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Dissertation Subject:

The Valuation of Floating Rate Mortgages
The case of Adjustable Rate Mortgages (A.R.Ms)
The Valuation of A.R.Ms indexed to the ECB lending rate

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Abstract

This dissertation attempts to investigate a consistent valuation algorithm for the valuation of adjustable rate mortgages indexed to the ECB lending rate. More specifically, the option of the mortgage holder to prepay his obligation at any point during the life of the loan is examined. Due to the fact that mortgage cash flows depend on the path that the interest rates will follow during the life of the loan, the Monte – Carlo simulation technique was used as a basic valuation tool. The market rate was assumed to follow the stochastic process introduced by Cox, Ingersoll and Ross (1985) while the index rate was assumed to have a linear relationship with the market rate and the first lag of the index rate. In order to value the prepayment option, the methodology used was similar to the one proposed by Longstaff and Schwartz (2001) for the valuation of American options with the use of Monte – Carlo simulation. The main conclusions of the empirical results are: a) there is a positive relationship between the value of the equivalent bond (the stream of promised payments, with no prepayment option) and the value of the prepayment option, b) the value of the prepayment option is critically dependent on the interaction between contract features, especially caps and margin levels. More specifically, caps pose an upper boundary on the prepayment option values while margins significantly increase the prepayment option value as a percentage of the remaining principal amount, c) the lag in the adjustment of the index rate with the market rate has a significant effect on the value of the prepayment option.

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Introduction

During the past years, and specifically ever since Greece joined the Euro-zone, credit extension has seen a dramatic increase. In addition, low ECB rates in conjunction with increasing house prices have extended mortgage lending as the purchase of a house has become a profitable investment. In this favorable environment, Greek banks have developed a variety of mortgages featuring different characteristics.

The purpose of this dissertation is to investigate an approach to the valuation of mortgages that nowadays occupy a large portion of the lending portfolio of Greek banks. The literature offers a variety of approaches used in the U.S. market, since with the process of securitization in 1980 – 1990's, many U.S. banks issued Mortgage Backed Securities (MBS) which were tradable instruments and therefore created the need for a consistent valuation approach.

The most interesting feature of these instruments, apart from behaving as regular bonds, is that the borrower has the right to prepay the mortgage at any time prior to the maturity of the mortgage. This option feature complicates the valuation procedure as the option to prepay has the characteristics of an American type option, which renders the mortgage a path-dependent security. Therefore, the valuation process differs from the valuation of regular fixed income instruments and consequently, various methods for valuing path-dependent securities have been examined by the literature.

Another feature that ought to be examined is the option of the borrower to default, which needs to be accounted for in order to properly price this type of instrument. Therefore these two option features pose difficulties in the valuation of mortgages as their exercise time is not fixed nor known beforehand and depends on various factors such as the level of interest rates and house prices.

The structure of this dissertation is as follows: in Section I, a general review of mortgage types as well as those offered by Greek banks is provided. In Section II, a review of the literature for the valuation of mortgages is demonstrated and in Section III a review of the methods that will be employed in this approach will be illustrated.

Section 1: Basic Characteristics of Mortgages

1.1 Types of mortgages

As mentioned in the introduction there are various types of mortgages offered by banks. In this Section, a brief review of these types will be presented, with a focus on the types used by Greek banks.

Mortgages can be divided into two broad categories:

a) Fixed-Rate Mortgages or FRMs
b) Floating Rate Mortgages or ARMs

A) FRMs have the following characteristics:

Coupon rate: The coupon rate on a mortgage determines the periodic cash flows the borrower must pay in order to amortize his obligation and reflects part interest and part principal over the life of the loan. The frequency of the cash flows can be monthly, semiannually or yearly. This coupon is fixed for the entire duration of the mortgage for FRMs

Margin: In FRMs a margin is added to the coupon rate depending on various factors such as the credit rating of the borrower, the value of the house, the face value of the mortgage, the duration of the mortgage and so on.

Life of the mortgage: The life of the mortgage is the amount of years in which the borrowers must repay his obligation.

B) ARMs have the following characteristics:

Coupon rate: As with FRMs, the coupon rate on a mortgage determines the periodic cash flows the borrower must pay in order to amortize his obligation and reflects part interest and part principal over the life of the loan. The frequency of the cash flows can be monthly, semiannually or yearly. However, the coupon rate is floating in an ARM. The floating coupon rate is reset periodically at each "reset date" and remains unchanged until the next "reset date".

Underlying Index: The adjustment rule for the coupon rate specifies an underlying index to which the rate is tied. This is only valid for ARMs and therefore there are ARMs indexed to various indices such as the 1 year T-Bill, the 1 year LIBOR, the Eleventh District Cost of Funds Index (EDCOFI) or the ECB lending rate, which will be examined more thoroughly in this study.

Margin: As with FRMs, similarly in ARMs a margin is added to the coupon rate depending on various factors such as the credit rating of the borrower, the value of the house, the face value of the mortgage, the duration of the mortgage and so on.

"Teaser" Rate: It is common for the initial coupon rate in ARMs to be lower than the fully indexed rate given by the margin to the initial level of the index. This initial rate is often referred to as a "teaser rate".

Annual Cap: ARM contracts usually specify a maximum adjustment in the coupon rate at each reset period, for example 2% per year.

Lifetime Cap: ARM contracts usually specify an overall maximum coupon rate over the life of the loan and a minimum coupon rate over the life of the loan.

Reset Frequency: The coupon rate on an ARM contract is adjusted at specific intervals. This interval can be the same as the frequency of the payments of the coupon rate or it can be different. For example, if the frequency of coupon rate payments is monthly, then the reset frequency can be monthly, semiannually or even yearly.

Life of the mortgage: The life of the mortgage is the amount of years in which the borrower must repay his obligation.

1.2 The Greek market

In the Greek market one can find both FRMs and ARMs. Moreover, Greek banks sometimes offer a combination of the two. For example, they offer mortgages with a fixed coupon rate for a certain number of years, and later on the coupon rate become floating. For ARM contracts the underlying index is the same across banks and this constitutes the ECB lending rate. The life of the mortgage varies and can be from 15 year to 40 years depending on the contract. However, there are some banks that offer lifetime caps for ARM while the majority of banks provide uncapped contracts. The reset frequency is not specified and adjustment to the coupon rate occurs only when the ECB decides to change its benchmark rate. The level of adjustment is 1:1 as stated in all mortgage contracts. The type of mortgage that will be examined in the present dissertation will be a 30 year fully floating non-defaultable ARM with monthly payments and with yearly index reset frequency. The ECB lending rate plus a margin of 2% will be used as an index rate and the 12-month Euribor rate as the market rate.

Section 2: Review of the Literature

The MBS securities mentioned in the introduction have attracted a great deal of interest and as a result, there is extensive literature concerned with their valuation. First of all, the problem of valuation needs to be divided into three main areas. Firstly, a model must be adopted for the term structure of the interest rates in order to valuate properly the cash flows coming from the mortgages, assuming that market interest rates are stochastic. Secondly, a model needs to be implemented in order to capture the behavior of the underlying index for ARMs as it determines the magnitude of the amounts of future cash flows. Finally, a pricing algorithm has to be developed in order to price path-dependent securities. Hence, this section is divided into three parts in order to review the literature concerning each area of the valuation problem.

2.1 Term Structure Models

The literature on term structure is overwhelming and therefore it will be attempted to present a concise but complete review of past research[1].Term structure models can be divided into many groups and the taxonomy can be confusing. For instance, there are continuous versus discrete models, single versus multi-factor models, fitted versus non fitted models and arbitrage free versus equilibrium models. The vast majority of term structure models are specified in continuous time framework since "the power of continuous time stochastic calculus allows more elegant derivations and proofs and provides an adequate framework to produce more precise theoretical solutions and more refined empirical hypothesis, unfortunately at the cost of a considerably higher degree of mathematical sophistication"[2].

For the purposes of this dissertation, focus will be mainly placed on single factor models where the single factor affecting the term structure of interest rates is assumed

[1]For an excellent and comprehensive review, the attentive reader can refer to the paper of R.Gibson, F.S.Lhabitant, D.Talay (2001): "Modeling the term structure of interest rates: a review of the literature", upon which this section is based on.
[2]Gibson, Lhabitant, Talay: "Modeling the term structure of interest rates: a review of the literature", 2001.

to be the instantaneous spot rate. Although this may seem over-simplified, multi-factor models such as the Brennan and Schwartz model[3] or empirical studies using a principal component analysis[4] will not be used. Instead, the single factor model was chosen for computational ease.

The most famous single factor models are those of Merton (1973), Vasicek (1977) and Cox, Ingersoll and Ross (1985). All of the above are equilibrium models meaning that they start from a description of the economy, - including the utility function[5] of a representative investor-, and derive the term structure of interest rates, the risk premium and other asset prices endogenously, based on the assumption that the market is in equilibrium. In particular, they conclude that the instantaneous spot rate follows a specific diffusion process, according to which assets dependent on the spot rate are priced analogously.

Merton (1973) was the first to propose a general stochastic process as a model for the short rate. The process for the short term rate is:

$$dr(t) = \mu_r dt + \sigma_r dW(t)$$

where μ_r and σ_r are constants and W(t) is a standard Brownian motion. Furthermore, Merton assumes a constant risk premium λ. Given the above, the short term rate is normally distributed, which means it can become negative.

Vasicek(1977) proposed to model the short term interest rate as an Ornstein – Uhlenbeck process:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma \cdot dW(t)$$

where κ, θ and σ are positive constants and W(t) is a standard Brownian motion. This defines an elastic random walk around a trend, with a mean reverting characteristic: when r(t) goes over or under θ, it tends to come back to its average long term level θ at an adjustment speed of κ. In addition, Vasicek postulates a constant risk premium λ. The spot rate, both in Merton and Vasicek, is normally distributed and as a consequence interest rates can become negative, which is incompatible with no arbitrage in the presence of cash in the economy. Nevertheless, the mean reversion

[3] Brennan M.J., Schwartz E.S. (1979), "A continuous time approach to the pricing of bonds", Journal of Banking and Finance, vol. 3, pp. 135 – 155
[4] see Wilson (1994) or Knez, Litterman, Schenkman (1994), "Explorations into factors explaining money market returns", Journal of Finance, vol. 49, pp. 1861 – 1882.
[5] Especially in the Cox, Ingersoll and Ross model this function is assumed to be logarithmic

process reduces the probability of unreasonably large or low interest rates. Under this model the term structure of interest rates can be positively shaped, negatively shaped or humped.

Cox, Ingersoll and Ross (1985b) have developed an equilibrium model in which interest rates are determined by the supply and demand of individuals having a logarithmic utility function. The result is a single factor model in which the short term rate satisfies:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

where κ, θ and σ are positive constants and W(t) is a standard Brownian motion and the risk premium at equilibrium is described as

$$\lambda(r,t) = \lambda\sqrt{r(t)}$$

The endogenously derived short term rate process, also known as the square root process, is similar to that of Vasicek (1977), but its variance is proportional to the short rate rather than constant. This means that as the short term interest rate increases, its standard deviation increases as well. Furthermore, if it hits the zero-boundary (which is only possible if $\sigma^2 > 2\kappa\theta$) , it will never become negative. According to this model, the value of r(t) determines the level of the term structure at time t, but not its shape. As in the case of Vasicek, upward sloping, downward sloping and humped yield curves are admissible. In reality, this is the model most commonly used in the literature of ARMs in order to model the market rate[6].

Apart from being equilibrium models, all of the above also constitute non fitted models. This means that they first specify the dynamics of the state variables and as a result of a particular specification they endogenously obtain a given term structure. In fact, this is one of their main disadvantages since they generally do not fit well with the initially observed term structure. Numerous scholars have tested the empirical application of these models and their results corroborate the aforementioned disadvantage[7].

[6] for example Titman and Torous , Stanton and Wallace.
[7] see Gibbons, Ramaswamy (1993), "A test of the Cox, Ingersoll and Ross model of the term structure", Review of Financial Studies, vol. 6, pp. 619 – 658 and Chan, Karolyi, Longstaff, Sanders (1992), "An empirical comparison of alternative models of the short term interest rate", Journal of Finance, vol. 47, pp. 1209 – 1227.

In order to overcome this discrepancy, fitted models have been proposed in the literature of term structure models. In fitted models, a term structure is determined exogenously, generally using market data, and the stochastic differential equation of certain state variables is specified in such a way that this term structure is obtained at a particular date. In a sense, the model is specifically built to fit an arbitrary (exogenous) initial term structure. A particularly interesting group of models are encountered in this category; they are the so-called forward curve models and rely on arbitrage free conditions. The first contribution to this approach was made by Ho and Lee (1986) in discrete time, but the most significant one was made by Heath, Jarrow, Morton (1992). The advantage of the approach is that if at time 0, the theoretical forward rate $f(0,T)$ is set to equal the observed one $f^*(0,T)$, there will be a perfect fit for the whole current term structure.

The discrete multi-period binomial model of Ho and Lee (1986) is particularly important since it was the first to model the movements of the entire term structure. Ho and Lee take the initial term structure as exogenously given at a point in time (by a set of zero coupon prices) and derive the subsequent feasible term structure movements so that they are compatible with no-arbitrage opportunity. They create a binomial lattice of the term structure rather than a binomial process for the bond price. Their result for the short term rate is the following:

$$r(t) = r(t-1) + (f(0,t) - f(0,t-1)) + \log\left(\frac{\pi + (1-\pi)\delta^{t}}{\pi + (1-\pi)\delta^{t-1}}\right) - (1-\pi)\log(\delta) + \varepsilon_t$$

This indicates that the interest rate at time t depends on the interest rate one period ago, the relevant slope of the yield curve (denoted by $(f(0,t) - f(0,t-1))$), a constant depending on time: $\log\left(\frac{\pi + (1-\pi)\delta^{t}}{\pi + (1-\pi)\delta^{t-1}}\right)$, π which is the probability of up and down movements and is constant through time, δ which is a parameter defining the magnitude of up and down movements and is also constant and some noise denoted by ε_t.

