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Thesis

Properties of the BDS test under GARCH(1,1) filtering

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Abstract

We examine the size of the BDS test in the flexible framework that the GARCH(1,1) process provides us in terms of moment, memory and heterogeneity properties. The validity of a number of assumptions about the functionality of the BDS test is under consideration. Using Monte Carlo simulations, we conclude to a new general assumption under which the asymptotic normality of the BDS statistic is verified.

Contents

1. Introduction

There is a common point that everyone agrees; our world is rather complex and in order for its aspects to be described correctly, employment of complex ways is a necessity. We generally talk of the existence of chaotic or nonlinear stochastic dynamics in financial data that must be modeled so as to increase our ability to understand and correctly forecast them. A variety of such models is available to the potential modeler bringing him to the last but most vital question…Which model describes the data more successfully and is there a way to find it? The answer for this question is quite simple. The researcher must find a test that would be able to detect or accept the best model and simply reject the others.

There are two categories of tests. The first category includes tests that attempt to diagnose the existence of a specific model. They check the data of whether they can be described by a model of certain properties. Such a test is the Engle test for detecting Autoregressive Conditional Heteroskedastic (ARCH) models. The other category includes tests that are of more general use and their null hypothesis is a very broad one. They are also called as portmanteau tests. A well known example for this category is the Ljung-Box statistic.

Another test of this category that was first introduced in 1987 and is now generally used as a test for detecting nonlinearity is the BDS test (after Brock, Dechert, Scheinkman). The test was initially intended to test for the existence or not of chaotic dynamics. It was based on the idea that a chaotic system is sensitive upon initial conditions and that close trajectories of the series remain close in the future. All this, lead to a particular metric, the correlation integral, a measure of spatial correlation of scattered points or particles in m-dimensional space, and its transformations to the BDS statistic. The null hypothesis is simple; if the series under scrutiny is iid then the BDS statistic asymptotically follow a standard normal distribution. The alternative hypothesis can be anything. Any hypotheses of chaos or nonlinearity are nested within the alternative hypothesis, which includes both nonwhite linear and nonwhite nonlinear processes. This gives the BDS test its generality. Nowadays it is mainly used as a test for the existence or not of nonlinear dynamics or as a goodness-of-fit test. That is, if we have correctly specified a model for a data series, after the estimation procedure we expect the residuals to be iid.

However the nature of the test and the mathematical tools that were used in order to provide the final result, requested a number of assumptions. Many of them concern the mathematical tools themselves, for example smoothness conditions for the kernel function that is used for the calculation of the correlation integral, and it is of little importance for the modeler. Others are about the properties of the series itself such as temporal dependence and existence of moments. It is vital to understand the nature of these assumptions and investigate the potential impact on the test if we relax some of them. In most cases we are not able to recognize the exact properties of a series. Knowing potential problems that may arise when we use the test in cases that the series fails to satisfy the appropriate assumptions can help us to evaluate our results correctly.

In order to investigate the assumptions of the BDS test, it would ease our task if we could have a flexible model for which we can alter its properties in a way that we wish. The GARCH(1,1) process can fit perfectly to this role. Nelson (1990) provided conditions for the existence or not of the moments of the unconditional model, depending on the innovation process and the relation between the coefficients of the model. Therefore by simply altering a distribution and tampering the values of the coefficients we could create a model with moment and memory characteristics that we prefer. Furthermore the GARCH(1,1) is very popular among the econometricians for modeling the conditional variance of financial time series. The results on the behavior of the test for this model will provide us data for its validity if it is used to check the residuals of an estimation of a GARCH(1,1) model for remaining nonlinearities.

The analysis of the behavior of the BDS test and especially its size for the various GARCH(1,1) processes will be made through a Monte Carlo simulation. This gives us the flexibility to study the validity of assumptions for small or large samples and therefore approach the asymptotic results. We expect either to state new assumptions or to relax some of the old. Besides that we will attempt to verify the results of Brock et al (1991) concerning the size of the test when it is applied directly to an iid series. Setting new assumptions about the existence or not of moments is expected. Therefore we use distributions with different moment structures in order to test all cases.

After we have evaluated the performance of the test, we will apply it to two data sets; a set of exchange rates and a set of stock indices. Our goal is to check whether filtering the data with a GARCH(1,1) process would remove all nonlinearities existing.

The thesis is structured as follows: in paragraph 2 the properties of the GARCH(1,1) models are presented analytically. Based on the work of Nelson (1990) we notice that moment and memory characteristics of the model are subject to the choice of the distribution for the iid innovations and especially the number of its moments. Additionally these characteristics change for different combinations of the coefficients of the model. We also study the moment properties as analyzed in He and Teräsvirta (1999) focusing on the conditions for the existence of 4^{th} order stationarity. Finally we examine in this paragraph the memory properties of the process as it is described by the concept of Near Epoch Dependence and L_0-approximability.

In paragraph 3 we proceed in the analysis of the BDS test itself. We initially provide a number of definitions for various mathematical tools and statistical concepts that are used in a later stage. Then we continue by defining the BDS test in the most formal manner and along certain assumptions are stated. The nuisance parameter free property is then examined and the necessary assumptions are provided. Finally we summarize all the assumptions mentioned and we examine the existing literature for the size of the BDS test.

In paragraph 4 we describe the Monte Carlo procedure that we will follow. The results are reported for both cases; the size of the test in the case of the iid series and the size of the test when it is applied to GARCH(1,1) models. We discuss the results and we check whether the assumptions that were made in paragraph 3 need revision or not. In paragraph 5 we describe our empirical study. We begin by examining the results of the existing literature. Then we describe our methodology and the data we will use. As last step we provide and discuss the results. Conclusion summarizes the course of the thesis and yields once more the plus interesting results that were derived. Finally in the Appendix we present all the results from the simulations and the empirical study.

2. Properties of GARCH (1, 1) models

2.1 Introduction

The vast use of conditional heteroskedasicity models can be attributed to their ability to include in terms of variance past information. Quoting Engle (1982) from his seminal paper for ARCH models:

"*Consider initially the first-order autoregression*

$$y_t = \gamma y_{t-1} + \varepsilon_t$$

where ε *is white noise with* $Var(\varepsilon)=\sigma^2$*. The conditional mean of* y_t *is* γy_{t-1} *while the unconditional mean is zero. Clearly, the vast improvement in forecasts due to time-series models stems from the use of the conditional mean….For real processes one might expect better forecast intervals if additional information from the past were allowed to affect the forecast variance; a more general class of models seems desirable.*"

These models have been widely used to model time-varying volatility and the persistence of shocks to volatility. Especially one member of the family, the GARCH(1,1) process, since its introduction by Bollerslev (1986) has been very popular in econometric modeling.

Definition of the model:

$$\{z_t\}_{t=-\infty,\infty} \sim \text{i.i.d, nondegenerate, } P[-\infty < z_t < \infty]=1, \qquad (2.1.1)$$

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\xi_{t-1}^2, \qquad (2.1.2)$$

$$\xi_t = \sigma_t \cdot z_t \ , \qquad (2.1.3)$$

where $\omega \geqslant 0, \beta \geqslant 0$ and $\alpha > 0$. In most papers a further restrictions has been placed on $\{z_t\}$:

$$E[z_t]=0 \text{ and } E[z_t^2]=1 \qquad (2.1.4)$$

We will not use the restriction (2.1.4) unless mentioned. Instead we will adopt the requirement that:

$$E[\ln(\beta+\alpha z_t^2)] \text{ exists} \qquad (2.1.5)$$

Note that (2.1.5) does not require that $E[\ln(\beta+\alpha z_t^2)]$ be finite, only that the expectations of the positive and negative parts of $\ln(\beta+\alpha z_t^2)$ are not both infinite. Relation (2.1.5) holds trivially for $\beta>0$.

We also define as the conditional model the $\{\sigma_t^2, \xi_t\}_{t=0,\infty}$ and as the unconditional model the process $\{{}_u\sigma_t^2, {}_u\xi_t\}_{t=-\infty,\infty}$.

If we denote as B the Borel sets on $[0, \infty)$, we define as μ_t, the probability measure for σ_t^2 : $\mu_t(\Gamma)=P[\sigma_t^2 \ \Gamma]$, $\forall\, \Gamma \ $ B.

2.2 Stationarity and ergodicity in GARCH(1,1) processes.

The analysis that follows comes directly from Nelson (1991). We keep only the basic results concerning stationarity. We must notice that the results concerning the properties of the GARCH(1,1) model rely on the relation between the coefficients α,β of the model. As we will see in detail later, combinations of α, β provide different stationarity characteristics and different moment results for the unconditional process.

Theorem 2. Let $\omega>0$. If $E[\ln(\beta+\alpha z_t^2)]<0$ then:

$$\omega/(1-\beta)\leq {}_u\sigma_t^2<\infty \text{ for all t a.s ,} \tag{2.2.1}$$

and ${}_u\sigma_t^2$ is strictly stationary and ergodic with a well-defined probability measure

$$\mu_\infty \text{ on } [\omega/(1-\beta),\infty] \quad \forall t, \tag{2.2.2}$$

$$ {}_u\sigma_t^2-\sigma_t^2 \to 0 \text{ a.s,} \tag{2.2.3}$$

$$\mu_t\to \mu_\infty \text{ and} \tag{2.2.4}$$

$$\mu_\infty \text{ is nondegenerate} \tag{2.2.5}$$

Corollary of theorem 3. Let $\omega>0$, $p>0$ and $E[\ln(\beta+\alpha z_t^2)]<0$.

$$E[\sigma_t^{-2p}]<\infty, \quad t\geqslant 1 \tag{2.2.6}$$

$$E[{}_u\sigma_t^{-2p}]<\infty, \quad \forall t \tag{2.2.7}$$

$$E[\sigma_t^{2p}]<\infty \text{ iff } E[\sigma_0^{2p}]<\infty \text{ and } E[(\beta+\alpha z_t^2)^p]<\infty \tag{2.2.8}$$

$$E[{}_u\sigma_t^{2p}]<\infty \text{ iff } E[(\beta+\alpha z_t^2)^p]<1 \tag{2.2.9}$$

$$\limsup_{t\to\infty} E[\sigma_t^{2p}]<\infty \text{ iff } E[\sigma_0^{2p}]<\infty \text{ and } E[(\beta+\alpha z_t^2)^p]<1 \tag{2.2.10}$$

$$\lim_{t\to\infty} E[\sigma_t^{2p}]=E[{}_u\sigma_t^{2p}] \text{ if } E[\sigma_0^{2p}]<\infty \tag{2.2.11}$$

Theorem 4. (a) Let $\omega>0$ and $E[\ln(\beta+\alpha z_t^2)]<0$. If $E[|z_t|^{2q}]<\infty$ for some $q>0$, then there exists a p, $0<p<q$, such that $E[(\beta+\alpha z_t^2)^p]<1$.(b) If, in addition, $E[(\beta+\alpha z_t^2)^r]<1$ for $0<r<q$, then exists a $\delta>0$ such that $E[(\beta+\alpha z_t^2)^{r+\delta}]<1$.

Theorem 4(a) says that if ${}_u\sigma_t^2$ is strictly stationary and z_t^2 has a finite moment of some (arbitrarily small, possibly fractional) order, then ${}_u\sigma_t^2$ has a finite (possibly fractional) moment as well. The existence of such a finite fractional moment implies, for example, that $E[\ln({}_u\sigma_t^2)]<\infty$. In addition we notice that in order for $E[(\beta+\alpha z_t^2)^p]<1$ to hold for p=1, the iid innovation must have at least a fractional moment of order larger than 2. Part (b) gives a condition for $E[{}_u\sigma_t^{2(p+\delta)}]<\infty$ for some $\delta>0$, given that $E[{}_u\sigma_t^{2p}]<\infty$. It says, for example, that if $E[(\beta+\alpha z_t^2)^{1/2}]<1$ and $E[|z_t|^{2p}]<\infty$ for some $p>\frac{1}{2}$, then not only is $E[|{}_u\xi_t|]<\infty$, but there is also a $\delta>0$ with $E[|{}_u\xi_t|^{1+\delta}]<\infty$.

Summarizing the above results Nelson (1991) produced the following figure for the case that z_t~NIID(0,1) :

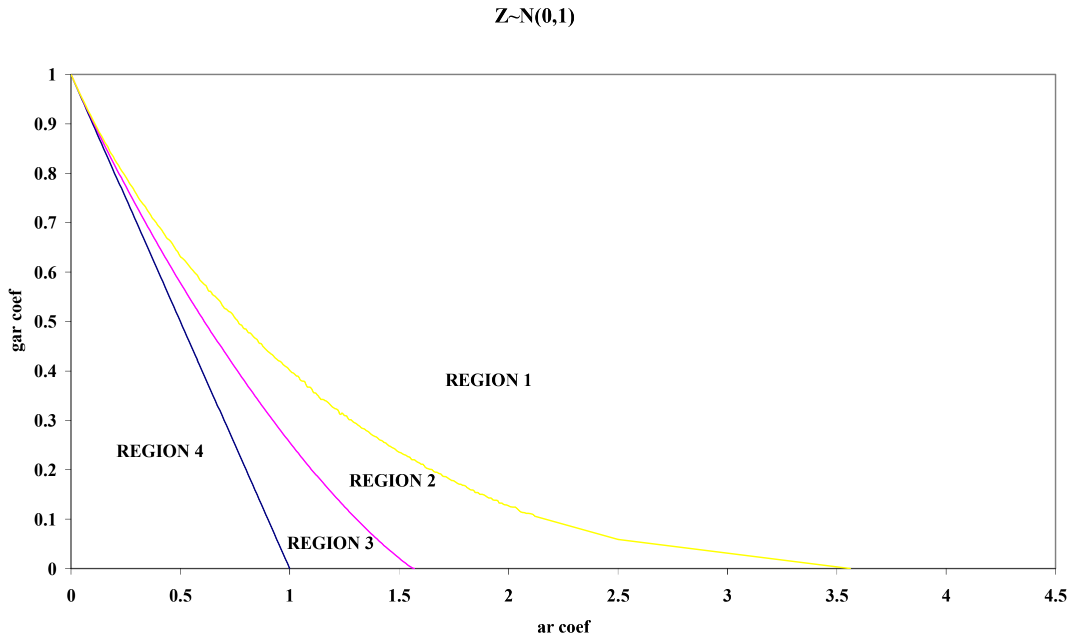

Figure 2.2.1

REGION 1	$E[\ln(\beta+\alpha z_t^2)]>0$
σ_t^2 is explosively nonstationary	
REGION 2	$E[\ln(\beta+\alpha z_t^2)]<0$
${}_u\sigma_t^2, {}_u\xi_t$ are strictly stationary and ergodic	
REGION 3	$E[\ln(\beta+\alpha z_t^2)]<0$ & $E[(\beta+\alpha z_t^2)^{1/2}]<1$
${}_u\sigma_t^2, {}_u\xi_t$ are strictly stationary and ergodic and $E[{}_u\xi_t]=0$, $E[{}_u\xi_t^2]=\infty$	
REGION 4	$E[\ln(\beta+\alpha z_t^2)]<0$ & $E[(\beta+\alpha z_t^2)^{1/2}]<1$ & $E[(\beta+\alpha z_t^2)]<1$
${}_u\sigma_t^2, {}_u\xi_t$ are strictly stationary and ergodic and $E[{}_u\xi_t]=0, E[{}_u\xi_t^2]<\infty$ (covariance stationarity as well)	

(the results and the relations contained in the table above, are the same for all distributions provided certain moment conditions hold)

Table 2.2.1

Note # 1: We must stress one more that according to theorem 4(a), in order for the region 4 to exist, and therefore the existence of covariance stationarity for some values of the coefficients α, β, the innovation process must have a least a fractional moment of order higher than 2. For example, for a t-student distribution with 3 degrees of freedom, even though there exists the second moment, the appropriate relation for the existence of region 4 can not be applied.

Note # 2: It is clear, even for the case of standard normal innovations, the change of the moment properties of the unconditional model as we move to different regions of the figure 2.2.1 for the α,β coefficients. For Region 4, the first and second moments exist, for region 3, there exists only the unconditional mean and for region 2, even though that the unconditional process is strictly stationary, no moments exist.

