\beta}\exp\left(-\tfrac{1}{1-\beta}\alpha\left(\tfrac{\beta-\beta^k}{1-\beta}\right)\right)\times E\exp\left(\tfrac{1-\beta^k}{1-\beta}\log\sigma^2_0\right)$$

$\times \left[\exp\left(\frac{1}{1-\beta}\alpha\right) E\exp\left(\overline{\delta}\left|Z_0\right|\right) Z_0^2\right]^{k-1}$,

which is bounded if and only if $\exp\left(\frac{1}{1-\beta}\alpha\right) E\left[Z_0^2\exp\left(\overline{\delta}\left|Z_0\right|\right)\right] = q^f < 1$. □

The constant c now takes the following form:

$$c = c^* = \frac{1}{(2\beta)^3}\overline{\delta}^3\exp\left(-\frac{3}{2}\underline{m}\right),$$

that is used in the following proof of Lemma 4.22:

Proof of Lemma 4.22. Recall the eq. (4.14). We have to calculate the bounds of: $E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^3\left|Z_{t-k}\right|\sigma_{t-k}\right]$, for which we have:

(1) $E\left[\left(\beta^{3(k-1)}c^*\sum_{i=1}^{n-1}\left|X_{t-i}\right|^3\right)\left|Z_{t-k}\right|\sigma_{t-k}\right] = \frac{1}{8}\beta^{3(k-2)}\overline{\delta}^3\exp\left(-\frac{3}{2}\underline{m}\right)E\left[\left(\sum_{i=1}^{n-1}\left|X_{t-i}\right|^3\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]$.

Examining the higher dependence, which is $E\left[\left(\left|X_{t-k+1}\right|^3\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]$, we have that:

$$\begin{aligned}\left(\left|X_{t-1}\right|^3\right)\left|Z_{t-2}\right|\sigma_{t-2} &= \left|Z_{t-1}\right|^3\left|\sigma_{t-1}^3\sigma_{t-2}Z_{t-2}\right| \\ &= \left|Z_{t-1}\right|^3\exp\left(\frac{3}{2}\alpha\right)\exp\left[\left(\frac{3}{2}\beta+\frac{1}{2}\right)\log\sigma_{t-2}^2\right]\left[\left|Z_{t-2}\right|\exp\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right].\end{aligned}$$

Hence,

$E\left[\left(\beta^{3(k-1)}c^*\sum_{i=1}^{n-1}\left|X_{t-i}\right|^3\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]$

$\leq (k-1)\frac{1}{8}\beta^{3(k-2)}\overline{\delta}^3\exp\left(-\frac{3}{2}\underline{m}\right)E\left|Z_0\right|^3\exp\left(\frac{3}{2}\alpha\right)E\exp\left[\left(\frac{3}{2}\beta+\frac{1}{2}\right)\log\sigma_0^2\right]\times E\left[\left|Z_0\right|\exp\left(\overline{\delta}\left|Z_0\right|\right)\right]$.

Hence,

$\sum_{k=1}^{\infty}(k-1)\frac{1}{8}\beta^{3(k-2)}\overline{\delta}^3\exp\left(-\frac{3}{2}\underline{m}\right)E\left|Z_0\right|^3\exp\left(\frac{3}{2}\alpha\right)E\exp\left[\left(\frac{3}{2}\beta+\frac{1}{2}\right)\log\sigma_0^2\right]\times E\left[\left|Z_0\right|\exp\left(\overline{\delta}\left|Z_0\right|\right)\right]$

is bounded if and only if: $E\left[\left|Z_0\right|\exp\left(\overline{\delta}\left|Z_0\right|\right)\right] < \infty$.

(2) $E\left[\left(\beta^{3(k-1)}c^{*(k-1)}\prod_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\left|Z_{t-k}\right|\sigma_{t-k}\right] =$

$= \frac{1}{8^{k-1}}\overline{\delta}^{3(k-1)}\exp\left(-\frac{3}{2}\underline{m}(k-1)\right)E\left[\left(\prod_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]$.

We need the following: $E\left[\left(\prod_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\left|Z_{t-k}\right|\sigma_{t-k}\right] =$

$= E\left(Z_{t-1}^3\sigma_{t-1}^3Z_{t-2}^3\sigma_{t-2}^3Z_{t-3}^3\sigma_{t-3}^3 ... Z_{t-k+1}^3\sigma_{t-k+1}^3\left|Z_{t-k}\right|\sigma_{t-k}\right) =$

$= E\exp\left(\frac{3}{2}\alpha\right)E\exp\left[\frac{3}{2}(\beta+1)\alpha\right]E\exp\left[\frac{3}{2}\left(\beta^2+\beta+1\right)\alpha\right]\times ...\times E\exp\left[\frac{3}{2}\left(\beta^{k-2}+...+1\right)\alpha\right]\times$

$\times E\exp\left(\frac{3}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right)Z_{t-2}^2 E\exp\left(\frac{3}{2}(\beta+1)\left(\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|\right)\right)Z_{t-3}^2\times ...$

$\times E\exp\left(\frac{3}{2}\left(\beta^{k-3}+...+1\right)\left(\gamma Z_{t-k+1}+\delta\left|Z_{t-k+1}\right|\right)\right)Z_{t-k+1}^2\times$
$\times E\left|\exp\left(\frac{3}{2}\left(\beta^{k-2}+...+1\right)\left(\gamma Z_{t-k}+\delta\left|Z_{t-k}\right|\right)\right)Z_{t-k}\right|\times E\exp\left(\left(\frac{3}{2}\left(\beta^{k-2}+...+1\right)+\frac{1}{2}\right)\log\sigma_{t-k}^2\right).$

Hence,

$$E\left[\left(\beta^{3(k-1)}c^{*(k-1)}\prod_{i=1}^{k-1}|X_{t-i}|^3\right)|Z_{t-k}|\,\sigma_{t-k}\right]\le\tfrac{1}{8^{k-1}}\overline{\delta}^{3(k-1)}\exp\left(-\tfrac{3}{2}\underline{m}\,(k-1)\right)\times$$
$$\times\exp\left[\tfrac{3}{2}\alpha\tfrac{1}{1-\beta}\left(k-1-\tfrac{\beta-\beta^k}{1-\beta}\right)\right]E\left[\exp\left(\tfrac{3}{2}\tfrac{1-\beta^{k-2}}{1-\beta}\overline{\delta}\,|Z_0|\right)Z_0^2\right]^{k-2}E\left[|Z_0|\exp\left(\tfrac{3}{2}\tfrac{1-\beta^{k-1}}{1-\beta}\overline{\delta}\,|Z_0|\right)\right]$$
$$\times E\exp\left(\left(\tfrac{3}{2}\tfrac{1-\beta^{k-1}}{1-\beta}+\tfrac{1}{2}\right)\log\sigma_0^2\right)$$
$$\le\tfrac{1}{8^{k-1}}\overline{\delta}^{3(k-1)}\exp\left(-\tfrac{3}{2}\underline{m}\,(k-1)\right)\exp\left[\tfrac{3}{2}\alpha\tfrac{1}{1-\beta}(k-1)\right]E\left[\exp\left(\tfrac{3}{2}\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)Z_0^2\right]^{k-1}$$
$$\times E\left[\exp\left(\tfrac{3}{2}\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)Z_0^2\right]^{-1}\exp\left(-\tfrac{3}{2}\alpha\tfrac{\beta}{(1-\beta)^2}\right)\times$$
$$\times E\left[|Z_0|\exp\left(\tfrac{3}{2}\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)\right]E\exp\left(\left(\tfrac{3}{2}\tfrac{1}{1-\beta}+\tfrac{1}{2}\right)\log\sigma_0^2\right)$$
$$\le\left[\tfrac{1}{8}\overline{\delta}^3\exp\left(-\tfrac{3}{2}\underline{m}\right)\exp\left(\tfrac{3}{2}\alpha\tfrac{1}{1-\beta}\right)E\left[\exp\left(\tfrac{3}{2}\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)Z_0^2\right]\right]^{k-1}$$
$$\times E\left[\exp\left(\tfrac{3}{2}\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)Z_0^2\right]^{-1}\exp\left(-\tfrac{3}{2}\alpha\tfrac{\beta}{(1-\beta)^2}\right)\times$$
$$\times E\left[|Z_0|\exp\left(\tfrac{3}{2}\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)\right]E\exp\left(\left(\tfrac{3}{2}\tfrac{1}{1-\beta}+\tfrac{1}{2}\right)\log\sigma_0^2\right),$$

where $\frac{1}{8}\overline{\delta}^3\exp\left[\frac{3}{2}\left(\alpha\frac{1}{1-\beta}-\underline{m}\right)\right]E\left[Z_0^2\exp\left(\frac{3}{2}\frac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)\right]<1.$

Hence:

$$\sum_k E\left[\left(\beta^{3(k-1)}c^{*(k-1)}\prod_{i=1}^{k-1}|X_{t-i}|^3\right)|Z_{t-k}|\,\sigma_{t-k}\right]=$$
$$=\tfrac{1}{1=q}E\left[\exp\left(\tfrac{3}{2}\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)Z_0^2\right]^{-1}\exp\left(-\tfrac{3}{2}\alpha\tfrac{\beta}{(1-\beta)^2}\right)\times$$
$$\times E\left[|Z_0|\exp\left(\tfrac{3}{2}\tfrac{1}{1-\beta}\overline{\delta}\,|Z_0|\right)\right]E\exp\left(\left(\tfrac{3}{2}\tfrac{1}{1-\beta}+\tfrac{1}{2}\right)\log\sigma_0^2\right).$$

Next:

$$E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\,|X_{t-i}|\right]\right)^3|Z_{t-k}|\,\sigma_{t-k}\,|X_{t-k}|^2\right]=E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\,|X_{t-i}|\right]\right)^3|X_{t-k}|^3\right],$$

for which we have:

(1) $E\left[\left(\beta^{3(k-1)}c^*\sum_{i=1}^{n-1}|X_{t-i}|^3\right)X_{t-k}^3\right]=\frac{1}{8}\beta^{3(k-2)}\overline{\delta}^3\exp\left(-\frac{3}{2}\underline{m}\right)E\left[\left(\sum_{i=1}^{n-1}|X_{t-i}|^3\right)|X_{t-k}|^3\right].$

Examining the higher dependence, which is $E\left[\left(|X_{t-k+1}|^3\right)|X_{t-k}|^3\right]$, we have that:

$$\begin{aligned}|X_{t-1}|^3|X_{t-2}|^3&=\left|Z_{t-1}^3\sigma_{t-1}^3\sigma_{t-2}^3Z_{t-2}^3\right|\\&=|Z_{t-1}|^3\exp\left(\frac{3}{2}\alpha\right)\exp\left[\frac{3}{2}(\beta+1)\log\sigma_{t-2}^2\right]\left[\left|Z_{t-2}^3\right|\exp\left(\frac{3}{2}\left(\gamma Z_{t-2}+\delta\,|Z_{t-2}|\right)\right)\right].\end{aligned}$$

Hence,

$$E\left[\left(\beta^{3(k-1)}c^*\sum_{i=1}^{n-1}|X_{t-i}|^3\right)|X_{t-k}|^3\right]$$
$$\le(k-1)\tfrac{1}{8}\beta^{3(k-2)}\overline{\delta}^3\exp\left(-\tfrac{3}{2}\underline{m}\right)\exp\left(\tfrac{3}{2}\alpha\right)E\exp\left[\tfrac{3}{2}(\beta+1)\log\sigma_0^2\right]E\left[\left|Z_0^3\right|\exp\left(\tfrac{3}{2}\left(\overline{\delta}\,|Z_0|\right)\right)\right].$$

Hence, $\sum_{k=1}^{\infty}(\cdot)$ is bounded if and only if: $E\left[\left|Z_0^3\right|\exp\left(\frac{3}{2}\left(\overline{\delta}\,|Z_0|\right)\right)\right]<\infty$.

(2) $E\left[\left(\beta^{3(k-1)}c^{*(k-1)}\prod_{i=1}^{k-1}|X_{t-i}|^3\right)X_{t-k}^3\right]=$

$=\frac{1}{8^{k-1}}\overline{\delta}^{3(k-1)}\exp\left(-\frac{3}{2}\underline{m}\,(k-1)\right)E\left[\left(\prod_{i=1}^{k-1}|X_{t-i}|^3\right)|X_{t-k}|^3\right].$

We need the following:

$E\left[\left(\prod_{i=1}^{k-1}|X_{t-i}|^3\right)|X_{t-k}|^3\right]$

$=E\exp\left(\frac{3}{2}\frac{1}{(1-\beta)}\alpha\left(k-1-\frac{\beta-\beta^k}{1-\beta}\right)\right)\times E\exp\left(\frac{3}{2}\frac{1-\beta^k}{1-\beta}\log\sigma_{t-k}^2\right)\times$

$\times E\left[\left|Z_{t-k}^3\right|\exp\left(\frac{3}{2}\frac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k}+\delta\left|Z_{t-k}\right|\right)\right)\right]\times\ldots$

$\times E\left[\left|Z_{t-2}^3\right|\exp\left(\frac{3}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right)\right].$

Hence,

$E\left[\left(\beta^{3(k-1)}c^{*(k-1)}\prod_{i=1}^{k-1}|X_{t-i}|^3\right)|X_{t-k}|^3\right]=\frac{1}{8^{k-1}}\overline{\delta}^{3(k-1)}\exp\left(-\frac{3}{2}\underline{m}\,(k-1)\right)\times$

$\times E\exp\left(\frac{3}{2}\frac{1}{(1-\beta)}\alpha\left(k-1-\frac{\beta-\beta^k}{1-\beta}\right)\right)\times E\exp\left(\frac{3}{2}\frac{1-\beta^k}{1-\beta}\log\sigma_{t-k}^2\right)\times$

$\times E\left[\left|z_{t-k}^3\right|\exp\left(\frac{3}{2}\frac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k}+\delta\left|Z_{t-k}\right|\right)\right)\right]\times\ldots$

$\times E\left[\left|Z_{t-2}^3\right|\exp\left(\frac{3}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right)\right]$

$\leq 8^{-(k-1)}\overline{\delta}^{3(k-1)}\exp\left(\left(\frac{3}{2}\left(\frac{\alpha}{(1-\beta)}-\underline{m}\right)\right)(k-1)\right)\exp\left(-\frac{3}{2}\frac{\alpha\beta}{(1-\beta)^2}\right)\times E\exp\left(\frac{3}{2}\frac{1}{1-\beta}\log\sigma_0^2\right)$

$\times\left(E\left|Z_0^3\right|\exp\left(\frac{3}{2}\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right)^{k-1}$

$\leq\left[8^{-1}\overline{\delta}^3\exp\left(\frac{3}{2}\left(\frac{\alpha}{(1-\beta)}-\underline{m}\right)\right)E\left|Z_0^3\right|\exp\left(\frac{3}{2}\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right]^{k-1}\exp\left(-\frac{3}{2}\frac{\alpha\beta}{(1-\beta)^2}\right)E\exp\left(\frac{3}{2}\frac{1}{1-\beta}\log\sigma_0^2\right),$

where $8^{-1}\overline{\delta}^3\exp\left(\frac{3}{2}\left(\frac{\alpha}{(1-\beta)}-\underline{m}\right)\right)E\left[\left|Z_0^3\right|\exp\left(\frac{3}{2}\frac{1}{1-\beta}\overline{\delta}\left|Z_0\right|\right)\right]=q^g<1.$

Hence:

$\sum_k E\left[\left(\beta^{3(k-1)}c^{*(k-1)}\prod_{i=1}^{k-1}|X_{t-i}|^3\right)|X_{t-k}|^3\right]=\frac{1}{1-q}\exp\left(-\frac{3}{2}\frac{\alpha\beta}{(1-\beta)^2}\right)E\exp\left(\frac{3}{2}\frac{1}{1-\beta}\log\sigma_0^2\right).$

Last:

$E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^3\left|Z_{t-k}\right|\sigma_{t-k}\left|\log h_{t-k}\right|^2\right]=$

$$=E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^3\left|Z_{t-k}\right|\sigma_{t-k}\left(\begin{array}{c}\left(\frac{\alpha}{1-\beta}\right)^2+2\left|\frac{\alpha}{1-\beta}\right|\sum_{k=0}^{\infty}\beta^k\overline{\delta}\left|X_{t-1-k}\right|\exp\left(-2^{-1}\underline{m}\right)\\+\sum_{k=0}^{\infty}\beta^{2k}\overline{\delta}^2\left|X_{t-1-k}\right|^2\exp\left(-\underline{m}\right)\end{array}\right)\right]=$$

$=\left(\frac{\alpha}{1-\beta}\right)^2E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^3\left|Z_{t-k}\right|\sigma_{t-k}\right]$

$+2\left|\frac{\alpha}{1-\beta}\right|\overline{\delta}\exp\left(-2^{-1}\underline{m}\right)E\left[\sum_{n=0}^{\infty}\left(\beta^k\left|X_{t-n-1}\right|\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^3\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]$

$+\overline{\delta}^2\exp\left(-\underline{m}\right)E\left[\sum_{n=0}^{\infty}\left(\beta^{2n}\left(\left|X_{t-n-1}\right|^2\right)\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^3\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]$, for which we have:

(1) $\left(\frac{\alpha}{1-\beta}\right)^2 E\left[\left(\beta^{k-1}\prod_{i=1}^{k-1}\left[1+c\left|X_{t-i}\right|\right]\right)^3 \left|Z_{t-k}\right|\sigma_{t-k}\right]$: see the conditions at the beginning of the proof.