Of course, the estimation of parameters π and δ have to be calculated from the data, meaning that a binomial tree has to be constructed and subsequently solved backwards for the values of π and δ. Obviously, this method becomes extremely complicated and requires numerous computations. Other disadvantages of the Ho and Lee model include that:

a) upward sloping term structures will always be generated,

b) interest rates may become infinite or negative at extreme points, since mean-reversion features are not incorporated at all,

c) it implies that all spot forward rates have the same instantaneous constant standard deviation σ_r,

d) it is not necessarily arbitrage-free due to b).

Heath, Jarrow and Morton (1992) have significantly extended the Ho and Lee (1986) model in three directions. Firstly, they consider forward rates rather than bond prices as their basic building blocks. Secondly, they allow for continuous trading, which results in a valuation formula that is independent of the pseudo probabilities π found in the Ho and Lee model. Thirdly, they extend the model from a one factor to allow for multiple factors.

Although it is not explicitly derived from an equilibrium framework, the Heath, Jarrow and Morton (1992) model is a model that completely explains the term structure dynamics in an arbitrage-free framework in the spirit of Harrison and Kreps (1979), and is fully compatible with an equilibrium model.

Heath, Jarrow and Morton set the forward rate for each fixed maturity T to evolve according to the following process:

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dW(t) \tag{1}$$

where μ_f(t,T) and σ_f(t,T) are adapted processes for each T. This specification is very general as the drifts μ_f(t,T) and volatilities σ_f(t,T) can in fact depend on the history of the Brownian motion W(t) and on the forward rate themselves up to time t. The model can be written in an integral form as

$$f(t,T) = f(0,T) + \int_0^t \mu_f(s,T)ds + \int_0^t \sigma_f(s,T)dW(s) \tag{2}$$

As a boundary value at time 0, we use the observed forward curve f *(0,T), that is for all T, and set

$$f(0,T) = f^*(0,T).$$

Owing to the manner of its construction, the model will perfectly fit the observed term structure.

It should be noted that a direct implication is that there is no such thing as "the" Heath, Jarrow, Morton model. Rather, there is a whole class of models, each characterized by specific functional forms of drifts and volatilities.

The major outcome of Heath, Jarrow, Morton is the following proposition which is essential for the existence of a unique equivalent martingale measure; namely, for computing the price of contingent claims by discounting their terminal expected values.

Proposition 7 *The following conditions are equivalent:*

1. *the market price of risk $\lambda(t)$ is independent of the maturity dates*
2. *a unique martingale measure exists*
3. *the parameters $\mu_f(t,T)$ and $\sigma_f(t,T)$ cannot be freely specified: drifts of forward rates under the risk-neutral probability are entirely determined by their volatility and by the market price of risk.*

The third segment of the proposition is probably the greatest contribution of the Heath, Jarrow, Morton model, as it allows the model to be arbitrage-free, a major improvement over the Ho and Lee (1986) and other similar models.

Proposition 8 *It is assumed that the family of forward rates is given by (1). Thus, in order to avoid arbitrage opportunities, there must be an adapted process $\lambda(t)$ which is independent of the maturity T such that*

$$\mu_f(t,T) = \sigma_f(t,T)\left[\int_t^T \sigma_f(t,s)ds - \lambda(t)\right] \tag{3}$$

It can be demonstrated that $\lambda(t)$ represents the instantaneous market price of risk, which is independent of the maturity T.

This proposition is similar to the arbitrage condition used in the one factor models. However, it implies that the choice of a particular model from the general specification of Heath, Jarrow and Morton can be reduced to the specification of the volatility coefficient. For a particular risk premium, the drift coefficient can be easily retrieved using (3). Furthermore, as volatility is the same under the risk neutral and the subjective probability, it can be estimated using historical data.

Accounting for this new no-arbitrage condition, equation (1) can be rewritten as

$$df(t,T) = \sigma_f(t,T)\left[\int_t^T \sigma_f(t,s)ds - \lambda(t)\right]dt + \sigma_f(t,T)dW(t) \tag{4}$$

under the subjective probability.

Under the risk neutral measure equation (3) can be rewritten as:

$$\mu_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,s)ds$$

and equation (4) as

$$df(t,T) = \sigma_f(t,T)\left[\int_t^T \sigma_f(t,s)ds\right]dt + \sigma_f(t,T)dW^*(t)$$

or equivalently in an integral form as

$$f(t,T) = f(0,T) + \int_0^t \sigma_f(s,T)\left[\int_s^t \sigma_f(s,u)du\right]ds + \int_0^t \sigma_f(s,T)dW^*(s) \tag{5}$$

where dW*(t) is the risk-neutral Brownian motion.

Since r(t) = f (t,t), the dynamics of the short rate under the risk neutral probability can be obtained from (5) in integral form as

$$r(t) = f(0,t) + \int_0^t \sigma_f(s,T)\left[\int_s^t \sigma_f(s,u)du\right]ds + \int_0^t \sigma_f(s,T)dW^*(s) \tag{6}$$

where dW*(t) is the Brownian motion generated by the risk-neutral probability measure. The principal difficulty of estimating a Heath, Jarrow, Morton model will arise because of the non-Markovian term in equation (6), which depends on the history of the process for time 0 to time t[8]. However, there are two volatility specifications under which the resulting models will be Markovian. The first one is the constant volatility model (Ho and Lee) for which $\sigma_f(t,T) = \sigma$ and the second is the exponential decay model for which $\sigma_f(t,T) = \sigma \cdot e^{-\lambda(T-t)}$.

[8] Hull and White (1993), Carverhill (1994, 1995) and Jeffrey (1995) have derived conditions for the volatility function that result in a Markovian short term interest rate.

According to Gibbons, Lhabitant and Talay the procedure for practical implementation of a Heath, Jarrow, Morton model is the following:

1. Specification of the volatilities $\sigma_f(t,T)$. The forward rate drifts do not need to be specified, as they will be uniquely determined by

$$\mu_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,s)ds$$

2. Observation of the effective forward rate structure $f^*(0,T)$ for $T \geq 0$.
3. Computation of the forward rate according to equation (2)
4. Computation of bond prices according to

$$B(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$$

For example, in the Ho and Lee model where volatility is assumed constant, the Heath, Jarrow, Morton terms are as follows:

The corresponding drift is: $\mu_f(t,T) = \sigma^2(T-t)$,

The forward rate is given by: $f(t,T) = f^*(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma W(t)$

And the spot rate: $r(t) = f(t,t) = f^*(0,t) + \sigma^2 \frac{t^2}{2} + \sigma W(t)$.

In conclusion, Gibbons, Lhabitant and Talay suggest that among different interest rate models the Heath, Jarrow, Morton model is the most appealing as it offers the following advantages:

a) the current term structure is matched by construction, without requiring an arbitrary time varying parameter
b) it does not make any presuppositions regarding investors' preferences, and it yields a pricing function that is uniquely determined by the specification of the variance structure of interest rate changes. Drifts estimates are not necessary; something which simplifies the estimation procedure.
c) All other interest rate models based on diffusion processes are nested into this general specification, including non-Markovian ones.

For these reasons, it is believed that currently the Heath, Jarrow and Morton class of term structure models is the most broadly defined yet unified framework for model comparison and model risk assessment.

2.2 Index Models

Although the underlying index plays an important role in the valuation of adjustable rate mortgages, past research does not thoroughly address this issue. Most of the literature concerning the pricing of ARMs assumes that the index rate is the same with the market rate used for discounting and therefore they both rates are modeled using the same process. However, Ott (1986) found that the index plays an important role in determining the price sensitivity of an ARM by examining the duration of such contracts. More specifically, he discovered that if the index responds quickly to market rate changes then the duration of an ARM is close to that of a fully floating ARM, while if the index is irresponsive to market rate changes then the duration of an ARM is close to that of a fixed rate mortgage.

To obtain a better understanding of this behavior he postulates the following model for the index[9]:

$$I_t = a + bR_t + cI_{t-1} + U_t \qquad (7)$$

where I is the specific index, R the market interest rate and U an error term. The coefficient b depicts the degree to which the market interest rate affects the index each period, while c is the adjustment coefficient relating how fast the index adjusts. If b equals 1 and c zero, the index adjusts perfectly to the market; while if b is zero and c one, the index is irresponsive to changes in the market interest rate. For incomplete adjustments to the market interest rate, the adjustment for j periods equals: $\sum_{i=1}^{j} bc^i$, while the total adjustment would be: $\frac{b}{1-c}$.

[9] This specification is also discussed by Cornell (1987): "Forecasting the eleventh district cost of funds", Passmore(1993): "Econometric models for the eleventh district cost of funs index" and Roll(1987): "Adjustable rate mortgages: The indexes".

However, most valuation approaches used in the literature consider that the market rate equals the index rate. One of the few exceptions are Stanton and Wallace (1994), (1996) that use the above model for various index rates as the one year LIBOR, the one year constant maturity Treasury yield, the Federal Housing Finance Board (FHFB) national average contract interest rate and the Eleventh District Cost of Funds Index (EDCOFI). According to their approach, if the error term is ignored and the index starts at a value I_0 while the interest rate remains at a constant level r, they demonstrated that the value of the index at any later time is given by:

$$I_t = (1 - c^t)\frac{a + br}{1 - c} + c^t I_0 \tag{8}$$

This is a weighted average of the long run value of I_t and its initial value. The speed of convergence is governed by the value of c. The half life, the number of periods required to reach half way between the two values is the solution to

$$c^{t_{1/2}} = \frac{1}{2}$$

which is

$$t_{1/2} = -\frac{\log(2)}{\log(c)}.$$

Like in Ott(1986) if we substitute $c = 0$ we have instantaneous adjustment, and equation (7) is reduced to :

$$I_t = a + br \tag{9}$$

According to the above findings by Ott (1986) and Stanton and Wallace (1994), (1996), the implementation of index dynamics is essential in order to correctly evaluate an ARM contract.

2.3 Valuation Models

After adopting the correct model for the term structure and the index rate the next, and most intriguing, step is the valuation method that should be employed in order to price an ARM or FRM contract. The main difficulty in this procedure, as mentioned in the introduction, is to take into account the option features of prepayment and default that are incorporated in these contracts. Titman and Torous (1989) start by examining non-prepayable fixed rate mortgages. In their approach, the mortgage value is determined by two state variables, the instantaneous risk-free interest rate and the value of the mortgaged building. Therefore, they assume two processes, one for the interest rate and one for the building price. The process for the interest rates is the Cox, Ingersoll and Ross (1985) model while for the value of the building they assume a lognormal diffusion process. Applying Ito's lemma they derive the partial differential equation for the value of the mortgage and solve it by using the hopscotch method. It should be noted that in their analysis rational default by the borrowers is assumed, meaning that default will occur if the value of the mortgaged building falls below the value of the mortgage, at any time prior to maturity. By comparing their estimated results with rates quoted by large insurance companies they found that their model provides an accurate explanation for market prices.

The next approach came from Schwartz and Torous(1989), Dunn and McConnell (1981) and McConnell and Singh(1993), that tried to integrate the prepayment feature in default-free mortgages. They used the Brennan and Schwartz (1979) two-factor interest rate model and tried to extract a prepayment function based on historical prepayment rates. Using maximum likelihood estimation, they derived a prepayment function and then implemented its dynamics on the valuation process. A main characteristic of the above approaches is that they try to model rational prepayment meaning that mortgagors will call their loans if the refinancing rate is less than the contract rate. Nevertheless, by implementing their model they are required to explicitly specify the factors that influence the prepayment decision, such as demographic and geographic explanatory variables, which poses difficulties in the practical implementation.

Kau, Keenan, Muller and Epperson (1990), (1993) published a series of articles for the valuation of FRMs and ARMs by taking into account both prepayment and default. The breakthrough of their approach was that they incorporated the prepayment and default option of the borrowers in their valuation procedure by adding an artificial state variable, the past contract rate. Therefore, their model is determined by 3 state variables: a) the spot interest rate, which is assumed to follow the process proposed by

Cox, Ingersoll and Ross (1985), b) the house price, which is assumed to follow the lognormal process proposed by Merton (1973) and c) the past contract rate that actually depends on a).

For non-defaultable mortgages, they divide the value of the contract as follows:

$$V = A - J$$

where A is the value of a floating rate default-free and non-prepayable bond providing the same payments as the mortgage and J is the option to terminate the mortgage due to prepayment.

Under this setting they came up with the partial differential equation:

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 X}{\partial r^2} + \gamma(\theta - r)\frac{\partial X}{\partial r} + \frac{\partial X}{\partial t} - rX = 0$$

where γ, θ and σ are the parameters from the interest rate model and X is the value of the mortgage. By imposing appropriate terminal and boundary conditions, they solved the above PDE numerically by using a standard explicit finite difference method to valuate each contract. In addition to the above, they tested the pricing results if the contract is subject to yearly and lifetime caps. One of their major conclusions was that prepayment is a decision endogenously determined as part of the valuation of a mortgage, since the value of the mortgage depends on its likely occurrence and its occurrence depends on the cost of the mortgage. Although they do not deal with index-amortized notes their approach is also applicable to these instruments.