2.3 Properties of moments of GARCH (1, 1) processes.

Following He & Teräsvirta (1999), we will provide a general condition for the existence of any integer moment of the absolute values of the observations. Additionally a general expression for this moment as a function of low-order moments is provided. First of all we introduce the notation:

$$c_{t-1}=\beta+\alpha\cdot z_{t-1}^2,\ g_{t-1}=\omega,\ v_i = E|z_t|^i,\ \gamma_{ci}=Ec_t^i,\ \gamma_{gi}=Eg_t^i,\ \tilde{\gamma}_{gi,cj}=Eg_t^i c_t^j,$$

where i, j positives integers. Let $\gamma_c=\gamma_{c1}$, $\gamma_g=\gamma_{g1}$, $\tilde{\gamma}_{gc}=\tilde{\gamma}_{g1,c1}$. We only require that $\gamma_g>0$.

<u>Theorem 1</u>. For the general GARCH (1, 1) model that started at some finite value infinitely many periods ago, the kmth unconditional moment exists if and only if :

$$\gamma_{cm} = Ec_t^m < 1 \tag{2.3.1}$$

Under this condition the kmth moment of ξ_t can be expressed recursively as

$$\mu_{km} = E|\xi_t|^{km} = \{v_{km}/(1-\gamma_{cm})\}\cdot\left\{\sum_{j=1}^{m}\binom{m}{j}\tilde{\gamma}_{gj,c(m-j)}(\mu_{k(m-j)}/v_{k(m-j)})\right\} \tag{2.3.2}$$

<u>Corollary 1.1</u>. For the general GARCH (1, 1) model with k=2 and $\gamma_{c2}<1$, the fourth unconditional moment of ξ_t is given by

$$\mu_4 = E\xi_t^4 = \frac{v_4\{\gamma_{g2}\cdot(1-\gamma_c)+2\cdot\tilde{\gamma}_{gc}\gamma_g\}}{(1-\gamma_c)\cdot(1-\gamma_{c2})} \tag{2.3.3}$$

We can summarize the above results for the unconditional model in what follows:

(v_2,v_4: 2nd and 4th moment of the innovation process respectively)

Existence of 2nd moment	
$\beta+\alpha\cdot v_2<1$	(2.3.4)
Existence of 4th moment	
$\beta^2+2\cdot\beta\cdot\alpha\cdot v_2+\alpha^2\cdot v_4<1$	(2.3.5)

We notice that condition (2.3.4) coincides with the relation that describes Region 4 in the analysis made by Nelson (1990). In both cases the second moment of the unconditional model exists and therefore we conclude that the process is covariance stationary. In the case that the model has a 4th moment we can speak of 4th order stationarity.

The existence of the fourth moment depends, as we observe from the relation (2.3.5), on the existence of the fourth moment of the original innovation process. For example the GARCH(1,1) process does not have a 4th moment in the cases of the t-student distributions with less than 5 degrees of freedom.

2.4 Memory properties of GARCH (1, 1) processes

In order to study the memory properties of the process we rely on the concept of Near Epoch Dependence that was introduced by Davidson (1994). We use the definitions and the notation as it is presented in Davidson (2001).

Definition #1 : x_t is said to be near-epoch dependent on $\{v_s\}$ in L_p-norm (or L_p-NED) for p>0 if

$$\left\| x_t - E_{t-m}^{t+m} x_t \right\|_p \leq d_t \nu(m)$$

where d_t is a sequence of positive constants, and $\nu(m) \to 0$ as $m \to \infty$. It is said to be L_p-NED of size $-\mu$ if $\nu(m)=O(m^{-\mu-\varepsilon})$ for $\varepsilon>0$.

$(x_t=(\ldots,v_{t-1},v_t,v_{t+1},\ldots)$, $\|\bullet\|_p$ denotes the L_p-norm $(E|\bullet|^p)^{1/p}$)

The NED property offers an approach to proving the FCLT for a range of time series models. The NED numbers fully determine the restrictions on the memory of the observed process. The CLT can be considered as a special case for size $\mu=\frac{1}{2}$.

According to Davidson (2002), for GARCH (1, 1) models with innovations with variance equal to unity, if the process is fourth-order stationary we can say it is geometrically L_2-NED on the underlying iid process, with respect to constants $d_t=1$. Additionally it can be proved that the process is L_1-NED if covariance stationarity exists.

However the NED measure of memory is unavailable without first moments and therefore we introduce an alternative approach with the notion of L_0-approximability. This is the condition that there exists a locally measurable (finite –lag) approximation to ξ_t, which is accordingly a uniform mixing process, given that z_t is independent.

In the case of the GARCH (1, 1) process, we call that that σ_t^2 is geometrically L_0 – approximable if the process ξ_t is strictly stationary and the condition for the coefficients to satisfy is:

$$\alpha < \frac{1}{\beta} - 1 \tag{2.4.1}$$

L_0-approximable is of importance because it shows a way in which short memory is a feature of the strictly stationary case, whether moments exist or not. However this property might be used in conjunction with mixing limit theorems to show that, for example, a law of large numbers applies to integrable transformations of the process.

Therefore we note that when we study a GARCH (1, 1) process we must be able to identify the appropriate regions for the coefficients α, β so as the L_0-approximability and L_1-L_2 NED to hold. For the latter this region is changed with respect to the distribution of the innovations z_t. For the case of L_0-approximability the condition (2.4.1) remains the same, depending on the innovations only for the second condition: the series to be strictly stationary.

3. The Brock, Dechert, Scheinkman (BDS) test

3.1 Introduction

This test has been widely applied to test the existence of potentially forecastable structure, nonstationarity, or hidden patterns. It has also been adapted to test for the adequacy of fit of one's forecasting model. One just applies the test on the forecast errors from the model and check for hidden structure. The construction of its null hypothesis gives this flexibility; the null hypothesis is that a time series is independent and identically distributed. The alternative remains unspecified. The test is based on the statistic derived by the use of the correlation integral and therefore its technique is a nonparametric one. However the test, in order to work, must satisfy a lot of conditions concerning the properties of the series under scrutiny and of the mathematical concepts that are used.

3.2 Some Definitions

Before we proceed with the description of the test and the analysis of the necessary assumptions for it to work, it is essential that we introduce certain mathematical and probabilistic concepts.

- **Strong mixing stochastic process**

A stochastic process y_t defined on the probability space $\{\Omega, F, P\}$ is **strong mixing** if the (mixing) coefficients

$$\lim_{k\to\infty} a(k) \overset{\text{def}}{=} \lim_{k\to\infty} \sup_{j} \sup_{\{A\in F_{-\infty}^{j}, B\in F_{j+k}^{+\infty}\}} \left|P(A\cap B) - P(A)P(B)\right| = 0 \qquad (3.2.1)$$

where: F_s^t is the Borel σ-algebra generated by $\{y_s, y_{s+1}, \ldots, y_t\}$, with $s<t$.

- **Absolutely regular stochastic process**

A stochastic process $\{y_t\}_{t\ N}$,that is strictly stationary with values in some countably generated measurable space (—,F) , is called **absolutely regular** if

$$\lim_{n\to\infty} \beta(n) = \lim_{n\to\infty} \sup_{\{a\in\mathbb{N}\}} E[\sup\{\left|P(A \mid \Im_1^a) - P(A)\right| \Big| A \in \Im_{a+n}^{\infty}\}] = 0 \qquad (3.2.2)$$

where : $\Im_a^b$ is the σ-algebra generated by $\{y_t|\ a \le t < b\}$ $(1 \le a \le b \le \infty)$

F the distribution of y_1

- **Uniformly mixing stochastic process**

A stochastic process $\{y_t\}_{t\ N}$,that is strictly stationary with values in some countably generated measurable space (—,F) , is called **uniformly mixing** if

$$\lim_{n\to\infty} \varphi(n) = \lim_{n\to\infty} \sup \max \{|P(A|B)-P(A)|,|P(B|A)-P(B)|\}=0 \qquad (3.2.3)$$

where the supremum extends over $n \in \mathbb{N}$, A $\Im_1^a$ and B $\Im_{a+n}^{\infty}$.

Note: <u>uniform mixing⇒absolutely regular ⇒strong mixing</u>

- **Ergodic stochastic process**

A second-order stationary stochastic process {y(t), t $\mathbb{T}$ } is said to be **ergodic** if

$$\lim_{T\to\infty}\left(\frac{1}{T}\sum_{\tau=1}^{T}\mathrm{Cov}(y(t),y(t+\tau))\right)=0 \qquad (3.2.4)$$

It is deduced that for the case of second-order stationary stochastic process strong mixing implies ergodicity.

- **Kernels, U-Statistics,V-Statistics**

- ✓ A measurable function h:—^{m}→ℝ is called a kernel if it is symmetric in its m arguments (i.e. for the functional θ, we have $\theta(F) := \int\ldots\int h(y_1,\ldots,y_m)\prod_{i=1}^{m}dF(y_i)$). The functional θ(F) is called an estimable parameter or a regular function of F over $\mathcal{F}$={F:|θ(F)|<∞} .
- ✓ A U-statistic U_N is then given by

$$U_N=\binom{N}{m}^{-1}\sum_{1\le t_1\le t_2<\ldots<t_m\le N}h(y_{t_1},\ldots,y_{t_m}) \ (N\geqslant m).$$

- ✓ A V-statistic V_N (v.Mises's functional) is defined to be

$$V_N=N^{-m}\sum_{t_1=1}^{N}\ldots\sum_{t_m=1}^{N}h(y_{t_1},\ldots,y_{t_m}) \ (N\geqslant 1)$$

- ✓ A kernel h is called degenerate (for the distribution F) if for all choices of a_i — (1≤i≤m) and all j (1,…, m)

$$E[h(a_1,\ldots,a_{j-1},y_1,a_{j+1},\ldots,a_m)]=0$$

- ✓ A U-statistic will be called degenerate if the corresponding kernel has this property (i.e. it is degenerate).The same applies and for the V-statistic.

3.3 Definition of the BDS test

Before we proceed with the description of the test, let us consider the definitions below.

Let {u_t} be a real-valued scalar time-series process with the m-history process to be:

$$u_t^m \overset{def}{=} (u_t,u_{t+1},\ldots,u_{t+m-1})$$

Definition: The correlation integral at embedding dimension m, for ε>0, is given by:

$$C_{m,\varepsilon}=\int\int\chi_\varepsilon(x^m,y^m)dF(x^m)dF(y^m) \qquad (3.3.1)$$

where χ_ε(.,.) is the symmetric indicator kernel with χ_ε(x,y)=1 if $\|x-y\|<\varepsilon$ and 0 otherwise, $\|\cdot\|$ is the max-norm, and F(.) is the distribution function of u_t^m.

An estimator of the correlation integral for a sample of size T for the process $\{u_t\}$ is given by the following U-statistic:

$$C_{m,\varepsilon} = \frac{1}{\binom{\bar{T}}{2}} \sum_{1 \le s \le t \le \bar{T}} \sum \chi_\varepsilon(u_t^m, u_s^m) \qquad (3.3.2)$$

$$\bar{T} = T - (m-1)$$

If $\{u_t\}$ is an iid process, then $C_{m,\varepsilon}=C_{1,\varepsilon}^{\ m}$, almost surely, for all $\varepsilon>0$, m=1,2,…Based on this Brock et al (1987) presented the following result :

<u>Theorem (Brock, Dechert, Scheinkman, 1987)</u>

If u_t is iid then,

$$V_{m,\varepsilon} = \sqrt{T} \cdot \frac{C_{m,\varepsilon} - (C_{1,\varepsilon})^m}{s_{m,\varepsilon}} \xrightarrow{d} N(0,1), \quad \forall\varepsilon>0,\ m=2, 3,\ldots \qquad (3.3.3)$$

$s_{m,\varepsilon}$ is an estimator of the asymptotic standard deviation, $\sigma_{m,\varepsilon}$, of $\sqrt{T} \cdot (C_{m,\varepsilon} - (C_{1,\varepsilon})^m)$ under the null of iid. The asymptotic variance $\sigma^2_{m,\varepsilon}$ is a continuous function of two constants C and K.

$$\sigma_m^2 = 4 \cdot \left[K^m + 2 \cdot \sum_{j=1}^{m-1} K^{m-j} C^{2j} + (m-1)^2 C^{2m} - m^2 \cdot K \cdot C^{2m-2} \right] \qquad (3.3.4)$$

$$C=C(\varepsilon)= \int [F(z+\varepsilon) - F(z-\varepsilon)] dF(z), \quad K=K(\varepsilon)= \int [F(z+\varepsilon) - F(z-\varepsilon)]^2 dF(z)$$

These two constants can be consistently estimated by the following V-statistics:

$$C = \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t=1}^{T} \chi_\varepsilon(u_s, u_t) \ ; \ K = \frac{1}{T^3} \sum_{r=1}^{T} \sum_{s=1}^{T} \sum_{t=1}^{T} \chi_\varepsilon(u_r, u_s) \chi_\varepsilon(u_s, u_t)$$

The above theorem can be generalized for the case of weakly dependent time series, using the conditions and the results provided by the Theorem 1 in Denker & Keller (1983).Before we precede to the definition of the more general case we provide this theorem:

<u>Theorem #1 {M. Denker & Keller G. (1983)}</u>

Let h:—^{m}→ℝ be a non-degenerate kernel. Then the asymptotic distribution of $\frac{N}{m \cdot \sigma_N}(U_N(h) - \theta)$ is N(0,1) provided one of the following conditions is satisfied :

a) $(y_n)_{n\geq 1}$ is uniformly mixing in both directions of time, $\sigma_N^2 \to \infty$ and for some $\delta>0$

$$\sup_{1\le t_1 \le t_2 < ... < t_m} E[|h(y_{t_1},...,y_{t_m})|^{2+\delta}] < \infty$$

b) $(y_n)_{n\geq 1}$ is uniformly mixing in both directions of time with mixing coefficients $\varphi(n)$ satisfying $\Sigma\varphi(n)<\infty$, $\sigma^2 \neq 0$ and

$$\sup_{1\le t_1 \le t_2 < ... < t_m} E[(h(y_{t_1},...,y_{t_m}))^2] < \infty$$

c) $(y_n)_{n\geq 1}$ is absolutely regular with coefficients $\beta(n)$ satisfying $\Sigma\beta(n)^{\delta/(2+\delta)}<\infty$ for some $\delta>0$, $\sigma^2 \neq 0$ and

$$\sup_{1\le t_1 \le t_2 < ... < t_m} E[|h(y_{t_1},...,y_{t_m})|^{2+\delta}] < \infty$$

The same statement holds for v.Mises's functionals when the supremum in a)- c) is replaced by the supremum over all choices of $1 \le t_i$ $(1 \le i \le m)$.

- It is showed by Sen (1972) that the Central Limit Theorem, for weakly dependent random variables, is valid for U-Statistics of mixing processes provided that 1(b) holds.

We can now provide the more general definition that includes the iid as a special case.

<u>Theorem 2.1 (Brock et al (1991))</u>

Suppose $\{u_t\}$ is a weakly dependent time series, satisfying conditions (a), (b) or (c) in the theorem above. Define:

$$w_{m,T}(\varepsilon) = \frac{C_{m,T}(\varepsilon) - (C_{1,T}(\varepsilon))^m}{\sigma_{m,T}(\varepsilon)} \tag{3.3.5}$$

Then,

$$w_{m,T}(\varepsilon) - \frac{C_m(\varepsilon) - (C(\varepsilon))^m}{\sigma_m(\varepsilon)} \tag{3.3.6}$$

is asymptotically normally distributed with mean zero and variance $[\omega_m(\varepsilon)/\sigma_m(\varepsilon)]^2$.

So far there has been no assumption regarding the moment structure of the original process u_t. As it is mentioned in Brock et al (1991) and in De Lima (1997) there exists no moment restriction for the BDS test to work. The only restriction posed is in the case that $\{u_t\}$ is iid, and is that the cumulative probability function F of the process exists, is nondegenate and twice differentiable.

3.4 Assumptions for the BDS test to be nuisance parameter free.

BDS test is one more statistical test concerning the presence of nonlinearities, instability and predictability in financial time series. It is in our best interest that when the test is applied its asymptotic distribution remains the same as if the true innovations were observed. However, even if the true innovations are iid, the estimation process generates residuals that exhibit a form of dependence.

