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Master thesis: Equilibrium Pricing in Thin Financial Markets

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Equilibrium Pricing in Thin Financial Markets

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Abstract: In this study, we analyze the equilibrium prices of securities in thin financial markets. We define what a thin market is and focus on the differences to competitive markets. Every agent is characterized by a mean variance utility function and risky endowments, that determine the way she acts in the risk sharing allocation with other agents. The considered market structure that we analyze is a complete one, where agents co-design the securities they trade according to their hedging needs. The allocation and prices of securities are an outcome of a game played by all agents in a form of a pure-strategy Nash equilibrium. We introduce a discrete time dynamic model and dedicate the analysis to bilateral transactions. The results of the so-called re-trading procedure depends on agents for-looking of future rounds. More precisely, behaving myopically or not, results in different outcomes of allocation and prices. In the first case, equilibrium converges to the Pareto optimal risk sharing allocation, while in the latter they stay at an ineffective equilibrium. Also, for some agent, gains may be higher than the optimal allocation of risk sharing. Finally, we examine the case of the evolving course of re-trading, including endogenously given transaction costs for each round.

Keywords: thin financial markets, Nash equilibrium, re-trading, risk sharing allocation

Περίληψη: Στη παρούσα εργασία, αναλύουμε τις τιμές ισορροπίας αξιογράφων (συμβόλαια) σε ρηχές χρηματοοικονομικές αγορές. Εισάγουμε τις έννοιες και το πλαίσιο των ρηχών αγορών και επικεντρωνόμαστε στις διαφορές που έχουν με τις ανταγωνιστικές αγορές. Οι πάικτες στην αγορά αυτή χαρακτηρίζονται από συναρτήσεις χρησιμότητας (τετραγωνικές μορφές) που αντικατοπτρίζουν τις προτημίσεις τους και το χαρτοφυλάκιο τους είναι μια τυχαία μεταβλητή. Έτσι, οι παίκτες ενεργούν στην αγορά με βάση τον κίνδυνο στον οποίο έχουν εκτεθεί και θέλουν αμφότεροι να τον μετριάσοουν. Η δομή της αγοράς που αναλύουμε είναι μια πλήρης αγορά, όπου οι παίκτες σχεδιάζουν από κοινού τα συμβόλαια που διαπραγματεύονται σύμφωνα με τις ανάγκες των κινδύνων που θέλουν να ανταλλάξουν. Ακόμη, η κατανομή και οι τιμές των συμβολαίων είναι αποτέλεσμα ενός παιγνίου υπό την έννοια μιας καθαρής στρατηγικής ισορροπίας Nash. Εμείς παρουσιάζουμε ένα διακτριτού χρόνου δυναμικτό υπόδειγμα και επικεντρωνόμαστε στις διμερείς συναλλαγές. Τα αποτελέσματα της διαδικασίας πολλών συναλλαγών (re-trading) εξαρτάται από την οξυδέρκεια των παικτών για μελλοντικές συναλλαγές. Συγκεκριμένα, είτε οι παίκτες συμπεριφέρονται μυωπικά είτε όχι, οδηγούμαστε σε διαφορετικά αποτελέσματα της κατανομής κινδύνου και των τιμών των συμβοαλίαων. Στην πρώτη περίπτωση, η ισορροπία συγκλίνει προς τη Pareto βέλτιστη κατανομή του κινδύνου, ενώ στη τελευταία αυτά παραμένουν σε μια ισορροπία Nash. Επιπλέον, η χρησιμότητα στην Ναση ισορροπία για κάποιο παίκτη μπορεί να είναι υψηλότερη από τη βέλτιστη κατανομή του καταμερισμού κινδύνου. Τέλος, εξετάζουμε την πορεία της ανωτέρω διαδικασίας, συμπεριλαμβανομένου του κόστους συναλλαγών (ενδογενώς δεδομένο) για κάθε γύρο.

Contents

1 Introduction to thin markets

1.1 The definition of thin financial markets

In order to set simply what a thin financial (OTC) market is we should keep in mind that it stands as the opposite of a liquid market. Thin markets (also called narrow markets) have small number of participants, low number of goods, commodities or financial products suppliers and users. Due to the small transactions either a single order on a financial security can affect the price significantly. The price volatility shifts in a very dramatic way and the spread between buyers and sellers is generally wider, as a consequence of the low trading volume and the few participants in the market.

The problem, that the literature deals with in thin markets (as this study does), is about agents strategic behavior and the liquidity issues it causes.