Stanton and Wallace (1994), (1996) examine the solution procedure of Kau, Keenan, Muller and Epperson (1990), (1993) for default free but index-amortized ARM for various indexes by modeling the index as in equation (7). First of all, they assume that interest rates are determined by the Cox, Ingersoll and Ross one factor model. Their next step is to estimate the parameters for the index rate model. Their methodology for pricing the ARM contract is similar to that used by Kau, Keenan, Muller and Epperson. The difference is that they extend their finite differences method to multiple state variables in order to take into account the dynamics of the index rate.

Furthermore, Downing, Stanton and Wallace (2005) take this one step further in order to include the effect of default in their valuation process. The interest rate process assumed is the Cox, Ingersoll and Ross model. In order to capture the dynamics of default, they also assumed that house prices evolve according to a geometric Brownian Motion. They proceeded in their analysis by using historical prepayment data in order to model the overall hazard rate governing mortgage termination. This specification changes the final partial differential that needs to be solved numerically and the corresponding boundary conditions. The above research yields statistical and economically significant improvements in its ability to match historical prepayment data.

Finally, two recent studies by J.Chen(2004) and S.G.Chastian, J.Chen(2005), focus on the same problem by using Monte-Carlo Simulation in their pricing procedure. Specifically, they simulate the interest rate process and a prepayment model in order to calculate the prices of Mortgage-Backed-Securities. They argue that their method is easier to implement than the finite differences method and produces efficient results.

Section 3: Methodology and Empirical Results

3.1.1 Estimation of the parameters for the Interest Rate process

As mentioned in the introduction, the instantaneous spot rate is assumed to be the 12-month Euribor. In order to model its behavior the Cox – Ingersoll - Ross (CIR) model shall be used. The Heath-Jarrow-Morton framework is avoided, although it may appear more appealing theoretically, since it allows interest rates to become negative and will thus produce ambiguous results. Moreover, the CIR model is widely used in the mortgage valuation literature. Therefore, in order to compare the results of the present dissertation with previous studies it would be more suitable to use the same model for the behavior of interest rates. Under the CIR model the general expression for the evolution of the short-term interest rate is:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t) \qquad (10)$$

In order to estimate the unknown parameters κ, θ the General Method of Moments will be used, according to the paper of Chan, Karolyi, Longstaff and Sanders[10]. For the parameter σ the implied volatility of interest rate options traded in the market will be used.

Following the paper of Chan, Karolyi, Longstaff and Sanders, Hansen's (1982) generalized method of moments (GMM) technique will be used in order to estimate model parameters, using first order moment restrictions. The goodness of fit between model and data is given by Hansen's *J* statistic, which measures the degree to which the moment conditions are satisfied.

The econometric specification that will be used is the following:

$$\begin{aligned} r_{t+1} - r_t &= \alpha + \beta r_t + \varepsilon_{t+1}, \\ E[\varepsilon_{t+1}] &= 0, \\ E[\varepsilon_{t+1}^2] &= \sigma^2 r_t \end{aligned} \qquad (11)$$

[10] K.C. Chan, A. Karolyi, F.A. Longstaff, A.B. Sanders : "An Empirical Comparison of Alternative Models of the Short – Term Interest Rate" , The Journal of Finance, Vol. 47, No.3, Jul. 1992.

where,
α is equivalent to $\kappa \cdot \theta$ and
β is equivalent to $-\kappa$.

It is important to acknowledge that the above discretized process is only an approximation of the continuous time specification. Given the continuity of the interest rate process, however, the amount of approximation error introduced can be shown to be of second-order importance if changes in *r* are measured over short periods of time.

The econometric approach is to test (11) as a set of over identifying restrictions on a system of moment equations using the Generalized Method of Moments (GMM) of Hansen (1982). This technique has a number of significant advantages that render it an intuitive and logical choice for the estimation of continuous time interest rate processes. First of all, the GMM approach does not require that the distribution of interest rate changes be normal; the asymptotic justification for the GMM procedure requires only that the relevant expectations exist. Secondly, the GMM estimators and their standard errors are consistent even if the disturbances, ε_{t+1}, are conditionally heteroskedastic. Since the temporal aggregation problem that arises from estimation of a continuous time process with discrete time data is likely to influence the distribution of the disturbances, the GMM approach should further alleviate the impact of this approximation error on the parameter estimates. For example, even though the CIR continuous time model assumes that changes in *r* are distributed as a random variable proportional to a non central χ^2, the discrete time version of the model may not. Finally, the GMM technique has also been used in other empirical tests of interest rate models by Gibbons and Ramaswamy (1986), Harvey (1988), and Longstaff (1989).

If θ defined as the parameter vector with elements α, β and σ^2, and given $\varepsilon_{t+1} = r_{t+1} - r_t - \alpha - \beta r_t$, then the vector $f_t(\theta)$ would be:

$$f_t(\theta) = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1} r_t \\ \varepsilon_{t+1}^2 - \sigma^2 r_t \\ \left(\varepsilon_{t+1}^2 - \sigma^2 r_t\right) \cdot r_t \end{bmatrix}. \tag{12}$$

Under the null hypothesis whereby the restrictions implied by (11) are true, $E[f_t(\theta)]=0$. The GMM procedure consists of replacing $E[f_t(\theta)]$ with its sample counterpart, $g_{\mathrm{T}}(\theta)$, using the T observations where

$$g_{\mathrm{T}}(\theta)=\frac{1}{T}\sum_{t=1}^{T}f_t(\theta), \tag{13}$$

and then choosing parameter estimates that minimize the quadratic form,

$$J_T(\theta)=g_T'(\theta)\cdot W_T(\theta)\cdot g_T(\theta), \tag{14}$$

where $W_T(\theta)$ is a positive definite symmetric weighting matrix. Matrix differentiation shows that minimizing $J_T(\theta)$ with respect to θ is equivalent to solving the homogeneous system of equations (orthogonality conditions),

$$D'(\theta)\cdot W_T(\theta)\cdot g_T(\theta)=0 \tag{15}$$

where $D(\theta)$ is the Jacobian matrix of $g_{\mathrm{T}}(\theta)$ with respect to θ.

The minimized value of the quadratic form in (14) is distributed χ^2 under the null hypothesis that the model is true with degrees of freedom equal to the number of orthogonality conditions net of the number of parameters to be estimated. This χ^2 measure provides a goodness of fit test for the model. A high value of this statistic means that the model is misspecified.

In order to apply the above methodology, 12-month Euribor rates from 15/10/2001 up to 22/02/2006 on a daily basis (1137 observations) were used. For the volatility parameter, the implied volatility for interest rate options on the 12-month Euribor as observed on 22/02/2006 was used.

3.1.2 Empirical Results

For the application of the methodology mentioned above, the econometric software EViews 4.1 was used in order to estimate the interest rate process parameters. For the volatility parameter σ, the implied volatility observed on 22/02/06 for at-the-money EURIBOR 12month caps and floors was, according to the Bloomberg database, 10.78% in annualized terms. In order to substitute this number in the econometric specification of equation (12), it was transformed to daily terms using the 255 days per year convention. The results can be summarized in the following table:

Table 1: Interest Rate Process Parameters

Estimation Method: Generalized Method of Moments

Sample: 10/15/2001 2/21/2006
Included observations: 1137
Total system (balanced) observations 4548
Kernel: Quadratic, Bandwidth: Variable Newey-West (7), Prewhitening
Iterate coefficients after one-step weighting matrix
Convergence achieved after: 1 weight matrix, 10 total coef iterations

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	0.087828	0.011132	7.890064	0.0000
C(2)	-3.171068	0.664285	-4.773655	0.0000

Determinant residual covariance	1.67E-31
J-statistic	0.000483

Table 1: Estimation Results. The parameter C(1) equals $\kappa \cdot \theta$ the coefficient of the reversion speed times the long run mean, and C(2) equals $-\kappa$ the coefficient of the reversion speed with a negative sign. The Quadratic Kernel was used with a variable Newey-West weighting matrix. The following instruments were used: a vector of ones and the first lag of the 12-month EURIBOR. In column (5) the probability of accepting the null hypothesis that the coefficient equals zero is illustrated.

It is important to note that because the J-Statistic is relatively small, it supports the hypothesis that the assumed model provides a good fit for the data[11]. Moreover, both parameters are statistically significant from the t-statistic tests. Hence, the values of the estimated parameters are deemed as satisfactory.

However, in order to perform the simulation of the risk neutral process, the market price of risk must also be estimated. Since this is a very bold attempt, it was considered wiser to use the same number for the market price of risk as that derived from the paper of Pearson and Sun (1989). As a result, the parameters that will be used in the Monte-Carlo simulations can be summarized in the following table:

[11] The same estimation procedure was performed for weekly data and the results showed a greater J-Statistic, therefore the daily data were considered more reliable.

Table 2: Final Interest Process Parameters in annualized terms

Parameters	
κ	3,171068
θ	2,77%
σ	10,78%
λ	-0,12165

Table 2: Interest Process Parameters. The reversion speed coefficient equals κ, the long – run mean coefficient equals θ, the volatility coefficient equals σ and the market price of risk equals λ. These are the parameters that will be used in the Monte – Carlo simulations for the market rate.

3.2.1 Estimation of the index rate process

As mentioned in Section 2, it will be assumed in this study that the ECB marginal lending rate which is the index rate of the mortgage follows the process:

$$I_t = a + bR_t + cI_{t-1} + U_t \qquad (16)$$

where,

I, is the specific index,

R, is the market interest rate and

U is an error term.

Therefore, in order to estimate the parameters *a, b* and *c* a regression will be performed using Ordinary Least Squares of the ECB marginal lending rate with the 12-month Euribor rate.

The data set consists of 371 weekly observations for the two rates from 15/10/2001 up to 22/02/2006.

3.2.2 Empirical Results

By using the specification in (16), the econometric software EViews 4.1 was employed for the estimation of the parameters *a, b* and *c.* The results can be summarized in the following table:

Table 3: Index Rate Process Parameters

Dependent Variable: ECB
Method: Least Squares

Sample(adjusted): 2 370
Included observations: 369 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	-0.026762	0.013392	-1.998313	0.0464
ECB(-1)	0.929250	0.009936	93.52750	0.0000
M12	0.072149	0.009613	7.504940	0.0000

R-squared	0.993787	Mean dependent var	2.918022
Adjusted R-squared	0.993753	S.D. dependent var	0.935353
S.E. of regression	0.073930	Akaike info criterion	-2.363307
Sum squared resid	2.000411	Schwarz criterion	-2.331511
Log likelihood	439.0301	Durbin-Watson stat	2.137532

Table 3: Estimation Results. The specification of equation (16) was used. Column 4 illustrates the t-statistic that was used to perform statistical significance tests. Column 5 illustrates the probability that the corresponding coefficient is statistically insignificant.

From table 3, it can be observed that the parameter α is negative, implying that the ECB marginal lending rate has been historically lower than the 12-month EURIBOR. Moreover, the parameter b is close to unity, implying the presence of unit roots in the time series. Augmented Dickey-Fuller tests were performed on the time series and found that the null hypothesis could not rejected, indicating that there are unit roots in the 12month EURIBOR and in the ECB lending rate. However, the results are consistent with previous studies by Stanton and Wallace (1994), (1996) who state that: *"because the series are relatively short, and it is well known that the low power of standard unit roots tests often leads to acceptance of the null hypothesis of a unit root in many economic time series, we rely on our strong priors that our interest rate series are mean reverting rather than explosive, and undertake all our estimation in levels of interest rates."*. Since the time series used in the present study are also relatively short the above opinion is shared in this dissertation.

Furthermore, certain residual tests were performed in order to examine the existence of autocorrelation and heteroscedasticity in the regression's residuals.

Table 4: Correlogram of Residuals

Sample: 2 370
Included observations: 369

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.069	-0.069	1.7692	0.183
		2	-0.017	-0.022	1.8773	0.391
		3	-0.079	-0.082	4.1956	0.241
		4	-0.075	-0.088	6.3322	0.176
		5	-0.044	-0.061	7.0498	0.217
		6	0.046	0.028	7.8547	0.249
		7	-0.069	-0.082	9.6549	0.209
		8	0.056	0.031	10.843	0.211
		9	-0.059	-0.061	12.154	0.205
		10	0.004	-0.012	12.160	0.274
		11	-0.055	-0.063	13.317	0.273
		12	0.091	0.074	16.526	0.168

Table 4: Correlogram of residuals. Column 1, AC, illustrates the autocorrelation coefficient starting from 1 lag up to 12 lags. Column 2, PAC, illustrates the partial autocorrelation coefficient for 1 lag up to 12 lags. Column 3, Q-Stat, illustrates the Q-Statistic that was used for statistical significance tests. Column 4, Prob, illustrates the probability that the corresponding coefficient is statistically insignificant.

Table 5: Correlogram of Squared Residuals

Sample: 2 370
Included observations: 369

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.016	-0.016	0.0974	0.755
		2	0.009	0.009	0.1300	0.937
		3	-0.017	-0.017	0.2411	0.971
		4	-0.019	-0.020	0.3784	0.984
		5	-0.014	-0.015	0.4559	0.994
		6	0.010	0.009	0.4913	0.998
		7	-0.022	-0.022	0.6746	0.999
		8	0.101	0.099	4.5260	0.807
		9	-0.025	-0.022	4.7567	0.855
		10	0.001	-0.002	4.7574	0.907
		11	-0.024	-0.021	4.9713	0.933
		12	0.032	0.034	5.3696	0.944

Table 5: Correlogram of squared residuals. Column 1, AC, illustrates the autocorrelation coefficient starting from 1 lag up to 12 lags. Column 2, PAC, illustrates the partial autocorrelation coefficient for 1 lag up to 12 lags. Column 3, Q-Stat, illustrates the Q-Statistic that was used for statistical significance tests. Column 4, Prob, illustrates the probability that the corresponding coefficient is statistically insignificant.