(2a) $2\left|\frac{\alpha}{1-\beta}\right|\overline{\delta}\exp\left(-2^{-1}\underline{m}\right)E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\beta^{3(k-1)}c^*\sum_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]=$

$$=\frac{1}{4}\left|\frac{\alpha}{1-\beta}\right|\overline{\delta}^4\exp\left(-2\underline{m}\right)\beta^{3(k-2)}E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\sum_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]=$$

$$=\frac{1}{4}\left|\frac{\alpha}{1-\beta}\right|\overline{\delta}^4\exp\left(-2\underline{m}\right)\beta^{3(k-2)}E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\sum_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\right)\left|X_{t-k}\right|\\+\sum_{n=k-1}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\sum_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\right)\left|X_{t-k}\right|\end{array}\right]=$$

$$=\frac{1}{4}\left|\frac{\alpha}{1-\beta}\right|\overline{\delta}^4\exp\left(-2\underline{m}\right)\beta^{3(k-2)}E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\left|X_{t-1}\right|^3+\left|X_{t-2}\right|^3+...+\left|X_{t-k+1}\right|^3\right)\right)\left|X_{t-k}\right|\\+\sum_{n=k-1}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\left|X_{t-1}\right|^3+\left|X_{t-2}\right|^3+...+\left|X_{t-k+1}\right|^3\right)\right)\left|X_{t-k}\right.\end{array}\right.$$

$$\leq\frac{1}{4}\left|\frac{\alpha}{1-\beta}\right|\overline{\delta}^4\exp\left(-2\underline{m}\right)\beta^{3(k-2)}E\left[\begin{array}{c}(k-1)\sum_{n=0}^{k-2}\beta^n\left|X_{t-k+1}\right|\left|X_{t-k+1}\right|^3\left|X_{t-k}\right|\\+(k-1)\beta^{k-1}\sum_{n=0}^{\infty}\beta^n\left|X_{t-k}\right|\left|X_{t-k+1}\right|^3\left|X_{t-k}\right|\end{array}\right]$$

$$\leq\frac{1}{4}\left|\frac{\alpha}{1-\beta}\right|\overline{\delta}^4\exp\left(-2\underline{m}\right)\beta^{3(k-2)}(k-1)\left[\frac{1-\beta^{k-1}}{1-\beta}E\left(\left|X_{t-1}\right|^4\left|X_{t-2}\right|\right)+\frac{\beta^{k-1}}{1-\beta}E\left(\left|X_{t-1}\right|^3\left|X_{t-2}\right|^2\right)\right]$$

$$\leq\frac{1}{4}\left|\frac{\alpha}{1-\beta}\right|\overline{\delta}^4\exp\left(-2\underline{m}\right)\beta^{3(k-2)}(k-1)\left[\begin{array}{c}\frac{1-\beta^{k-1}}{1-\beta}EZ_0^4\exp\left(2\alpha\right)E\exp\left(\left(2\beta+\frac{1}{2}\right)\log\sigma_{t-2}^2\right)\times\\E\left[\left|Z_0\right|\exp\left(2\overline{\delta}\left|Z_0\right|\right)\right]\\+\frac{\beta^{k-1}}{1-\beta}E\left|Z_0\right|^3\exp\left(\frac{3}{2}\alpha\right)E\exp\left(\left(\frac{3}{2}\beta+1\right)\log\sigma_{t-2}^2\right)\times\\E\left[Z_0^2\exp\left(\frac{3}{2}\overline{\delta}\left|Z_0\right|\right)\right]\end{array}\right],$$

where $E\left[\left|Z_0\right|\exp\left(2\overline{\delta}\left|Z_0\right|\right)\right]<\infty$ and $E\left[Z_0^2\exp\left(\frac{3}{2}\overline{\delta}\left|Z_0\right|\right)\right]<\infty$.

(2b) $2\left|\frac{\alpha}{1-\beta}\right|\overline{\delta}\exp\left(-2^{-1}\underline{m}\right)E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\beta^{3(k-1)}c^{*(k-1)}\prod_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]=$

$$=\left|\frac{\alpha}{1-\beta}\right|2^{4-3k}\overline{\delta}^{3k-2}\exp\left(-\underline{m}\left(1-\frac{3}{2}k\right)\right)E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\prod_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\right)\left|Z_{t-k}\right|\sigma_{t-k}\right]=$$

$$=\left|\frac{\alpha}{1-\beta}\right|2^{4-3k}\overline{\delta}^{3k-2}\exp\left(-\underline{m}\left(1-\frac{3}{2}k\right)\right)E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\prod_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\right)\left|X_{t-k}\right|\\+\sum_{n=k-1}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\prod_{i=1}^{k-1}\left|X_{t-i}\right|^3\right)\right)\left|X_{t-k}\right|\end{array}\right]=$$

$$=\left|\frac{\alpha}{1-\beta}\right|2^{4-3k}\overline{\delta}^{3k-2}\exp\left(-\underline{m}\left(1-\frac{3}{2}k\right)\right)E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\left|X_{t-1}\right|^3\left|X_{t-2}\right|^3...\left|X_{t-k+1}\right|^3\right)\right)\left|X_{t-k}\right|\\+\sum_{n=k-1}^{\infty}\left(\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\left|X_{t-1}\right|^3\left|X_{t-2}\right|^3...\left|X_{t-k+1}\right|^3\right)\right)\left|X_{t-}\right.\end{array}\right.$$

$$\leq \left|\frac{\alpha}{1-\beta}\right| 2^{4-3k}\overline{\delta}^{3k-2}\exp\left(-\underline{m}\left(1-\tfrac{3}{2}k\right)\right) E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^{n}\left(\left|X_{t-n-1}\right|\right)\left(\left|X_{t-1}\right|^{3}\left|X_{t-2}\right|^{3}...\left|X_{t-k+1}\right|^{3}\right)\right)\left|X_{t-k}\right| \\ +\beta^{k-1}\sum_{n=0}^{\infty}\beta^{n}\left|X_{t-1}\right|^{3}\left|X_{t-2}\right|^{3}...\left|X_{t-k+1}\right|^{3}\left|X_{t-k}\right|^{2}\end{array}\right]$$

$$\leq \left|\frac{\alpha}{1-\beta}\right| 2^{4-3k}\overline{\delta}^{3k-2}\exp\left(-\underline{m}\left(1-\tfrac{3}{2}k\right)\right)\left[\begin{array}{c} cov\left(\left|X_{t-1}\right|^{3}\left|X_{t-2}\right|^{3}\left|X_{t-3}\right|^{3}...\left|X_{t-k+1}\right|^{3}\left|X_{t-k}\right|,\left|X_{t-j}\right|\right) \\ +\frac{\beta^{k-1}}{1-\beta}E\prod_{i=1}^{k-1}\left|X_{t-i}\right|^{3}\left|X_{t-k}\right|^{2}\end{array}\right],$$

where $E\prod_{i=1}^{k-1}\left|X_{t-i}\right|^{3}\left|X_{t-k}\right|^{2}=\exp\left(\frac{3}{2}\frac{1}{1-\beta}\alpha\left(k-1-\frac{\beta-\beta^{k}}{1-\beta}\right)\right)\times E\exp\left(\left(\frac{3}{2}\frac{\beta-\beta^{k}}{1-\beta}+1\right)\log\sigma_{t-k}^{2}\right)\times$
$\times E\exp\left(\frac{3}{2}\frac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k}+\delta\left|Z_{t-k}\right|\right)\right)Z_{t-k}^{2}\times E\exp\left(\frac{3}{2}\frac{1-\beta^{k-2}}{1-\beta}\left(\gamma Z_{t-k+1}+\delta\left|Z_{t-k+1}\right|\right)\right)Z_{t-k+1}^{3}\times$
$...\times$
$\times E\exp\left(\frac{3}{2}\frac{1-\beta^{2}}{1-\beta}\left(\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|\right)\right)Z_{t-3}^{3}\times E\exp\left(\frac{3}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right)Z_{t-2}^{3}.$

We have also that:

$cov\left(\left|X_{t-1}\right|^{3}\left|X_{t-2}\right|^{3}\left|X_{t-3}\right|^{3}...\left|X_{t-k+1}\right|^{3}\left|X_{t-k}\right|,\left|X_{t-j}\right|\right),\qquad j=1,...,k-1$
$cov\left(\left|X_{t-1}\right|^{3}\left|X_{t-2}\right|^{3}\left|X_{t-3}\right|^{3}...\left|X_{t-k+1}\right|^{3}\left|X_{t-k}\right|,\left|X_{t-j}\right|\right)\leq E\left(Z_{0}^{4}\right)\times$
$\times\exp\left[\frac{1}{(1-\beta)}\alpha\left[2\left(1-\beta^{k-1}\right)+\frac{3}{2}\left(k-2-\beta\frac{1-\beta^{k-2}}{1-\beta}\right)\right]\right]\times$
$\times\left[E\left(\left|Z_{0}\right|^{3}\exp\left\{\frac{3}{2}\overline{\delta}\left|Z_{0}\right|\right\}\right)\right]^{k-2}E\left[\left|Z_{0}\right|\exp\left(\left(\frac{3}{2}\frac{1-\beta^{k-1}}{1-\beta}+\frac{1}{2}\right)\overline{\delta}\left|Z_{0}\right|\right)\right]\times$
$\times E\exp\left[\left(\frac{3}{2}\frac{1-\beta^{k}}{1-\beta}+\frac{1}{2}\beta-1\right)\log\sigma_{t-k}^{2}\right],$
as $\quad\left|X_{t-1}\right|^{4}\left|X_{t-2}\right|^{3}...\left|X_{t-k+1}\right|^{3}\left|X_{t-k}\right|=$
$=Z_{t-1}^{4}\exp\left\{2\alpha\right\}\exp\left\{\left(\frac{3}{2}+2\beta\right)\alpha\right\}\exp\left\{\left(\frac{3}{2}\left(1+\beta\right)+2\beta^{2}\right)\alpha\right\}\times...$
$\times\exp\left\{\left(\frac{3}{2}\left(1+\beta+...+\beta^{k-3}\right)+2\beta^{k-2}\right)\alpha\right\}\times$
$\times\exp\left[2\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right]Z_{t-2}^{3}\exp\left[\left(\frac{3}{2}+2\beta\right)\left(\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|\right)\right]Z_{t-3}^{3}\times...$
$\times\exp\left[\left(\frac{3}{2}\left(1+\beta+...+\beta^{k-4}\right)+2\beta^{k-3}\right)\left(\gamma Z_{t-k+1}+\delta\left|Z_{t-k+1}\right|\right)\right]Z_{t-k+1}^{3}\times$
$\times\left|\exp\left[\left(\frac{3}{2}\left(1+\beta+...+\beta^{k-3}\right)+2\beta^{k-2}\right)\left(\gamma Z_{t-k}+\delta\left|Z_{t-k}\right|\right)\right]Z_{t-k}\right|\times$
$\times\exp\left[\left(\frac{1}{2}+\frac{3}{2}\left(\beta+...+\beta^{k-2}\right)+2\beta^{k-1}\right)\log\sigma_{t-k}^{2}\right]$, hence:

$$\begin{aligned} E\left|X_{t-1}\right|^{4}\left|X_{t-2}\right|^{3}...\left|X_{t-k+1}\right|^{3}\left|X_{t-k}\right| \leq\ & E\left(Z_{0}^{4}\right)\times \\ & \times\exp\left[\frac{1}{(1-\beta)}\alpha\left[\begin{array}{c}2\left(1-\beta^{k-1}\right)\\ +\frac{3}{2}\left(k-2-\beta\frac{1-\beta^{k-2}}{1-\beta}\right)\end{array}\right]\right]\times \\ & \times\left[E\left|Z_{0}\right|^{3}\exp\left(\left(\frac{3}{2}+2\beta\right)\overline{\delta}\left|Z_{0}\right|\right)\right]^{k-2}\times \\ & \times E\left[\left|Z_{0}\right|\exp\left(\left(\frac{3}{2}\frac{1-\beta^{k-1}}{1-\beta}+\frac{1}{2}\beta^{k-2}\right)\overline{\delta}\left|Z_{0}\right|\right)\right]\times \\ & \times E\exp\left[\left(\frac{3}{2}\frac{1-\beta^{k}}{1-\beta}+\frac{1}{2}\beta^{k-1}-1\right)\log\sigma_{t-k}^{2}\right]. \end{aligned}$$

Second $\quad |X_{t-1}|^3 |X_{t-2}|^4 ... |X_{t-k+1}|^3 |X_{t-k}| =$

$= |Z_{t-1}|^3 \exp\left\{\frac{3}{2}\alpha\right\} \exp\left\{\left(\frac{3}{2}\beta+2\right)\alpha\right\} \exp\left\{\left(\frac{3}{2}+2\beta+\frac{3}{2}\beta^2\right)\alpha\right\} \times ...$

$\times \exp\left\{\left(\frac{3}{2}\left(1+\beta+...+\beta^{k-4}\right)+2\beta^{k-3}+\frac{3}{2}\beta^{k-2}\right)\alpha\right\} \times$

$\times \exp\left(\gamma Z_{t-2}+\delta|Z_{t-2}|\right) Z_{t-2}^4 \exp\left[\left(\frac{3}{2}\beta+2\right)\left(\gamma Z_{t-3}+\delta|Z_{t-3}|\right)\right] Z_{t-3}^3 \times ...$

$\times \exp\left[\left(\frac{3}{2}\left(1+\beta+...+\beta^{k-5}\right)+2\beta^{k-4}+\frac{3}{2}\beta^{k-3}\right)\left(\gamma Z_{t-k+1}+\delta|Z_{t-k+1}|\right)\right] Z_{t-k+1}^3 \times$

$\times \left|\exp\left[\left(\frac{3}{2}\left(1+\beta+...+\beta^{k-4}\right)+2\beta^{k-3}+\frac{3}{2}\beta^{k-2}\right)\left(\gamma Z_{t-k}+\delta|Z_{t-k}|\right)\right] Z_{t-k}\right| \times$

$\times \exp\left[\left(\frac{1}{2}+\frac{3}{2}\left(\beta+...+\beta^{k-3}\right)+2\beta^{k-2}+\frac{3}{2}\beta^{k-1}\right)\log\sigma_{t-k}^2\right]$, hence:

$$
\begin{aligned}
E|X_{t-1}|^3 |X_{t-2}|^4 ... |X_{t-k+1}|^3 |X_{t-k}| \quad \leq \quad & E\left(Z_0^3\right) \times \\
& \times \exp\left[\frac{1}{(1-\beta)}\alpha\left[\begin{array}{c} 2\left(1-\beta^{k-2}\right) \\ +\frac{3}{2}\left(k-2-\beta^{k-1}-\beta\frac{1-\beta^{k-3}}{1-\beta}\right)\end{array}\right]\right] \\
& \times\left[E|Z_0|^3 \exp\left(\left(\frac{3}{2}\beta+2\right)\overline{\delta}|Z_0|\right)\right]^{k-2} \times \\
& \times E\left[|Z_0| \exp\left(\left(\frac{3}{2}\frac{1-\beta^{k-1}}{1-\beta}+\frac{1}{2}\beta^{k-3}\right)\overline{\delta}|Z_0|\right)\right] \times \\
& \times E \exp\left[\left(\frac{3}{2}\frac{1-\beta^{k}}{1-\beta}+\frac{1}{2}\beta^{k-2}-1\right)\log\sigma_{t-k}^2\right].
\end{aligned}
$$

And, last $\quad |X_{t-1}|^3 |X_{t-2}|^3 |X_{t-3}|^3 ... |X_{t-k+1}|^4 |X_{t-k}| =$

$= |Z_{t-1}|^3 \exp\left\{\frac{3}{2}\alpha\right\} \exp\left\{\frac{3}{2}\left(\beta+1\right)\alpha\right\} \exp\left\{\frac{3}{2}\left(\beta^2+\beta+1\right)\alpha\right\} \times ...$