As the main problem is that we estimate the statistical model and we apply the test to the remaining residuals, we must be certain that the final asymptotic distribution is not affected. Tests that carry this property are called nuisance-parameter free tests. Therefore, we provide the definition below:

A test is called nuisance-parameter free, if the intermediate step of estimating parameters of a given model does not affect the asymptotic distribution of this test.

In mathematical terms,
If S($\hat{\theta}$) is a statistic that depends upon some consistently estimated parameter, $\hat{\theta}$, and assuming that at the true value θ,

$$\sqrt{T} \cdot \frac{(S(\theta) - \mu(\theta))}{\sigma(\theta)} \xrightarrow{d} N(0,1) \tag{3.4.1}$$

then S($\hat{\theta}$) is a nuisance-parameter free statistic if

$$\sqrt{T} \cdot (S(\hat{\theta}) - S(\theta)) \xrightarrow{p} 0 \tag{3.4.2}$$

$$\mu(\lambda) = \lim_{T\to\infty} E[S_T(\lambda)] \ , \sigma(\lambda) = \lim_{T\to\infty} E[S_T(\lambda)^2]$$

(when the actual value of the parameter is θ)

In order to show that the asymptotic distribution of $S_T(\hat{\theta})$ is the same as the one corresponding to $S_T(\theta)$, we note that the Mean Value Theorem guarantees that

$$\sqrt{T} \cdot (S(\hat{\theta}) - \mu(\theta)) = \sqrt{T} \cdot (S(\theta) - \mu(\theta)) + \sqrt{T} \cdot (\hat{\theta} - \theta)' \left[\frac{\partial}{\partial \lambda} S(\lambda) \bigg|_{\lambda=\theta^*} \right] \tag{3.4.3}$$

,

where θ* is a point between θ and $\hat{\theta}$.

Slutsky's theorem guarantees that S($\hat{\theta}$) and S(θ) are asymptotically equivalent, provided that :
a) $\sqrt{T} \cdot (\hat{\theta} - \theta) = O_P(1)$, i.e. $\hat{\theta}$ is a $\sqrt{T}$-consistent estimator of θ
b) $\lim_{T\to\infty} E\left[\frac{\partial}{\partial \lambda} S(\lambda) \bigg|_{\lambda=\theta^*} \right] = 0$
Let us concentrate in condition b).It is clear that is we are not able to assume the differentiability of S(λ) the above approach to determine the asymptotic distribution for a nuisance-parameter statistic is invalid .
Under smoothness conditions on the kernel, moment conditions on the interaction of the kernel with the data generating process, and regularity conditions on the strength of temporal dependence to be detailed in later sections, the nuisance parameter free property is valid for the BDS test.
In order to prove this property we must provide a theorem guaranteeing that the limiting distribution of a bounded U-statistic with a non-differentiable kernel is not changed when estimated parameters are present.

Theorem
Under the assumption A-D, the following holds:

$$Q(\hat{\theta}) = \sqrt{T} \cdot [(S(\hat{\theta}) - \mu(\hat{\theta})) - (S(\theta) - \mu(\theta))] \xrightarrow{p} 0 \qquad (3.4.4)$$

Assumptions
A. (Data Generating Process)

$$y_t = G(Y_{t-1}, \theta) + u_t$$

where $\{u_t\}$ is iid and $Y_{t-1}=\{y_{t-1},\ldots,y_{t-p}\}$; $\{y_t\}$ is a strong mixing process with mixing coefficients that satisfy the summability condition $\sum_{k=1}^{\infty} a(k)^{1/2} < \infty$; G is a measurable function of Y_{t-1}.

The Residual function,

$$u_t(\lambda) \overset{def}{=} y_t - G(Y_{t-1},\lambda) = u_t + G(Y_{t-1},\theta) - G(Y_{t-1},\lambda) \Leftrightarrow u_t(\lambda) = u_t + G^*(Y_{t-1},\theta,\lambda)$$

satisfies the following three properties :
1. $u_t(\theta)=u_t$
2. $u_t(\hat{\theta})=\hat{u}_t$
3. If λ is a constant, $u_t(\lambda)$ is a strong mixing process of size γ, for some $\gamma>2$.

G* is a measurable function. Further assume that Y_{t-1} is a finite-order vector (i.e. p is finite). If y_t is mixing such that the strong mixing coefficient a(k) is $O(k^{-\gamma})$, for some $\gamma>0$ τhen, we have that $u_t(\lambda)$ is mixing such that a(k) is $O(k^{-\gamma})$, for each λ. The summability condition on the mixing coefficients implies that a(.) must decline faster than $1/k^2$.

Let $S(\theta)$ be a U-statistic with a bounded symmetric kernel, $h(u_1(\theta),\ldots,u_m(\theta))<B$, and $\mu(\theta)= E[h(u_1(\theta),\ldots,u_m(\theta))]^2$. Define as:

$$W(u_{j_1}(\lambda),\ldots,u_{j_m}(\lambda)) = h(u_{j_1}(\lambda),\ldots,u_{j_m}(\lambda)) - \mu(\lambda) - \left[h(u_{j_1}(\theta),\ldots,u_{j_m}(\theta)) - \mu(\theta)\right]$$

and let

$$Q(\lambda) \overset{def}{=} \frac{\sqrt{T}}{\binom{T}{m}} \sum_j W(u_{j_1}(\lambda),\ldots,u_{j_m}(\lambda))$$

where Σ_j denotes summation over the $\binom{T}{m}$ combinations of k distinct elements $\{j_1,\ldots,j_m\}$ from $\{1,\ldots T\}$. The kernel W represents the difference between the kernels h evaluated at two different points, λ and θ, of the residual function.

B.

$$E\left[\sup_{\theta_1 \in K(\theta,d)} \left| h(u_{s_1}(\theta_1),\ldots,u_{s_m}(\theta_1)) - h(u_{s_1}(\theta),\ldots,u_{s_m}(\theta)) \right| \right] < M \cdot d \qquad (3.4.5)$$

where M is a constant and $K(\lambda,d)=\{\lambda_1 \ \mathbb{R}^p : \|\lambda_1 - \lambda\| \le d\}$, d>0 and $\|\cdot\|$ is the max-norm.

C.

$$\lim_{d\to 0} E\left[\sup_{\theta_1\in K(\theta,d)} \left|h(u_{s_1}(\theta_1),...,u_{s_m}(\theta_1)) - h(u_{s_1}(\theta),...,u_{s_m}(\theta))\right|^2\right] = 0 \qquad (3.4.6)$$

For a bounded kernel this assumption is automatically satisfied.

D. ($\sqrt{T}$ - consistency)

$$\sqrt{T}\cdot(\hat{\theta}-\theta) = O_P(1) \qquad (3.4.7)$$

Usually this assumption is valid for the case of stationary and ergodic processes.

We are now able to prove the nuisance parameter free property for BDS test. As the BDS statistic is a function of U-statistics, we expect that as long as the assumptions A-D hold, then the test will be nuisance-parameter free. In particular if we define as $S_{m,\lambda}(\lambda) \overset{\text{def}}{=} C_{m,\varepsilon}(\lambda) - [C_{1,\varepsilon}(\lambda)]^m$, then the BDS test is nuisance parameter free if we show that:

$$\begin{cases} \sqrt{T}\cdot[S_{m,\varepsilon}(\hat{\theta}) - S_{m,\varepsilon}(\theta)] \xrightarrow{p} 0 \\ s_{m,\varepsilon}(\hat{\theta}) - s_{m,\varepsilon}(\theta) \xrightarrow{p} 0 \end{cases} \qquad (3.4.8)$$

<u>Proposition</u>

If

$$\left.\begin{matrix} \sqrt{T}\cdot[C_{m,\varepsilon}(\hat{\theta}) - C_{m,\varepsilon}(\theta)] \xrightarrow{p} 0 \\ \sqrt{T}\cdot[C_{1,\varepsilon}(\hat{\theta}) - C_{1,\varepsilon}(\theta)] \xrightarrow{p} 0 \end{matrix}\right\} \Rightarrow \sqrt{T}\cdot[S_{m,\varepsilon}(\hat{\theta}) - S_{m,\varepsilon}(\theta)] \xrightarrow{p} 0$$

<u>Note</u>

As $\sigma_{m,\varepsilon}$ is a continuous function on C and K then, if

$$\left.\begin{matrix} K(\hat{\theta}) - K(\theta) \xrightarrow{p} 0 \\ C(\hat{\theta}) - C(\theta) \xrightarrow{p} 0 \end{matrix}\right\} \Rightarrow s_{m,\varepsilon}(\hat{\theta}) - s_{m,\varepsilon}(\theta) \xrightarrow{p} 0$$

The second assumption is derived by the fact that C is the V-statistic of $C_{m,\varepsilon}$. The first assumption is derived only if the assumption B is valid for the kernel. In particular

$$E\left[\sup_{\lambda\in K(\theta,d)} \left|\chi_\varepsilon(u_r(\lambda),...,u_s(\lambda)) - \chi_\varepsilon(u_r(\theta),...,u_s(\theta))\right|\right] \le M\cdot d \Rightarrow K(\hat{\theta}) - K(\theta) \xrightarrow{p} 0$$

The assumption above cannot be satisfied when we apply the BDS test on the standardized residuals of a GARCH(1,1) model because the residual function G ($=\sigma^2(Y_{t-1},\theta)$) is not measurable, depending on previous values of the series and of the conditional variance. Therefore we follow a transformation of the residuals so as the kernel satisfy assumption B.

The appropriate transformation was proposed by Brock & Potter (1992).Instead of using the standardized residuals after a prefiltering with a GARCH(1,1) model, use their natural logarithms,

$$N_t \overset{\text{def}}{=} \ln U_t^2 = \ln y_t^2 - \ln(\sigma^2(Y_{t-1},\theta)) \tag{3.4.9}$$

Under the null hypothesis of correct specification, u_t is iid; it follows that N_t is also iid ($u_t \sim IID \Rightarrow N_t \sim IID$). Moreover if we set $\tilde{y}_t = \ln y_t^2$ & $\tilde{\sigma}(Y_{t-1},\theta) = \ln(\sigma^2(Y_{t-1},\theta))$ then we can rewrite the model as $N_t = \tilde{y}_t - \tilde{\sigma}(Y_{t-1},\theta)$, this implies that the asymptotic distribution of the BDS statistic is the same when applied to the estimated residuals $\hat{N}_t$ or to N_t, provided that $\tilde{\sigma}(Y_{t-1},\theta)$ satisfies the conditions below. The gradient of the residuals function is :

$$\nabla\tilde{\sigma}(Y_{t-1},\theta) = \frac{1}{\sigma^2(Y_{t-1},\theta)}(1, y_{t-1}^2)'$$

if $\omega>0$ and $\alpha>0$ (sufficient and necessary conditions) then $\sigma(Y_{t-1},\theta) > \omega^*$ and

$$E\{\nabla\tilde{\sigma}(Y_{t-1},\theta)\} \le \frac{1}{\omega^*} E\{(1, y_{t-1}^2)'\}$$

$$\omega^* = \frac{1}{1-\beta}$$

the above expectation are finite if the original series y_t is covariance stationary i.e. $\alpha+\beta<1$.

3.5 Summary of assumptions

We summarize the basic assumptions that the BDS test must hold in order for the asymptotic distribution to exist and not to be altered if the estimated residuals are used.

The stochastic process is given by:

$$\mathbf{y_t = G(Y_{t-1},\theta) + u_t}$$

1. $\{u_t\}$ is IID with a non-degenerate cumulative distribution F
2. There exists the unconditional distribution of $\{u_t^m\}$
3. $\{y_t\}$ is stationary and ergodic
4. The parameters can be consistently estimated
5. $\{u_t\}$ must satisfy conditions (a)-(c) in Theorem # 1(Denker & Keller (1983)).
6. G is a measurable function.
7. Kernels must be nondegenerate (their variance must be positive).
8. The choice of m, ε, are essential. Large values of m require a large sample size when large values of ε may present that the points of the series are closer than in reality. Monte Carlo evidence suggested that a choice for ε to be 0.5,1,1.5 of the standard deviation of the data and for m to satisfy the condition that T/m>200 is sufficient.

3.6 Literature Review for the size of BDS test.

As far as we have seen, BDS test can be used as a general diagnostic tool for the presence or not of nonlinear structure or chaotic dynamics. Therefore it is necessary to evaluate its performance in the means of size and power. The literature review reveals papers that consider these properties in contest with other nonlinear tests and they are focused in the power properties. However the main task of this thesis is to explore the size properties of the test so we will present only the analogous papers.

We can classify the literature into two categories. The first is concerned about the size performance when the test is applied to iid series of various distributions. The second focuses in the size of the test when it is applied to residuals of a fitted model; estimation takes place using the correct model that generated the original data. It also tries to investigate whether the nuisance free property of the test can be verified or not.

We present the two categories:

- The main reference for the first category is the Brock et al (1991) .Data were generated from the distributions: Standard Normal, t-student with 3 degrees of freedom, Double exponential, $\chi^2(4)$, uniform and bimodal mixture of normals. All distributions were scaled so that their standard deviations equal 1 regarding the standard normal as the base model and the others as departures from it. The authors performed Monte Carlo simulations and they reported that the size is reasonable for all distributions provided the sample size is adequate, i.e. larger than 500, and that the embedding dimension is smaller than 5. Similar results for the same distributions and the same sample size yielded Brooks (1999). He also computed the size for 50 observations; he found again that the test is reasonably sized. The critical values for performing the test were drawn from the asymptotic distribution of the test under the null (N(0,1)) for all cases mentioned above. Ashley & Patterson (2001) derived the size of the BDS test for a variety of distribution and they used critical values obtained from a bootstrapping procedure. Despite the small size of the sample, the size was close to nominal.
- The second category of papers is about the size of BDS test, when we apply it to the residuals of a correctly specified model. As our thesis aim is to explore the BDS test size for the cases of GARCH(1,1) process, we will focus only on the relevant results. Again Brock et al (1991) explored the size properties for a stationary GARCH(1,1) process. They generated an adequately sized sample from such a process and they filtered it with the correct model. Then they used the test on the standardized residuals. The size turned out to be rather different from that it was expected using the asymptotic theory. Following the same procedure, Brooks & Henry (1999) derived similar results. The test was under-sized and it became more conservative as the sample size increased. In the presence of this evidence the authors of the above papers suggested bootstrapping methods or Monte Carlo simulations for obtaining the appropriate critical values.

  De Lima (1996) instead of applying the test on the standardized residuals he applied it on the natural logarithms of them as Brock & Potter (1992) suggested. He used a stationary GARCH(1,1) model with N(0,1) innovations. The results were exceptionally good; size was close to nominal.

4. Monte Carlo procedure for the BDS test

4.1 Description of the procedure

We used Monte Carlo simulations in order to obtain three different sets of results.

i) The size of the BDS test when it is applied directly to an iid series.

We generated series of sample size of 250, 1000 and 1500 observations from 6 different distributions.

Cauchy	t-student(2)	t-student(3)
t-student(4)	t-student(5)	N(0,1)

This choice was due to the stylized fact that the t-student distribution has the property that their moments exist only for order equal of its degrees of freedom minus 1. The Cauchy distribution is regarded as a t-student with 1 degree of freedom. Therefore we examine the behavior of the test for iid series that possess different moment properties in order to inquiry the validity of the assumption that there are no moment restrictions for the test to function. We summarize them in the table below:

Distribution	Existing Moments	Distribution	Existing Moments
Cauchy	none	**t-student(4)**	1^{st}, 2^{nd}, 3^{rd}
t-student(2)	1^{st}	**t-student(5)**	1^{st}, 2^{nd}, 3^{rd} ,4^{th}
t-student(3)	1^{st}, 2^{nd}	**N(0,1)**	∞

Table 4.1.1

After the series has been generated, BDS test was applied directly to it. The BDS statistic was evaluated for embedding dimensions m=2,…,5 and ε equal to one standard deviation of the series. For the acceptance or not of the null hypothesis the critical value was drawn for the standard normal distribution for significance level of 5%[1]. We remind that the null hypothesis is that the series under scrutiny is independently and identically distributed.