From the Q-Statistics of Table 4, it can be observed that the null-hypothesis is maintained, implying that the regression's residuals are uncorrelated.

The correlogram of squared residuals in Table 5, also shows that the null hypothesis of the squared residuals being uncorrelated holds true. Therefore, from the above tables it can be deduced that the model assumed for the ECB lending rate is adequate for the purposes of this dissertation.

3.3.1 The Valuation Procedure

After estimating the dynamics of the short-rate and the index rate, the final action is to proceed with the valuation of the mortgage. Following the paper of J.Chen[12], Monte – Carlo simulation will be used in the pricing procedure. In this section, the implementation of the algorithm to price an Adjustable – Rate Mortgage is going to be presented in detail.

Generally the price of any security can be written as the net present value (NPV) of its discounted cash flows. More specifically:

$$P = E[V] = E\left[\sum_{t=0}^{N} PV(t)\right] = E\left[\sum_{t=0}^{N} d(t)c(t)\right],$$

where,

P is the price of the mortgage,

V is the value of the mortgage, which is a random variable, dependent on the realization of the economic scenario,

$PV(t)$ is the present value for cash flow at time t,

$d(t)$ is the discount factor at time t,

$c(t)$ is the cash flow at time t,

N is the maturity of the mortgage.

Monte Carlo simulation is used to generate cash flows for many paths. By the strong law of large numbers, the following holds:

$$E[V] = \lim_{M \to \infty} \frac{1}{M} \sum_{i=1}^{M} V_i ,$$

[12] J.Chen : "Simulation Based Pricing of Mortgage – Backed Securities", Proceeding of the 2004 Winter Simulation Conference.

where V_i is the value calculated from path *i.*

The calculation of *d(t)* is derived from the short term interest rate process,

$$d(t) = d(0,1)d(1,2)....d(t-1,t) = \prod_{i=0}^{t-1} \exp\left(-r(i)\Delta t\right) = \exp\left\{-\left[\sum_{i=0}^{t-1} r(i)\right]\Delta t\right\},$$

where,

d(i,i+1) is the discounting factor for the end of period *i+1* at the end of period *i*;

r(i) is the short term rate used to generate *d(i,i+1),* observed at the end of period *i*;

Δt is the time step in simulation, generally monthly, i.e. *Δt* = 1/12.

The CIR interest rate model will be used to generate the short term rate *r(i)*; then, *d(t)* is instantly available when the short rate path is generated.

In order to generate *c(t),* the path dependent cash flow of the mortgage for month *t* , the index rate that determines this amount must also be simulated. As mentioned earlier, the following relation between the index and the short rate is assumed:

$$I_i = -0.00027 + 0.072149\ r(i) + 0.92925\ I_{i-1} \quad (17)$$

Therefore, once the short rate path is generated then the index rate is instantly available. The index rate determines *c(t)* according to the amortization formula:

$$c(t) = B(t-1)\left(\frac{I_{t-1}/12}{1-\left(1+I_{t-1}/12\right)^{N-t}}\right), \quad (18)$$

$$IP(t) = B(t-1)\frac{I_{t-1}}{12}, \quad (19)$$

$$SP(t) = c(t) - IP(t), \quad (20)$$

$$B(t) = B(t-1) - SP(t). \quad (21)$$

where,

I_t , is the value of the index rate at month *t,*

B(i) , is the remaining balance at the end of month *i,*

IP(i), is the interest payment for month *i,*

SP(i), is Scheduled Principal Prepayment for month *i,*

N, is the maturity of the mortgage.

3.3.2 Interest Rate Model

In the one-factor Cox-Ingersoll-Ross interest rate model, the underlying process for the short-term rate is given by:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t).$$

The risk-neutral process to be used for the simulations is:

$$r_{t+1} = r_t + \left(\kappa(\theta - r_t) - \lambda\sigma\sqrt{r_t}\right)\Delta t + \sigma\sqrt{r_t}\sqrt{\Delta t}\cdot\varepsilon \qquad (22)$$

And by substituting the parameters from Table 2:

$$r_{t+1} = r_t + \left(3.171(0.0277 - r_t) - (-0.12165)(0.1078)\sqrt{r_t}\right)\left(\tfrac{1}{12}\right) + (0.1078)\sqrt{r_t}\sqrt{\tfrac{1}{12}}\cdot\varepsilon \qquad (23)$$

where,

$\varepsilon \sim N(0,1)$

Therefore, after creating a path for the short-rate, which is assumed to be the 12month Euribor, all the necessary variables that were mentioned earlier can be calculated in order to find the non-prepayable cash flows occurring from the mortgage.

Figure 1: Sample Paths for the 12-month EURIBOR and ECB rate

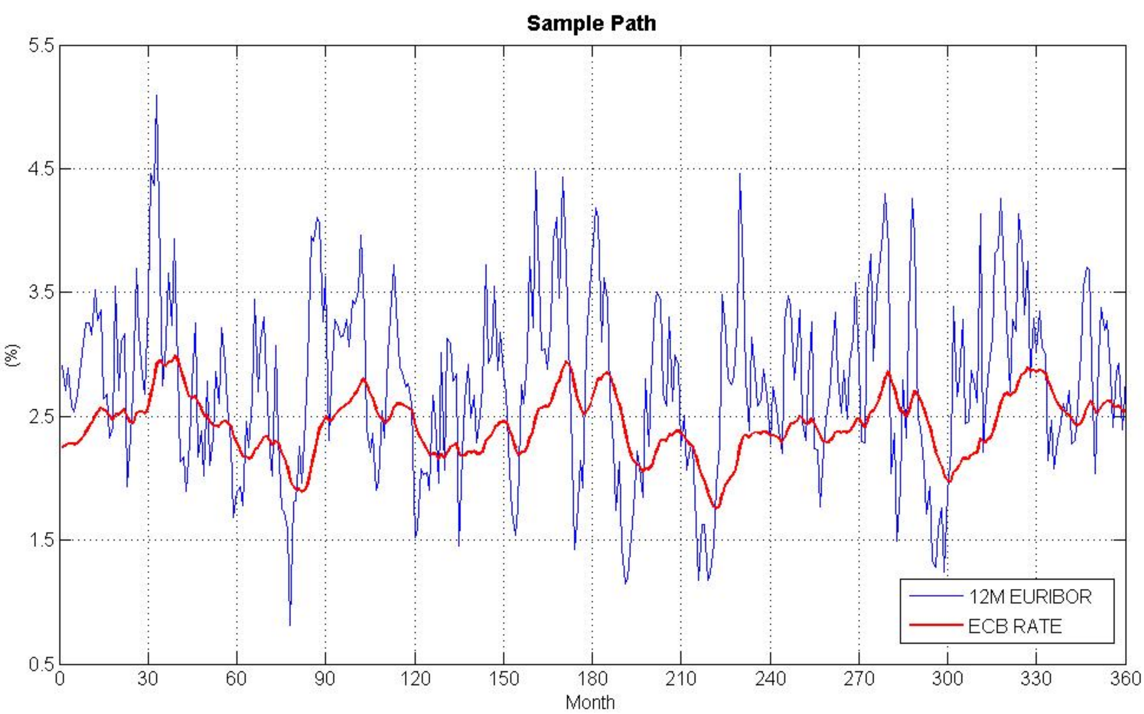

Figure 1: Sample paths for the 12M EURIBOR and ECB rate. The 12M EURIBOR was simulated using equation (23) while the ECB rate using equation (17). Starting values were 2.912% and 2.25% for the 12M EURIBOR and ECB rate respectively.

Figure 2: Evolution of Monthly Mortgage Payments

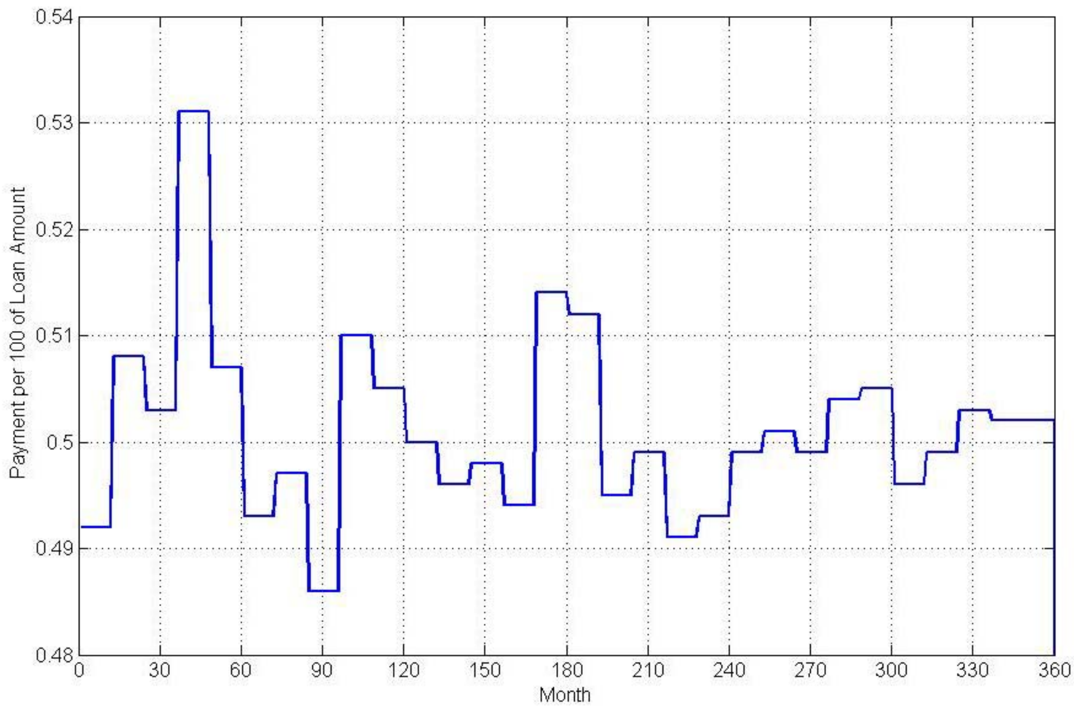

Figure 2: Evolution of Monthly Mortgage Payments. The payment is adjusted on a yearly basis, which means that on the first month of each year the amount of payment is calculated through equation (18) using the prevailing value of the ECB rate on that month. For the next 11 months of the year the amount of payment remains constant.

Figure 3: Evolution of Outstanding Balance, Cumulative Interest and Cumulative Principal

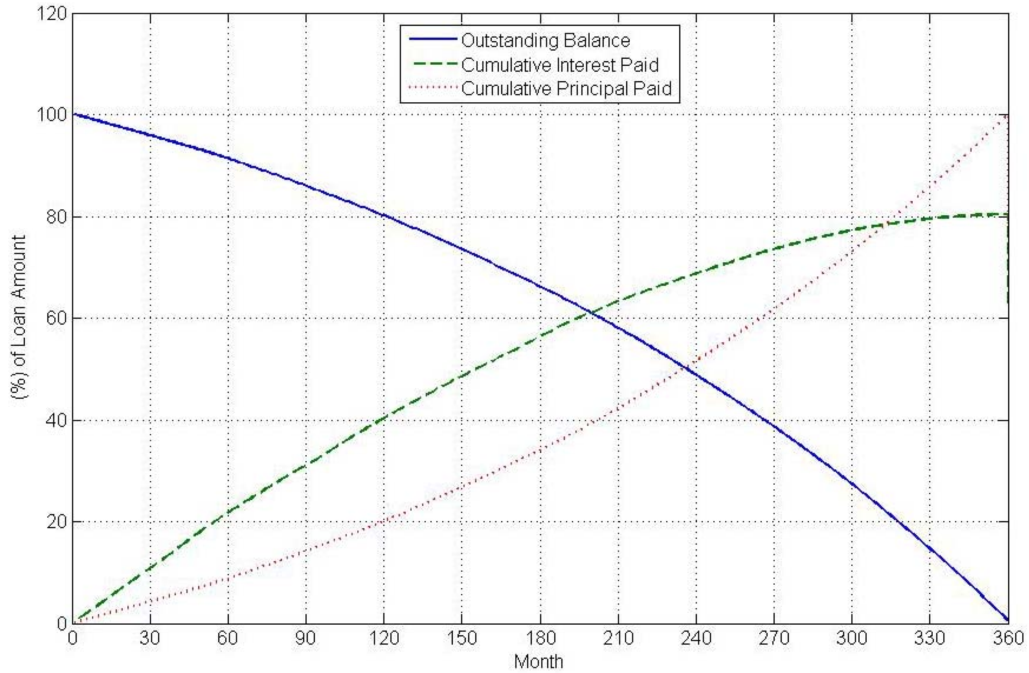

Figure 3: Evolution of Outstanding Balance, Cumulative Interest and Cumulative Principal. Outstanding Balance was calculated using equation (21). Interest paid was calculated using equation (19) and the cumulative sum over each period is presented in the plot. Principal paid was calculated using equation (20) and the cumulative sum over each period is presented in the plot.