$\times \exp\left\{\left(\frac{3}{2}\left(\beta^{k-2}+\beta^{k-3}+...+\beta\right)+2\right)\alpha\right\}$

$\times \left|\exp\left\{\frac{3}{2}\left(\gamma Z_{t-2}+\delta|Z_{t-2}|\right)\right\} Z_{t-2}^3\right| \left|\exp\left\{\frac{3}{2}\left(\beta+1\right)\left(\gamma Z_{t-3}+\delta|Z_{t-3}|\right)\right\} Z_{t-3}^3\right| \times ...$

$\times \exp\left\{\frac{3}{2}\left(\beta^{k-3}+\beta^{k-4}+...+1\right)\left(\gamma Z_{t-k+1}+\delta|Z_{t-k+1}|\right)\right\} Z_{t-k+1}^4$

$\times \left|\exp\left[\left(\frac{3}{2}\left(\beta^{k-2}+\beta^{k-3}+...+\beta\right)+2\right)\left(\gamma Z_{t-k}+\delta|Z_{t-k}|\right)\right] Z_{t-k}\right| \times$

$\times \exp\left\{\left(\frac{3}{2}\left(\beta^{k-1}+\beta^{k-2}+...+\beta^2\right)+2\beta+\frac{1}{2}\right)\log\sigma_{t-k}^2\right\}$, hence:

$$
\begin{aligned}
E|X_{t-1}|^3 |X_{t-2}|^3 ... |X_{t-k+1}|^4 |X_{t-k}| \quad \leq \quad & E\left(Z_0^3\right) \times \\
& \times \exp\left[\frac{1}{(1-\beta)}\alpha\left[2\left(1-\beta\right)+\frac{3}{2}\left(k-2+\beta-\beta\frac{1-\beta^{k-1}}{1-\beta}\right)\right]\right] \\
& \times\left[E|Z_0|^3 \exp\left\{\frac{3}{2}\overline{\delta}|Z_0|\right\}\right]^{k-2} \times \\
& \times E\left[|Z_0| \exp\left(\left(\frac{3}{2}\frac{1-\beta^{k-1}}{1-\beta}+\frac{1}{2}\right)\overline{\delta}|Z_0|\right)\right] \times \\
& \times E \exp\left[\left(\frac{3}{2}\frac{1-\beta^{k}}{1-\beta}+\frac{1}{2}\beta-1\right)\log\sigma_{t-k}^2\right].
\end{aligned}
$$

$$(3a)\ \overline{\delta}^2 \exp(-\underline{m}) E\left[\sum_{n=0}^{\infty}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(\beta^{3(k-1)}c^*\sum_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|Z_{t-k}|\,\sigma_{t-k}\right]=$$

$$=\tfrac{1}{8}\overline{\delta}^5\exp\left(-\tfrac{5}{2}\underline{m}\right)\beta^{3(k-2)}E\left[\sum_{n=0}^{\infty}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|Z_{t-k}|\,\sigma_{t-k}\right]=$$

$$=\tfrac{1}{8}\overline{\delta}^5\exp\left(-\tfrac{5}{2}\underline{m}\right)\beta^{3(k-2)}E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|X_{t-k}|\\+\sum_{n=k-1}^{\infty}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|X_{t-k}|\end{array}\right]=$$

$$=\tfrac{1}{8}\overline{\delta}^5\exp\left(-\tfrac{5}{2}\underline{m}\right)\beta^{3(k-2)}E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(|X_{t-1}|^3+|X_{t-2}|^3+...+|X_{t-k+1}|^3\right)\right)|X_{t-k}|\\+\sum_{n=k-1}^{\infty}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(|X_{t-1}|^3+|X_{t-2}|^3+...+|X_{t-k+1}|^3\right)\right)|X_{t-k}|\end{array}\right]$$

$$\leq\tfrac{1}{8}\overline{\delta}^5\exp\left(-\tfrac{5}{2}\underline{m}\right)\beta^{3(k-2)}E\left[\begin{array}{c}(k-1)\sum_{n=0}^{k-2}\beta^{2n}|X_{t-k+1}|^2|X_{t-k+1}|^3|X_{t-k}|\\+(k-1)\beta^{k-1}\sum_{n=0}^{\infty}\beta^{2n}|X_{t-k}|^2|X_{t-k+1}|^3|X_{t-k}|\end{array}\right]$$

$$\leq\tfrac{1}{8}\overline{\delta}^5\exp\left(-\tfrac{5}{2}\underline{m}\right)\beta^{3(k-2)}(k-1)\left[\sum_{n=0}^{k-2}\beta^{2n}E\left(|X_{t-1}|^5|X_{t-2}|\right)+\beta^{k-1}\sum_{n=0}^{\infty}\beta^{2n}E\left(|X_{t-1}|^3|X_{t-2}|^3\right)\right]$$

$$\leq\tfrac{1}{8}\overline{\delta}^5\exp\left(-\tfrac{5}{2}\underline{m}\right)\beta^{3(k-2)}(k-1)\left[\begin{array}{c}\sum_{n=0}^{k-2}\beta^{2n}E|Z_0|^5\exp\left(\tfrac{5}{2}\alpha\right)E\exp\left(\left(\tfrac{5}{2}\beta+\tfrac{1}{2}\right)\log\sigma_{t-2}^2\right)\times\\E\left[|Z_0|\exp\left(\tfrac{5}{2}\overline{\delta}\,|Z_0|\right)\right]\\+\beta^{k-1}\sum_{n=0}^{\infty}\beta^{2n}E|Z_0|^3\exp\left(\tfrac{3}{2}\alpha\right)E\exp\left(\left(\tfrac{3}{2}(\beta+1)\right)\log\sigma_{t-2}^2\right)\times\\E\left[Z_0^3\exp\left(\tfrac{3}{2}\overline{\delta}\,|Z_0|\right)\right]\end{array}\right],$$

where $E\left[|Z_0|\exp\left(\frac{5}{2}\overline{\delta}\,|Z_0|\right)\right]<\infty$ and $E\left[Z_0^3\exp\left(\frac{3}{2}\overline{\delta}\,|Z_0|\right)\right]<\infty$.

$$(3b)\ \overline{\delta}^2\exp(-\underline{m})E\left[\sum_{n=0}^{\infty}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(\beta^{3(k-1)}c^{*(k-1)}\prod_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|Z_{t-k}|\,\sigma_{t-k}\right]=$$

$$=8^{-(k-1)}\overline{\delta}^{3k-1}\exp\left(\underline{m}\left(\tfrac{1}{2}-\tfrac{3}{2}k\right)\right)E\left[\sum_{n=0}^{\infty}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|Z_{t-k}|\,\sigma_{t-k}\right]=$$

$$=8^{-(k-1)}\overline{\delta}^{3k-1}\exp\left(\underline{m}\left(\tfrac{1}{2}-\tfrac{3}{2}k\right)\right)E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|X_{t-k}|\\+\sum_{n=k-1}^{\infty}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|X_{t-k}|\end{array}\right]=$$

$$=8^{-(k-1)}\overline{\delta}^{3k-1}\exp\left(\underline{m}\left(\tfrac{1}{2}-\tfrac{3}{2}k\right)\right)E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(|X_{t-1}|^3|X_{t-2}|^3...|X_{t-k+1}|^3\right)\right)|X_{t-k}|\\+\sum_{n=k-1}^{\infty}\left(\beta^{2n}\left(|X_{t-n-1}|^2\right)\left(|X_{t-1}|^3|X_{t-2}|^3...|X_{t-k+1}|^3\right)\right)|X_{t-k}|\end{array}\right]$$

$$\leq 8^{-(k-1)}\overline{\delta}^{3k-1}\exp\left(\underline{m}\left(\tfrac{1}{2}-\tfrac{3}{2}k\right)\right)E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^{2n}\left(\left|X_{t-n-1}\right|^{2}\right)\left(\left|X_{t-1}\right|^{3}\left|X_{t-2}\right|^{3}...\left|X_{t-k+1}\right|^{3}\right)\right)\left|X_{t-k}\right| \\ +\beta^{k-1}\sum_{n=0}^{\infty}\beta^{2n}\left|X_{t-1}\right|^{3}\left|X_{t-2}\right|^{3}...\left|X_{t-k+1}\right|^{3}\left|X_{t-k}\right|^{3}\end{array}\right]$$

$$\leq 8^{-(k-1)}\overline{\delta}^{3k-1}\exp\left(\underline{m}\left(\tfrac{1}{2}-\tfrac{3}{2}k\right)\right)\left[\begin{array}{c} cov\left(\left|X_{t-1}\right|^{3}\left|X_{t-2}\right|^{3}\left|X_{t-3}\right|^{3}...\left|X_{t-k+1}\right|^{3}\left|X_{t-k}\right|,\left|X_{t-j}\right|^{2}\right) \\ +\beta^{k-1}E\sum_{n=0}^{\infty}\beta^{2n}\prod_{i=1}^{k}\left|X_{t-i}\right|^{3}\end{array}\right],$$

where $E\prod_{i=1}^{k}\left|X_{t-i}\right|^{3}=\exp\left(\frac{3}{2}\frac{1}{1-\beta}\alpha\left(k-1-\frac{\beta-\beta^{k}}{1-\beta}\right)\right)\times\exp\left(\frac{3}{2}\frac{1-\beta^{k}}{1-\beta}\log\sigma_{t-k}^{2}\right)\times$

$\times\left|\exp\left(\frac{3}{2}\frac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k}+\delta\left|Z_{t-k}\right|\right)\right)Z_{t-k}^{3}\right|\times\left|\exp\left(\frac{3}{2}\frac{1-\beta^{k-2}}{1-\beta}\left(\gamma Z_{t-k+1}+\delta\left|Z_{t-k+1}\right|\right)\right)Z_{t-k+1}^{3}\right|\times$

$...\times$

$\times\left|\exp\left(\frac{3}{2}\frac{1-\beta^{2}}{1-\beta}\left(\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|\right)\right)Z_{t-3}^{3}\right|\times\left|\exp\left(\frac{3}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right)Z_{t-2}^{3}\right|.$

We have also that:

$$\begin{aligned} cov\left(\left|X_{t-1}\right|^{4}\left|X_{t-2}\right|^{3}\left|X_{t-3}\right|^{3}...\left|X_{t-k+1}\right|^{3}\left|X_{t-k}\right|,\left|X_{t-j}\right|\right) \leq\ & E\left(Z_{0}^{5}\right)\times \\ & \times\exp\left[\frac{1}{(1-\beta)}\alpha\left[\begin{array}{c}\frac{5}{2}\left(1-\beta^{k-1}\right) \\ +\frac{3}{2}\left(k-2-\beta\frac{1-\beta^{k-2}}{1-\beta}\right)\end{array}\right.\right. \\ & \times\left[E\left|Z_{0}^{3}\right|\exp\left\{\frac{3}{2}\overline{\delta}\left|Z_{0}\right|\right\}\right]^{k-2}\times \\ & \times E\left[\left|Z_{0}\right|\exp\left(\left(\frac{3}{2}\frac{1-\beta^{k-1}}{1-\beta}+1\right)\overline{\delta}\left|Z_{0}\right|\right)\right. \\ & \times E\exp\left[\left(\frac{3}{2}\frac{1-\beta^{k}}{1-\beta}+\beta-1\right)\log\sigma_{t-k}^{2}\right]. \end{aligned}$$

□

C.3 Proof of the Main Theorem 4.10

Proof of the Main Theorem. The relations $E\left\|(\log h_0)'\right\|<\infty$ (see Lemma 4.18), $E\left\|(\log h_0)''\right\|<\infty$ (see Lemma 4.23) and eq. (4.15) show that $E\left\|l_0'\right\|<\infty$, $E\left\|l_0''\right\|<\infty$ and $E\left|(\log h_0)'(\boldsymbol{\theta}_0)\right|^2<\infty$ (see also the conditions in Lemma 4.13).

For the asymptotic covariance matrix $\mathbf{V}_0$, we have first the SRE for $(\log h_t)'$ evaluated at the true parameter value:

$$\begin{aligned}(\log h_1)'(\boldsymbol{\theta}_0) &= \frac{\partial g}{\partial \boldsymbol{\theta}}\left(X_0, \log \sigma_0^2\right)|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} + \frac{\partial g}{\partial s}\left(X_0, \log \sigma_0^2\right)|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} (\log h_0)'(\boldsymbol{\theta}_0) \\ &= (1, \log h_0, Z_0, |Z_0|) + \beta_0 - \frac{1}{2}(\gamma_0 Z_0 + \delta_0 |Z_0|)(\log h_0)'(\boldsymbol{\theta}_0).\end{aligned}$$

Taking the expectation on both sides, gives:

$$E\left[(\log h_0)'(\boldsymbol{\theta}_0)\right] = \left[\left(1, \frac{\alpha_0}{1-\beta_0}, 0, E|Z_0|\right) + \beta_0\right] \times \left(1 + \frac{1}{2}\delta_0 E|Z_0|\right)^{-1}.$$

Likewise, squaring the first equation of $(\log h_1)'(\boldsymbol{\theta}_0)$ and taking expectations on both sides yields the value of $E\left[\left((\log h_0)'(\boldsymbol{\theta}_0)\right)^T (\log h_0)'(\boldsymbol{\theta}_0)\right]$, that is:

$E\left[(1, \log h_0, Z_0, |Z_0|)^T (1, \log h_0, Z_0, |Z_0|)\right] + \beta_0^2 + \frac{1}{4}E(\gamma_0 Z_0 + \delta_0 |Z_0|)^2 E\left[(\log h_0)'(\boldsymbol{\theta}_0)\right]^2$
$+2\beta_0 E(1, \log h_0, Z_0, |Z_0|) - E\left[\left((1, \log h_0, Z_0, |Z_0|)(\gamma_0 Z_0 + \delta_0 |Z_0|)\right)^T\right] E\left[(\log h_0)'(\boldsymbol{\theta}_0)\right] =$
$= E\left[(1, \log h_0, Z_0, |Z_0|)^T (1, \log h_0, Z_0, |Z_0|)\right] + \beta_0^2 + \frac{1}{4}E\left(\gamma_0^2 + \delta_0^2 + 2\gamma_0\delta_0 E(Z_0|Z_0|)\right) E\left[(\log h_0)'(\boldsymbol{\theta}_0)\right]^2$
$+2\beta_0\left(1, \frac{\alpha_0}{1-\beta_0}, 0, E|Z_0|\right)$
$-\left(\delta_0 E|Z_0|, \delta_0\frac{\alpha_0}{1-\beta_0}E|Z_0|, \gamma_0 + \delta_0 E(Z_0|Z_0|), \gamma_0 E(Z_0|Z_0|) + \delta_0\right)^T \left[\left(1, \frac{\alpha_0}{1-\beta_0}, 0, E|Z_0|\right) + \beta_0\right] \times$
$\left(1 + \frac{1}{2}\delta_0 E|Z_0|\right)^{-1}$.