1.2 The price impact in thin markets

The main feature that describes the frame of a thin market is price impact of the traders. More precisely, it is estimated that more than 70% of the daily trade volume on the NYSE[1] is traded by pension funds, mutual funds, money managers, insurance companies, investment banks, commercial trusts, endowment funds, hedge funds, and some hedge fund investors, namely the institutional investors. See the empirical studies of Chan and Lakonishok [(1993, 1995)], and Keim and Madhavan [(1995, 1996, 1998)]. Also, according to these studies, only 20% of the buying and selling orders of the elephants[2] are completed within one day. In contrast to this, 50% of a typical elephant trading orders are implemented after four days.

Intuitively, taking the position of the broker or a dealer, imagine that the trading orders look like small windows-commands. Based on the aforementioned studies, these orders of the large investors are not taking place at once, despite that they are executed sequencially and divided into smaller pieces (windows). This happens for the moderation of the price impact, which may occur due to the size of the investor who gives the orders in a transaction on the NYSE. In addition to this, transaction costs that are related to the price impact can overwhelm the explicit costs of trade, such as commission, order processing and brokerage fees. That is why institutional investors are referred as elephant traders and the way they trade as iceberg orders. It is very important to mention that due to the technological boom the transactions process has been automated the last thirty years. Therefore, to best estimate this price effect, novel models are developed (usually called *market impact models*).

[1] New York Stock Exchange.

[2] We call elephants the big guys on the market who can move large blocks of shares and have appalling influence on the stock market's movements.

Other significant factors that affect the price of a security are the exogenous supply or demand liquidity shocks (anticipated or unanticipated) such as an IPO (initial public offering), issuance of new debt, inclusions of new stock in the stock market or index weight changes. The prices overreaction passes through temporary and permanent effects and thus makes difficult to estimate the fundamental value[3] of a security. For instance, if some big investor orders high amount of XYZ stock and keep it to her portfolio, the price will move steeply upward before it will shift its long-run level (new equilibrium price) in the following period. This permanent and transitory price deviations are modeled in Rostek and Weretka [(2015)], where it is shown that the effects in exogenous shocks are divided in fundamental and liquidity effects. The fundamental effect depicts the new equilibrium price level, which aggregately reflects the permanent result in the value of the security in the following period. The temporary effect reaches its peak the time that the transaction takes place[4].

The frame of the Brunnermeier and Pedersen [(2005)], gives also a novel explanation for the price overreactions. Their model is based on the Cournot oligopoly market structure. The concept of this study responds to the prices overreaction, where a large trader incurs shortage of liquidity and she needs to gain cash by selling an asset, that holds in her portfolio. In the spirit of their seminal study, the long position traders are assumed to be as the predators while the short position ones, namely those who need to liquidate their investment position, as the preys. In other words, the predator drains liquidity from the market, when it is necessary at maximum, while she sells simultaneously the same asset and moves downward the price. In an afterward period, the predator takes long position in the asset gaining both the asset and the cash from the price volatility.

Remark 1. *The Cournot[5] model is the first duopoly model that interprets the equilibrium in oligopolies. Specifically, Cournot inspired his model after the observation of how competitive a spring water duopoly behaved. In this model the game is played among the firms that all produce the same-homogeneous product (not an asset). Given that each firm can predict the other firms output strategy, it chooses a profit-maximization strategy (best response strategy). The fundamental difference between thin market price impact modeling and Cournot model structure is, that the latter refers to markets of goods, where the sellers develop games by choosing as parameter their strategies, which are about the supply of the good rather than the price.*

[3]The fundamental value of a security in essence depicts the fair value of it under the assumptions of market efficiency. Specifically, the investors, have the same beliefs, are well informed, thus the value of the security equilibrium price is fair. Due to thinness of the market, the security prices may deviate from their fundamental values since use of market power leads to price impact.

[4]For more information you see in Rostek and Weretka [(2015)], where an extended discussion is stated for the transitory and permanent price effects.

[5]See in Varian, Hal R. [(2006)], intermediate microeconomics, a modern approach (7th ed.), published by W. W. Norton & Company.

Remark 2. *The fact that thin markets present slow moving capital, because of the price volatility, distinguishes them from the competitive capital markets and consequently, we can not use CAPM to exact the equilibrium security price. In general, the clear message is that big blocks of orders are managed with the so-called order break-up strategy, which constitutes the common strategy to trade in thin markets. This strategy is illustrated by Weretka and Rostek [(2015)] and Vayanos, in [(2001)] as the equilibrium strategy, where all traders optimizing dynamically. Hence, the price impact can be moderated.*

1.3 How do we model thin markets?

Modeling in thin markets is separated in three different categories: information asymmetry or private information models, inventory effects and non-equilibrium mechanisms. The topics that are listed in thin market modeling varying among concepts such as market power effects[6], predatory trading[7] and market manipulation[8], which explains the price behavior of the securities. Below we present each category with typical references of the related literature.