3.3.3 Prepayment Model

After calculating the value of a mortgage with no prepayment, the value of the prepayment option must be deducted in order to obtain the "clean" price of the mortgage. This is an American-Style option and in fact it corresponds to a call on the mortgage, on the part of the mortgage holder, with a strike price equal to the unpaid balance. Hence,

$$O(t) = \max\left(MG(t) - B(t)\right) \qquad (24)$$

where,

O(t) is the intrinsic value of the prepayment option at time t

MG(t) is the value of the mortgage when no prepayment occurs at time t

B(t) is the unpaid balance at time t.

In order to valuate this option the methodology used is similar to that of Longstaff and Schwartz (2001) for valuing American options by simulation. To this extent, it is assumed that the exercise of the prepayment option is only possible at the end of each month. Consequently, this means that there are 360 exercise points for a 30 year mortgage[13]. Using the cross-section of the simulations the algorithm works backwards from month 359[14] to month 1 comparing the immediate exercise value with the expected cash flows from waiting to exercise the option at a later time. The conditional expectation function that will be used is of the form:

$$E\left[Y|X\right] = a + bX + cX^2, \qquad (25)$$

where,

a, b, c are constants and

X, is the mortgage price at month 358.

Specifically, since the immediate exercise value on month 359 is known for every path, it is discounted to month 358 using the prevailing interest rate for each path. Hence, a vector *Y* with dimensions Nx1 is created, with N denoting the number of paths. In order to estimate the expected cash flow from continuing the option's life

[13] In reality, American-Style options can be exercised at any point in time, therefore by making this assumption we are technically examining a Bermudan-Style option that can be only exercised at certain points in time.
[14] Since by definition at the maturity of the mortgage the value of the prepayment option is zero, we omit month 360.

conditional on the mortgage price at month 358, *Y* was regressed using three variables: a constant, the mortgage price at month 358 and the square of the mortgage price at month 358.

The value of immediate exercise equals the intrinsic value from equation (24) while the value of continuation is given by substituting *X* into the conditional expectation function (25). If the intrinsic value is greater than the continuation value, then the option is exercised and the cash flow for month 359 is substituted by 0. If the continuation value is greater than the intrinsic value then the cash flow for month 358 is substituted by 0 and the cash flow for month 359 is the intrinsic value for that month.

Proceeding recursively, the same steps were used for months 357 down to month 1. In this manner, the optimal stopping times for each simulation path were identified and their corresponding cash flow. Having identified the cash flows generated by the prepayment option at each month along each path, the option can now be valued by discounting each cash flow back to time zero and averaging over all paths[15].

After calculating both the cash flows from the mortgage with no prepayment and the value of the prepayment option, the "clean" price of the mortgage equals:

$$V(t) = MG(t) - O(t)$$

where,

V (t), is the mortgage price at time *t*

MG (t), the value of an equivalent bond producing the same promised payments with the mortgage

O (t), the value of the prepayment option at time *t.*

[15] For a more intuitive example, the reader is advised to consult the study of F.Longstaff, E.S. Schwartz : "Valuing American Oprions by Simulation: A Simple Least-Squares Approach", *The Review of Financial Studies, Vol.14, No.1 (Spring 2001), pp.113 – 147 .*

3.3.4 Empirical Results

The algorithm was first applied to mortgages with no caps, whose coupon adjusts annually to the prevailing value of the ECB lending rate using the software Matlab 7.0.4. All simulations were run on an AMD Athlon™ XP 1700+ running at 1.43GHz with 512MB of RAM. For this type of mortgage, the value of the prepayment option can only be a function of the lag in the index, plus the annual reset frequency. Figure 4 shows the value per 100€ of remaining principal of the equivalent bond (the stream of promised payments, with no prepayment option), for different values of the instantaneous risk - free interest rate, the 12-month EURIBOR, and the index, the ECB lending rate. For every point plotted, the current coupon rate is assumed to equal the current value of the ECB lending rate. Figures 5 and 6 depict the market value of the mortgage and the mortgage holder's prepayment option respectively. These figures illustrate that, for certain combinations of the 12-month EURIBOR and the ECB lending rate, this option is extremely valuable. However, most of these combinations (high ECB lending rate together with low 12-month EURIBOR) are unlikely to occur in practice.

Figure 4: Equivalent Bond Value

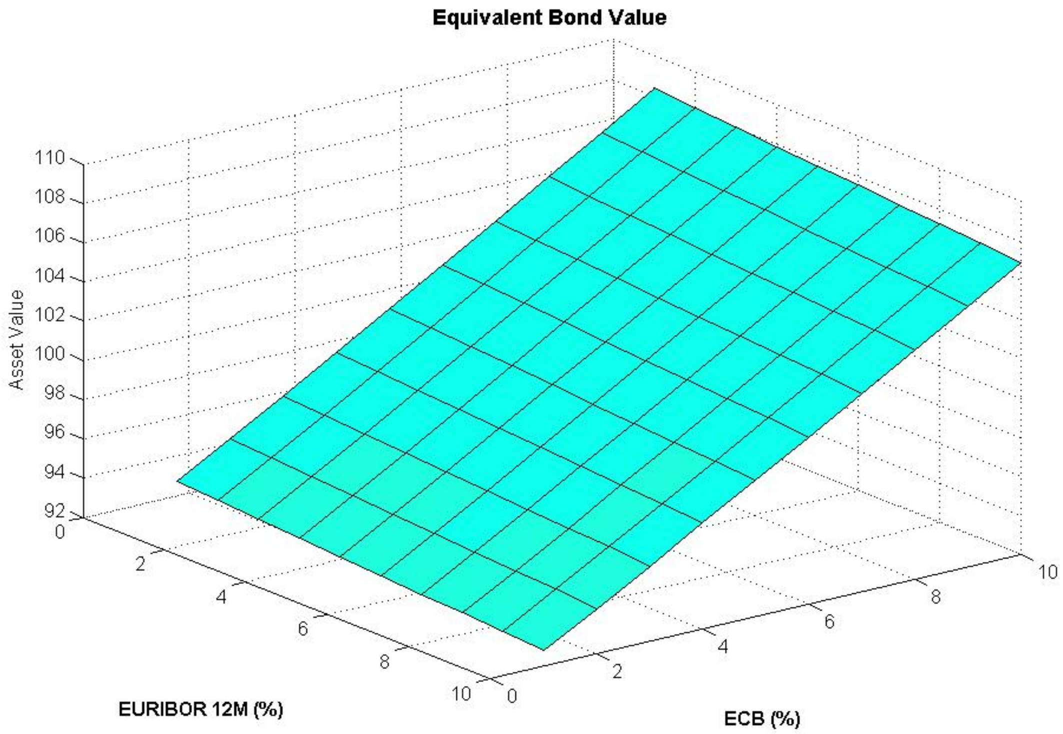

Figure 4: Equivalent bond values for different values of the 12 - month EURIBOR and ECB rate. Coupon rate equals the value of the ECB rate without any margin added, and resets annually to the prevailing value of the ECB rate. There are no caps on coupon movements. Remaining principal is 100.

Figure 5: Mortgage Value

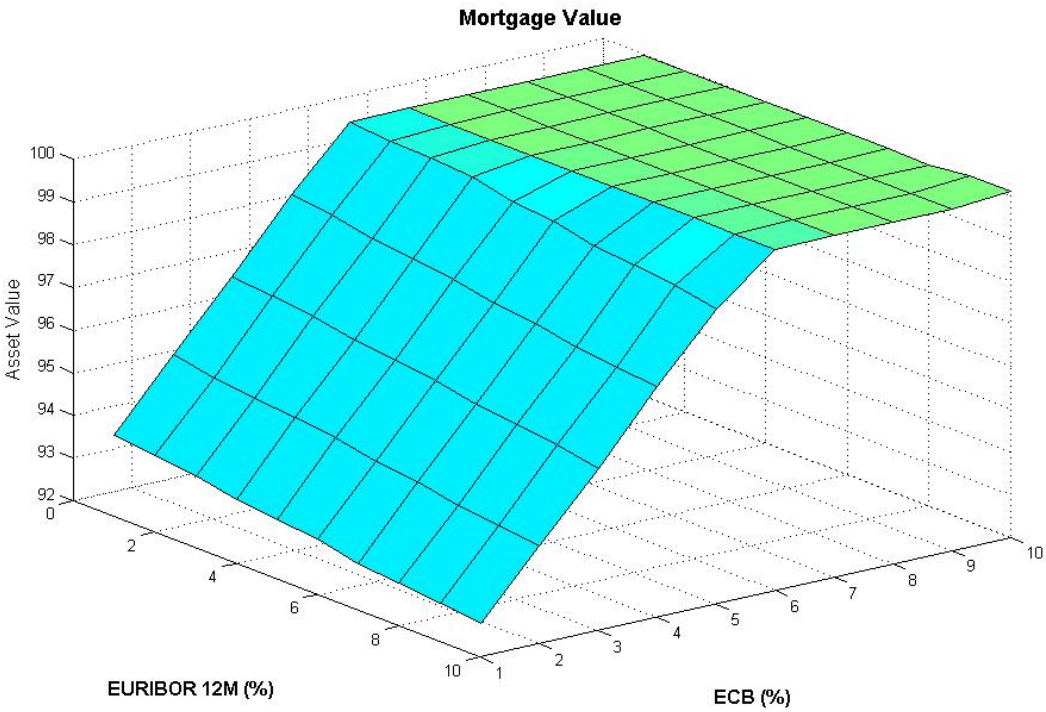

Figure 5: Mortgage values for different values of the 12 - month EURIBOR and ECB rate. Coupon rate equals the value of the ECB rate without any margin added, and resets annually to the prevailing value of the ECB rate. There are no caps on coupon movements. Remaining principal is 100.

Figure 6: Prepayment Option Value

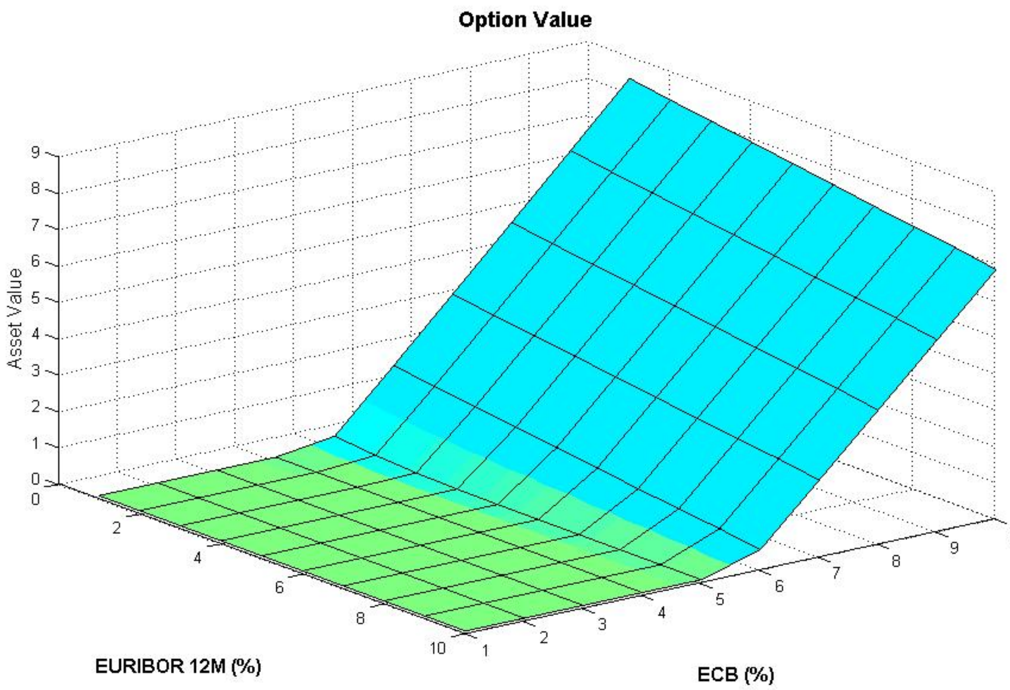

Figure 6: Prepayment option values for different values of the 12 - month EURIBOR and ECB rate. Coupon rate equals the value of the ECB rate without any margin added, and resets annually to the prevailing value of the ECB rate. There are no caps on coupon movements. Remaining principal is 100.

In order to examine the value of the prepayment option in more likely interest environments, and to see how this value would be affected by fluctuations in the level of interest rates, figures 7 and 8 demonstrate the values of the equivalent bond, the mortgage and the prepayment option for different values of the 12-month EURIBOR, with the current value of the ECB lending rate set to 2%. For values of the 12-month EURIBOR between 0% and about 2%, the value of the option drops from almost 1.6% of remaining principal to below 0.2%. The graph of the equivalent bond and the mortgage values shows that, in this region, it is optimal for mortgage holders to prepay immediately. For values of the 12-month EURIBOR above 2% the prepayment option value decreases but does not become equal to zero. This is mostly to due to the fact that the higher the 12-month EURIBOR is today, the faster the ECB lending rate will rise in the near future and thus there is a non-zero probability that prepayment maybe optimal. This means that the higher the current value of the ECB lending rate (and the current coupon rate on the mortgage), the higher the value of the equivalent bond and, subsequently, the value of the prepayment option. Figures 9 and 10 depict the values of the equivalent bond, the mortgage and the prepayment option for different values of the ECB lending rate, with the current value of the 12-month EURIBOR set to 2%.These results indicate that the prepayment option embedded in an Adjustable Rate Mortgage indexed to the ECB lending rate may have significant value.

Figure 7: Equivalent Bond and Mortgage Values for different values of the 12-month EURIBOR.