Hence, the covariance matrix is equal to
$\mathbf{V}_0 = 4^{-1}E\left(Z_0^4 - 1\right) E\left[\left((\log h_0)'(\boldsymbol{\theta}_0)\right)^T (\log h_0)'(\boldsymbol{\theta}_0)\right]^{-1} =$
$= 4^{-1}E\left(Z_0^4 - 1\right)\left(1 - \beta_0^2 - \frac{1}{4}E\left(\gamma_0^2 + \delta_0^2 + 2\gamma_0\delta_0 E(Z_0|Z_0|)\right) - 2\beta_0\left(1, \frac{\alpha_0}{1-\beta_0}, 0, E|Z_0|\right)\right) \times$
$\{\mathbf{U}_0 - \mathbf{W}_0\}^{-1}$,
where $\mathbf{U}_0 = E\left[(1, \log h_0, Z_0, |Z_0|)^T (1, \log h_0, Z_0, |Z_0|)\right]$,
$\mathbf{W}_0 = \left(\delta_0 E|Z_0|, \delta_0\frac{\alpha_0}{1-\beta_0}E|Z_0|, \gamma_0 + \delta_0 E(Z_0|Z_0|), \gamma_0 E(Z_0|Z_0|) + \delta_0\right)^T \left[\left(1, \frac{\alpha_0}{1-\beta_0}, 0, E|Z_0|\right) + \beta_0\right] \times$
$\left(1 + \frac{1}{2}\delta_0 E|Z_0|\right)^{-1}$. □

C.4 Dependence Results and Useful Expressions for the Analysis of the First Order Derivative

The following lemmas are useful for the first order derivative of the log-variance function. They comprise from tractable expressions for products of the observed sequence, and also from dependence results that are used for the proof of the main Theorem. To begin with, let the following:

Lemma C.1. *Find a tractable expression of* $\prod_{i=1}^{k} |X_{t-i}|$ *and evaluate its expectation.*

Proof. $\prod_{i=1}^{k} |X_{t-i}| = |Z_{t-1}\sigma_{t-1}Z_{t-2}\sigma_{t-2}Z_{t-3}\sigma_{t-3}Z_{t-4}\sigma_{t-4}...Z_{t-k}\sigma_{t-k}| =$

$$= |Z_{t-1}| \left|\exp\left(\tfrac{1}{2}\log\sigma_{t-1}^2\right) Z_{t-2}\exp\left(\tfrac{1}{2}\log\sigma_{t-2}^2\right) Z_{t-3}\exp\left(\tfrac{1}{2}\log\sigma_{t-3}^2\right)...Z_{t-k}\exp\left(\tfrac{1}{2}\log\sigma_{t-k}^2\right)\right| =$$

$$= |Z_{t-1}| \left|\begin{array}{c} \exp\left(\tfrac{1}{2}\left(\alpha+\gamma Z_{t-2}+\delta|Z_{t-2}|+\beta\log\sigma_{t-2}^2\right)\right) Z_{t-2}\exp\left(\tfrac{1}{2}\log\sigma_{t-2}^2\right) \\ Z_{t-3}\exp\left(\tfrac{1}{2}\log\sigma_{t-3}^2\right)...Z_{t-k}\exp\left(\tfrac{1}{2}\log\sigma_{t-k}^2\right) \end{array}\right| =$$

$$= |Z_{t-1}| \left|\begin{array}{c} \exp\left(\tfrac{1}{2}\alpha\right)\exp\left(\tfrac{1}{2}\left(\gamma Z_{t-2}+\delta|Z_{t-2}|\right)\right) Z_{t-2}\exp\left(\tfrac{1}{2}(\beta+1)\log\sigma_{t-2}^2\right) \\ Z_{t-3}\exp\left(\tfrac{1}{2}\log\sigma_{t-3}^2\right)...Z_{t-k}\exp\left(\tfrac{1}{2}\log\sigma_{t-k}^2\right) \end{array}\right| =$$

$$= ... =$$

$$= \exp\left(\tfrac{1}{2}\alpha\right)\exp\left(\tfrac{1}{2}(\beta+1)\alpha\right)\exp\left(\tfrac{1}{2}(\beta(\beta+1)+1)\alpha\right)...\exp\left(\tfrac{1}{2}\left(\beta^{k-2}+...+1\right)\alpha\right)\times$$

$$\times\exp\left(\tfrac{1}{2}\left(\beta^{k-1}+...+1\right)\log\sigma_{t-k}^2\right)\times$$

$$\times|Z_{t-k}|\exp\left(\tfrac{1}{2}\left(\beta^{k-2}+...+1\right)\left(\gamma Z_{t-k}+\delta|Z_{t-k}|\right)\right)\times...\times|Z_{t-2}|\exp\left(\tfrac{1}{2}\left(\gamma Z_{t-2}+\delta|Z_{t-2}|\right)\right) =$$

$$= \exp\left(\frac{1}{2(1-\beta)}\alpha\left(k-1-\beta\frac{1-\beta^{k-1}}{1-\beta}\right)\right)\times\exp\left(\frac{1}{2}\frac{1-\beta^k}{1-\beta}\log\sigma_{t-k}^2\right)\times$$

$$\times|Z_{t-k}|\exp\left(\frac{1}{2}\frac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k}+\delta|Z_{t-k}|\right)\right)\times|Z_{t-k+1}|\exp\left(\frac{1}{2}\frac{1-\beta^{k-2}}{1-\beta}\left(\gamma Z_{t-k+1}+\delta|Z_{t-k+1}|\right)\right)\times$$

$$...\times$$

$$\times|Z_{t-3}|\exp\left(\frac{1}{2}\frac{1-\beta^2}{1-\beta}\left(\gamma Z_{t-3}+\delta|Z_{t-3}|\right)\right)\times|Z_{t-2}|\exp\left(\tfrac{1}{2}\left(\gamma Z_{t-2}+\delta|Z_{t-2}|\right)\right).$$

Hence,

$$E\prod_{i=1}^{k}|X_{t-i}| = E\exp\left(\frac{1}{2(1-\beta)}\alpha\left(k-1-\beta\frac{1-\beta^{k-1}}{1-\beta}\right)\right)\times E\exp\left(\frac{1}{2}\frac{1-\beta^k}{1-\beta}\log\sigma_{t-k}^2\right)\times$$

$$\times E\left[|Z_{t-k}|\exp\left(\frac{1}{2}\frac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k}+\delta|Z_{t-k}|\right)\right)\right]\times...\times E\left[|Z_{t-2}|\exp\left(\tfrac{1}{2}\left(\gamma Z_{t-2}+\delta|Z_{t-2}|\right)\right)\right].$$

□

Lemma C.2. *The expansion of* $\prod_{i=1}^{k-1}\left(1+c|X_{t-i}|\right)$.

Proof. The term $\prod_{i=1}^{k-1}\left(1+c|X_{t-i}|\right)$ is expanded in the following way:

$$\prod_{i=1}^{k-1}\left(1+c|X_{t-i}|\right) = 1+c\sum_{i=1}^{k-1}|X_{t-i}|+c^2|X_{t-1}|\sum_{i=2}^{k-1}|X_{t-i}|+c^2|X_{t-2}|\sum_{i=3}^{k-1}|X_{t-i}|+...+$$

$$c^2|X_{t-k+2}|\sum_{i=k-1}^{k-1}|X_{t-i}|$$

$$+c^3|X_{t-1}||X_{t-2}|\sum_{i=3}^{k-1}|X_{t-i}|+c^3|X_{t-1}||X_{t-3}|\sum_{i=4}^{k-1}|X_{t-i}|+...+c^3|X_{t-1}||X_{t-k+2}|\sum_{i=k-1}^{k-1}|X_{t-i}|$$

$$+c^3\left|X_{t-2}\right|\left|X_{t-3}\right|\sum_{i=4}^{k-1}\left|X_{t-i}\right|+...+c^3\left|X_{t-2}\right|\left|X_{t-k+2}\right|\sum_{i=n-1}^{k-1}\left|X_{t-i}\right|+...+c^3\left|X_{t-k+3}\right|\left|X_{t-k+2}\right|\sum_{i=k-1}^{k-1}\left|X_{t-i}\right|$$
$$+c^4\left|X_{t-1}\right|\left|X_{t-2}\right|\left|X_{t-3}\right|\sum_{i=4}^{k-1}\left|X_{t-i}\right|+c^4\left|X_{t-1}\right|\left|X_{t-2}\right|\left|X_{t-4}\right|\sum_{i=5}^{k-1}\left|X_{t-i}\right|+...+$$
$$+c^4\left|X_{t-1}\right|\left|X_{t-2}\right|\left|X_{t-k+2}\right|\sum_{i=k-1}^{k-1}\left|X_{t-i}\right|+c^4\left|X_{t-1}\right|\left|X_{t-3}\right|\left|X_{t-4}\right|\sum_{i=5}^{k-1}\left|X_{t-i}\right|+...+$$
$$+c^4\left|X_{t-1}\right|\left|X_{t-3}\right|\left|X_{t-k+2}\right|\sum_{i=k-1}^{k-1}\left|X_{t-i}\right|+...+c^4\left|X_{t-2}\right|\left|X_{t-4}\right|\left|X_{t-k+2}\right|\sum_{i=k-1}^{k-1}\left|X_{t-i}\right|+...+$$
$$+c^4\left|X_{t-k+4}\right|\left|X_{t-k+3}\right|\left|X_{t-k+2}\right|\sum_{i=k-1}^{k-1}\left|X_{t-i}\right|+...+c^{k-1}\prod_{i=1}^{k-1}\left|X_{t-i}\right|.$$

If $\sum_{i=1}^{k-1}\left|X_{t-i}\right|$ and $\prod_{i=1}^{k-1}\left|X_{t-i}\right|$ are both finite, then all other terms in the expansion converge, as well. This is verified by the fact that the condition for the boundedness of the term $\prod_{i=1}^{k-1}\left|X_{t-i}\right|$ is stronger than the condition needed for the middle terms in the expansion to be finite. □

Lemma C.3. *Find a tractable expression for the bound of* $\left|\log h_t\right|$ *and* $\left|\log h_t\right|^2$:

Proof. We have that

$$\begin{aligned}
\log h_t &= \alpha+\gamma Z_{t-1}+\delta\left|Z_{t-1}\right|+\beta\log h_{t-1}\\
&= \alpha+\left(\gamma X_{t-1}+\delta\left|X_{t-1}\right|\right)\exp\left(-2^{-1}\log h_{t-1}\right)+\beta\log h_{t-1}\\
&= ...=\frac{\alpha}{1-\beta}+\sum_{k=0}^{\infty}\beta^k\left(\gamma X_{t-1-k}+\delta\left|X_{t-1-k}\right|\right)\exp\left(-2^{-1}\log h_{t-1-k}\right)\\
&\Rightarrow\\
\left|\log h_t\right| &\leq \left|\frac{\alpha}{1-\beta}\right|+\sum_{k=0}^{\infty}\beta^k\overline{\delta}\left|X_{t-1-k}\right|\exp\left(-2^{-1}\underline{m}\right),\\
\left|\log h_t\right|^2 &\leq \left(\frac{\alpha}{1-\beta}\right)^2+2\left|\frac{\alpha}{1-\beta}\right|\sum_{k=0}^{\infty}\beta^k\overline{\delta}\left|X_{t-1-k}\right|\exp\left(-2^{-1}\underline{m}\right)\\
&\quad+\sum_{k=0}^{\infty}\beta^{2k}\overline{\delta}^2\left|X_{t-1-k}\right|^2\exp\left(-\underline{m}\right).
\end{aligned}$$

□

Lemma C.4. *Find a tractable expression for* $E\left[\sum_{n=0}^{\infty}\beta^n\left(\left|X_{t-n-1}\right|\right)\left(\sum_{i=1}^{k-1}\left|X_{t-i}\right|\right)\right]$:

Proof. Start with:

$$\sum_{n=0}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)=\sum_{n=0}^{k-2}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)+\sum_{n=k-1}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)=$$

$$=\sum_{n=0}^{k-2}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)+\sum_{n=k-1}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right).$$

Now

$$\sum_{n=0}^{k-2}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)=|X_{t-1}|\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)+\beta|X_{t-2}|\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)$$

$$+\beta^2|X_{t-3}|\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)+....+\beta^{k-2}|X_{t-k+1}|\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)=$$

$$=|X_{t-1}|\left(|X_{t-1}|+|X_{t-2}|+|X_{t-3}|+...+|X_{t-k+1}|\right)$$

$$+\beta|X_{t-2}|\left(|X_{t-1}|+|X_{t-2}|+|X_{t-3}|+...+|X_{t-k+1}|\right)$$

$$+\beta^2|X_{t-3}|\left(|X_{t-1}|+|X_{t-2}|+|X_{t-3}|+...+|X_{t-k+1}|\right)+....$$

$$+\beta^{k-2}|X_{t-k+1}|\left(|X_{t-1}|+|X_{t-2}|+|X_{t-3}|+...+|X_{t-k+1}|\right)=$$

$$=\sum_{i=0}^{k-2}\beta^i|X_{t-1-i}|^2+(1+\beta)|X_{t-1}||X_{t-2}|+\left(1+\beta^2\right)|X_{t-1}||X_{t-3}|+....+\left(1+\beta^{k-2}\right)|X_{t-1}||X_{t-k+1}|+$$

$$+\beta(1+\beta)|X_{t-2}||X_{t-3}|+\beta\left(1+\beta^2\right)|X_{t-2}||X_{t-4}|+...+\beta\left(1+\beta^{k-3}\right)|X_{t-2}||X_{t-k+1}|+$$

$$+\beta^2(1+\beta)|X_{t-3}||X_{t-4}|+\beta^2\left(1+\beta^2\right)|X_{t-3}||X_{t-5}|+...+\beta^2\left(1+\beta^{k-4}\right)|X_{t-3}||X_{t-k+1}|+$$

$$...+$$

$$+\beta^{k-3}(1+\beta)|X_{t-k+2}||X_{t-k+1}|=$$

$$=\sum_{i=0}^{k-2}\beta^i|X_{t-1-i}|^2+\sum_{i=1}^{k-2}\left(1+\beta^i\right)|X_{t-1}||X_{t-1-i}|+\sum_{i=1}^{k-3}\beta\left(1+\beta^i\right)|X_{t-2}||X_{t-2-i}|$$

$$+\sum_{i=1}^{k-4}\beta^2\left(1+\beta^i\right)|X_{t-3}||X_{t-3-i}|+...+\beta^{k-3}(1+\beta)|X_{t-k+2}||X_{t-k+1}|=$$

$$=\sum_{i=0}^{k-2}\beta^i|X_{t-1-i}|^2+\sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}|.$$

Hence $E\left[\sum_{n=0}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)\right]=$

$$=E\left[\sum_{n=0}^{k-2}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)\right]+E\left[\sum_{n=k-1}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)\right]=$$

$$=E\left[\sum_{i=0}^{k-2}\beta^i|X_{t-1-i}|^2+\sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}|\right]$$

$$+E\left[\sum_{n=k-1}^{\infty}\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)\right].$$

□

Lemma C.5. *Moment bounds of* $\sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}|$

Proof. $\sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}|=$

$$=(1+\beta)|X_{t-1}||X_{t-2}|+\left(1+\beta^2\right)|X_{t-1}||X_{t-3}|+...+\left(1+\beta^{k-2}\right)|X_{t-1}||X_{t-k+1}|$$

$$+\beta(1+\beta)|X_{t-2}||X_{t-3}|+\beta\left(1+\beta^2\right)|X_{t-2}||X_{t-4}|+...+\beta\left(1+\beta^{k-3}\right)|X_{t-2}||X_{t-k+1}|$$

$$+\beta^2(1+\beta)|X_{t-3}||X_{t-4}|+\beta^2\left(1+\beta^2\right)|X_{t-3}||X_{t-5}|+...+\beta^2\left(1+\beta^{k-4}\right)|X_{t-3}||X_{t-k+1}|+$$

$...+$

$+\beta^{k-4}(1+\beta)|X_{t-k+3}||X_{t-k+2}|+\beta^{k-4}\left(1+\beta^2\right)|X_{t-k+3}||X_{t-k+1}|+...+\beta^{k-3}(1+\beta)|X_{t-k+2}||X_{t-k+1}|.$

Hence:

$\sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}| =$

$= \frac{1-\beta^{k-2}}{1-\beta}(1+\beta)|X_{t-1}||X_{t-2}|+\frac{1-\beta^{k-3}}{1-\beta}\left(1+\beta^2\right)|X_{t-1}||X_{t-3}|+...+\left(1+\beta^{k-2}\right)|X_{t-1}||X_{t-k+1}|.$

Examining the higher dependence (that of the most recent past), we have that:

$\beta^{k-2}E\sum_{j=0}^{k-3}\sum_{i=1}^{k-2-j}\beta^j\left(1+\beta^i\right)|X_{t-1-j}||X_{t-1-j-i}| \le \beta^{k-2}(k-2)\frac{1-\beta^{k-2}}{1-\beta}(1+\beta)E|X_{t-1}||X_{t-2}|$

$\le \beta^{k-2}(k-2)\frac{1-\beta^{k-2}}{1-\beta}(1+\beta)E|Z_0|E\exp\left(\frac{1}{2}\alpha\right)E\exp\left(\frac{1}{2}(\beta+1)\log\sigma_{t-2}^2\right)\times$

$\times E\left[|Z_{t-2}|\exp\left(\frac{1}{2}(\gamma Z_{t-2}+\delta|Z_{t-2}|)\right)\right]$

$\le \beta^{k-2}(k-2)\frac{1-\beta^{k-2}}{1-\beta}(1+\beta)E|Z_0|E\exp\left(\frac{1}{2}\alpha\right)E\exp\left(\frac{1}{2}(\beta+1)\log\sigma_0^2\right)E\left[|Z_0|\exp\left(\frac{1}{2}\overline{\delta}|Z_0|\right)\right].$

□

Lemma C.6. *Moment bounds of* $\sum_{n=k-1}^{\infty}\beta^n(|X_{t-n-1}|)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)$

Proof. $\sum_{n=k-1}^{\infty}\beta^n(|X_{t-n-1}|)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right) =$