We repeated this procedure 1000 times and 1500 times and we stored the BDS statistic each time and whether the null was rejected or not. Then the mean, standard deviation, skewness and kurtosis were calculated for the statistic. The size of the test was the percentage of rejections of the null over all replications.

ii) Derivation of the boundaries among the different regions of the α, β coefficients of a GARCH(1,1) process for various iid innovations.

As we have mentioned in paragraph 2, the moment properties of the unconditional GARCH(1,1) model change with respect to the innovations process and the relation between the coefficients α, β. Since it is our intention to evaluate the BDS test size for a variety of innovations and for different regions it is necessary to provide the appropriate boundaries (i.e. Figure 2.2.1).

We begin by mentioning the innovation's distributions. Besides the standard normal distribution, we use the t-student distributions with 3, 4, 5 degrees of freedom after we multiply them with the inverse of their standard deviation so that the variance equals unity. From this point forward we denote these series as sd-t-student.

The following table summarizes the properties of the distributions:

Distribution	**variance** v_2	**4^{th} moment** v_4	**Distribution**	**variance** v_2	**4^{th} moment** v_4
t-stud(3)	3	-	**sd-t-stud(3)**	1	-
t-stud(4)	2	-	**sd-t-stud(4)**	1	-
t-stud(5)	5/3	25	**sd-t-stud(5)**	1	9

Table 4.1.2

Recall that for a t-student distribution with f degrees of freedom we have:

$$v_2=\frac{f}{f-2}\ ,\quad f>2 \qquad\qquad v_4=3\cdot\frac{f^2}{f-2}\cdot(f-4)\ ,\quad f>4$$

For the case of $X_t=\sqrt{\frac{1}{Var(Z_t)}}\cdot Z_t$, Z_t~t-student with f degrees of freedom:

$$v_2=1,\quad f>2 \qquad\qquad v_4=3\cdot(f-2)\cdot(f-4)\ ,\quad f>4$$

We provide again the boundary conditions for each region according to Nelson (1990):

Boundary	
Region 1-Region 2	$E[\ln(\beta+\alpha z_t^2)]=0$
Region 2-Region 3	$E[(\beta+\alpha z_t^2)^{1/2}]=1$
Region 3-Region 4	$E[(\beta+\alpha z_t^2)]=1$

Table 4.1.3

For the derivation of each boundary for each distribution a possible, and the correct way, is to solve the equations analytically and take the results for the coefficients α, β. However this task is much too complicated and is escaping the purpose of this thesis. Instead of the analytical solution we will proceed with a numerical solution to our problem.

We generated a series of 125.000 observations from each of the standardized t-student distributions that we already mentioned. Then using a simple program, we kept fixed the value of the coefficient α and we started an iteration procedure for the estimation of β that satisfied each of the conditions of table 4.1.3. This way we were able to determine the boundaries for every region and for each distribution that we used as innovation process.

iii) The size of the BDS test when it is applied to residuals of a correctly fitted GARCH(1,1) model.

The next step in our Monte Carlo procedure brings us closer to the purpose of this thesis. The derivation of the size of the BDS test and the estimated distribution of its statistic if we generate a GARCH(1,1) model and we estimate the series with a GARCH(1,1) model. The procedure is similar to the first case that the series were iid. In more details the steps that we followed were:

-We chose a distribution function for the innovation process to follow; a standard normal and the standardized t-student with 3, 4, 5 degrees of freedom. Our primary concern was that the variance was equal to unity. This way the estimation process will be consistent and the results of Davidson (2002) would be valid. The appropriate regions for the coefficients α, β had already been calculated.

-We generated a GARCH(1,1) model with sample size of 350, 1100 and 1600 observations. The coefficients α, β were chosen to be in different region each time.

-We estimated a GARCH(1,1) model on the data after discarding the first 100 observations of each series. Using the natural logarithms of the standardized residuals of this procedure we proceeded with the application of the BDS test. The embedding dimension was set to be m=2,3,4,5 and the ε to be equal to one standard deviations[2]. The test provided the BDS statistic for each dimension and an answer to whether the null hypothesis that the residuals were clear of any remaining nonlinearities had been rejected or not.

-We repeated the procedure 1000 and 1500 times and we stored the results. Then for every embedding dimension we calculated the basic distributional characteristics (mean, standard deviation, skewness and kurtosis) for the BDS statistic. The percentage of rejections of the null hypothesis was in each case the size of the test.

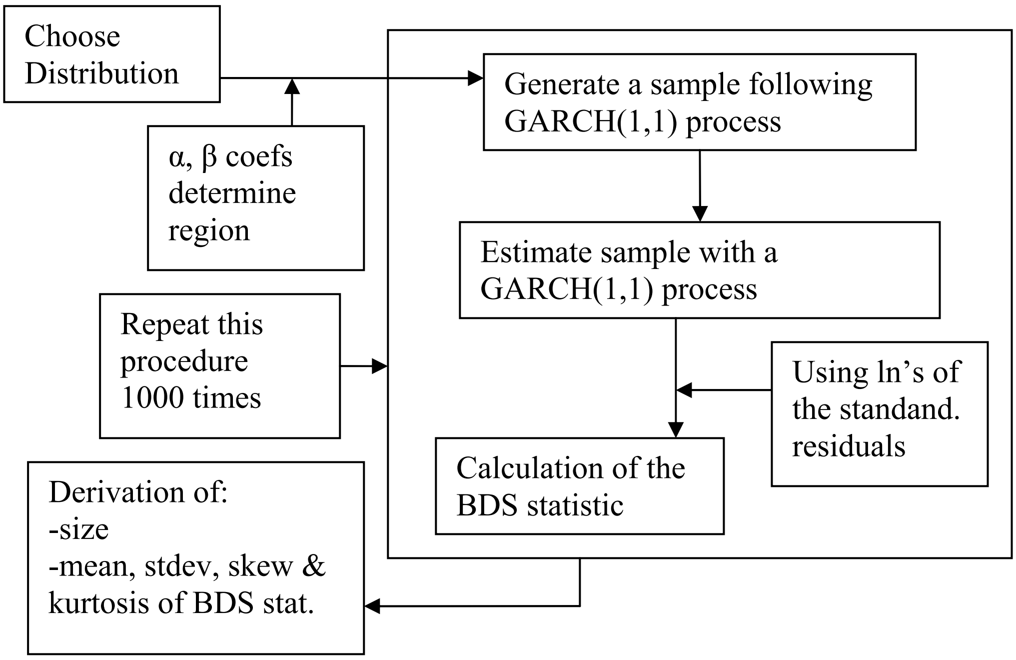

Figure 4.1.1

-If during the estimation step, the coefficients were found negative, in contrast with the available theory, or they took much larger values than the true, we started the procedure again by generating a new series. This problem did not lead to a large increase of the number of replications finally performed[3]. Instead the quality of the coefficient estimates was improved and the distributional characteristics were of greater sense[4].

4.2 Size of the BDS test: the case of iid series.

Using the procedure we described above we calculated the distributional characteristics of the BDS statistic and the size of the test, for each iid process. The results given in the table below are for embedding dimension m=2[5].

Distribution:		**Cauchy**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.054911	1.034848	1.868273	18.72275	0.0520000
1000		0.000451	1.058239	1.501240	13.16928	0.0680000
1500	**2000**	-0.004222	1.012158	2.875441	21.08473	0.0540000
Distribution:		**t-student(2)**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.033302	1.047797	0.346193	2.879772	0.0600000
1000		0.017934	1.056212	0.179230	3.092861	0.0720000
1500	**2000**	-0.021783	1.029198	0.203046	3.006934	0.0565000
Distribution:		**t-student(3)**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.063450	1.060971	0.366295	3.588656	0.0690000
1000		-0.037556	0.985328	-0.016446	3.017958	0.0480000
1500	**2000**	-0.045178	1.012401	0.183272	3.248233	0.0590000
Distribution:		**t-student(4)**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.067248	1.033512	0.148684	3.335028	0.0720000
1000		-0.031777	0.997771	0.218548	3.399237	0.0510000
1500	**2000**	-0.026404	1.036300	0.236622	3.037243	0.0575000
Distribution:		**t-student(5)**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.057494	1.061156	0.195169	3.041910	0.0660000
1000		-0.080395	1.012596	0.091908	2.846425	0.0500000
1500	**2000**	-0.013680	1.001899	0.202866	2.845676	0.0490000
Distribution:		**N(0,1)**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.037955	1.119189	0.310305	3.458254	0.0690000
1000		-0.033040	1.049061	-0.115106	3.166985	0.0640000
1500	**2000**	-0.043328	1.041246	0.070926	3.095671	0.0610000

Table 4.2.1

As we can notice, for all cases but the Cauchy, the BDS statistic's distribution can be well approximated by the standard normal. Additionally the size of the test is very close to the nominal value of 5%. We must emphasize the fact that this result is more prominent for the case of more than 250 observations. For this case the test is little oversized and there are small deviations for all the four distributional characteristics we report. Therefore a researcher that intends to use the test must keep in mind that his sample size must be as large as possible. So far our results coincide with that of the literature.

In the case of the Cauchy distribution, we can no longer consider that the statistic's distribution is standard normal. Even though the mean and the standard deviation are zero and unity respectively, as in the N (0, 1) case, the values that the skewness and the kurtosis take are quite larger. Especially the large value of the kurtosis reveals a very leptokurtic distribution. However the size remains close to the nominal.

The main point that must be noticed from the analysis above is that one of the assumptions we have made so far has failed. It was noted in paragraph 3 that there are no moment restrictions for the iid series under scrutiny in order for the BDS test to function. The case of the Cauchy distribution proves the opposite. The main property of the Cauchy distribution is that despite the existence of its density function, it has no moments at all. Here the BDS test behaves as it was assumed for the case of distributions that at least their first moment, the mean, exists.

Therefore we must restate the moment assumption for the case of the iid series; instead of only the existence of the density function, we must pose the restriction that the first moment of the distribution exists. Yet we must notice that even the test does not approximate the standard normal, size is close to nominal. This creates the possibility that the first moment existence assumption may be relaxed and the critical values could be drawn from the standard normal.

4.3 Size of the BDS test: the case of GARCH(1,1) series.

We have mentioned that the Monte Carlo simulation for this case would contain results for different distributions for the innovations and for different sets of coefficients α, β so that we cover all possible regions. In addition we consider the cases that the process is 2nd or 4th order stationary and the case of L_0-approximability whenever it is possible; i.e. wherever the fourth moment of the innovation exists or not for the case of the fourth order stationarity.

The results will be report in the following manner: for each innovation distribution it will be provided a figure that demonstrates the regions after their boundaries have been numerically calculated as described in paragraph 4.1.Insted of the L_0-approximability boundary it is given the boundary that the relation (2.4.1) provides. We remind that L_0-approximability besides the validity of the relation (2.4.1) requires that the series is strict stationary. The boundary lines appear truncated. This is due to the numerical procedure and to the approximation we chose for the zero equality. However this does not concern us, as we take points that are "deep" inside the regions and away from the boundary lines.

Then we will present the pairs of α, β coefficients that are used in the simulation and the regions they belong, along with the moment characteristics for the unconditional process for each case. Finally we will report for embedding dimension m=2, the distributional characteristics of the BDS test statistic and its size for each coefficient for 250, 1000, 1500 observations and 1000 and 2000 replications.

The choice for m=2 was made because we consider the case of 250 observations. On key assumption is that the embedding dimension one would use for the test depends primarily on the length of the series under check. If the observations are few the choice of m=2 is the most appropriate. The results for larger embedding dimension are omitted in this paragraph and they are reported in full in the Appendix. Another important issue that may be raised is whether the results are point specific. Even though their selection was not made in order to match any criterion, unreported results for different points from the same regions and therefore identical properties for the GARCH process reached the same conclusions.

- Results for IID innovations $X_t = \sqrt{\frac{1}{Var(Z_t)}} \cdot Z_t$, $Z_t \sim$ t-student (3)

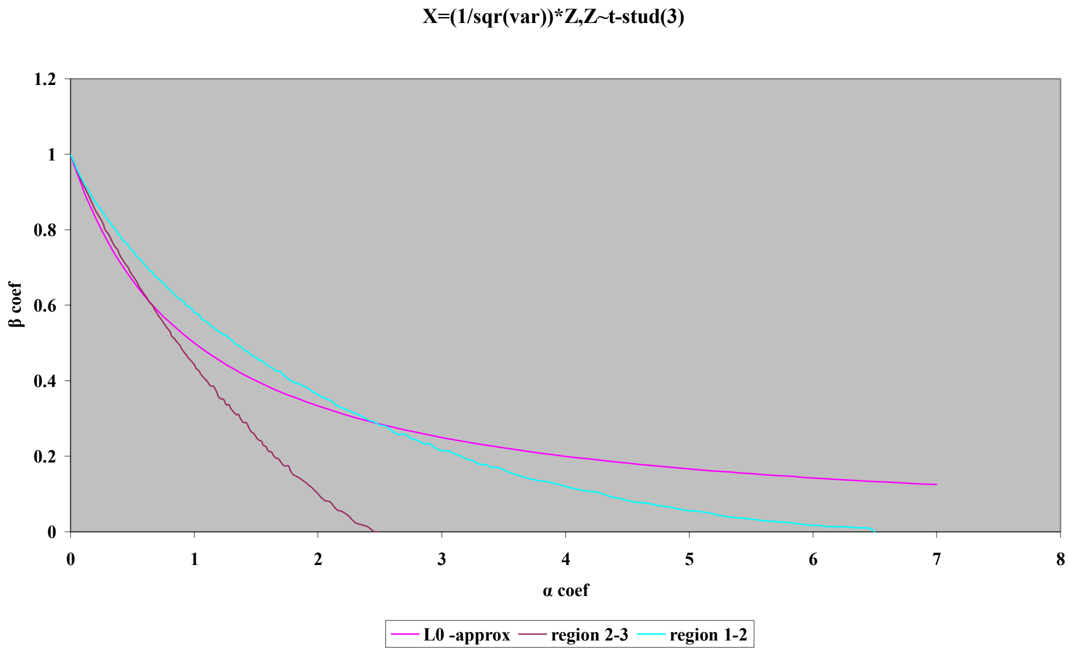

Figure 4.3.1

# observ	# replic	Point	size	Distributional Characteristics			
				μ	s	a_3	a_4
250	**1000**	**A**	0.085	0.048790	1.118076	0.265967	2.83240
	1000	**B**	0.087	0.077560	1.143551	0.395657	3.45682
	1000	**C**	0.056	0.033520	1.072474	0.290596	3.12098
1000	**1000**	**A**	0.058	0.092502	1.083316	0.347621	3.88416
	1000	**B**	0.079	0.183223	1.132336	1.188402	12.5768
	1000	**C**	0.063	0.050424	1.073817	0.236518	3.14306
1500	**2000**	**A**	0.066	0.096628	1.103159	0.839001	7.42989
	2000	**B**	0.073	0.172289	1.185840	2.018296	25.5164
	2000	**C**	0.065	0.086955	1.110019	1.029108	8.26192

Table 4.3.1

Point	Coordinates	Region	Moment-Memory properties	L_0-approximability
A	**(0.52,0.30)**	**3**	strict stationarity 1^{st} moment	YES
B	**(1.70,0.30)**	**2**	strict stationarity	YES
C	**(0.80,0.60)**	**2**	strict stationarity	NO

Table 4.3.2

- Results for IID innovations $X_t=\sqrt{\frac{1}{Var(Z_t)}}\cdot Z_t$, Z_t~ t-student (4)