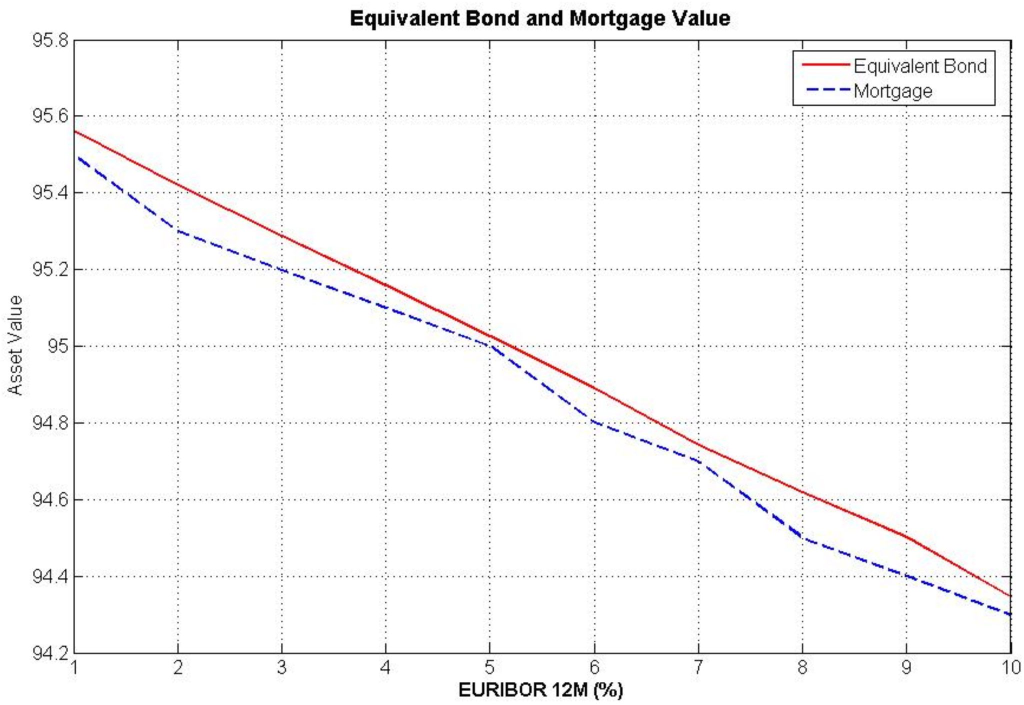

Figure 7: Bond and Mortgage values for different values of the 12 – month EURIBOR. Current ECB rate is 2%. Coupon rate equals 2% and adjusts annually to the prevailing value of the ECB rate. There are no caps on coupon movements. Remaining principal is 100.

Figure 8: Prepayment Option Values for different values of the 12-month EURIBOR.

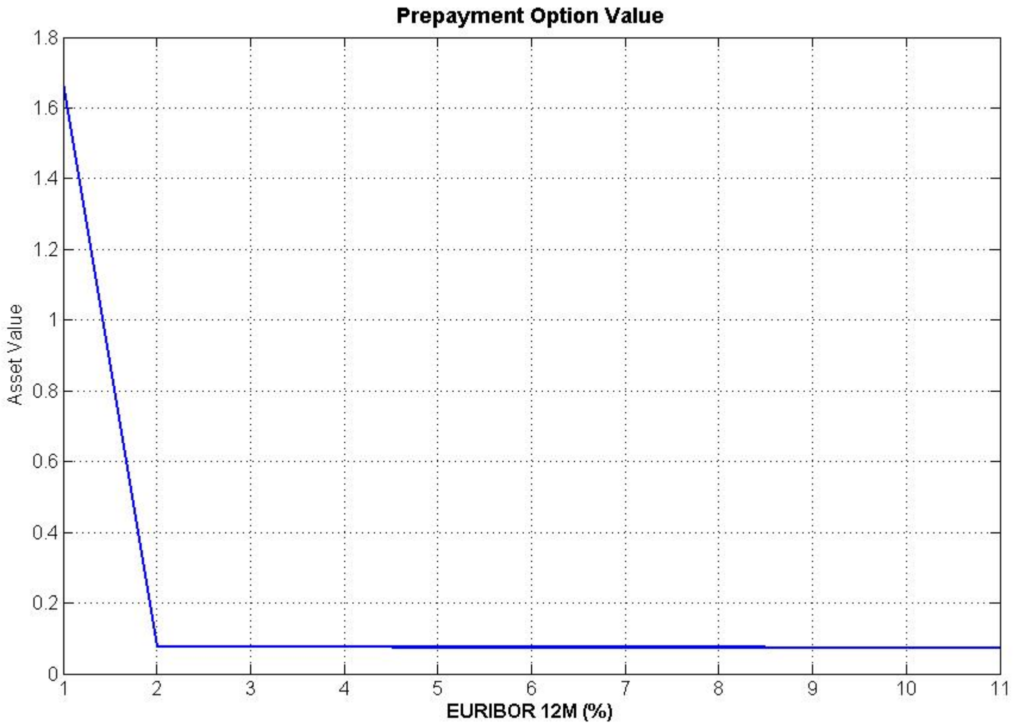

Figure 8: Prepayment option values for different values of the 12 – month EURIBOR. Current ECB rate is 2%. Coupon rate equals 2% and adjusts annually to the prevailing value of the ECB rate. There are no caps on coupon movements. Remaining principal is 100.

Figure 9: Equivalent Bond and Mortgage Values for different values of the ECB lending rate.

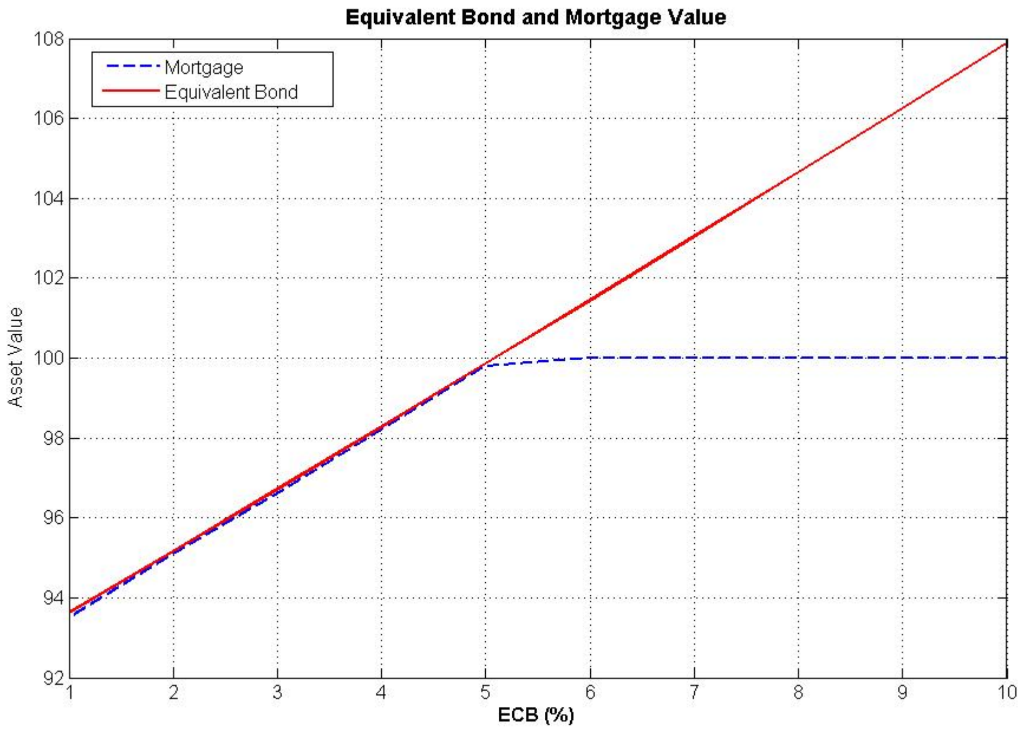

Figure 9: Bond and Mortgage values for different values of the ECB rate. Current 12 – month EURIBOR is 2%. Coupon rate equals 2% and adjusts annually to the prevailing value of the ECB rate. There are no caps on coupon movements. Remaining principal is 100.

Figure 10: Prepayment Option Values for different values of the ECB lending rate.

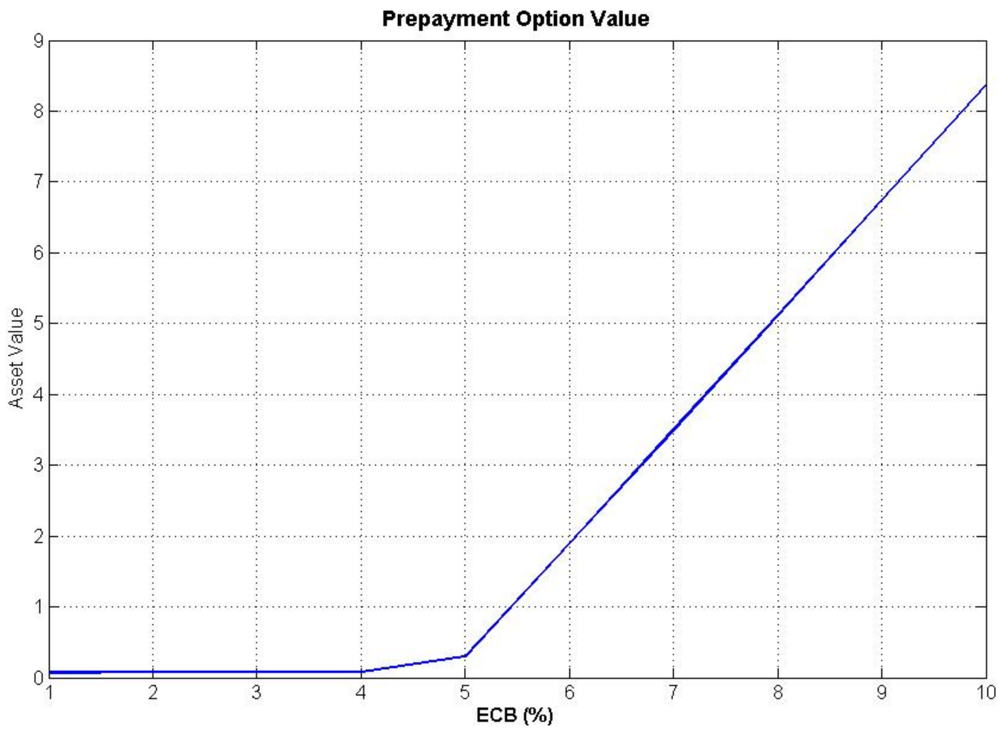

Figure 10: Prepayment option values for different values of the ECB rate. Current 12 – month EURIBOR is 2%. Coupon rate equals 2% and adjusts annually to the prevailing value of the ECB rate. There are no caps on coupon movements. Remaining principal is 100.

Another important feature of ARM contracts is the presence of caps on movements in the coupon rate and the addition of a margin to the index rate. It is possible that much of the option value found above stems from the possibility that the coupon rate may become very high at some time in the future. Figure 11 shows the value per 100€ of remaining principal of the equivalent bond (the stream of promised payments, with no prepayment option), for different values of the instantaneous risk - free interest rate, the 12-month EURIBOR, and the index, the ECB lending rate. For every point plotted, the current coupon rate is assumed to equal the current value of the ECB lending rate plus a margin of 0.5%. Coupon rate has a lifetime cap of 4%. Figures 12 and 13 depict the market value of the mortgage and the mortgage holder's prepayment option respectively.

The main difference from the previous analysis is that, both the equivalent bond and prepayment option values are higher. This is mainly due to the addition of a 0.5% margin over the value of the ECB rate. Moreover, in Figure 13 the value of the prepayment option does not have the same behavior as that observed in Figure 6. As the ECB rate increases towards the lifetime cap of 4% the value of the prepayment option also increases. However, from that point and further the rate of increase in the value of the prepayment option decreases and the plot becomes more flat. This means that the effect of a lifetime cap acts as a boundary in the value of the prepayment option.

Figure 11: Equivalent Bond Value with a lifetime cap of 4% and a 0.5% margin

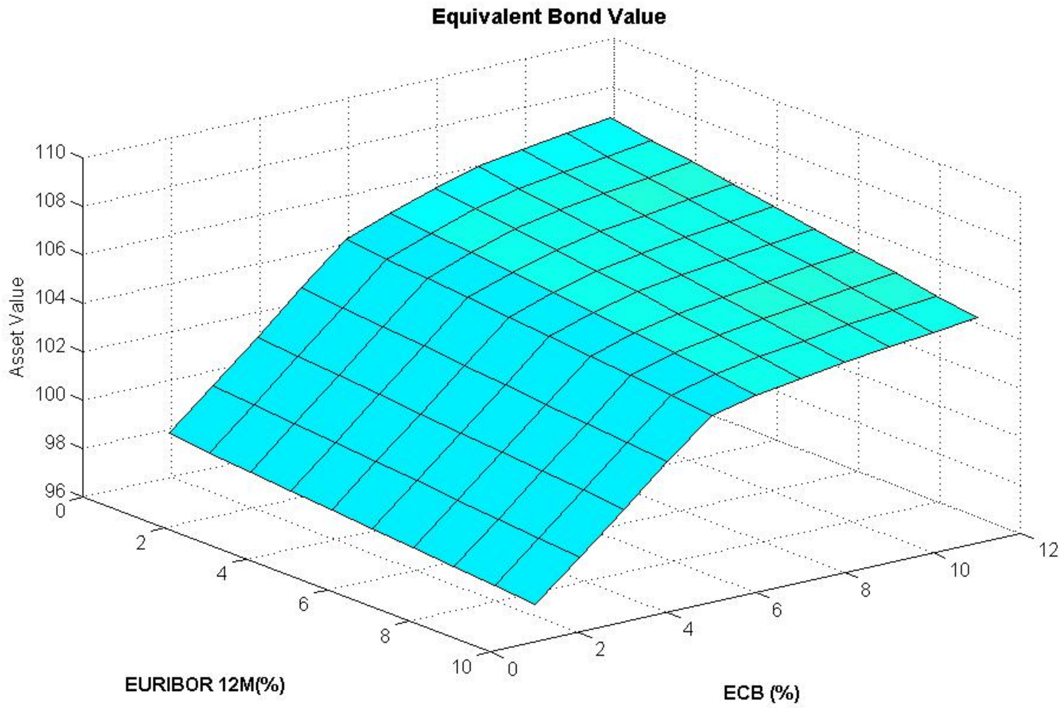

Figure 11: Equivalent bond values for different values of the 12 - month EURIBOR and ECB rate. Coupon rate equals the value of the ECB rate plus a margin of 0.5%, and resets annually to the prevailing value of the ECB rate. Coupon rate has a lifetime cap of 4%. Remaining principal is 100.