$= \left(\beta^{k-1}|X_{t-k}|+\beta^k|X_{t-k-1}|+\beta^{k+1}|X_{t-k-2}|+...\right)(|X_{t-1}|+|X_{t-2}|+...+|X_{t-k+1}|) =$

$= \beta^{k-1}|X_{t-k}|(|X_{t-1}|+|X_{t-2}|+...+|X_{t-k+1}|)+\beta^k|X_{t-k-1}|(|X_{t-1}|+|X_{t-2}|+...+|X_{t-k+1}|)$

$+\beta^{k+1}|X_{t-k-2}|(|X_{t-1}|+|X_{t-2}|+...+|X_{t-k+1}|)+...$

Hence,

$E\left[\beta^{k-2}\sum_{n=k-1}^{\infty}\beta^n(|X_{t-n-1}|)\left(\sum_{i=1}^{k-1}|X_{t-i}|\right)\right] = \beta^{k-2}(k-1)\sum_{n=k-1}^{\infty}\beta^nE|X_t||X_{t-n+k}|$

$\le \beta^{k-2}(k-1)\sum_{n=k-1}^{\infty}\beta^nE|X_{t-1}||X_t|$

$= \beta^{k-2}(k-1)E|X_{t-1}||X_t|\beta^{k-1}\sum_{n=0}^{\infty}\beta^n = \beta^{k-2}(k-1)E|X_{t-1}||X_t|\frac{\beta^{k-1}}{1-\beta}$

$= \beta^{k-2}(k-1)\frac{\beta^{k-1}}{1-\beta}E|Z_0|E\exp\left(\frac{1}{2}\alpha\right)E\exp\left(\frac{1}{2}(\beta+1)\log\sigma_{t-1}^2\right)E\left[|Z_{t-1}|\exp\left(\frac{1}{2}(\gamma Z_{t-1}+\delta|Z_{t-1}|)\right)\right]$

$= \beta^{k-2}(k-1)\frac{\beta^{k-1}}{1-\beta}E|Z_0|E\exp\left(\frac{1}{2}\alpha\right)E\exp\left(\frac{1}{2}(\beta+1)\log\sigma_0^2\right)E\left[|Z_0|\exp\left(\frac{1}{2}(\gamma Z_0+\delta|Z_0|)\right)\right]$

$\le \beta^{k-2}(k-1)\frac{\beta^{k-1}}{1-\beta}E|Z_0|E\exp\left(\frac{1}{2}\alpha\right)E\exp\left(\frac{1}{2}(\beta+1)\log\sigma_0^2\right)E\left[|Z_0|\exp\left(\frac{1}{2}\overline{\delta}|Z_0|\right)\right].$ □

Lemma C.7. *Find a tractable expression for* $\sum_{j=0}^{k-2}\beta^j|X_{t-j-1}|\left(\prod_{i=1}^{k-1}|X_{t-i}|\right)$ *and calculate its moment bounds.*

Proof. $\sum_{j=0}^{k-2}\beta^j\left|X_{t-j-1}\right|\left(\prod_{i=1}^{k-1}\left|X_{t-i}\right|\right)=$
$=\left(\left|X_{t-1}\right|+\beta\left|X_{t-2}\right|+...+\beta^{k-2}\left|X_{t-k+1}\right|\right)\times\left(\left|X_{t-1}\right|\left|X_{t-2}\right|\left|X_{t-3}\right|...\left|X_{t-k+1}\right|\right)$
$\leq\sum_{j=0}^{k-2}\beta^j cov\left(\left|X_{t-1}\right|\left|X_{t-2}\right|\left|X_{t-3}\right|...\left|X_{t-k+1}\right|,\left|X_{t-j}\right|\right),\qquad j=1,...,k-1,$
where

$$\begin{aligned}cov\left(\left|X_{t-1}\right|\left|X_{t-2}\right|\left|X_{t-3}\right|...\left|X_{t-k+1}\right|,\left|X_{t-j}\right|\right) &\leq \exp\left[\frac{1}{2\left(1-\beta\right)}\alpha\left[k-1-\beta^{k-2}-\beta\frac{1-\beta^{k-2}}{1-\beta}\right]\right]\times\\ &\times\left[E\left|Z_0\right|\exp\left\{\frac{1}{2}\overline{\delta}\left|Z_0\right|\right\}\right]^{k-3}\\ &\times E\left[Z_0^2\exp\left\{\frac{1}{2}\left(\frac{1}{1-\beta}+1\right)\overline{\delta}\left|Z_0\right|\right\}\right]\\ &\times E\exp\left\{\frac{1}{2}\left(\frac{1-\beta^{k-1}}{1-\beta}+1\right)\log\sigma_{t-k+1}^2\right\},\end{aligned}$$

by taking into account the upper bounds of its term in the covariance between $\left|X_{t-1}\right|\left|X_{t-2}\right|\left|X_{t-3}\right|...\left|X_{t-k+1}\right|$ and $\left|X_{t-j}\right|$. The above result comes from the following:

Notice that:

$\left|X_{t-1}\right|^2\left|X_{t-2}\right|\left|X_{t-3}\right|...\left|X_{t-k+1}\right|=Z_{t-1}^2\exp\left(\log\sigma_{t-1}^2\right)\left|\sigma_{t-2}Z_{t-2}\sigma_{t-3}Z_{t-3}...\sigma_{t-k+1}Z_{t-k+1}\right|=$
$=Z_{t-1}^2\exp\left(\alpha+\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|+\beta\log\sigma_{t-2}^2\right)\left|\exp\left(\frac{1}{2}\log\sigma_{t-2}^2\right)Z_{t-2}\sigma_{t-3}Z_{t-3}...\sigma_{t-k+1}Z_{t-k+1}\right|=$
$=Z_{t-1}^2\exp\left\{\alpha\right\}\exp\left\{\left(\frac{1}{2}+\beta\right)\log\sigma_{t-2}^2\right\}\left|\exp\left\{\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right\}Z_{t-2}\right|\left|\sigma_{t-3}Z_{t-3}...\sigma_{t-k+1}Z_{t-k+1}\right|=$
$=Z_{t-1}^2\exp\left\{\alpha\right\}\exp\left\{\left(\frac{1}{2}+\beta\right)\left(\alpha+\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|+\beta\log\sigma_{t-3}^2\right)\right\}\left|\exp\left\{\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right\}Z_{t-2}\right|\times$
$\left|\exp\left(\frac{1}{2}\log\sigma_{t-3}^2\right)Z_{t-3}...\sigma_{t-k+1}Z_{t-k+1}\right|=$
$=Z_{t-1}^2\exp\left\{\alpha\right\}\exp\left\{\left(\frac{1}{2}+\beta\right)\alpha\right\}\exp\left\{\left(\beta\left(\frac{1}{2}+\beta\right)+\frac{1}{2}\right)\log\sigma_{t-3}^2\right\}\left|\exp\left\{\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right\}Z_{t-2}\right|\times$
$\times\left|\exp\left\{\left(\frac{1}{2}+\beta\right)\left(\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|\right)\right\}Z_{t-3}\right|\left|\sigma_{t-4}Z_{t-4}...\sigma_{t-k+1}Z_{t-k+1}\right|=$
$=...=$
$=Z_{t-1}^2\exp\left\{\alpha\right\}\exp\left\{\left(\frac{1}{2}+\beta\right)\alpha\right\}\exp\left\{\left(\frac{1}{2}+\frac{1}{2}\beta+\beta^2\right)\alpha\right\}\times...\times\exp\left\{\left(\frac{1}{2}+\frac{1}{2}\beta+\frac{1}{2}\beta^2+...+\beta^{k-2}\right)\alpha\right\}\times$
$\times\left|\exp\left\{\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right\}Z_{t-2}\right|\left|\exp\left\{\left(\frac{1}{2}+\beta\right)\left(\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|\right)\right\}Z_{t-3}\right|\times...$
$\times\left|\exp\left\{\left(\frac{1}{2}+\frac{1}{2}\beta+\frac{1}{2}\beta^2+...+\beta^{k-3}\right)\left(\gamma Z_{t-k+1}+\delta\left|Z_{t-k+1}\right|\right)\right\}Z_{t-k+1}\right|\times$
$\times\exp\left\{\left(\frac{1}{2}+\frac{1}{2}\beta+\frac{1}{2}\beta^2+...+\beta^{k-2}\right)\log\sigma_{t-k+1}^2\right\}.$

Hence,

$$
\begin{aligned}
E\left|X_{t-1}\right|^2\left|X_{t-2}\right|\left|X_{t-3}\right|...\left|X_{t-k+1}\right| \;\le\; & \exp\left[\frac{1}{2\left(1-\beta\right)}\alpha\left[k-1-\beta^{k-2}-\beta\frac{1-\beta^{k-2}}{1-\beta}\right]\right]\times \\
& \times\left[E\left|Z_{t-2}\right|\exp\left\{\frac{1}{2}\left(\frac{1-\beta}{1-\beta}+1\right)\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right\}\right]^{k-2} \\
& \times E\exp\left\{\frac{1}{2}\left(\frac{1-\beta^{k-1}}{1-\beta}+\beta^{k-2}\right)\log\sigma^2_{t-k+1}\right\}.
\end{aligned}
$$

Second:

$$
\begin{aligned}
&\left|X_{t-1}\right|\left|X_{t-2}\right|^2\left|X_{t-3}\right|...\left|X_{t-k+1}\right| = \left|Z_{t-1}\right|\left|\sigma_{t-1}\sigma^2_{t-2}Z^2_{t-2}\sigma_{t-3}Z_{t-3}...\sigma_{t-k+1}Z_{t-k+1}\right| = \\
&= \left|Z_{t-1}\right|\left|\exp\left\{\tfrac{1}{2}\left(\alpha+\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|+\beta\log\sigma^2_{t-2}\right)\right\}\exp\left\{\log\sigma^2_{t-2}\right\}Z^2_{t-2}\sigma_{t-3}Z_{t-3}...\sigma_{t-k+1}Z_{t-k+1}\right| = \\
&= \left|Z_{t-1}\right|\exp\left\{\tfrac{1}{2}\alpha\right\}\exp\left\{\tfrac{1}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right\}Z^2_{t-2}\exp\left\{\left(\tfrac{1}{2}\beta+1\right)\log\sigma^2_{t-2}\right\}\left|\sigma_{t-3}Z_{t-3}...\sigma_{t-k+1}Z_{t-k+1}\right| = \\
&= \left|Z_{t-1}\right|\exp\left\{\tfrac{1}{2}\alpha\right\}\exp\left\{\tfrac{1}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right\}Z^2_{t-2}\exp\left\{\left(\tfrac{1}{2}\beta+1\right)\left(\alpha+\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|+\beta\log\sigma^2_{t-3}\right)\right\}\times \\
&\times\exp\left\{\tfrac{1}{2}\log\sigma^2_{t-3}\right\}\left|Z_{t-3}...\sigma_{t-k+1}Z_{t-k+1}\right| = \\
&= \left|Z_{t-1}\right|\exp\left\{\tfrac{1}{2}\alpha\right\}\exp\left\{\left(\tfrac{1}{2}\beta+1\right)\alpha\right\}\exp\left\{\tfrac{1}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right\}Z^2_{t-2}\left|\exp\left\{\left(\tfrac{1}{2}\beta+1\right)\left(\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|\right)\right\}\right. \\
&\times\exp\left\{\left(\beta\left(\tfrac{1}{2}\beta+1\right)+\tfrac{1}{2}\right)\log\sigma^2_{t-3}\right\}\left|\sigma_{t-4}Z_{t-4}...\sigma_{t-k+1}Z_{t-k+1}\right| = \\
&= ... = \\
&= \left|Z_{t-1}\right|\exp\left\{\tfrac{1}{2}\alpha\right\}\exp\left\{\left(\tfrac{1}{2}\beta+1\right)\alpha\right\}\exp\left\{\left(\tfrac{1}{2}\beta^2+\beta+\tfrac{1}{2}\right)\alpha\right\}\times...\times\exp\left\{\left(\tfrac{1}{2}\beta^{k-2}+\beta^{k-3}+\tfrac{1}{2}\beta^{k-4}+...+\tfrac{1}{2}\right.\right. \\
&\times\exp\left\{\tfrac{1}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right\}Z^2_{t-2}\left|\exp\left\{\left(\tfrac{1}{2}\beta+1\right)\left(\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|\right)\right\}Z_{t-3}\right|\times... \\
&\times\left|\exp\left\{\left(\tfrac{1}{2}\beta^{k-3}+\beta^{k-4}+\tfrac{1}{2}\beta^{k-5}+...+\tfrac{1}{2}\right)\left(\gamma Z_{t-k+1}+\delta\left|Z_{t-k+1}\right|\right)\right\}Z_{t-k+1}\right| \\
&\times\exp\left\{\left(\tfrac{1}{2}\beta^{k-2}+\beta^{k-3}+\tfrac{1}{2}\beta^{k-4}+...+\tfrac{1}{2}\right)\log\sigma^2_{t-k+1}\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E\left|X_{t-1}\right|\left|X_{t-2}\right|^2\left|X_{t-3}\right|...\left|X_{t-k+1}\right| \;\le\; & E\left|Z_0\right|\exp\left[\frac{1}{2\left(1-\beta\right)}\alpha\left[k-1-\beta^{k-3}-\beta\frac{1-\beta^{k-2}}{1-\beta}\right]\right]\times \\
& \times E\left[Z^2_{t-2}\exp\left\{\frac{1}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right\}\right]\times \\
& \left[E\left|Z_{t-3}\right|\exp\left\{\frac{1}{2}\left(\frac{1-\beta^2}{1-\beta}+1\right)\left(\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|\right)\right\}\right]^{k-3}\times \\
& \times E\exp\left\{\frac{1}{2}\left(\frac{1-\beta^{k-1}}{1-\beta}+\beta^{k-3}\right)\log\sigma^2_{t-k+1}\right\}.
\end{aligned}
$$

As well as:

$$
\begin{aligned}
&\left|X_{t-1}\right|\left|X_{t-2}\right|\left|X_{t-3}\right|^2...\left|X_{t-k+1}\right| = \left|Z_{t-1}\right|\left|\sigma_{t-1}\sigma_{t-2}Z_{t-2}\sigma^2_{t-3}Z^2_{t-3}...\sigma_{t-k+1}Z_{t-k+1}\right| = \\
&= \left|Z_{t-1}\right|\exp\left\{\tfrac{1}{2}\log\sigma^2_{t-1}\right\}\left|\sigma_{t-2}Z_{t-2}\sigma^2_{t-3}Z^2_{t-3}...\sigma_{t-k+1}Z_{t-k+1}\right| =
\end{aligned}
$$

$= |Z_{t-1}| \exp\left\{\frac{1}{2}\left(\alpha + \gamma Z_{t-2} + \delta |Z_{t-2}| + \beta \log \sigma_{t-2}^2\right)\right\} \exp\left\{\frac{1}{2}\log \sigma_{t-2}^2\right\} \left|Z_{t-2}\sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1} Z_{t-k+1}\right| =$

$= |Z_{t-1}| \exp\left\{\frac{1}{2}\alpha\right\} \left|\exp\left\{\frac{1}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right\} Z_{t-2}\right| \exp\left\{\left(\frac{1}{2}(\beta+1)\right)\log \sigma_{t-2}^2\right\} \left|\sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1} Z_{t-k+1}\right.$

$= |Z_{t-1}| \exp\left\{\frac{1}{2}\alpha\right\} \left|\exp\left\{\frac{1}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right\} Z_{t-2}\right| \exp\left\{\left(\frac{1}{2}(\beta+1)\right)\left(\alpha + \gamma Z_{t-3} + \delta |Z_{t-3}| + \beta \log \sigma_{t-3}^2\right)\right.$

$\times \exp\left\{\log \sigma_{t-3}^2\right\} Z_{t-3}^2 \left|\sigma_{t-4} Z_{t-4} ... \sigma_{t-k+1} Z_{t-k+1}\right| =$

$= |Z_{t-1}| \exp\left\{\frac{1}{2}\alpha\right\} \exp\left\{\left(\frac{1}{2}(\beta+1)\right)\alpha\right\} \left|\exp\left\{\frac{1}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right\} Z_{t-2}\right| \times$

$\times \exp\left\{\left(\frac{1}{2}(\beta+1)\right)\left(\gamma Z_{t-3} + \delta |Z_{t-3}|\right)\right\} Z_{t-3}^2 \times$

$\times \exp\left\{\left(\beta\left(\frac{1}{2}(\beta+1)\right) + 1\right)\log \sigma_{t-3}^2\right\} \left|\sigma_{t-4} Z_{t-4} ... \sigma_{t-k+1} Z_{t-k+1}\right| =$