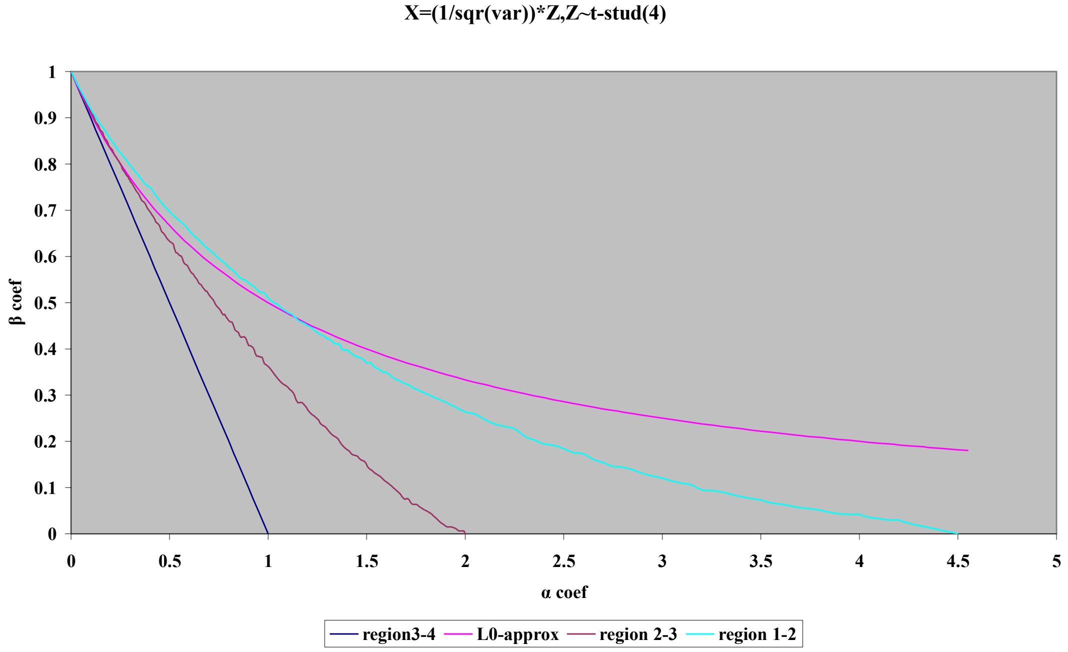

Figure 4.3.2

# observ	#replic	Point	size	Distributional Characteristics			
				μ	**s**	**a_3**	**a_4**
250	**1000**	**A**	0.070	-0.07616	1.109222	0.182149	3.402947
	1000	**B**	0.066	0.001489	1.055650	0.247877	3.149775
	1000	**C**	0.064	-0.05662	1.092364	0.185671	3.185553
	1000	**D**	0.061	-0.01348	1.059562	0.267870	2.874518
1000	**1000**	**A**	0.049	-0.08986	0.995256	0.064094	2.802747
	1000	**B**	0.060	-0.00699	1.054675	0.138975	2.876385
	1000	**C**	0.071	0.112161	1.178033	1.249408	9.510047
	1000	**D**	0.072	-0.05949	1.066752	0.195612	2.940678
1500	**2000**	**A**	0.058	0.020479	1.025831	0.063884	3.041970
	2000	**B**	0.063	-0.01415	1.051548	0.229238	3.240956
	2000	**C**	0.100	0.226400	1.376441	1.855231	12.34100
	2000	**D**	0.059	-0.0086	1.074656	0.787757	9.379962

Table 4.3.3

Point	Coordinates	Region	Moment-Memory properties	L_0-approximability
A	**(0.40,0.50)**	**4**	2^{nd} order stationary 1^{st} ,2^{nd} moment	YES
B	**(0.85,0.20)**	**3**	strict stationary 1^{st} moment	YES
C	**(1.29,0.40)**	**2**	strict stationary	YES
D	**(0.5,0.68)**	**2**	strict stationary	NO

Table 4.3.4

- Results for IID innovations $X_t = \sqrt{\frac{1}{Var(Z_t)}} \cdot Z_t$, Z_t~ t-student (5)

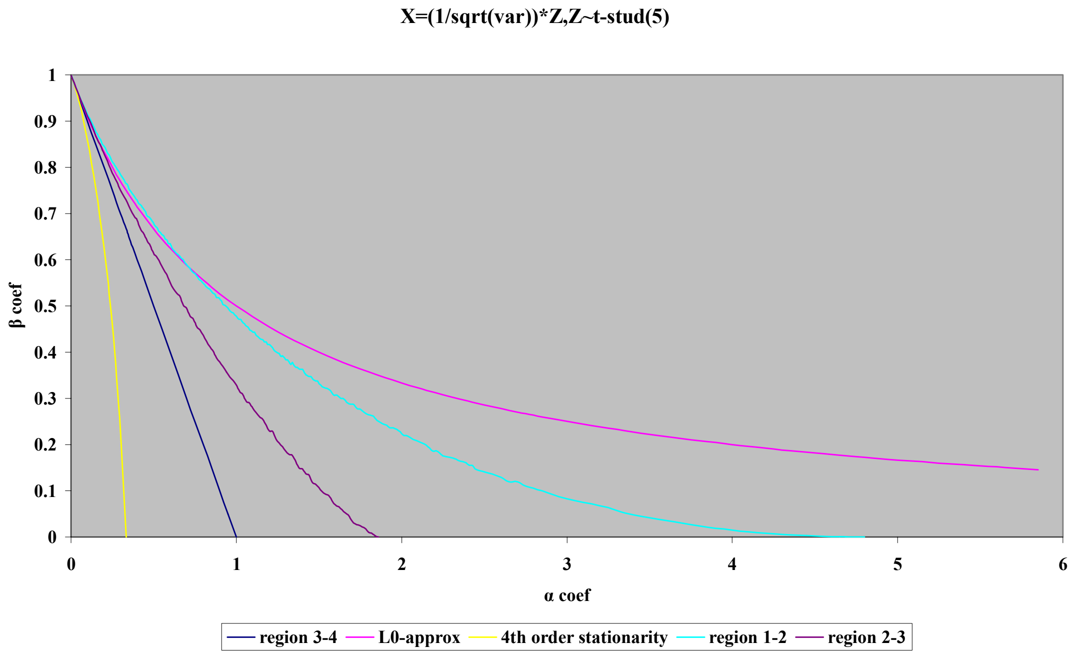

Figure 4.3.3

Point	Coordinates	Region	Moment-Memory properties	L_0-approximability
A	**(0.13,0.40)**	**4**	4^{th} order stationary 1^{st} ,2^{nd} ,3^{rd} ,4^{th} moment	YES
B	**(0.33,0.50)**	**4**	2^{nd} order stationary 1^{st} ,2^{nd} moment	YES
C	**(0.90,0.20)**	**3**	strict stationary 1^{st} moment	YES
D	**(1.31,0.30)**	**2**	strict stationary	YES

Table 4.3.5

# observ	#replic	Point	size	Distributional Characteristics			
				μ	s	a_3	a_4
250	1000	A	0.058	-0.13396	1.066728	0.280347	3.115136
	1000	B	0.065	0.025224	1.057137	0.228024	2.790781
	1000	C	0.050	-0.09345	1.042572	0.105024	2.808112
	1000	D	0.060	-0.04461	1.066005	0.048409	2.945696
1000	1000	A	0.048	-0.01434	1.021235	0.231608	3.192480
	1000	B	0.052	-0.02634	1.022998	0.095895	2.997172
	1000	C	0.047	0.006391	0.995318	0.249750	2.927890
	1000	D	0.052	-0.04827	1.013248	0.034611	2.814745
1500	2000	A	0.050	-0.02162	1.006946	0.173131	3.178304
	2000	B	0.043	-0.06572	0.986474	0.142582	3.088077
	2000	C	0.066	-0.02219	1.056432	0.107103	2.956959
	2000	D	0.063	-0.02778	1.067078	0.245327	3.664950

Table 4.3.6

- Results for IID innovations $Z_t \sim N(0,1)$

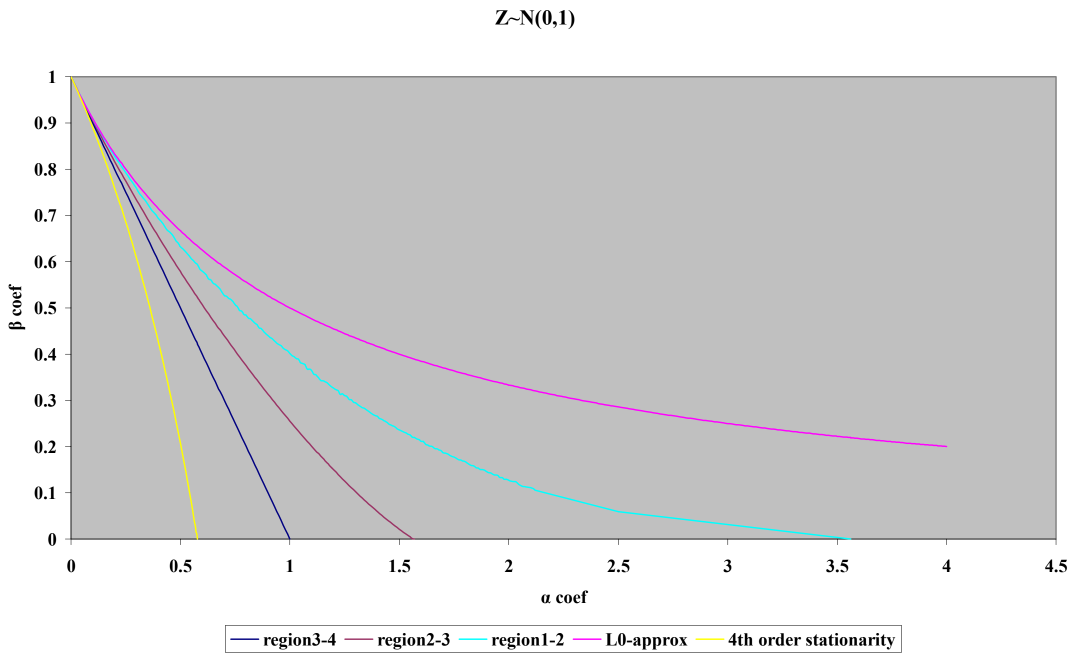

Figure 4.3.4

Point	Coordinates	Region	Moment-Memory properties	L_0-approximability
A	(0.1,0.60)	4	4^{th} order stationary 1^{st} ,2^{nd} ,3^{rd} ,4^{th} moment	YES
B	(0.5,0.40)	4	2^{nd} order stationary 1^{st} ,2^{nd} moment	YES
C	(0.8,0.30)	3	strict stationary 1^{st} moment	YES
D	(1.0,0.34)	2	strict stationary	YES

Table 4.3.7

# observ	#replic	Point	size	Distributional Characteristics			
				μ	s	a_3	a_4
250	**1000**	**A**	0.055	-0.16110	1.035908	0.230125	2.940034
	1000	**B**	0.058	-0.14458	1.030191	0.180871	3.240725
	1000	**C**	0.073	-0.09671	1.070953	0.214965	2.975537
	1000	**D**	0.058	-0.1657	1.043602	0.326253	3.034485
1000	**1000**	**A**	0.043	-0.1055	0.988685	0.059269	3.037820
	1000	**B**	0.057	-0.06053	1.032951	0.094806	2.873307
	1000	**C**	0.051	-0.00422	1.026620	0.153928	2.948097
	1000	**D**	0.052	-0.07823	0.994844	0.047307	3.141048
1500	**2000**	**A**	0.050	-0.02916	1.012702	0.098904	2.897881
	2000	**B**	0.048	-0.0563	0.992907	-0.00692	2.978359
	2000	**C**	0.048	-0.03936	0.987956	0.160154	3.055161
	2000	**D**	0.054	-0.0454	1.032204	0.428811	5.147213

Table 4.3.8

It is now time to evaluate our results and see whether the procedure revealed any possible flows to the assumptions we have made so far. We will examine each distribution specifically.

-for the sd-t-stud(3) we notice that the BDS statistic's asymptotic distribution can not be approximated by the standard normal for all points examined. Even though the mean and the standard deviation take values close to those expected, it is the skewness and the kurtosis that cause the problem. Yet in the case of point C where we have strict stationarity but no L_0-approximability, the estimations for skewness and kurtosis are reduced. The test is slightly oversized; it takes values from ~6%-8%, when the nominal is 5%. What are the special features for case? The innovations process has only 2 moments and there is no2nd order stationarity for the GARCH process.

-for the sd-t-stud(4) we have asymptotic normality for the test for the points that belongs to regions 3, 4 (A, B) and for the other 2 points (C, D) that belong to region 2 we have a large deviation from N(0,1).Again for points C, D the mean and the standard deviation are as expected. However skewness and kurtosis take large values that surprisingly are reduced, not dramatically, when we estimate from point C to point D. We remind that the sole difference between these points is that in point D there is no L_0-approximability. The size of the test is around 8%, not very far from the nominal. In this case the innovation process has 3 moments and there exists an area of 2nd order stationarity for the corresponding GARCH model.

-for the sd-t-stud(5) we have similar results with the case for the 4 degrees of freedom. For coefficient values in regions 3, 4, regardless of the existence or not of 4th order stationarity for the GARCH process the BDS statistic distribution reaches asymptotically the standard normal and the size of the test is very close to nominal. For the point D, the one belonging in region 2, there is a deviation in terms of the skewness and kurtosis that appears to be larger than expected. Despite this difference, the size is not distant from 5%. The main features of this innovation process are the existence of the first four moments and the existence for the corresponding GARCH process of a region that has fourth order stationarity.

-for the N(0,1) the results are the same as for those for the sd-t-stud(5) case. In regions 3, 4 we have the BDS test statistic to have an asymptotic standard normal distribution and in region 2 there is a deviation from that in terms of skewness and kurtosis. However, even in the last case, the size is close to the nominal values. In this case, the innovations have an infinite number of moments and there is no restriction in considering the notions of 4^{th} and 2^{nd} order stationarity for the unconditional GARCH process. We must also notice the movement of the line that describes the coefficients relation for the L_0-approximability to hold. For sd-t-stud with 3 degrees of freedom this line left outside a large part of region 2. As the degrees of freedom increased this left out area was decreasing and as the degrees become infinitely large (t-stud(f) $\xrightarrow{f\to\infty}$ N(0,1)) is vanishing. However Davidson (2002) states that even in the case of the N(0,1), this area exists.

Considering that the test has as an asymptotic the N(0,1) distribution, for points in region 3, in all cases but the standardized t-student with 3 degrees of freedom, can lead us to the conclusion that the existence of more than 2 moments for the innovation process or equivalently the existence of a region where 2^{nd} order stationarity is valid for the GARCH(1,1) model is a necessary condition for the test to work.

Another aspect of the results that must be considered is the reason that the test fails so heavily in region 2. A further clue to this inquiry comes from the mean values of the estimators as reported in the Appendix. It is reported that when the estimation was made in region 2, the procedure failed to estimate correctly the constant term of the model in all cases. Furthermore the large values of the skewness and kurtosis were deteriorated with the increase of the sample. We remind that in this region, the GARCH model does not have any moments at all and for the case of the iid series we concluded that the existence of first moment was necessary. In fact, if there is a close similarity of the results for the case of the Cauchy iid series and of those for the region 2 points. In conclusion we can say that the failure in this region can be a failure of the consistently estimated parameter assumption and the nonexistence of moments for the unconditional model.

Finally we summarize in order to derive any further assumption if necessary or to relax some. So far the test resulted as the theory described in regions

where at least the first moment of the unconditional process existed and the innovation distributions had moments, even fractional, of order higher than 2.

This relaxes a lot the assumption mentioned in De Lima (1996) considering that the test would work only for coefficients that satisfy the 2^{nd} order stationarity condition; for innovations z_t~iid(0,1), $\alpha+\beta<1$.Probably the estimated conditional variance, that enters the procedure of the test, has similar behavior for cases that $\alpha+\beta>1$.

In every other case the BDS statistic had mean and standard deviation close to zero and unity respectively but much larger skewness and kurtosis. The size in all case remained fairly close to nominal. A potential user of the test for real data can rely on it regardless of the estimations of the coefficients α, β as long has a valid hypothesis about the distribution of the innovation.