Figure 12: Mortgage Value with a lifetime cap of 4% and a 0.5% margin

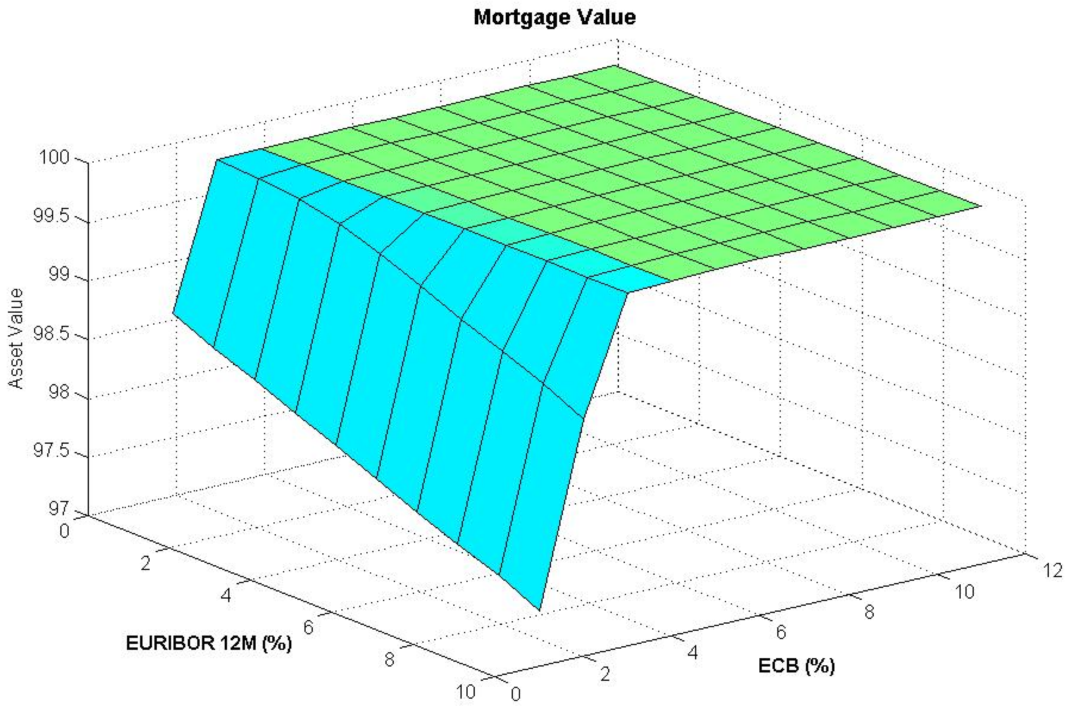

Figure 12: Mortgage values for different values of the 12 - month EURIBOR and ECB rate. Coupon rate equals the value of the ECB rate plus a margin of 0.5%, and resets annually to the prevailing value of the ECB rate. Coupon rate has a lifetime cap of 4%. Remaining principal is 100.

Figure 13: Prepayment Option Value with a lifetime cap of 4% and a 0.5% margin

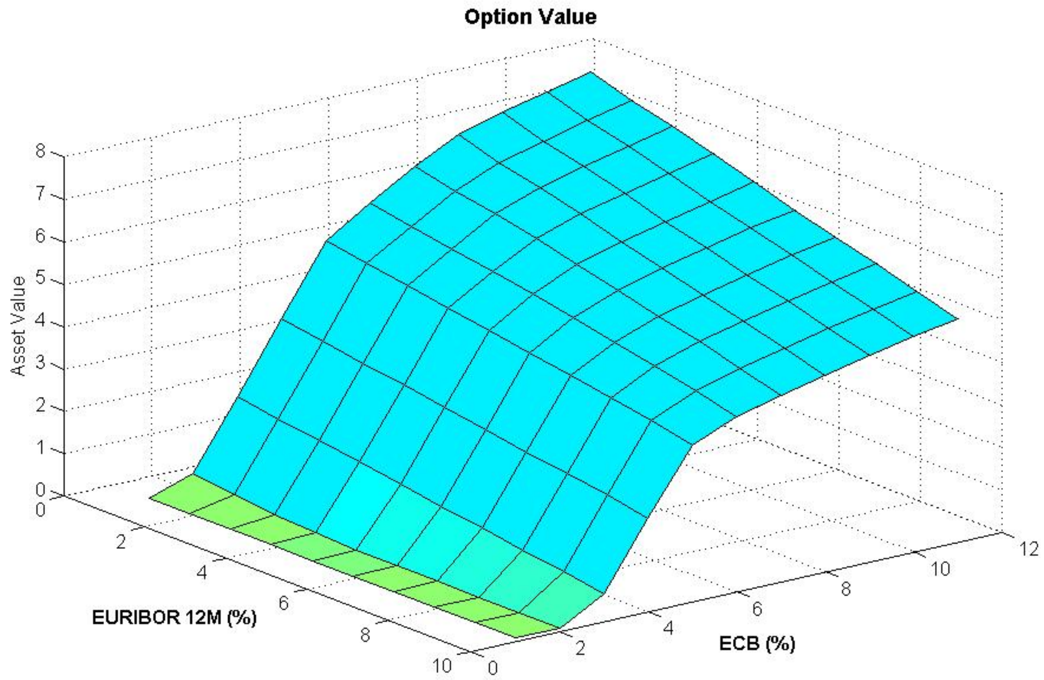

Figure 13: Prepayment option values for different values of the 12 - month EURIBOR and ECB rate. Coupon rate equals the value of the ECB rate plus a margin of 0.5%, and resets annually to the prevailing value of the ECB rate. Coupon rate has a lifetime cap of 4%. Remaining principal is 100.

Figures 14 and 15 demonstrate the values of the equivalent bond, the mortgage and the prepayment option for different values of the 12-month EURIBOR, with the current value of the ECB lending rate set to 2%. It can be observed that in this case the equivalent bond value is significantly higher than mortgage value. The prepayment option value in figure 15 behaves in the same fashion as in figure 8, however the values are significantly higher. Figures 16 and 17 depict the values of the equivalent bond, the mortgage and the prepayment option for different values of the ECB lending rate, with the current value of the 12-month EURIBOR set to 2%.

It can be observed from figures 15 and 17 that the addition of a margin has a strong effect on the value of the prepayment option, which is now worth approximately 1% of the remaining principal amount over a wide range of values for the 12 – month EURIBOR. In figures 8 and 10, in which no margin was added to the index rate, the prepayment option was worth approximately less than 0.1% of the remaining principal amount over the same range of values for the 12 – month EURIBOR. These figures show that the value of the prepayment option is critically dependent on the interaction between contract features, especially caps and margin levels.

Figure 14: Equivalent Bond and Mortgage Values for different values of the 12-month EURIBOR with the presence of a 4% lifetime cap and a 0.5% margin.

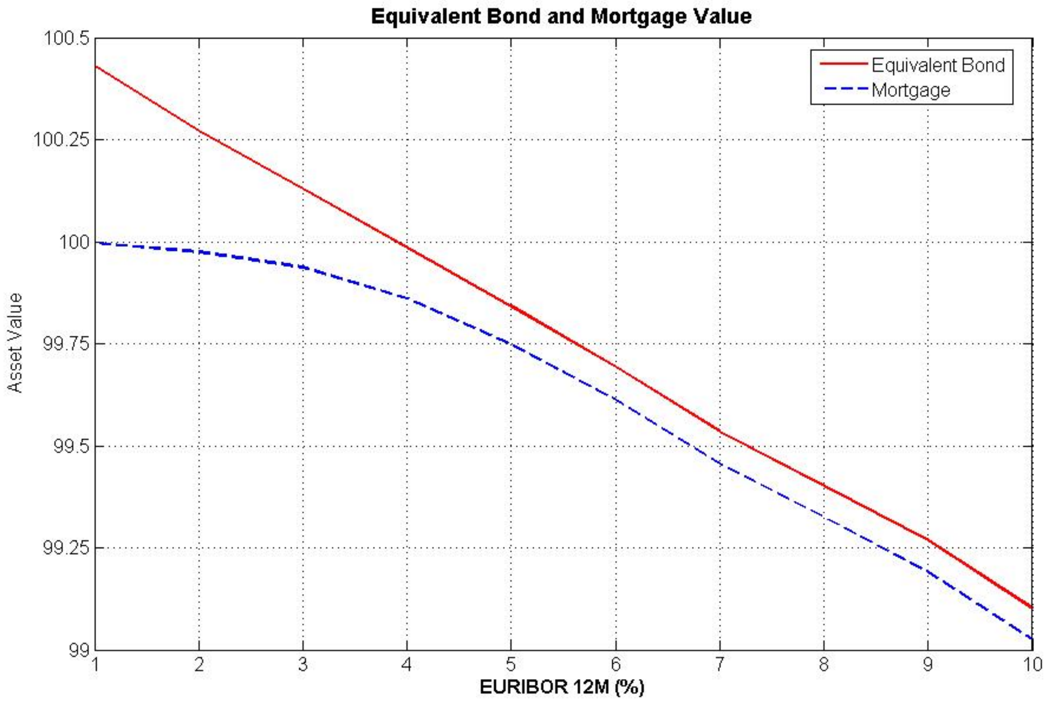

Figure 14: Bond and Mortgage values for different values of the 12 – month EURIBOR. Current ECB rate is 2%. Coupon rate equals 2.5% and adjusts annually to the prevailing value of the ECB rate plus a margin of 0.5%. Coupon rate has a lifetime cap of 4%. Remaining principal is 100.

Figure 15: Prepayment Option Values for different values of the 12-month EURIBOR with the presence of a 4% lifetime cap and a 0.5% margin.

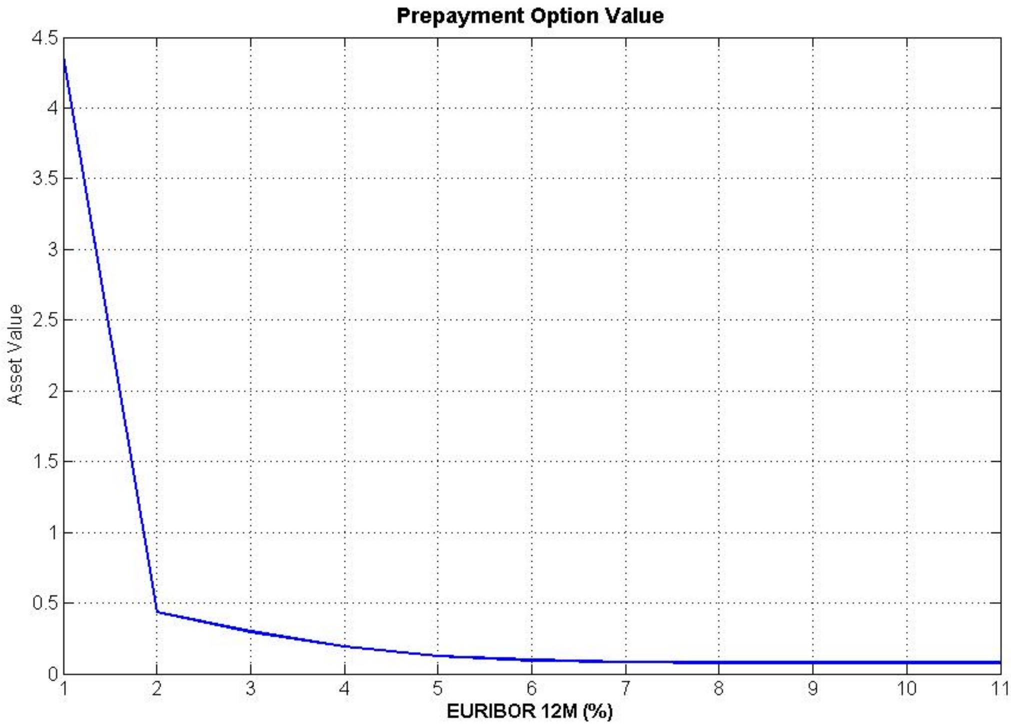

Figure 15: Prepayment option values for different values of the 12 – month EURIBOR. Current ECB rate is 2%. Coupon rate equals 2.5% and adjusts annually to the prevailing value of the ECB rate plus a margin of 0.5%. Coupon rate has a lifetime cap of 4%. Remaining principal is 100.