$= ... =$

$= |Z_{t-1}| \exp\left\{\frac{1}{2}\alpha\right\} \exp\left\{\left(\frac{1}{2}(\beta+1)\right)\alpha\right\} \exp\left\{\left(\frac{1}{2}\beta^2 + \frac{1}{2}\beta + 1\right)\alpha\right\} \times ...$

$\times \exp\left\{\left(\frac{1}{2}\beta^{k-2} + \frac{1}{2}\beta^{k-3} + \beta^{k-4} + \frac{1}{2}\beta^{k-5} ... + \frac{1}{2}\right)\alpha\right\} \times$

$\times \exp\left\{\frac{1}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right\} Z_{t-2} \left|\exp\left\{\left(\frac{1}{2}(\beta+1)\right)\left(\gamma Z_{t-3} + \delta |Z_{t-3}|\right)\right\} Z_{t-3}^2\right| \times ...$

$\times \left|\exp\left\{\left(\frac{1}{2}\beta^{k-3} + \frac{1}{2}\beta^{k-4} + \beta^{k-5} + \frac{1}{2}\beta^{k-6} ... + \frac{1}{2}\right)\left(\gamma Z_{t-k+1} + \delta |Z_{t-k+1}|\right)\right\} Z_{t-k+1}\right|$

$\times \exp\left\{\left(\frac{1}{2}\beta^{k-2} + \frac{1}{2}\beta^{k-3} + \beta^{k-4} + \frac{1}{2}\beta^{k-5} ... + \frac{1}{2}\right)\log \sigma_{t-k+1}^2\right\}.$

Hence,

$$
\begin{aligned}
E|X_{t-1}||X_{t-2}||X_{t-3}|^2 ... |X_{t-k+1}| \leq\ & E|Z_0| \exp\left[\frac{1}{2(1-\beta)}\alpha\left[k - 1 - \beta^{k-4} - \beta\frac{1-\beta^{k-2}}{1-\beta}\right]\right] \times \\
& \times E\left[|Z_{t-2}| \exp\left\{\frac{1}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right\}\right] \times \\
& \times E\left[Z_{t-3}^2 \exp\left\{\frac{1}{2}(\beta+1)\left(\gamma Z_{t-3} + \delta |Z_{t-3}|\right)\right\}\right] \times \\
& \times \left[E|Z_{t-4}| \exp\left\{\frac{1}{2}\left(\frac{1-\beta^3}{1-\beta} + 1\right)\left(\gamma Z_{t-4} + \delta |Z_{t-4}|\right)\right\}\right]^{k-4} \\
& \times E \exp\left\{\frac{1}{2}\left(\frac{1-\beta^{k-1}}{1-\beta} + \beta^{k-4}\right)\log \sigma_{t-k+1}^2\right\}.
\end{aligned}
$$

And, last:

$$|X_{t-1}||X_{t-2}||X_{t-3}| ... |X_{t-k+1}|^2 = |Z_{t-1}| \left|\sigma_{t-1}\sigma_{t-2} Z_{t-2}\sigma_{t-3} Z_{t-3} ... \sigma_{t-k+1}^2 Z_{t-k+1}^2\right| =$$

$= |Z_{t-1}| \exp\left\{\frac{1}{2}\log \sigma_{t-1}^2\right\} \left|\sigma_{t-2} Z_{t-2}\sigma_{t-3} Z_{t-3} ... \sigma_{t-k+1}^2 Z_{t-k+1}^2\right| =$

$= |Z_{t-1}| \exp\left\{\frac{1}{2}\left(\alpha + \gamma Z_{t-2} + \delta |Z_{t-2}| + \beta \log \sigma_{t-2}^2\right)\right\} \exp\left\{\frac{1}{2}\log \sigma_{t-2}^2\right\} \left|Z_{t-2}\sigma_{t-3} Z_{t-3} ... \sigma_{t-k+1}^2 Z_{t-k+1}^2\right| =$

$= |Z_{t-1}| \exp\left\{\frac{1}{2}\alpha\right\} \left|\exp\left\{\frac{1}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right\} Z_{t-2}\right| \exp\left\{\left(\frac{1}{2}(\beta+1)\right)\log \sigma_{t-2}^2\right\} \left|\sigma_{t-3} Z_{t-3} ... \sigma_{t-k+1}^2 Z_{t-k+1}^2\right.$

$= |Z_{t-1}| \exp\left\{\frac{1}{2}\alpha\right\} \left|\exp\left\{\frac{1}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right\} Z_{t-2}\right| \exp\left\{\left(\frac{1}{2}(\beta+1)\right)\left(\alpha + \gamma Z_{t-3} + \delta |Z_{t-3}| + \beta \log \sigma_{t-3}^2\right)\right.$

$\times \exp\left\{\frac{1}{2}\log \sigma_{t-3}^2\right\} Z_{t-3} \left|\sigma_{t-4} Z_{t-4} ... \sigma_{t-k+1}^2 Z_{t-k+1}^2\right| =$

$= |Z_{t-1}| \exp\left\{\frac{1}{2}\alpha\right\} \exp\left\{\left(\frac{1}{2}(\beta+1)\right)\alpha\right\} \left|\exp\left\{\frac{1}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right\} Z_{t-2}\right| \times$

$\times \exp\left\{\left(\frac{1}{2}(\beta+1)\right)\left(\gamma Z_{t-3} + \delta |Z_{t-3}|\right)\right\} Z_{t-3} \times$

$\times \exp\left\{\left(\beta\left(\frac{1}{2}(\beta+1)\right)+\frac{1}{2}\right)\log\sigma_{t-3}^2\right\}\left|\sigma_{t-4}Z_{t-4}...\sigma_{t-k+1}^2 Z_{t-k+1}^2\right| =$

$= ... =$

$= |Z_{t-1}|\exp\left\{\frac{1}{2}\alpha\right\}\exp\left\{\left(\frac{1}{2}(\beta+1)\right)\alpha\right\}\exp\left\{\left(\frac{1}{2}\beta^2+\frac{1}{2}\beta+\frac{1}{2}\right)\alpha\right\}\times ...$

$\times \exp\left\{\left(\frac{1}{2}\beta^{k-2}+\frac{1}{2}\beta^{k-3}+\frac{1}{2}\beta^{k-4}+\frac{1}{2}\beta^{k-5}...+1\right)\alpha\right\}\times$

$\times \exp\left\{\frac{1}{2}(\gamma Z_{t-2}+\delta|Z_{t-2}|)\right\}Z_{t-2}\left|\exp\left\{\left(\frac{1}{2}(\beta+1)\right)(\gamma Z_{t-3}+\delta|Z_{t-3}|)\right\}Z_{t-3}\right|\times ...$

$\times\left|\exp\left\{\left(\frac{1}{2}\beta^{k-3}+\frac{1}{2}\beta^{k-4}+\frac{1}{2}\beta^{k-5}+\frac{1}{2}\beta^{k-6}...+1\right)(\gamma Z_{t-k+1}+\delta|Z_{t-k+1}|)\right\}Z_{t-k+1}\right|$

$\times \exp\left\{\left(\frac{1}{2}\beta^{k-2}+\frac{1}{2}\beta^{k-3}+\frac{1}{2}\beta^{k-4}+\frac{1}{2}\beta^{k-5}...+1\right)\log\sigma_{t-k+1}^2\right\}.$

Hence,

$$\begin{aligned}E|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|^2 \leq\ & E|Z_0|\exp\left[\frac{1}{2(1-\beta)}\alpha\left[k-1-\beta-\beta\frac{1-\beta^{k-2}}{1-\beta}\right]\right]\times\\ & \times\left[E|Z_{t-2}|\exp\left\{\frac{1}{2}(\gamma Z_{t-2}+\delta|Z_{t-2}|)\right\}\right]^{k-3}\times\\ & \times E\left[Z_{t-k+1}^2\exp\left\{\frac{1}{2}\left(\frac{1-\beta^{k-2}}{1-\beta}+1\right)(\gamma Z_{t-k+1}+\delta|Z_{t-k+1}|)\right\}\right]\\ & \times E\exp\left\{\frac{1}{2}\left(\frac{1-\beta^{k-1}}{1-\beta}+1\right)\log\sigma_{t-k+1}^2\right\}.\end{aligned}$$

□

Lemma C.8. *Moment bounds of* $\sum_{n=k-1}^{\infty}\beta^n(|X_{t-n-1}|)\left(\prod_{i=1}^{k-1}|X_{t-i}|\right)$

Proof. $\sum_{n=k-1}^{\infty}\beta^n(|X_{t-n-1}|)\left(\prod_{i=1}^{k-1}|X_{t-i}|\right) = \left(\beta^{k-1}|X_{t-k}|+\beta^k|X_{t-k-1}|+\beta^{k+1}|X_{t-k-2}|+...\right)\times$ $(|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|) =$

$= \beta^{k-1}|X_{t-k}|(|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|)+\beta^k|X_{t-k-1}|(|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|)+$ $\beta^{k+1}|X_{t-k-2}|(|X_{t-1}||X_{t-2}||X_{t-3}|...|X_{t-k+1}|)+...$

Hence,

$E\left[\sum_{n=k-1}^{\infty}\beta^n(|X_{t-n-1}|)\left(\prod_{i=1}^{k-1}|X_{t-i}|\right)\right] \leq \sum_{n=k-1}^{\infty}\beta^n E\prod_{i=1}^{k}|X_{t-i}| = \beta^{k-1}\sum_{n=0}^{\infty}\beta^n E\prod_{i=1}^{k}|X_{t-i}| =$ $\frac{\beta^{k-1}}{1-\beta}E\prod_{i=1}^{k}|X_{t-i}|.$

That is,

$E\left[\sum_{n=k-1}^{\infty}\beta^n(|X_{t-n-1}|)\left(\prod_{i=1}^{k-1}|X_{t-i}|\right)\right] \leq \frac{\beta^{k-1}}{1-\beta}E\exp\left(\frac{1}{2(1-\beta)}\alpha\left(k-1-\frac{\beta-\beta^k}{1-\beta}\right)\right)E\exp\left(\frac{1}{2}\frac{1-\beta^k}{1-\beta}\log\sigma_{t-k}^2\right)$

$\times E\left[|Z_{t-k}|\exp\left(\frac{1}{2}\frac{1-\beta^{k-1}}{1-\beta}(\gamma Z_{t-k}+\delta|Z_{t-k}|)\right)\right]\times...\times E\left[|Z_{t-2}|\exp\left(\frac{1}{2}(\gamma Z_{t-2}+\delta|Z_{t-2}|)\right)\right]$

$\leq \frac{\beta^{k-1}}{1-\beta}\exp\left(\frac{1}{2(1-\beta)}\alpha\left(k-1-\frac{\beta-\beta^k}{1-\beta}\right)\right)E\exp\left(\frac{1}{2}\frac{1}{1-\beta}\log\sigma_0^2\right)\times\left(E|Z_0|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\bar{\delta}|Z_0|\right)\right)^{k-1}$

$$\leq \frac{\beta^{k-1}}{1-\beta} \exp\left(\frac{1}{2(1-\beta)}\alpha(k-1)\right) \exp\left(-2^{-1}\frac{\alpha\beta}{(1-\beta)^2}\right) E \exp\left(\frac{1}{2}\frac{1}{1-\beta}\log\sigma_0^2\right) \times \left(E|z_0|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right)^{k-1}$$
$$\leq \left(\beta\exp\left(\frac{1}{2(1-\beta)}\alpha\right) E|Z_0|\exp\left(\frac{1}{2}\frac{1}{1-\beta}\overline{\delta}|Z_0|\right)\right)^{k-1} \frac{1}{1-\beta}\exp\left(-2^{-1}\frac{\alpha\beta}{(1-\beta)^2}\right) E\exp\left(\frac{1}{2}\frac{1}{1-\beta}\log\sigma_0^2\right).$$

□

C.5 Dependence Results and Useful Expressions for the Analysis of the Second Order Derivative

We proceed with useful inequalities and moment bounds that are used for the establishment of the second order derivative finiteness.

Lemma C.9. *Find tractable expressions for* $\prod_{i=1}^{k}\|A_{t-i}\|^{\eta}$ *and* $E\left\|(\log h_t)'\right\|^{\eta}$, $\forall\eta > 1$.

Proof. Recall that

$$\begin{aligned}\|A_t\| &\leq \beta + 2^{-1}\overline{\delta}|X_t|\exp\left(-2^{-1}\underline{m}\right)\\ &= \beta\left[1 + \frac{1}{\beta}2^{-1}\overline{\delta}|X_t|\exp\left(-2^{-1}\underline{m}\right)\right]\end{aligned}$$

Hence,

$$\begin{aligned}\|A_t\|^{\eta} &\leq \beta^{\eta}\left[1 + \frac{1}{\beta}2^{-1}\overline{\delta}|X_t|\exp\left(-2^{-1}\underline{m}\right)\right]^{\eta}\\ &\leq \beta^{\eta}\left[2^{\eta}\left(1 + \frac{1}{(2\beta)^{\eta}}\overline{\delta}^{\eta}|X_t|^{\eta}\exp\left(-2^{-1}\underline{m\eta}\right)\right)\right]\\ &\leq (2\beta)^{\eta} + \overline{\delta}^{\eta}|X_t|^{\eta}\exp\left(-2^{-1}\underline{m\eta}\right)\end{aligned}$$

and

$$\prod_{i=1}^{k}\|A_{t-i}\|^{\eta} \leq (2\beta)^{\eta k}\prod_{i=1}^{k}\left(1 + \frac{1}{(2\beta)^{\eta}}\overline{\delta}^{\eta}|X_{t-i}|^{\eta}\exp\left(-2^{-1}\underline{m\eta}\right)\right)$$

Also

$$\begin{aligned}\|B_t\|^{\eta} &\leq (1 + |X_t| + |\log h_t|)^{\eta}\\ &\leq 2^{\eta}\left[1 + (|X_t| + |\log h_t|)^{\eta}\right]\\ &\leq 2^{\eta}\left[1 + 2^{\eta}\left(|X_t|^{\eta} + |\log h_t|^{\eta}\right)\right]\\ &\leq 2^{\eta} + 2^{2\eta}\left(|X_t|^{\eta} + |\log h_t|^{\eta}\right),\end{aligned}$$

making use of

$$(x+y)^{\eta} \leq 2^{\eta}\left(x^{\eta}+y^{\eta}\right).$$

We have:

$$\begin{aligned}
E\left\|\left(\log h_t\right)'\right\|^{\eta} &\leq E\left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\left\|A_{t-i}\right\|\right)\left\|B_{t-k}\right\|\right]^{\eta} \\
&\leq c_1 E\left[\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\left\|A_{t-i}\right\|^{\eta}\left(1+\left|X_{t-k}\right|+\left|\log h_{t-k}\right|\right)^{\eta}\right] \\
&\leq c_1 E\left[\sum_{k=1}^{\infty}(2\beta)^{\eta(k-1)}\prod_{i=1}^{k-1}\left(1+\frac{1}{(2\beta)^{\eta}}\bar{\delta}^{\eta}\left|X_{t-i}\right|^{\eta}\exp\left(-2^{-1}\underline{m\eta}\right)\right)\right]\times \\
&\quad \times\left[2^{\eta}+2^{2\eta}\left(\left|X_{t-k}\right|^{\eta}+\left|\log h_{t-k}\right|^{\eta}\right)\right]
\end{aligned}$$

Hence:

$$\begin{aligned}
\left\|\left(\log h_t\right)'\right\|^{2} &\leq \left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\left\|A_{t-i}\right\|\right)\left\|B_{t-k}\right\|\right]^{2} \\
&\leq c_1\left[\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\left\|A_{t-i}\right\|^{2}\left(1+\left|X_{t-k}\right|+\left|\log h_{t-k}\right|\right)^{2}\right] \\
&\leq c_1\left[\sum_{k=1}^{\infty}(2\beta)^{2(k-1)}\prod_{i=1}^{k-1}\left(1+\frac{1}{(2\beta)^{2}}\bar{\delta}^{2}\left|X_{t-i}\right|^{2}\exp\left(-\underline{m}\right)\right)\right]\left[4+16\left(\left|X_{t-k}\right|^{2}+\left|\log h_{t-k}\right|^{2}\right)\right. \\
&\leq c_1^{*}\left[\sum_{k=1}^{\infty}(2\beta)^{2(k-1)}\prod_{i=1}^{k-1}\left(1+c\left|X_{t-i}\right|^{2}\right)\right]\left[1+4\left(\left|X_{t-k}\right|^{2}+\left|\log h_{t-k}\right|^{2}\right)\right], \\
\text{where } c &= \frac{1}{(2\beta)^{2}}\bar{\delta}^{2}\exp\left(-\underline{m}\right)
\end{aligned}$$

or (more general):

$$\begin{aligned}\left\|(\log h_t)'\right\|^2 &\leq \left[\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\|A_{t-i}\|\right)\|B_{t-k}\|\right]^2\\ &\leq c_1\left[\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\|A_{t-i}\|^2\left(1+|X_{t-k}|+|\log h_{t-k}|\right)^2\right]\\ &\leq c_1\left[\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\|A_{t-i}\|^2\right]\left[4+16\left(|X_{t-k}|^2+|\log h_{t-k}|^2\right)\right]\\ &\leq c_1^*\left[\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\|A_{t-i}\|^2\right]\left[1+4\left(|X_{t-k}|^2+|\log h_{t-k}|^2\right)\right],\\ \text{where } \sum_{k=1}^{\infty}\prod_{i=1}^{k-1}\|A_{t-i}\|^2 &= \sum_{k=1}^{\infty}(2\beta)^{2(k-1)}\prod_{i=1}^{k-1}\left(1+c\,|X_{t-i}|^2\right)\\ \text{and } c &= \frac{1}{(2\beta)^2}\overline{\delta}^2\exp\left(-\underline{m}\right).\end{aligned}$$