5. Use of the BDS test on financial data

5.1 Existing Literature

BDS test is widely used for determining the existence of low-order chaos or more generally the existence of nonlinearities and nonstationarities in the series under scrutiny. As the null hypothesis is that the series or the residuals after an estimation procedure are iid, it can also be used as a general portmanteau test. The majority of the existing literature uses the BDS test both ways; as a portmanteau test for checking whether the imposed model is the correct one, and as specific test at the original series for uncovering chaos or nonlinearities. The series that have been examined varies from exchange rates to unemployment. In particular:

- Gallegati M. & Mignacca D. (1994) examined whether US Real GNP is chaotic or not. They applied the BDS test to the residuals that were obtained after the series was filtered with an appropriate ARMA model. This way all linear dependence was removed. The sample they used was rather small therefore bootstrapping for the derivation of the critical values was used. The result was that for post WWII year we can not accept that the residuals are iid and that nonlinear or chaotic structures exist.
- Hsieh D.A (1989) tested daily foreign exchange rates for the existence of nonlinear dependence. He used daily quotes for British Pound, Canadian Dollar, Japanese Yen, Deutch Mark, Swiss Franc from January 2nd ,1974 to December 30th ,1983;a total of 2510 observations. First Hsieh prefiltered the data with a linear model and applied the BDS test to the residuals. He detected evidence of strong nonlinear dependence. Then he filtered again the residuals with a GARCH(1,1) model with various innovations processes (N(0,1) , t-student with 3 degrees of freedom, generalized error distribution) and applied the BDS test to the standardized residuals. The critical values were obtained from Monte Carlo simulations. Hsieh concluded that conditional heteroskedasticity, in the form of GARCH models, accounts for a large part of the nonlinearity in daily exchange rates.
- Johnson B. & McClelland R (1995) applied the BDS test to US unemployment rate after they prefiltered it with an AR (10) model that provided the best value for the Akaike Criterion. The test rejected the null hypothesis that the residuals were iid even after a SETAR model was posed to counter for the existing nonlinearity.
- Mahajan A. & Wagner A. J. (1999) examined for the existence of nonlinear dynamics in foreign exchange rates. They used ten (10) exchange rates for 3 sub-periods; the first was from January 1974 to December 1978, the second from January 1980 to December 1985 and the last from January 1986 to November 1991. The BDS test was applied directly to the returns of series and it provided different results from period to period. It appeared that the nonlinearity was evidently stronger in the first period, rejection of the iid, and was reducing to second and vanishing in the third period. The authors considered this as evidence that the random walk model can fairly characterize the movement of exchange rates.
- Panas E. & Ninni V. (2000) used the test in oil markets seeking for evidence pro the existing of chaotic or more generally nonlinear structure. They prefiltered their data with an AR(s)-GARCH(p,q) model that was chosed to

minimize the Akaike Criterion. Afterwards they applied the BDS test to the standardized residuals and they concluded that there existed further nonlinearity that was not captured.

- Adrangi B., Chatrath A., Duhard K.K, Raffiee K. (2001) examined the existence of chaos in oil markets yet they used daily prices from futures contracts of NYMEX that were seasonally adjusted. Using the BDS test after filtering the data with ARCH-type models suggested that there was no further evidence of nonlinearity, and therefore chaos. In most cases GARCH(1,1) models captured successfully the majority of the existing nonlinear structure. The authors obtained the appropriate critical values using Monte-Carlo simulations.
- Kosfeld R. & Robé S. (2001) considered the case of German bank stock returns. Data ranged from the 3rd week of March 1987 to the 2nd week of February 1998.Again prefiltering with an appropriate ARMA model was used in order to remove possible linear dependencies. The application of the BDS test to these residuals provided strong evidence of nonlinearity. As a second step they applied the BDS test in standardized residuals after the fit of low-order GARCH processes: mainly (1, 1) or (2, 1) .The test suggested that these models captured the existing nonlinearities.
- Cecen A. A. & Erkal C. (1996) searched for nonlinear dependence in the form of chaos for hourly data, of a period of seven months, of four (4) exchange rates: British Pound, Deutche Mark, Japanese Yen and Swiss Franc. BDS test strongly rejected an iid behavior in the data and thereby implying nonlinear dependence.
- Brock W.A., Hsieh D. A. & LeBaron Blake (1991) used the BDS test to detect nonlinearities in stock index data and specifically for the S&P 500 index and the CRSP index.In the case of CRSP index they split the data in two periods: the first from July 1962 to April 1974 and the second from May 1974 to December 1985. They filtered their data with a GARCH(1,1) and they applied the test to the standardized residuals. GARCH models captured all nonlinear structure for the second period and they failed to do so for the first period. In the case of the S&P 500 index, data were split to a period from 1928 to 1939 and to a period from 1950 to 1962. The BDS test was applied to the standardized residuals of a GARCH(1,1) model and showed no evidence for the existence of further nonlinear structure.

We must stress out the point that through the entire literature the BDS test is used merely as a general test that provides evidence for nonlinear structure and it can not point to any direction for an appropriate model. The critical values that are used after a prefiltering with a GARCH model in most cases are derived using a Monte Carlo simulation. This is correct for the cases that the test is applied to the standardized residuals. However if someone used, as we have already mentioned, the natural logarithm of them, the asymptotic distribution of the test remains the N(0,1) and there is no further need for simulations or bootstrapping methods.

5.2 Application to data

5.2.1 Methodology

Before we use the BDS test on the data we prefiltered them with an ARMA(p, q)-GARCH(1,1) process in order to remove all linear and possible nonlinear structure. Afterwards we applied the test to the natural logarithm of the standardized residuals. In particular the procedure was:

- An appropriate ARMA(p,q) model was chosen so that it was as parsimonious as possible and that it was able to remove all linear dependence of the logarithmic returns of original data. The diagnostic check was made by examining the serial correlation of the residuals after the estimation of the ARMA(p, q) process upon the data.
- After the ARMA model was found, we estimated an ARMA (p, q)-GARCH(1,1) process directly. We took the standardized residuals and we used their natural logarithms as input for the BDS test. The test was applied for values of the embedding dimension m=2, 3, 4, 5 and ε =1 standard deviation of the data.
- It is possible to filter the original data with the ARMA process and then estimate a GARCH(1,1) process applied to the residuals. However adjustments are required to the associated covariance matrix so as to obtain least squares estimates for the GARCH coefficients.
- The critical values for the test were obtained from the standard normal distribution N (0, 1); for 5% significance level are ± 1.96.
- If the BDS test statistic resulted a number larger than 1.96 or smaller than -1.96 we rejected the null hypothesis that the residuals were iid and that the GARCH (1, 1) removed all nonlinear dependence.

5.2.2 Data Description

The methodology discussed in the previous section was used in two different groups of data sets. The first group was comprised by stock indices of 35 different countries[6].

1	Argentina	**11**	France	**21**	New Zealand
2	Australia	**12**	Germany	**22**	Pakistan
3	Austria	**13**	Greece	**23**	Peru
4	Belgium	**14**	Hong-Kong	**24**	Philippines
5	Brazil	**15**	India	**25**	Portugal
6	Chile	**16**	Indonesia	**26**	Russia
7	China	**17**	Japan	**27**	Singapore
8	ChezhRep	**18**	Malaysia	**28**	South Korea
9	Denmark	**19**	Mexico	**29**	Sri-Lanka
10	Finland	**20**	Netherlands	**30**	Switzerland

31	Taiwan
32	Thailand
33	United Kingdom
34	United States (S&P500)
35	Venezuela

Table 5.2.2.1

The data ranged from May 1998 to May 2002, a total of 975 observations. We have already seen in the simulation that the asymptotic distribution of the BDS statistic can well be approximated from the N (0, 1) distribution for 1000 observations. There is not any particular characteristic for these indices. We notice that there exist both developed and developing countries.

The second group is comprised from 27 foreign exchange rate[7]. All rates are against US dollar.

1	Austria	**11**	Greece	**21**	Singapore
2	Belgium	**12**	Australia	**22**	Switzerland
3	Finland	**13**	Canada	**23**	United Kingdom
4	France	**14**	Denmark	**24**	India
5	Germany	**15**	Hong Kong	**25**	Sri-Lanka
6	Ireland	**16**	Japan	**26**	Taiwan
7	Italy	**17**	Malaysia	**27**	Thailand
8	Netherlands	**18**	Norway		
9	Portugal	**19**	New Zealand		
10	Spain	**20**	Sweden		

Table 5.2.2.2

The rates were from January 1997 to April 2002, a total of 1330 observations. Again the number is sufficient to ensure that the asymptotic distribution of the BDS statistic can be approximated by the standard normal. Again we observe that the exchange rates are against the currencies of both developed and developing countries. In the case of the countries that participate in the European Union, the national currency is replaced by the Euro exchange rate, multiplied by the fixed rate of this country's currency against it.

5.2.3 Results

Following the methodology described above for the data sets we have, we derived the following tables that contain the percentage of the series that the ARMA (p,q)-GARCH(1,1) was unable to remove all nonlinear dependence:

Stock Indices			
Remaining nonlinear structure (%)			
Embedding dimension			
m=2	m=3	m=4	m=5
5.71	8.57	5.71	8.57

Table 5.2.3.1

In particular we notice that the ARMA (p,q)-GARCH(1, 1) process can not capture all the existing nonlinearity for all embedding dimension for the case of two stock indices: the Austrian stock index and the index of Singapore. We may therefore assume that the employment of higher order GARCH models or nonlinear models of another kind could possibly counter for this form of nonlinearity. However for the vast majority of stock indices (33 out of 35) we see that the GARCH(1,1) model can account for all nonlinear structure and therefore describe with success these kind of dynamics.

In the case of foreign exchange rates:

Exchange Rate			
Remaining nonlinear structure (%)			
Embedding dimension			
m=2	m=3	m=4	m=5
18.52	22.22	18.52	14.81

Table 5.2.3.2

We notice that in this case the ARMA (p,q)-GARCH(1,1) fails in more cases to capture the nonlinearity in the data. In particular in 5 cases out of 27, BDS test detects further nonlinear structures; for Hong Kong $, Japanese Yen, Malaysia Ringgit, Taiwan NT$ and Thailand Baht. The results for the Sri- Lanka Rupee are ambiguous as the remaining nonlinear structure is rejected for m=4,5 and is accepted for m=2,3. However we notice again that the simple ARMA (p,q)-GARCH(1,1) model can account for the existing nonlinearity in most of the cases.

All results are reported in detail in Appendix 2.

6. Conclusion

Our main objective for this thesis was to discover and evaluate the properties of the BDS test through the flexible structure of a GARCH(1,1) process by employing Monte Carlo simulations. In order to do show, we first explored the moment and memory properties of a GARCH model. It was found that depending on the innovation distribution, the moment characteristics of the process are a function of the relation of the coefficients α, β.Then we study the BDS test and the underlying assumptions. The majority of them concerned smoothness conditions on the kernel, moment conditions on the interaction of the kernel with the data generating process, and regularity conditions on the strength of temporal dependence for the series under scrutiny.

Our application of the test for deriving its size in the case of GARCH (1,1) process with different innovations lead us to propose a further assumption that can complement the other already existing. We discovered through the simulation procedure that necessary condition for the test to function properly, is the existence of the first moment of the unconditional model (existence of region 3) and the existence of moment of order higher than 2 for the iid innovation. Finally we use the test to discover whether a ARMA(p,q)-GARCH(1,1) process can remove nonlinear structures from real data. We considered 27 exchange rates and 35 stock indices. The results was that for the majority of the rates and indices, a AR(p)-GARCH(1,1) process can remove all nonlinearities.

Notes:

[1] The critical value is 1.96
[2]The choice was made accordingly to Brock et al (1991).
[3] For example in the case of 2000 replications, this problem appeared only 20 times.
[4] Without the correction, even though the size was close to nominal, the mean of the BDS statistic took values even in the range of 100.
[5] The results for the other embedding dimensions were similar. Note that in the case of the 250 observations , due to the small sample size, we can basically rely on the results for embedding dimension m=2.
[6] Data were taken by the web page: http://www.yahoo.com. The initial source of the data as it is mentioned at the site is Reuters.
[7] Exchange rates data were available at the internet web page of Federal Reserve Bank of U.S.A

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APPENDIX

1. BDS test: Size for IID processes. Monte Carlo Results.

1.1 Embedding dimension m=3

Distribution:		Cauchy				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	**1000**	-0.109908	1.010630	1.657947	11.93448	0.0540000
1000		0.005796	1.026499	1.438528	9.094557	0.0590000
1500	**2000**	-0.017227	1.035487	1.412205	21.72654	0.0500000
Distribution:		**t-student(2)**				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	**1000**	-0.053941	1.062612	0.349986	3.181309	0.0660000
1000		0.006231	1.053698	0.134340	3.068604	0.0660000
1500	**2000**	-0.026093	1.031726	0.195678	2.957093	0.0615000
Distribution:		**t-student(3)**				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	**1000**	-0.041375	1.045138	0.361682	3.606686	0.0580000
1000		-0.050518	1.000165	0.028756	2.955727	0.0530000
1500	**2000**	-0.060629	1.006797	0.160484	3.117441	0.0530000
Distribution:		**t-student(4)**				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	**1000**	-0.071531	1.025730	0.133983	2.993705	0.0530000
1000		-0.041430	1.021438	0.183374	3.179697	0.0530000
1500	**2000**	-0.023209	1.021886	0.203344	3.062869	0.0575000
Distribution:		**t-student(5)**				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	**1000**	-0.056939	1.031650	0.273322	3.048220	0.0500000
1000		-0.106686	1.017581	0.129562	2.818850	0.0460000
1500	**2000**	-0.030957	1.032041	0.238426	3.134125	0.0590000
Distribution:		**N(0,1)**				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	**1000**	-0.049574	1.106696	0.372387	3.275693	0.0730000
1000		-0.023882	1.031308	0.015258	3.247357	0.0710000
1500	**2000**	-0.049918	1.026727	0.103056	3.181306	0.0565000

1.2 Embedding dimension m=4

Distribution:		**Cauchy**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.114347	0.987871	1.456823	9.227464	0.0470000
1000		0.014240	1.029150	1.330394	6.659900	0.0620000
1500	**2000**	-0.017821	1.042666	1.377189	13.65991	0.0500000
Distribution:		**t-student(2)**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.073057	1.066761	0.318541	3.225002	0.0740000
1000		-0.009529	1.064467	0.091860	3.002785	0.0630000
1500	**2000**	-0.027344	1.021978	0.172364	2.984876	0.0565000
Distribution:		**t-student(3)**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.025602	1.078902	0.434066	3.712323	0.0680000
1000		-0.045312	0.996404	0.044098	3.063518	0.0500000
1500	**2000**	-0.076708	0.998518	0.158821	2.984852	0.0520000
Distribution:		**t-student(4)**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.066680	1.006092	0.174308	2.783134	0.0470000
1000		-0.036784	1.024213	0.208156	3.206473	0.0490000
1500	**2000**	-0.027669	1.015443	0.257665	3.034386	0.0535000
Distribution:		**t-student(5)**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.056543	1.035270	0.363265	3.078482	0.0550000
1000		-0.117204	1.010606	0.206546	2.926262	0.0500000
1500	**2000**	-0.031948	1.020744	0.200138	3.067414	0.0580000
Distribution:		**N(0,1)**				
# observ.	**# replic**	**mean μ**	**std. dev. s**	**skewness a_3**	**kurtosis a_4**	**size**
250	**1000**	-0.056241	1.105812	0.451667	3.306924	0.0700000
1000		-0.018486	1.030061	0.195283	3.151076	0.0580000
1500	**2000**	-0.059546	1.030110	0.183236	3.219839	0.0580000

1.3 Embedding dimension m=5

Distribution:		Cauchy				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	1000	-0.121888	1.011192	0.995364	7.357102	0.0470000
1000		0.011722	1.036544	1.245629	6.034662	0.0620000
1500	2000	-0.025174	1.053798	1.057712	12.98944	0.0495000
Distribution:		t-student(2)				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	1000	-0.088067	1.060145	0.390565	3.285407	0.0580000
1000		-0.011854	1.046911	0.097917	2.919875	0.0630000
1500	2000	-0.029577	1.021674	0.150610	3.064483	0.0545000
Distribution:		t-student(3)				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	1000	-0.026294	1.081933	0.504971	3.916710	0.0620000
1000		-0.036463	1.002481	0.053869	2.994835	0.0480000
1500	2000	-0.085811	0.993938	0.178438	3.004568	0.0515000
Distribution:		t-student(4)				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	1000	-0.077844	1.014841	0.221400	2.785330	0.0470000
1000		-0.033400	1.019247	0.240631	3.157904	0.0530000
1500	2000	-0.034607	1.009759	0.311697	3.147411	0.0545000
Distribution:		t-student(5)				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	1000	-0.072144	1.039504	0.468974	3.170226	0.0500000
1000		-0.112284	1.005547	0.241365	2.977387	0.0550000
1500	2000	-0.030957	1.032041	0.238426	3.134125	0.0590000
Distribution:		N(0,1)				
# observ.	# replic	mean μ	std. dev. s	skewness a_3	kurtosis a_4	size
250	1000	-0.064071	1.125752	0.573815	3.488355	0.0770000
1000		-0.011496	1.042179	0.316721	3.249111	0.0680000
1500	2000	-0.061967	1.025990	0.261202	3.252851	0.0580000