Figure 16: Equivalent Bond and Mortgage Values for different values of the ECB lending rate with the presence of a 4% lifetime cap and a 0.5% margin.

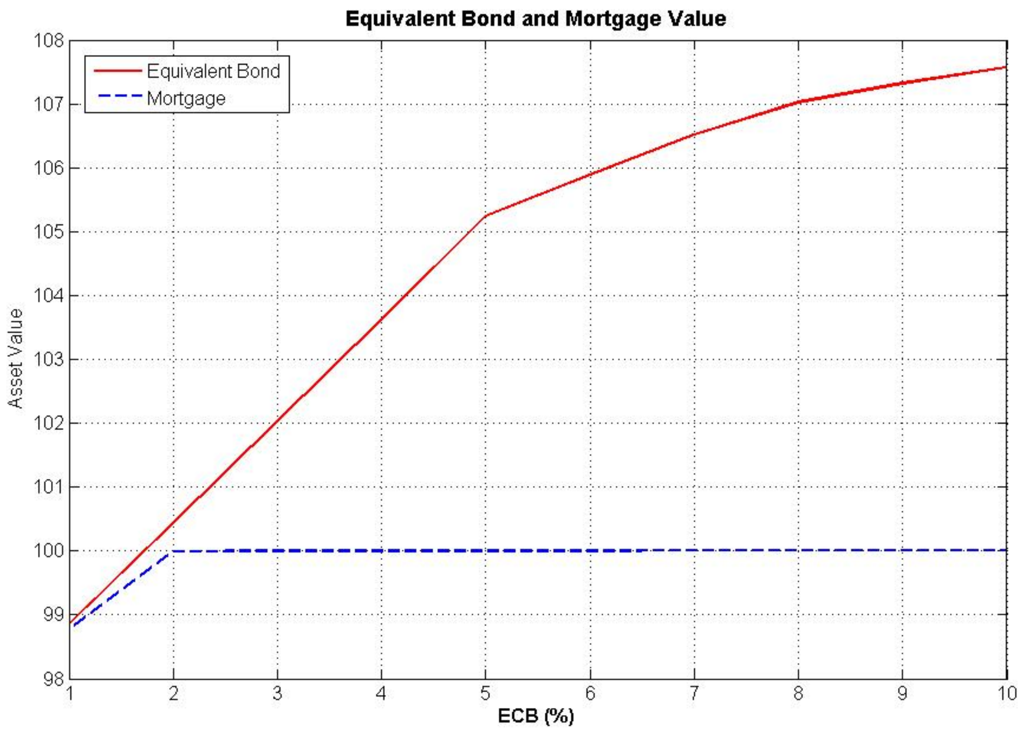

Figure 16: Bond and Mortgage values for different values of the ECB rate. Current 12 – month EURIBOR is 2%. Coupon rate equals 2.5% and adjusts annually to the prevailing value of the ECB rate plus a margin of 0.5%. Coupon rate has a lifetime cap of 4%. Remaining principal is 100.

Figure 17: Prepayment Option Values for different values of the ECB lending rate with the presence of a 4% lifetime cap and a 0.5% margin.

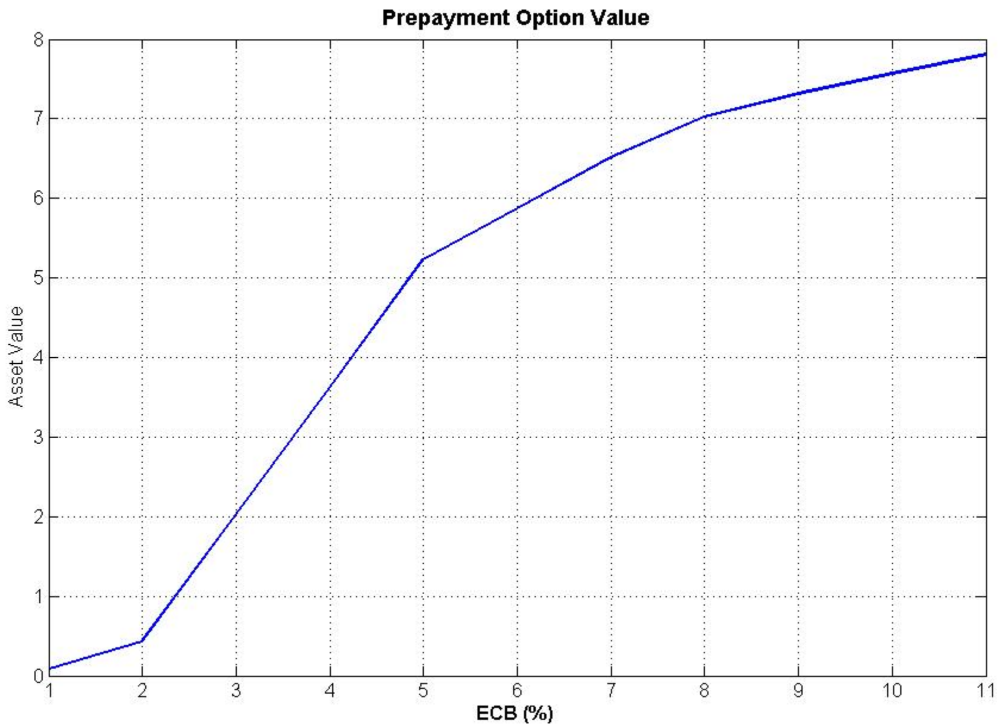

Figure 17: Prepayment option values for different values of the ECB rate. Current 12 – month EURIBOR is 2%. Coupon rate equals 2.5% and adjusts annually to the prevailing value of the ECB rate plus a margin of 0.5%. Coupon rate has a lifetime cap of 4%. Remaining principal is 100.

Finally, the above algorithm was implemented in order to valuate certain mortgage contracts similar to the ones offered by Greek banks. In order to show the convergence of the algorithm mortgages were priced using 100, 1.000 and 10.000 paths for the 12-month EURIBOR and the ECB lending rate.

Table 6: Range of Values for the Interest Rate Paths

EUR12M		
# Paths	**Min**	**max**
100	0.004%	6.56%
1000	0.001%	7.26%
10000	0.001%	8.83%

ECB lending Rate		
# Paths	**Min**	**max**
100	1.38%	4.06%
1000	1.25%	4.09%
10000	1.19%	4.99%

The valuation algorithm was applied for 5 different types of mortgage contracts: a) a fully-floating 30 year mortgage indexed to the ECB lending rate plus a margin of 2%, b) a fully-floating 30 year mortgage indexed to the 12-month Euribor plus a margin of 2%,, c) a 6% fixed rate 30 year mortgage , d) a 30 year mortgage indexed to the ECB lending rate plus a margin of 2%, with a 4.25% lifetime cap and e) a 30 year mortgage indexed to the 12-month Euribor plus a margin of 2%, with a 4.25% lifetime cap. Starting values for the ECB lending rate and 12-month EURIBOR were 2.25% and 2.91% respectively as observed on 22/02/2006. The results are summarized in Table 7.

Table 7: 30 year Mortgages on a 100€ Loan

30 Year Mortgages, Loan Amount 100

ECB + margin 2% FULLY FLOATING ARM

Number of Paths	Equivalent Bond	95% Conf. Interval		Prepayment Option	95% Conf. Interval		Clean Price	Time
100	122.7964	122.7298	122.8629	22.6891	22.6328	22.7455	100.1073	5 sec
1000	122.8135	122.7900	122.8370	22.7582	22.7347	22.7816	100.0553	74 sec
10000	122.8096	122.8021	122.8171	22.7543	22.7468	22.7618	100.0553	2322 sec

E12M + margin 2% FULLY FLOATING ARM

Number of Paths	Equivalent Bond	95% Conf. Interval		Prepayment Option	95% Conf. Interval		Clean Price	Time
100	128.2220	127.8493	128.5947	27.9641	27.6179	28.3102	100.2579	9 sec
1000	128.0613	127.954	128.1686	27.9588	27.8557	28.062	100.1025	114 sec
10000	127.9661	127.9328	127.9994	27.8631	27.8312	27.8951	100.1030	2230 sec

6% FIXED RATE MORTGAGE

Number of Paths	Equivalent Bond	95% Conf. Interval		Prepayment Option	95% Conf. Interval		Clean Price	Time
100	146.7748	146.3418	147.2078	46.6138	46.1843	47.0432	100.1610	8 secs
1000	146.8007	146.6594	146.942	46.6395	46.5037	46.7753	100.1612	116 secs
10000	146.9004	146.8553	146.9454	46.5124	46.464	46.5609	100.3880	2731 sec

E12M + margin 2% FULLY FLOATING ARM with lifetime Cap 4.25%

Number of Paths	Equivalent Bond	95% Conf. Interval		Prepayment Option	95% Conf. Interval		Clean Price	Time
100	128.1710	127.8083	128.5337	27.8758	27.4937	28.2580	100.2952	8 sec
1000	128.0239	127.9166	128.1313	27.9239	27.8205	28.0273	100.1000	117 sec
10000	127.9305	127.8974	127.9636	27.8279	27.7961	27.8597	100.1026	2150 sec

ECB + margin 2% FULLY FLOATING ARM with lifetime Cap 4.25%

Number of Paths	Equivalent Bond	95% Conf. Interval		Prepayment Option	95% Conf. Interval		Clean Price	Time
100	122.7964	122.7298	122.8629	22.6891	22.6328	22.7455	100.1073	5 sec
1000	122.8135	122.7900	122.8370	22.7582	22.7347	22.7816	100.0553	74 sec
10000	122.8095	122.802	122.817	22.7542	22.7468	22.7617	100.0553	2488 sec

From Table 7, it can be observed for all mortgages that as the number of paths increases, the 95% confidence intervals for the prices of both the equivalent bond and the prepayment option narrow. However, computational time also increases exponentially. Moreover, one can also notice that the equivalent bond has a positive correlation with the prepayment option. The financial intuition behind this outcome is that a higher price for the equivalent bond means that the coupon rate is greater than the market rate, which in turn results in a higher probability that the option to prepay will be exercised, and this will result in a higher option price. In effect, this leads to small fluctuations in the "clean price" of all mortgages, as illustrated in column 8, Table 7.

Nevertheless, fluctuations in the equivalent bond and the prepayment option are more obvious. As one would expect, the 6% fixed rate mortgage has the greatest equivalent bond and the greatest prepayment option value, in comparison to the fully-floating mortgages. By comparing the two fully floating mortgages it was noticed that both the equivalent bond and the prepayment option are greater for the one indexed to the 12-month EURIBOR. This result is in agreement with the mortgage market since the margin charged by Greek banks for mortgages indexed to the ECB lending rate is always greater than the margin charged when the 12-month EURIBOR is used as an index. Apart from this, the greater volatility of the 12-month EURIBOR results in a higher prepayment option price.

The effect of a lifetime cap of 4.25% is almost negligible with respect to mortgages indexed to the ECB lending rate, as expected from the range of values of Table 6. However, this is more noticeable in the contract where the 12-month EURIBOR is used as index. The result is that both the equivalent bond and the prepayment option have smaller values when a lifetime cap is used. Nevertheless, column 8 of Table 7 shows that the "clean price" is identical to the uncapped contract.

Summary and conclusions

This dissertation attempts to analyze Adjustable Rate Mortgages based on the ECB lending rate, a common practice in the Greek market. The relationship between movements in the ECB rate and market rates was examined in order to identify the existence of lags between the two. Moreover, it was attempted to model the lagged behavior of the ECB rate by using a linear model. The partial adjustment model estimated, using the 12-month EURIBOR as a contemporaneous interest rate variable and a single lagged value of the ECB rate, was found to adequately describe the dynamics of the ECB rate.

The lag in the index means that, if interest rates fall substantially, mortgage holders may want to refinance their loans to avoid paying above market interest rates. The commonly used Monte – Carlo simulation technique was applied for the valuation algorithm and was used for various types of mortgage contracts. The algorithm can capture the effects of lags in the index, caps and floors on coupon adjustment, discrete coupon adjustment frequency and teaser rates.

For the special feature of the prepayment option in mortgage contracts, the Least – Squares approach for American Options was used as proposed in the article of Longstaff and Schwartz (2001). In this fashion, it was attempted to calculate an optimal prepayment strategy for mortgage holders, leading to the valuation of their prepayment options. It was found that the lag in the ECB rate contributes significantly to the value of the mortgage holder's prepayment option under realistic assumptions regarding contract terms and interest rates.

The main conclusions of the empirical results are:

a) there is a positive relationship between the value of the equivalent bond (the stream of promised payments, with no prepayment option) and the value of the prepayment option.

b) the value of the prepayment option is critically dependent on the interaction between contract features, especially caps and margin levels. More specifically, caps pose an upper boundary on the prepayment option values while margins significantly increase the prepayment option value as a percentage of the remaining principal amount.

c) the lag in the adjustment of the index rate with the market rate has a significant effect on the value of the prepayment option.

Future work could be focused in the field of interest rate model employment (for example applications using a Heath – Jarrow – Morton interest rate model), the valuation algorithm (perhaps using a certain finite difference scheme instead of crude Monte – Carlo simulation) and the valuation of the prepayment option (using an empirical prepayment function). Moreover, the valuation algorithm could be extended in order to include the occurrence of default in mortgage payments by using the methodology proposed by Downing, Stanton and Wallace in the paper "An Empirical Test of a Two-Factor Mortgage Valuation Model: How Much Do House Prices Matter?" , April(2005).

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