□

Lemma C.10. *Moment bounds of* $E\left[\left(\prod_{i=1}^{k-1}|X_{t-i}|^2\right)|Z_{t-k}|\,\sigma_{t-k}\right]$

Proof. $E\left[\left(\prod_{i=1}^{k-1}|X_{t-i}|^2\right)|Z_{t-k}|\,\sigma_{t-k}\right]=E\left(Z_{t-1}^2\sigma_{t-1}^2Z_{t-2}^2\sigma_{t-2}^2Z_{t-3}^2\sigma_{t-3}^2...Z_{t-k+1}^2\sigma_{t-k+1}^2\,|Z_{t-k}|\,\sigma_{t-k}\right)=$
$=E\exp(\alpha)\,E\exp\left[(\beta+1)\,\alpha\right]E\exp\left[\left(\beta^2+\beta+1\right)\alpha\right]\times...\times E\exp\left[\left(\beta^{k-2}+...+1\right)\alpha\right]\times$
$\times E\exp\left(\left(\gamma Z_{t-2}+\delta\,|Z_{t-2}|\right)\right)Z_{t-2}^2E\exp\left((\beta+1)\left(\gamma Z_{t-3}+\delta\,|Z_{t-3}|\right)\right)Z_{t-3}^2\times...\times$
$\times E\exp\left(\left(\beta^{k-3}+...+1\right)\left(\gamma Z_{t-k+1}+\delta\,|Z_{t-k+1}|\right)\right)Z_{t-k+1}^2\times$
$\times E\left|\exp\left(\left(\beta^{k-2}+...+1\right)\left(\gamma Z_{t-k}+\delta\,|Z_{t-k}|\right)\right)Z_{t-k}\right|\times E\exp\left(\left(\beta^{k-2}+...+1+\tfrac{1}{2}\right)\log\sigma_{t-k}^2\right).$

□

Lemma C.11. *Moment bounds of* $\sum_{n=0}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\beta^{2(k-1)}c\sum_{i=1}^{k-1}|X_{t-i}|^2\right)\right)|Z_{t-k}|\,\sigma_{t-k}$

Proof. $E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\beta^{2(k-1)}c\sum_{i=1}^{k-1}|X_{t-i}|^2\right)\right)|Z_{t-k}|\,\sigma_{t-k}\right]=$
$=2^{-2}\overline{\delta}^2\exp\left(-\underline{m}\right)\beta^{2(k-2)}E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|^2\right)\right)|Z_{t-k}|\,\sigma_{t-k}\right]=$

$$= 2^{-2}\overline{\delta}^3 \exp\left(-\tfrac{3}{2}\underline{m}\right)\beta^{2(k-2)} E\left[\begin{array}{c} \sum_{n=0}^{k-2}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|^2\right)\right)|X_{t-k}| \\ +\sum_{n=k-1}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\sum_{i=1}^{k-1}|X_{t-i}|^2\right)\right)|X_{t-k}| \end{array}\right] =$$

$$= 2^{-2}\overline{\delta}^3 \exp\left(-\tfrac{3}{2}\underline{m}\right)\beta^{2(k-2)} E\left[\begin{array}{c} \sum_{n=0}^{k-2}\left(\beta^n\left(|X_{t-n-1}|\right)\left(|X_{t-1}|^2+|X_{t-2}|^2+...+|X_{t-k+1}|^2\right)\right)|X_{t-k}| \\ +\sum_{n=k-1}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(|X_{t-1}|^2+|X_{t-2}|^2+...+|X_{t-k+1}|^2\right)\right)|X_{t-k}| \end{array}\right]$$

$$\leq 2^{-2}\overline{\delta}^3 \exp\left(-\tfrac{3}{2}\underline{m}\right)\beta^{2(k-2)} E\left[\begin{array}{c} (k-1)\sum_{n=0}^{k-2}\beta^n|X_{t-k+1}|\,|X_{t-k+1}|^2|X_{t-k}| \\ +(k-1)\beta^{k-1}\sum_{n=0}^{\infty}\beta^n|X_{t-k}|\,|X_{t-k+1}|^2|X_{t-k}| \end{array}\right]$$

$$\leq 2^{-2}\overline{\delta}^3 \exp\left(-\tfrac{3}{2}\underline{m}\right)\beta^{2(k-2)}(k-1)\left[\tfrac{1-\beta^{k-1}}{1-\beta}E\left(|X_{t-1}|^3|X_{t-2}|\right)+\tfrac{\beta^{k-1}}{1-\beta}E\left(|X_{t-1}|^2|X_{t-2}|^2\right)\right]$$

$$\leq 2^{-2}\overline{\delta}^3 \exp\left(-\tfrac{3}{2}\underline{m}\right)\beta^{2(k-2)}(k-1)\left[\begin{array}{c} \tfrac{1-\beta^{k-1}}{1-\beta}E|Z_0|^3\exp\left(\tfrac{3}{2}\alpha\right)E\exp\left(\left(\tfrac{3}{2}\beta+\tfrac{1}{2}\right)\log\sigma_{t-2}^2\right)\times \\ E\left[|Z_0|\exp\left(\tfrac{3}{2}\overline{\delta}|Z_0|\right)\right] \\ +\tfrac{\beta^{k-1}}{1-\beta}\exp(\alpha)E\exp\left((\beta+1)\log\sigma_{t-2}^2\right)\times \\ E\left[Z_0^2\exp\left(\overline{\delta}|Z_0|\right)\right] \end{array}\right].$$

□

Lemma C.12. *Find a tractable expression for* $\sum_{j=0}^{k-2}\left(\beta^j\left(|X_{t-j-1}|\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|^2\right)\right)|X_{t-k}|$ *and calculate its moment bounds.*

Proof. $E\left[\sum_{j=0}^{k-2}\left(\beta^j\left(|X_{t-j-1}|\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|^2\right)\right)|X_{t-k}|\right]$

$$\leq \sum_{j=0}^{k-2} cov\left(|X_{t-1}|^2|X_{t-2}|^2|X_{t-3}|^2...|X_{t-k+1}|^2|X_{t-k}|,|X_{t-j}|\right), \qquad j=1,...,k-1,$$

where

$$\begin{aligned} cov\left(|X_{t-1}|^2|X_{t-2}|^2|X_{t-3}|^2...|X_{t-k+1}|^2|X_{t-k}|,|X_{t-j}|\right) \leq\ & E|Z_{t-1}|^3\times \\ & \times\exp\left[\frac{1}{(1-\beta)}\alpha\left[\begin{array}{c}\tfrac{1}{2}\left(1-\beta^{k-1}\right) \\ +k-1-\beta\tfrac{1-\beta^{k-1}}{1-\beta}\end{array}\right]\right] \\ & \times\left[E\left(\left|\exp\left\{\overline{\delta}|Z_0|\right\}Z_0^2\right|\right)\right]^{k-2} \\ & \times E\left[|Z_0|\exp\left(\left(\frac{1-\beta^{k-1}}{1-\beta}+\frac{1}{2}\right)\overline{\delta}|Z_0|\right)\right] \\ & \times E\exp\left\{\left[\frac{1-\beta^k}{1-\beta}+\frac{1}{2}(\beta-1)\right]\log\sigma_{t-k}^2\right\} \end{aligned}$$

by taking into account the upper bounds of its term in the covariance between $|X_{t-1}|^2 |X_{t-2}|^2 |X_{t-3}|^2 ... |X_{t-k+1}|^2 |X_{t-k}|$ and $|X_{t-j}|$. The above result comes from the following:

Notice that:

$$|X_{t-1}|^3 |X_{t-2}|^2 ... |X_{t-k+1}|^2 |X_{t-k}| =$$

$$= |Z_{t-1}|^3 \exp\left(\tfrac{3}{2}\log\sigma_{t-1}^2\right) \sigma_{t-2}^2 Z_{t-2}^2 \sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1}^2 Z_{t-k+1}^2 |\sigma_{t-k} Z_{t-k}| =$$

$$= |Z_{t-1}|^3 \exp\left(\tfrac{3}{2}\left(\alpha + \gamma Z_{t-2} + \delta |Z_{t-2}| + \beta \log\sigma_{t-2}^2\right)\right) \exp\left(\log\sigma_{t-2}^2\right) Z_{t-2}^2 \sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1}^2 Z_{t-k+1}^2 |\sigma_{t-k} Z$$

$$= |Z_{t-1}|^3 \exp\left\{\tfrac{3}{2}\alpha\right\} \exp\left\{\left(1+\tfrac{3}{2}\beta\right)\log\sigma_{t-2}^2\right\} \times$$

$$\times \exp\left[\tfrac{3}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right] Z_{t-2}^2 \sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1}^2 Z_{t-k+1}^2 |\sigma_{t-k} Z_{t-k}| =$$

$$= |Z_{t-1}|^3 \exp\left\{\tfrac{3}{2}\alpha\right\} \exp\left\{\left(1+\tfrac{3}{2}\beta\right)\left(\alpha + \gamma Z_{t-3} + \delta |Z_{t-3}| + \beta\log\sigma_{t-3}^2\right)\right\} \times$$

$$\times \exp\left[\tfrac{3}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right] Z_{t-2}^2 \exp\left(\log\sigma_{t-3}^2\right) \times Z_{t-3}^2 ... \sigma_{t-k+1}^2 Z_{t-k+1}^2 |\sigma_{t-k} Z_{t-k}| =$$

$$= |Z_{t-1}|^3 \exp\left\{\tfrac{3}{2}\alpha\right\} \exp\left\{\left(1+\tfrac{3}{2}\beta\right)\alpha\right\} \exp\left\{\left(\beta\left(1+\tfrac{3}{2}\beta\right)+1\right)\log\sigma_{t-3}^2\right\} \exp\left[\tfrac{3}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right] Z_{t-2}^2 \times$$

$$\times \exp\left[\left(1+\tfrac{3}{2}\beta\right)\left(\gamma Z_{t-3} + \delta |Z_{t-3}|\right)\right] Z_{t-3}^2 \sigma_{t-4}^2 Z_{t-4}^2 ... \sigma_{t-k+1}^2 Z_{t-k+1}^2 |\sigma_{t-k} Z_{t-k}| =$$

$$= ... =$$

$$= |Z_{t-1}|^3 \exp\left\{\tfrac{3}{2}\alpha\right\} \exp\left\{\left(1+\tfrac{3}{2}\beta\right)\alpha\right\} \exp\left\{\left(1+\beta+\tfrac{3}{2}\beta^2\right)\alpha\right\} \times ... \times \exp\left\{\left(1+\beta+...+\beta^{k-3}+\tfrac{3}{2}\beta^{k-2}\right)\alpha\right\}$$

$$\times \exp\left[\tfrac{3}{2}\left(\gamma Z_{t-2} + \delta |Z_{t-2}|\right)\right] Z_{t-2}^2 \exp\left[\left(1+\tfrac{3}{2}\beta\right)\left(\gamma Z_{t-3} + \delta |Z_{t-3}|\right)\right] Z_{t-3}^2 \times ... \times$$

$$\times \exp\left[\left(1+\beta+...+\beta^{k-4}+\tfrac{3}{2}\beta^{k-3}\right)\left(\gamma Z_{t-k+1} + \delta |Z_{t-k+1}|\right)\right] Z_{t-k+1}^2 \times$$

$$\times \left|\exp\left[\left(1+\beta+...+\beta^{k-3}+\tfrac{3}{2}\beta^{k-2}\right)\left(\gamma Z_{t-k} + \delta |Z_{t-k}|\right)\right] Z_{t-k}\right| \exp\left[\left(\tfrac{1}{2}+\beta+...+\beta^{k-2}+\tfrac{3}{2}\beta^{k-1}\right)\log\sigma_{t-}^2\right.$$

Hence,

$$\begin{aligned} E|X_{t-1}|^3 |X_{t-2}|^2 ... |X_{t-k+1}|^2 |X_{t-k}| \quad \leq \quad & E|Z_{t-1}|^3 \exp\left[\frac{1}{(1-\beta)}\alpha\left[\frac{1}{2}\left(1-\beta^{k-1}\right)+k-1-\beta\frac{1-\beta^{k-1}}{1-\beta}\right.\right. \\ & \times \left[E Z_0^2 \exp\left\{\left(1+\frac{3}{2}\beta\right)\bar{\delta}|Z_0|\right\}\right]^{k-2} \times \\ & \times E\left[|Z_0| \exp\left(\left(\frac{1-\beta^{k-1}}{1-\beta}+\frac{1}{2}\beta^{k-2}\right)\bar{\delta}|Z_0|\right)\right] \times \\ & \times E\exp\left\{\left[\frac{1-\beta^k}{1-\beta}+\frac{1}{2}\left(\beta^{k-1}-1\right)\right]\log\sigma_{t-k}^2\right\}. \end{aligned}$$

Second:

$$|X_{t-1}|^2 |X_{t-2}|^3 ... |X_{t-k+1}|^2 |X_{t-k}| = Z_{t-1}^2 \exp\left(\log\sigma_{t-1}^2\right) \sigma_{t-2}^3 Z_{t-2}^3 \sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1}^2 Z_{t-k+1}^2 |\sigma_{t-k} Z_{t-}$$

$$= Z_{t-1}^2 \exp\left(\alpha + \gamma Z_{t-2} + \delta|Z_{t-2}| + \beta\log\sigma_{t-2}^2\right) \exp\left[\tfrac{3}{2}\left(\log\sigma_{t-2}^2\right)\right] Z_{t-2}^3 \sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1}^2 Z_{t-k+1}^2 |\sigma_{t-k} Z_{t-}$$

$$= Z_{t-1}^2 \exp\{\alpha\} \exp\left\{\left(\beta + \tfrac{3}{2}\right)\log\sigma_{t-2}^2\right\} \exp\left(\gamma Z_{t-2} + \delta|Z_{t-2}|\right) Z_{t-2}^3 \sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1}^2 Z_{t-k+1}^2 |\sigma_{t-k} Z_{t-k}| =$$

$$= Z_{t-1}^2 \exp\{\alpha\} \exp\left\{\left(\beta + \tfrac{3}{2}\right)\left(\alpha + \gamma Z_{t-3} + \delta|Z_{t-3}| + \beta\log\sigma_{t-3}^2\right)\right\} \left|\exp\left(\gamma Z_{t-2} + \delta|Z_{t-2}|\right) Z_{t-2}^3\right| \times$$

$$\times \exp\left(\log\sigma_{t-3}^2\right) Z_{t-3}^2 ... \sigma_{t-k+1}^2 Z_{t-k+1}^2 |\sigma_{t-k} Z_{t-k}| =$$

$$= Z_{t-1}^2 \exp\{\alpha\} \exp\left\{\left(\beta + \tfrac{3}{2}\right)\alpha\right\} \exp\left\{\left(\beta\left(\beta + \tfrac{3}{2}\right) + 1\right)\log\sigma_{t-3}^2\right\} \left|\exp\left[\left(\gamma Z_{t-2} + \delta|Z_{t-2}|\right)\right] Z_{t-2}^3\right| \times$$

$$\times \exp\left[\left(\beta + \tfrac{3}{2}\right)\left(\gamma Z_{t-3} + \delta|Z_{t-3}|\right)\right] Z_{t-3}^2 \sigma_{t-4}^2 Z_{t-4}^2 ... \sigma_{t-k+1}^2 Z_{t-k+1}^2 |\sigma_{t-k} Z_{t-k}| =$$

$$= ... =$$

$$= Z_{t-1}^2 \exp\{\alpha\} \exp\left\{\left(\beta + \tfrac{3}{2}\right)\alpha\right\} \exp\left\{\left(1 + \tfrac{3}{2}\beta + \beta^2\right)\alpha\right\} \times ... \times \exp\left\{\left(1 + \beta + ... + \beta^{k-4} + \tfrac{3}{2}\beta^{k-3} + \beta^{k-2}\right) \alpha\right.$$

$$\times \left|\exp\left(\gamma Z_{t-2} + \delta|Z_{t-2}|\right) Z_{t-2}^3\right| \exp\left[\left(\beta + \tfrac{3}{2}\right)\left(\gamma Z_{t-3} + \delta|Z_{t-3}|\right)\right] Z_{t-3}^2 \times ... \times$$

$$\times \exp\left[\left(1 + \beta + ... + \beta^{k-5} + \tfrac{3}{2}\beta^{k-4} + \beta^{k-3}\right)\left(\gamma Z_{t-k+1} + \delta|Z_{t-k+1}|\right)\right] Z_{t-k+1}^2 \times$$

$$\times \left|\exp\left[\left(1 + \beta + ... + \beta^{k-4} + \tfrac{3}{2}\beta^{k-3} + \beta^{k-2}\right)\left(\gamma Z_{t-k} + \delta|Z_{t-k}|\right)\right] Z_{t-k}\right| \times$$

$$\times \exp\left[\left(\tfrac{1}{2} + \beta + ... + \beta^{k-3} + \tfrac{3}{2}\beta^{k-2} + \beta^{k-1}\right)\log\sigma_{t-k}^2\right].$$