2. Monte Carlo Simulation Results

2.1 Results for IID innovations $X_t = \sqrt{\frac{1}{Var(Z_t)}} \cdot Z_t$, $Z_t \sim$ t-student (3)

Point A: [α=0.52, β=0.30], Region 3

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var))* tstud(3)	1	0.52	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.048790	1.118076	0.265967	2.832408	0.0850000
dim m=3	0.060956	1.129461	0.255831	2.977675	0.0880000
dim m=4	0.026017	1.128779	0.328250	3.104233	0.0730000
dim m=5	0.022783	1.128315	0.421475	3.349418	0.0730000
				mean estimation	
			ct coef	ar coef	gar coef
			0.8559678	0.4929254	0.3493224

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(3)	1	0.52	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.092502	1.083316	0.347621	3.884160	0.0580000
dim m=3	0.107144	1.091297	0.361266	3.754446	0.0790000
dim m=4	0.107268	1.083859	0.381333	3.573656	0.0670000
dim m=5	0.110857	1.082081	0.351579	3.375122	0.0670000
				mean estimation	
			ct coef	ar coef	gar coef
			0.9651676	0.5310945	0.3014914

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(3)	1	0.52	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.096628	1.103159	0.839001	7.429893	0.0655000
dim m=3	0.085002	1.104682	0.732227	6.082692	0.0615000
dim m=4	0.077509	1.090954	0.659845	5.515135	0.0655000
dim m=5	0.067254	1.083597	0.624355	5.157481	0.0635000
				mean estimation	
			ct coef	ar coef	gar coef
			0.9718186	0.5386050	0.2958768

2.1 Results for IID innovations $X_t = \sqrt{\dfrac{1}{Var(Z_t)}} \cdot Z_t$, $Z_t \sim$ t-student (3)

Point B: [α=1.70, β=0.30], Region 2

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(3)	1	1.70	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.077560	1.143551	0.395657	3.456820	0.0870000
dim m=3	0.085322	1.135119	0.421527	3.593589	0.0830000
dim m=4	0.069401	1.130245	0.555387	3.775787	0.0750000
dim m=5	0.059590	1.157635	0.684647	3.955845	0.0840000
				mean estimation	
			ct coef	ar coef	gar coef
			1.2707386	1.5414582	0.2830949

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(3)	1	1.70	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.183223	1.132336	1.188402	12.57687	0.0790000
dim m=3	0.173744	1.131372	1.404144	15.23948	0.0780000
dim m=4	0.165139	1.124291	1.584052	17.80555	0.0730000
dim m=5	0.150786	1.130593	1.769087	20.77378	0.0660000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0100246	1.6418799	0.2972994

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(3)	1	1.70	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.172289	1.185840	2.018296	25.51642	0.0730000
dim m=3	0.185987	1.197951	2.256547	27.85095	0.0755000
dim m=4	0.179648	1.201742	2.121340	25.21875	0.0785000
dim m=5	0.169268	1.199446	1.974531	22.68868	0.0765000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0492448	1.6780973	0.2941117

2.1 Results for IID innovations $X_t = \sqrt{\dfrac{1}{Var(Z_t)}} \cdot Z_t$, Z_t~ t-student (3)

Point C: [α=0.80, β=0.60], Region 2, non L_0-approximable

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(3)	1	0.8	0.6
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.033520	1.072474	0.290596	3.120982	0.0560000
dim m=3	0.018346	1.085746	0.426448	3.375117	0.0720000
dim m=4	0.001977	1.093824	0.534508	3.830682	0.0700000
dim m=5	-0.003681	1.110777	0.576152	4.178022	0.0710000
				mean estimation	
			ct coef	ar coef	gar coef
			1.9623387	0.7355162	0.5702226

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(3)	1	0.8	0.6
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.050424	1.073817	0.236518	3.143067	0.0630000
dim m=3	0.059444	1.082404	0.133827	2.968537	0.0730000
dim m=4	0.055594	1.083503	0.167383	2.950700	0.0730000
dim m=5	0.060521	1.087799	0.283321	2.970982	0.0720000
				mean estimation	
			ct coef	ar coef	gar coef
			1.3481877	0.8120357	0.5825205

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(3)	1	0.8	0.6
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.086955	1.110019	1.029108	8.261921	0.0645000
dim m=3	0.097938	1.121268	1.033070	8.267141	0.0690000
dim m=4	0.090956	1.128254	1.026196	7.998514	0.0675000
dim m=5	0.089487	1.130355	1.015983	7.856202	0.0655000
				mean estimation	
			ct coef	ar coef	gar coef
			1.2920093	0.8123267	0.5869759

2.2 Results for IID innovations $X_t = \sqrt{\frac{1}{Var(Z_t)}} \cdot Z_t$, Z_t~ t-student (4)

Point A: [α=0.40, β=0.50], Region 4, 2nd order stationarity

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	0.40	0.50
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.076161	1.109222	0.182149	3.402947	0.0700000
dim m=3	-0.080161	1.094096	0.321763	3.515127	0.0720000
dim m=4	-0.084935	1.102157	0.359307	3.502276	0.0720000
dim m=5	-0.087561	1.112006	0.454073	3.424977	0.0750000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0993281	0.4016346	0.4776559

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	0.40	0.50
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.089864	0.995256	0.064094	2.802747	0.0490000
dim m=3	-0.090665	1.009033	0.035574	2.917648	0.0480000
dim m=4	-0.079114	1.011570	0.052176	2.904328	0.0520000
dim m=5	-0.061542	1.015782	0.139074	2.911779	0.0540000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0476198	0.4118625	0.4845287

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	0.40	0.50
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.020479	1.025831	0.063884	3.041970	0.0575000
dim m=3	0.016178	1.018133	0.168601	3.311867	0.0565000
dim m=4	0.001940	1.016641	0.223093	3.364376	0.0535000
dim m=5	-0.001394	1.018832	0.269129	3.428145	0.0545000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0289806	0.4024324	0.4938709

2.2 Results for IID innovations $X_t = \sqrt{\frac{1}{Var(Z_t)}} \cdot Z_t$, Z_t~ t-student (4)

Point B: [α=0.85, β=0.20], Region 3

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	0.85	0.2
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.001489	1.055650	0.247877	3.149775	0.0660000
dim m=3	-0.022186	1.048798	0.250708	2.963945	0.0610000
dim m=4	-0.035855	1.037256	0.302043	3.033981	0.0490000
dim m=5	-0.042482	1.043959	0.424178	3.259338	0.0540000
				mean estimation	
			ct coef	ar coef	gar coef
			0.9670181	0.8163563	0.2141785

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	0.85	0.2
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.006986	1.054675	0.138975	2.876385	0.0600000
dim m=3	-0.021154	1.043118	0.175389	2.928806	0.0590000
dim m=4	-0.032608	1.040246	0.201144	3.118316	0.0610000
dim m=5	-0.035869	1.048327	0.221061	3.057067	0.0620000
				mean estimation	
			ct coef	ar coef	gar coef
			0.9964614	0.8355175	0.2006862

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	0.85	0.2
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.014150	1.051548	0.229238	3.240956	0.0625000
dim m=3	-0.021654	1.038483	0.262759	3.128387	0.0535000
dim m=4	-0.016688	1.029952	0.289733	3.009733	0.0500000
dim m=5	-0.022476	1.020324	0.346259	3.145052	0.0500000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0113866	0.8459238	0.1967340

2.2 Results for IID innovations $X_t=\sqrt{\dfrac{1}{Var(Z_t)}}\cdot Z_t$, Z_t~ t-student (4)

Point C: [α=1.29, β=0.40], Region 2

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	1.29	0.4
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.056623	1.092364	0.185671	3.185553	0.0640000
dim m=3	-0.086538	1.093089	0.207466	3.078820	0.0680000
dim m=4	-0.097246	1.083802	0.326444	3.166987	0.0780000
dim m=5	-0.093749	1.089049	0.505390	3.500497	0.0660000
				mean estimation	
			ct coef	ar coef	gar coef
			2.4243430	1.1926402	0.3949307

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	1.29	0.4
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.112161	1.178033	1.249408	9.510047	0.0710000
dim m=3	0.111335	1.163736	1.168382	8.668880	0.0740000
dim m=4	0.101543	1.145625	1.127116	8.009106	0.0720000
dim m=5	0.092564	1.139790	1.063256	7.424577	0.0690000
				mean estimation	
			ct coef	ar coef	gar coef
			1.7840752	1.3042153	0.3866656

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	1.29	0.4
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.226400	1.376441	1.855231	12.34100	0.1000000
dim m=3	0.238109	1.369951	1.840045	12.13320	0.0940000
dim m=4	0.229133	1.332455	1.795875	11.85104	0.0895000
dim m=5	0.222272	1.319029	1.699119	11.08828	0.0905000
				mean estimation	
			ct coef	ar coef	gar coef
			2.1570813	1.3917367	0.3727349

2.2 Results for IID innovations $X_t = \sqrt{\frac{1}{Var(Z_t)}} \cdot Z_t$, Z_t~ t-student (4)

Point D: [α=1.29, β=0.40], Region 2, non L_0-approximable

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	0.5	0.68
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.013482	1.059562	0.267870	2.874518	0.0610000
dim m=3	-0.044343	1.083091	0.315418	2.834571	0.0640000
dim m=4	-0.040602	1.075117	0.354149	2.927834	0.0620000
dim m=5	-0.037858	1.076167	0.425708	3.138457	0.0640000
				mean estimation	
			ct coef	ar coef	gar coef
			3.1191124	0.4775429	0.6655495

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	0.5	0.68
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.059489	1.066752	0.195612	2.940678	0.0720000
dim m=3	-0.061066	1.054519	0.173998	2.888580	0.0650000
dim m=4	-0.047015	1.034894	0.240567	3.007807	0.0660000
dim m=5	-0.041291	1.031319	0.286292	3.081016	0.0610000
				mean estimation	
			ct coef	ar coef	gar coef
			1.6894625	0.5049376	0.6733662

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(4)	1	0.5	0.68
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.008598	1.074656	0.787757	9.379962	0.0590000
dim m=3	-0.009054	1.073677	0.719520	8.710452	0.0640000
dim m=4	-0.010911	1.066276	0.598599	7.298687	0.0575000
dim m=5	-0.014917	1.057316	0.531098	6.496824	0.0565000
				mean estimation	
			ct coef	ar coef	gar coef
			1.5062953	0.5097454	0.6735539

2.3 Results for IID innovations $X_t = \sqrt{\frac{1}{Var(Z_t)}} \cdot Z_t$, Z_t~ t-student (5)

Point A: [α=0.13, β=0.40], Region 4, 4th order stationarity

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	0.13	0.40
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.133963	1.066728	0.280347	3.115136	0.0580000
dim m=3	-0.156573	1.049225	0.455072	3.268548	0.0600000
dim m=4	-0.158807	1.050293	0.458017	3.460869	0.0630000
dim m=5	-0.153050	1.045207	0.471976	3.418080	0.0610000
				mean estimation	
			ct coef	ar coef	gar coef
			0.7358867	0.1471815	0.5163079

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	0.13	0.40
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.014342	1.021235	0.231608	3.192480	0.0480000
dim m=3	-0.049082	1.032126	0.262028	3.337051	0.0520000
dim m=4	-0.043373	1.020432	0.252464	3.390413	0.0560000
dim m=5	-0.037612	1.024977	0.278413	3.534521	0.0600000
				mean estimation	
			ct coef	ar coef	gar coef
			0.9303826	0.1281931	0.4341355

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	0.13	0.40
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.021618	1.006946	0.173131	3.178304	0.0500000
dim m=3	-0.013527	1.008954	0.158890	3.116670	0.0545000
dim m=4	-0.015749	1.004008	0.180376	2.993443	0.0535000
dim m=5	-0.020181	1.004799	0.218535	2.982283	0.0505000
				mean estimation	
			ct coef	ar coef	gar coef
			0.9779385	0.1312999	0.4113732

2.3 Results for IID innovations $X_t = \sqrt{\frac{1}{Var(Z_t)}} \cdot Z_t$, Z_t~ t-student (5)

Point B: [α=0.33, β=0.50], Region 4, 2nd order stationarity

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	0.33	0.50
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.025224	1.057137	0.228024	2.790781	0.0650000
dim m=3	-0.027727	1.049129	0.285397	2.865999	0.0610000
dim m=4	-0.045963	1.057050	0.437981	3.220840	0.0510000
dim m=5	-0.063134	1.067812	0.564770	3.505293	0.0550000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0701763	0.3350759	0.4830175

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	0.33	0.50
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.026339	1.022998	0.095895	2.997172	0.0520000
dim m=3	-0.048791	1.017901	0.183778	3.058798	0.0540000
dim m=4	-0.054416	1.001329	0.243515	3.126572	0.0450000
dim m=5	-0.062027	0.993066	0.286066	3.224432	0.0540000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0396302	0.3302249	0.4870841

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	0.33	0.50
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.065715	0.986474	0.142582	3.088077	0.0425000
dim m=3	-0.066735	0.992991	0.165593	3.156270	0.0485000
dim m=4	-0.065044	0.999445	0.199014	3.244880	0.0515000
dim m=5	-0.059776	1.009801	0.242001	3.171501	0.0515000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0194680	0.3298474	0.4948654

2.3 Results for IID innovations $X_t=\sqrt{\dfrac{1}{Var(Z_t)}}\cdot Z_t$, Z_t~ t-student (5)

Point C: [α=0.90, β=0.20], Region 3

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	0.90	0.20
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.093449	1.042572	0.105024	2.808112	0.0500000
dim m=3	-0.105405	1.047942	0.165602	2.837803	0.0560000
dim m=4	-0.125568	1.041734	0.267912	3.018015	0.0590000
dim m=5	-0.137110	1.038499	0.372837	3.130171	0.0480000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0176372	0.8626131	0.2043834

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	0.90	0.20
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	0.006391	0.995318	0.249750	2.927890	0.0470000
dim m=3	-0.013821	1.016929	0.232166	3.092567	0.0550000
dim m=4	-0.029765	1.012028	0.242274	3.221756	0.0540000
dim m=5	-0.027930	1.003236	0.279562	3.321289	0.0560000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0136319	0.8882204	0.1986084

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	0.90	0.20
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.022184	1.056432	0.107103	2.956959	0.0660000
dim m=3	-0.028110	1.045289	0.203320	3.051216	0.0635000
dim m=4	-0.033323	1.039051	0.227813	3.106398	0.0590000
dim m=5	-0.028432	1.039814	0.256066	3.074264	0.0630000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0089707	0.8916463	0.1996798

2.3 Results for IID innovations $X_t = \sqrt{\frac{1}{Var(Z_t)}} \cdot Z_t$, Z_t~ t-student (5)

Point D: [α=1.31, β=0.30], Region 2

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	1.31	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.044606	1.066005	0.048409	2.945696	0.0600000
dim m=3	-0.062545	1.093208	0.212558	2.981967	0.0810000
dim m=4	-0.069519	1.095146	0.301778	3.066757	0.0760000
dim m=5	-0.078795	1.083032	0.415173	3.201443	0.0690000
				mean estimation	
			ct coef	ar coef	gar coef
			1.4493486	1.2836868	0.2920909

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	1.31	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.048267	1.013248	0.034611	2.814745	0.0520000
dim m=3	-0.075184	1.033179	0.139662	2.928540	0.0590000
dim m=4	-0.078239	1.025684	0.215675	3.024484	0.0630000
dim m=5	-0.077663	1.027632	0.267496	3.019599	0.0600000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0649278	1.3144789	0.2977638

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		(1/sqr(var)) *tstud(5)	1	1.31	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.027781	1.067078	0.245327	3.664950	0.0625000
dim m=3	-0.018592	1.068316	0.371788	4.250156	0.0620000
dim m=4	-0.017889	1.058930	0.507884	5.027676	0.0620000
dim m=5	-0.027246	1.052851	0.587560	5.205914	0.0590000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0825922	1.3200758	0.2964028

2.4 Results for IID innovations Z_t ~N (0,1)

Point A: [α=0.10, β=0.60], Region 4, 4th order stationarity

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	0.10	0.60
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.161104	1.035908	0.230125	2.940034	0.0550000
dim m=3	-0.189218	1.022552	0.378041	3.074274	0.0560000
dim m=4	-0.177177	1.025704	0.461999	3.344374	0.0530000
dim m=5	-0.165145	1.027865	0.505798	3.443775	0.0530000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0277727	0.1142300	0.5832208

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	0.10	0.60
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.105499	0.988685	0.059269	3.037820	0.0430000
dim m=3	-0.114689	0.971283	0.074801	3.058635	0.0410000
dim m=4	-0.103182	0.979407	0.096407	2.937809	0.0430000
dim m=5	-0.100174	0.989073	0.126425	3.005334	0.0510000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0473065	0.0993089	0.5866018

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	0.10	0.60
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.029163	1.012702	0.098904	2.897881	0.0495000
dim m=3	-0.040964	1.010190	0.098122	2.916016	0.0505000
dim m=4	-0.041054	1.021773	0.125431	3.082282	0.0560000
dim m=5	-0.040918	1.015694	0.187169	3.155526	0.0515000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0406047	0.0995432	0.5874852

2.4 Results for IID innovations Z_t ~N (0,1)

Point B: [α=0.50, β=0.40], Region 4, 2nd order stationarity

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	0.50	0.40
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.144580	1.030191	0.180871	3.240725	0.0580000
dim m=3	-0.138957	1.018385	0.313513	3.427933	0.0560000
dim m=4	-0.138592	1.009322	0.366454	3.185046	0.0480000
dim m=5	-0.123727	1.023385	0.492412	3.318776	0.0520000
				mean estimation	
			ct coef	ar coef	gar coef
			1.1050305	0.4857606	0.3887670

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	0.50	0.40
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.060525	1.032951	0.094806	2.873307	0.0570000
dim m=3	-0.063277	1.020477	0.143555	3.183173	0.0590000
dim m=4	-0.063992	1.006673	0.219170	3.201986	0.0500000
dim m=5	-0.066034	0.995806	0.279318	3.131026	0.0480000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0170427	0.4963973	0.3987045

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	0.50	0.40
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.056295	0.992907	-0.006919	2.978359	0.0480000
dim m=3	-0.059907	0.999069	0.022985	3.063781	0.0490000
dim m=4	-0.050540	1.008951	0.084046	3.258199	0.0495000
dim m=5	-0.041386	1.011779	0.131449	3.393801	0.0520000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0148083	0.4986828	0.3974963

2.4 Results for IID innovations Z_t ~N (0,1)

Point C: [α=0.80, β=0.30], Region 3

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	0.80	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.096714	1.070953	0.214965	2.975537	0.0730000
dim m=3	-0.130475	1.064871	0.344881	3.234771	0.0620000
dim m=4	-0.142105	1.057611	0.375741	3.195843	0.0640000
dim m=5	-0.125018	1.063908	0.423758	3.379619	0.0640000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0946081	0.7825050	0.2938575

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	0.80	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.004214	1.026620	0.153928	2.948097	0.0510000
dim m=3	-0.008839	1.025092	0.119083	3.042091	0.0550000
dim m=4	0.001969	1.017115	0.158720	3.219249	0.0510000
dim m=5	0.006020	1.010241	0.188020	3.335991	0.0520000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0245120	0.7925008	0.2982350

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	0.80	0.30
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.039359	0.987956	0.160154	3.055161	0.0480000
dim m=3	-0.053974	1.010678	0.185778	3.068852	0.0545000
dim m=4	-0.053936	1.022660	0.183654	3.110689	0.0575000
dim m=5	-0.051065	1.024411	0.218920	3.130555	0.0550000
				mean estimation	
			ct coef	ar coef	gar coef
			1.0154990	0.7952178	0.3001044

2.4 Results for IID innovations Z_t ~N (0,1)

Point D: [α=1.00, β=0.34], Region 3

# repl	1000		true	GARCH(1,1)	process
# observ	250	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	1	0.34
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.165695	1.043602	0.326253	3.034485	0.0580000
dim m=3	-0.200306	1.057524	0.219015	2.914052	0.0630000
dim m=4	-0.212188	1.033205	0.242518	3.022589	0.0620000
dim m=5	-0.217004	1.039837	0.347264	2.990336	0.0720000
				mean estimation	
			ct coef	ar coef	gar coef
			1.4474064	0.9671329	0.3419771

# repl	1000		true	GARCH(1,1)	process
# observ	1000	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	1	0.34
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.078226	0.994844	0.047307	3.141048	0.0520000
dim m=3	-0.075185	0.988351	0.076116	3.344309	0.0530000
dim m=4	-0.094698	1.002853	0.112271	3.210673	0.0390000
dim m=5	-0.093677	1.004770	0.141433	3.239775	0.0450000
				mean estimation	
			ct coef	ar coef	gar coef
			1.1182915	0.9981305	0.3372530

# repl	2000		true	GARCH(1,1)	process
# observ	1500	iid innov	ct coef	arch coef	garch coef
		N(0,1)	1	1	0.34
BDS stat	characteristics				
	mean	stdev	skewness	kyrtosis	size
dim m=2	-0.045400	1.032204	0.428811	5.147213	0.0540000
dim m=3	-0.049568	1.034269	0.445519	4.665604	0.0495000
dim m=4	-0.055605	1.042664	0.459902	4.557064	0.0540000
dim m=5	-0.051536	1.038628	0.405062	4.320967	0.0515000
				mean estimation	
			ct coef	ar coef	gar coef
			1.1046518	1.0016466	0.3381617

3. Empirical Results

Stock Indices			Foreign Exchange Rate	
Number	**Country**		**Number**	**Country**
1	Argentina		**1**	Austria
2	Australia		**2**	Belgium
3	Austria		**3**	Finland
4	Belgium		**4**	France
5	Brazil		**5**	Germany
6	Chile		**6**	Ireland
7	China		**7**	Italy
8	Chezh Rep		**8**	Netherlands
9	Denmark		**9**	Portugal
10	Finland		**10**	Spain
11	France		**11**	Greece
12	Germany		**12**	Australia
13	Greece		**13**	Canada
14	Hong-Kong		**14**	Denmark
15	India		**15**	Hong Kong
16	Indonesia		**16**	Japan
17	Japan		**17**	Malaysia
18	Malaysia		**18**	Norway
19	Mexico		**19**	New Zealand
20	Netherlands		**20**	Sweden
21	New Zealand		**21**	Singapore
22	Pakistan		**22**	Switzerland
23	Peru		**23**	United Kingdom
24	Philippines		**24**	India
25	Portugal		**25**	Sri-Lanka
26	Russia		**26**	Taiwan
27	Singapore		**27**	Thailand
28	South Korea			
29	Sri-Lanka			
30	Switzerland			
31	Taiwan			
32	Thailand			
33	United Kingdom			
34	United States (S&P500)			
35	Venezuela			

Using the table above we correspond to each country a specific number that we use as a key in order to read the results that follow.

3.1 Empirical Results (Stock Indices)

country	prefiltering model	ct coef of pref.mod	ar(1) coef	ar(2) coef	ma(1) coef
1	AR(1)	-0.0004	0.0958		
2	AR(1)	0.0005	-0.0100		
3	AR(2)	0.0003	0.0945	-0.0126	
4	AR(2)	0.0001	0.1753	-0.0762	
5	AR(1)	0.0007	0.0385		
6	AR(1)	-0.0017	-0.2285		
7	AR(1)	0.0004	0.0318		
8	AR(1)	0.0004	0.0832		
9	AR(1)	0.0003	0.1079		
10	AR(1)	0.0013	0.0558		
11	AR(1)	0.0002	0.0324		
12	AR(1)	0.0001	0.0289		
13	AR(1)	0.0002	0.1657		
14	AR(1)	0.0004	0.0603		
15	AR(1)	0.0005	0.0582		
16	AR(1)	0.0003	0.1455		
17	AR(1)	-0.0001	-0.0069		
18	AR(1)	0.0007	0.1964		
19	AR(1)	0.0009	0.1475		
20	AR(2)	0.0000	0.0194	-0.0218	
21	AR(1)	0.0002	0.0232		
22	AR(1)	-0.0001	0.0585		
23	AR(1)	-0.0001	0.1826		
24	AR(1)	-0.0005	0.1706		
25	AR(1)	0.0000	0.1378		
26	AR(1)	0.0019	0.0537		
27	AR(1)	0.0006	0.0812		
28	AR(1)	0.0011	0.0826		
29	AR(2)	-0.0001	0.3820	-0.0478	
30	AR(1)	0.0002	0.0395		
31	AR(1)	0.0002	0.0799		
32	AR(1)	0.0081	-0.1246		
33	AR(1)	-0.0001	0.0189		
34	AR(1)	0.0002	0.0091		
35	ARMA(1,1)	-0.0005	0.2689		0.0246

The ARMA(p,q) model that is reported here, is the appropriate one that removes all linear dependence with the employment of as few parameters as possible.

3.1 Empirical Results (Stock Indices) (cont)

country	GARCH ct coef	arch coef	garch coef	m=2 non IID	m=3 non IID	m=4 non IID	m=5 non IID
1	0.0000	0.1234	0.8450	YES	NO	NO	NO
2	0.0000	0.0706	0.8569	NO	NO	NO	NO
3	0.0000	0.0841	0.8896	YES	YES	YES	YES
4	0.0000	0.1785	0.8110	NO	NO	NO	NO
5	0.0000	0.1755	0.7670	NO	NO	NO	NO
6	0.0001	1.7680	-0.0091	NO	NO	NO	NO
7	0.0000	0.2056	0.7773	NO	YES	NO	NO
8	0.0000	0.0890	0.8507	NO	NO	NO	NO
9	0.0000	0.0909	0.8577	NO	NO	NO	NO
10	0.0000	0.0392	0.9566	NO	NO	NO	NO
11	0.0000	0.0667	0.9070	NO	NO	NO	NO
12	0.0000	0.0936	0.8751	NO	NO	NO	NO
13	0.0001	0.2573	0.6353	NO	NO	NO	NO
14	0.0000	0.0521	0.9298	NO	NO	NO	NO
15	0.0001	0.0907	0.7006	NO	NO	NO	YES
16	0.0000	0.1618	0.7898	NO	NO	NO	NO
17	0.0000	0.0765	0.8763	NO	NO	NO	NO
18	0.0000	0.2145	0.6749	NO	NO	NO	NO
19	0.0000	0.0870	0.8707	NO	NO	NO	NO
20	0.0000	0.1146	0.8611	NO	NO	NO	NO
21	0.0000	0.1740	0.8034	NO	NO	NO	NO
22	0.0000	0.1277	0.8435	NO	NO	NO	NO
23	0.0000	0.2644	0.6606	NO	NO	NO	NO
24	0.0000	0.1510	0.7219	NO	NO	NO	NO
25	0.0000	0.1948	0.7271	NO	NO	NO	NO
26	0.0001	0.1164	0.8140	NO	NO	NO	NO
27	0.0000	0.1474	0.7209	NO	YES	YES	YES
28	0.0001	0.0772	0.7752	NO	NO	NO	NO
29	0.0000	0.6102	0.4381	NO	NO	NO	NO
30	0.0000	0.1358	0.8329	NO	NO	NO	NO
31	0.0000	0.1425	0.7838	NO	NO	NO	NO
32	0.0003	2.4060	0.0150	NO	NO	NO	NO
33	0.0000	0.0960	0.8546	NO	NO	NO	NO
34	0.0000	0.0892	0.8405	NO	NO	NO	NO
35	0.0001	0.4891	0.5351	NO	NO	NO	NO

If the indication in last columns is “YES”, then we may accept that the data are not clear from nonlinear structure after we filtered them with an AR (p)- GARCH(1, 1) model.

3.2 Empirical Results (Foreign Exchange Rates)

country	prefiltering model	ct coef of pref.mod	ar(1) coef	ar(2) coef	ma(1) coef
1	AR(1)	0.0003	0.0188		
2	AR(1)	0.0002	0.0098		
3	AR(1)	0.0003	0.0126		
4	AR(1)	0.0002	0.0262		
5	AR(1)	0.0002	0.0272		
6	AR(1)	-0.0003	0.0266		
7	AR(1)	0.0003	0.0295		
8	AR(1)	0.0002	0.0268		
9	AR(1)	0.0003	0.0274		
10	AR(1)	0.0003	0.0268		
11	AR(1)	0.0002	0.0667		
12	AR(1)	-0.0003	0.0232		
13	AR(1)	0.0001	0.0641		
14	AR(1)	0.0003	0.0281		
15	ARMA(1,1)	0.0000	0.4599		-0.7220
16	AR(1)	0.0003	0.0241		
17	AR(2)	0.0000	0.0892	0.0923	
18	AR(1)	0.0002	0.0188		
19	AR(1)	-0.0003	-0.0263		
20	AR(1)	0.0003	0.0338		
21	AR(1)	0.0001	0.0079		
22	AR(1)	0.0002	0.0226		
23	AR(1)	-0.0001	0.0436		
24	AR(1)	0.0002	-0.2652		
25	AR(2)	0.0005	-0.3247	-0.1366	
26	AR(2)	0.0000	-0.0473	-0.1880	
27	AR(2)	0.0003	0.0191	-0.0289	

The ARMA(p,q) model that is reported here, is the appropriate one that removes all linear dependence with the employment of as few parameters as possible.

3.2 Empirical Results (Foreign Exchange Rates) (cont)

country	GARCH ct coef	arch coef	garch coef	m=2 non IID	m=3 non IID	m=4 non IID	m=5 non IID
1	0.0000	0.0328	0.9127	NO	NO	NO	NO
2	0.0000	0.0192	0.9669	NO	NO	NO	NO
3	0.0000	0.0160	0.9750	NO	NO	NO	NO
4	0.0000	0.0178	0.9700	NO	NO	NO	NO
5	0.0000	0.0169	0.9705	NO	NO	NO	NO
6	0.0000	0.0122	0.9731	NO	NO	NO	NO
7	0.0000	0.0182	0.9709	NO	NO	NO	NO
8	0.0000	0.0190	0.9674	NO	NO	NO	NO
9	0.0000	0.0177	0.9667	NO	NO	NO	NO
10	0.0000	0.0142	0.9768	NO	NO	NO	NO
11	0.0000	0.1902	0.1026	NO	NO	NO	NO
12	0.0000	0.0489	0.9207	NO	NO	NO	NO
13	0.0000	0.0482	0.9268	NO	NO	NO	NO
14	0.0000	0.1418	0.1228	NO	NO	NO	NO
15	0.0000	0.1449	0.8656	YES	YES	YES	YES
16	0.0000	0.0540	0.9229	YES	YES	YES	NO
17	0.0000	0.3026	0.7556	YES	YES	YES	YES
18	0.0000	0.1164	0.7226	NO	NO	NO	NO
19	0.0000	0.0471	0.8898	NO	NO	NO	NO
20	0.0000	0.0325	0.9270	NO	NO	NO	NO
21	0.0000	0.0650	0.9315	NO	NO	NO	NO
22	0.0000	0.0218	0.7872	NO	NO	NO	NO
23	0.0000	0.0287	0.9432	NO	NO	NO	NO
24	0.0000	0.2671	0.7876	NO	NO	NO	NO
25	0.0000	0.1247	0.8583	YES	YES	NO	NO
26	0.0000	0.5398	0.7768	YES	YES	YES	YES
27	0.0000	0.1579	0.8654	NO	YES	YES	YES

If the indication in last columns is “YES”, then we may accept that the data are not clear from nonlinear structure after we filtered them with an AR (p)- GARCH(1, 1) model.