Hence,

$$\begin{aligned} E|X_{t-1}|^2 |X_{t-2}|^3 ... |X_{t-k+1}|^2 |X_{t-k}| \leq\ & \exp\left[\frac{1}{(1-\beta)}\alpha\left[\frac{1}{2}\left(1 - \beta^{k-2}\right) + k - 1 - \beta\frac{1-\beta^{k-1}}{1-\beta}\right]\right] \times \\ & \times \left[E Z_0^2 \exp\left\{\left(\beta + \frac{3}{2}\right)\bar{\delta}|Z_0|\right\}\right]^{k-2} \times \\ & \times E\left[|Z_0| \exp\left(\left(\frac{1-\beta^{k-1}}{1-\beta} + \frac{1}{2}\beta^{k-3}\right)\bar{\delta}|Z_0|\right)\right] \times \\ & \times E\exp\left\{\left[\frac{1-\beta^k}{1-\beta} + \frac{1}{2}\left(\beta^{k-2} - 1\right)\right]\log\sigma_{t-k}^2\right\}. \end{aligned}$$

And, last:

$$|X_{t-1}|^2 |X_{t-2}|^2 |X_{t-3}|^2 ... |X_{t-k+1}|^3 |X_{t-k}| = Z_{t-1}^2 \sigma_{t-1}^2 \sigma_{t-2}^2 Z_{t-2}^2 \sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1}^3 Z_{t-k+1}^3 |\sigma_{t-k} Z_{t-k}|$$

$$= Z_{t-1}^2 \exp\left\{\log\sigma_{t-1}^2\right\} \sigma_{t-2}^2 Z_{t-2}^2 \sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1}^3 Z_{t-k+1}^3 |\sigma_{t-k} Z_{t-k}| =$$

$= Z_{t-1}^2 \exp\left(\alpha + \gamma Z_{t-2} + \delta |Z_{t-2}| + \beta \log \sigma_{t-2}^2\right) \exp\left\{\log \sigma_{t-2}^2\right\} Z_{t-2}^2 \sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1}^3 Z_{t-k+1}^3 |\sigma_{t-k} Z_{t-k}| =$

$= Z_{t-1}^2 \exp\{\alpha\} \exp\{(\gamma Z_{t-2} + \delta |Z_{t-2}|)\} Z_{t-2}^2 \exp\left\{(\beta+1) \log \sigma_{t-2}^2\right\} \sigma_{t-3}^2 Z_{t-3}^2 ... \sigma_{t-k+1}^3 Z_{t-k+1}^3 |\sigma_{t-k} Z_{t-k}|$

$= Z_{t-1}^2 \exp\{\alpha\} \exp\{(\gamma Z_{t-2} + \delta |Z_{t-2}|)\} Z_{t-2}^2 \exp\left\{(\beta+1)\left(\alpha + \gamma Z_{t-3} + \delta |Z_{t-3}| + \beta \log \sigma_{t-3}^2\right)\right\} \times$

$\times \exp\left\{\log \sigma_{t-3}^2\right\} Z_{t-3}^2 \left|\sigma_{t-4}^2 Z_{t-4}^2 ... \sigma_{t-k+1}^3 Z_{t-k+1}^3\right| |\sigma_{t-k} Z_{t-k}| =$

$= Z_{t-1}^2 \exp\{\alpha\} \exp\{(\beta+1)\alpha\} \exp\{(\gamma Z_{t-2} + \delta |Z_{t-2}|)\} Z_{t-2}^2 \exp\{((\beta+1))(\gamma Z_{t-3} + \delta |Z_{t-3}|)\} Z_{t-3}^2 \times$

$\times \exp\left\{(\beta(\beta+1)+1) \log \sigma_{t-3}^2\right\} \left|\sigma_{t-4}^2 Z_{t-4}^2 ... \sigma_{t-k+1}^3 Z_{t-k+1}^3\right| |\sigma_{t-k} Z_{t-k}| =$

$= ... =$

$= Z_{t-1}^2 \exp\{\alpha\} \exp\{((\beta+1))\alpha\} \exp\left\{\left(\beta^2+\beta+1\right)\alpha\right\} \times ... \times \exp\left\{\left(\beta^{k-2} + \beta^{k-3} + ... + \beta + \frac{3}{2}\right)\alpha\right\} \times$

$\times \exp\{(\gamma Z_{t-2} + \delta |Z_{t-2}|)\} Z_{t-2}^2 \exp\{(\beta+1)(\gamma Z_{t-3} + \delta |Z_{t-3}|)\} Z_{t-3}^2 \times ...$

$\times \exp\left\{\left(\beta^{k-3} + \beta^{k-4} + ... + 1\right)(\gamma Z_{t-k+1} + \delta |Z_{t-k+1}|)\right\} Z_{t-k+1}^2 \times$

$\times \left|\exp\left[\left(\beta^{k-2} + \beta^{k-3} + ... + \beta + \frac{3}{2}\right)(\gamma Z_{t-k} + \delta |Z_{t-k}|)\right] Z_{t-k}\right| \times$

$\times \exp\left\{\left(\beta^{k-1} + \beta^{k-2} + ... + \beta^2 + \frac{3}{2}\beta + \frac{1}{2}\right) \log \sigma_{t-k}^2\right\}.$

Hence,

$$\begin{aligned} E |X_{t-1}|^2 |X_{t-2}|^2 ... |X_{t-k+1}|^3 |X_{t-k}| \quad &\leq \quad \exp\left[\frac{1}{(1-\beta)}\alpha\left[\frac{1}{2}(1-\beta) + k - 1 - \beta\frac{1-\beta^{k-1}}{1-\beta}\right]\right] \times \\ &\times \left[E Z_0^2 \exp\left\{\bar{\delta} |Z_0|\right\}\right]^{k-2} \\ &\times E\left[|Z_0| \exp\left(\left(\frac{1-\beta^{k-1}}{1-\beta} + \frac{1}{2}\right)\bar{\delta} |Z_0|\right)\right] \\ &\times E \exp\left\{\left[\frac{1-\beta^k}{1-\beta} + \frac{1}{2}(\beta-1)\right] \log \sigma_{t-k}^2\right\}. \end{aligned}$$

□

Lemma C.13. *Find a tractable expression for* $\sum_{n=k-1}^{\infty}\left(\beta^n (|X_{t-n-1}|)\left(\prod_{i=1}^{k-1} |X_{t-i}|^2\right)\right) |X_{t-k}|$ *and calculate its moment bounds.*

Proof. $\sum_{n=k-1}^{\infty}\left(\beta^n (|X_{t-n-1}|)\left(\prod_{i=1}^{k-1} |X_{t-i}|^2\right)\right) |X_{t-k}| =$

$= \sum_{n=k-1}^{\infty}\left(\beta^n (|X_{t-n-1}|)\left(|X_{t-1}|^2 |X_{t-2}|^2 ... |X_{t-k+1}|^2\right)\right) |X_{t-k}|$

$\leq \beta^{k-1} \sum_{n=0}^{\infty} \beta^n |X_{t-1}|^2 |X_{t-2}|^2 \ldots |X_{t-k+1}|^2 |X_{t-k}|^2 \leq \frac{\beta^{k-1}}{1-\beta} \prod_{i=1}^{k} |X_{t-i}|^2,$

where

$E \prod_{i=1}^{k} X_{t-i}^2 = \exp\left(\frac{1}{1-\beta}\alpha\left(k-1-\frac{\beta-\beta^k}{1-\beta}\right)\right) \times \exp\left(\frac{1-\beta^k}{1-\beta}\log\sigma_{t-k}^2\right) \times$

$\times \exp\left(\frac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k} + \delta|Z_{t-k}|\right)\right) Z_{t-k}^2 \times \exp\left(\frac{1-\beta^{k-2}}{1-\beta}\left(\gamma Z_{t-k+1} + \delta|Z_{t-k+1}|\right)\right) Z_{t-k+1}^2 \times$

$\ldots\times$

$\times \exp\left(\frac{1-\beta^2}{1-\beta}\left(\gamma Z_{t-3} + \delta|Z_{t-3}|\right)\right) Z_{t-3}^2 \times \exp\left(\gamma Z_{t-2} + \delta|Z_{t-2}|\right) Z_{t-2}^2.$ □

Lemma C.14. *Moment bounds of* $E\left[\left(\prod_{i=1}^{k-1} |X_{t-i}|^3\right)|Z_{t-k}|\,\sigma_{t-k}\right]$

Proof. $E\left[\left(\prod_{i=1}^{k-1} |X_{t-i}|^3\right)|Z_{t-k}|\,\sigma_{t-k}\right] = E\left(Z_{t-1}^3\sigma_{t-1}^3 Z_{t-2}^3\sigma_{t-2}^3 Z_{t-3}^3\sigma_{t-3}^3 \ldots Z_{t-k+1}^3\sigma_{t-k+1}^3 |Z_{t-k}|\,\sigma_{t-k}\right) =$

$= E\exp\left(\frac{3}{2}\alpha\right) E\exp\left[\frac{3}{2}(\beta+1)\alpha\right] E\exp\left[\frac{3}{2}\left(\beta^2+\beta+1\right)\alpha\right]\times\ldots\times E\exp\left[\frac{3}{2}\left(\beta^{k-2}+\ldots+1\right)\alpha\right]\times$

$\times E\exp\left(\frac{3}{2}\left(\gamma Z_{t-2}+\delta|Z_{t-2}|\right)\right) Z_{t-2}^2 E\exp\left(\frac{3}{2}(\beta+1)\left(\gamma Z_{t-3}+\delta|Z_{t-3}|\right)\right) Z_{t-3}^2 \times\ldots\times$

$\times E\exp\left(\frac{3}{2}\left(\beta^{k-3}+\ldots+1\right)\left(\gamma Z_{t-k+1}+\delta|Z_{t-k+1}|\right)\right) Z_{t-k+1}^2\times$

$\times E\left|\exp\left(\frac{3}{2}\left(\beta^{k-2}+\ldots+1\right)\left(\gamma Z_{t-k}+\delta|Z_{t-k}|\right)\right) Z_{t-k}\right|\times E\exp\left(\left(\frac{3}{2}\left(\beta^{k-2}+\ldots+1\right)+\frac{1}{2}\right)\log\sigma_{t-k}^2\right).$

□

Lemma C.15. *Moment bounds of* $\left[\sum_{n=0}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\beta^{3(k-1)}c^{*(k-1)}\prod_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|Z_{t-k}|\,\sigma_{t-k}\right]$:

Proof. $E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\beta^{3(k-1)}c^{*(k-1)}\prod_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|Z_{t-k}|\,\sigma_{t-k}\right] =$

$= 2^{-3(k-1)}\overline{\delta}^{3(k-1)}\exp\left(-\frac{3}{2}\underline{m}(k-1)\right) E\left[\sum_{n=0}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|Z_{t-k}|\,\sigma_{t-k}\right] =$

$$= 2^{-3(k-1)}\overline{\delta}^{3(k-1)}\exp\left(-\frac{3}{2}\underline{m}(k-1)\right) E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|X_{t-k}| \\ +\sum_{n=k-1}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(\prod_{i=1}^{k-1}|X_{t-i}|^3\right)\right)|X_{t-k}|\end{array}\right] =$$

$$= 2^{-3(k-1)}\overline{\delta}^{3(k-1)}\exp\left(-\frac{3}{2}\underline{m}(k-1)\right) E\left[\begin{array}{c}\sum_{n=0}^{k-2}\left(\beta^n\left(|X_{t-n-1}|\right)\left(|X_{t-1}|^3|X_{t-2}|^3\ldots|X_{t-k+1}|^3\right)\right)|X_{t-k}| \\ +\sum_{n=k-1}^{\infty}\left(\beta^n\left(|X_{t-n-1}|\right)\left(|X_{t-1}|^3|X_{t-2}|^3\ldots|X_{t-k+1}|^3\right)\right)|X_{t-k}|\end{array}\right.$$

$$\le 2^{-3(k-1)}\overline{\delta}^{3(k-1)} \exp\left(-\tfrac{3}{2}\underline{m}\,(k-1)\right) E \left[\begin{array}{r} \sum_{n=0}^{k-2} \left(\beta^n \left(|X_{t-n-1}|\right)\left(|X_{t-1}|^3 |X_{t-2}|^3 \ldots |X_{t-k+1}|^3\right)\right) |X_{t-k}| \\ +\beta^{k-1} \sum_{n=0}^{\infty} \beta^n |X_{t-1}|^3 |X_{t-2}|^3 \ldots |X_{t-k+1}|^3 |X_{t-k}|^2 \end{array} \right]$$

$$\le 2^{-3(k-1)}\overline{\delta}^{3(k-1)} \exp\left(-\tfrac{3}{2}\underline{m}\,(k-1)\right) E \left[\begin{array}{r} (k-1) \sum_{n=0}^{k-2} \beta^n \left(|X_{t-k+1}|\right) \prod_{i=1}^{k-1} |X_{t-i}|^3 |X_{t-k}| \\ +\frac{\beta^{k-1}}{1-\beta} \prod_{i=1}^{k-1} |X_{t-i}|^3 |X_{t-k}|^2 \end{array} \right]$$

$$\le 2^{-3(k-1)}\overline{\delta}^{3(k-1)} \exp\left(-\tfrac{3}{2}\underline{m}\,(k-1)\right) E \left[\begin{array}{r} (k-1) \sum_{n=0}^{k-2} \beta^n \left(|X_{t-1}|^3 |X_{t-2}|^3 \ldots |X_{t-k+1}|^4\right) |X_{t-k}| \\ +\frac{\beta^{k-1}}{1-\beta} \prod_{i=1}^{k-1} |X_{t-i}|^3 |X_{t-k}|^2 \end{array} \right],$$

where

$$E \prod_{i=1}^{k-1} |X_{t-i}|^3 |X_{t-k}|^2 = \exp\left(\tfrac{3}{2}\tfrac{1}{1-\beta}\alpha\left(k-1-\tfrac{\beta-\beta^k}{1-\beta}\right)\right) \times E\exp\left(\left(\tfrac{3}{2}\tfrac{\beta-\beta^k}{1-\beta}+1\right)\log\sigma_{t-k}^2\right) \times$$

$$\times E\exp\left(\tfrac{3}{2}\tfrac{1-\beta^{k-1}}{1-\beta}\left(\gamma Z_{t-k}+\delta\left|Z_{t-k}\right|\right)\right) Z_{t-k}^2 \times E\left[\left|Z_{t-k+1}^3\right| \exp\left(\tfrac{3}{2}\tfrac{1-\beta^{k-2}}{1-\beta}\left(\gamma Z_{t-k+1}+\delta\left|Z_{t-k+1}\right|\right)\right)\right] \times$$

$$\ldots\times$$

$$\times E\left[\left|Z_{t-3}^3\right| \exp\left(\tfrac{3}{2}\tfrac{1-\beta^2}{1-\beta}\left(\gamma Z_{t-3}+\delta\left|Z_{t-3}\right|\right)\right)\right] \times E\left[\left|Z_{t-2}^3\right| \exp\left(\tfrac{3}{2}\left(\gamma Z_{t-2}+\delta\left|Z_{t-2}\right|\right)\right)\right]. \quad \square$$

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