(a) Information asymmetry or private information models remain the basic category of thin market modeling with plenty of literature such as *Glosten and Milgrom [(1985)], Kyle [(1985, 1989)], Easley and O'Hara [(1987)], Back [(1992)], Foster and Viswanathan [(1996)], Holden and Subrahmanyam [(1996)].* These models provide price overreaction explanation through the signaling (or expectations) channel. More precisely, the informed traders through a purchase or sale order give the signal (or shape expectations) to the rest of the participants to move in the same direction. This leads to price volatility of a security and the value of an asset reduces or increases dramatically in dependence with the order that is given (sale or purchase respectively). However, information asymmetry contributes partially in the understanding of price impact, while other factors such as investors preferences,

[6]Someone who has the knowledge that she has price impact on a security in the market is said to own market power. She can affect the equilibrium price, even if her size as an agent is small. For instance, information asymmetry gives market power and someone can use it to gain profits. Similarly, the government with the tax rate system that is applied can affect the price of a good or a financial product.

[7]It differs from market power and as we have already mentioned the predator exploits the needs of someone who wants to liquidate position. For example, we can thought about predatory trading as the exact opposite concept of fierce sale. Namely, recalling the Lehman Brother collapse period each investor had the need to get rid of whatever security was related to Lehman. So the predators knew that, and applied their strategies to take advantage of the others needs. Also, you can see the LTCM collapse case on September 23 in1998, which is characterized as a predatory trading looks like arbitrage, it is not since the predator strategies are risky.

[8]Someone who uses her market power to make price impact of a security through strategic choices is defined to apply market manipulation. However, the use of this term is more general and in order to understand it better simply suppose the cartels, namely it is something like a cooperative game strategy on behalf of a small group of people who control the selling prices of their goods to make profit maximization.

equally influence the price volatility and thus, they are developed more forcefully inventory models.

(b) Inventory effects (our study is based in such a model) are also argued by numerous of papers such as *Ho and Stoll [(1981)] Grossman and Miller [(1988)], Vayanos [(2001)], Attari, Mello and Ruckes [(2005)], Brunnermeier and Pedersen [(2005)], Pritsker [(2005)], DeMarzo and Urošević [(2006)] extended by Urošević.* In such models, information asymmetry is combined with the portfolio analysis and constitute the component of price impact. There are conditions when traders accomplish their transactions, that can not based only in the information asymmetry. For instance, when a company decides an IPO there are intermediaries such as an investment bank that will buy a huge block of shares from the newly introduced company to the stock market, aiming to sell them in a subsequent period and gain a profit from the bid ask spread. Because of the market thinness if the intermediaries face huge orders of sale or purchase for the shares from third parties, they ask for minus or plus a risk premium respectively. Given that, the third parties behave in a risk averse way and will hold the shares in their portfolio the intermediaries (according to price shifts) ask for this premium as a compensation for the investment risk they undertake from the IPO.

    This type of models are developed under the setting and fit in both game theoretic and general equilibrium. This means that we can make an instant comparison between strategic models (Nash equilibrium) with the competitive (non-strategic) ones.

(c) Non-equilibrium mechanisms is the last category of novel papers such as *Bertsimas and Lo [(1998)], Almgren and Chriss [(2000)], Subramanian and Jarrow [(2001)], Dubil [(2002)], Almgren [(2003)], Huberman and Stanzl [(2004)], Almgren et al. [(2005)], Engle and Ferstenberg [(2007)]* that enhance the price impact through empirical analysis of the market dynamics.

2 Description of the basic model and its findings

In principal, the pricing in thin financial markets is the milestone that we conquer with this investigation. Before we set its conversation goals, we argue the main assumptions and the findings of Anthropelos [(2017)], on which we base our model. In the following sections, we present the expansion of this model in order to succeed in pricing of financial trade-able securities. Furthermore, we set the agents' preferences on which the model is built. In this point, our conversations will focus in the competitive equilibrium price allocation among the agents.

2.1 The mean variance preferences

We introduce a static market model and let n be the agents, who participate in this market, while each one posses a *random endowment*. Every agent is exposed to a random risk based on her endowment and hence she tries to hedge it. Note that the payoff, that each one gains, is discounted and measured in standard monetary units (i.e. in dollars). Since the model is one shot, we can assume so without loss of generality. Hereafter, we denote the endowments as random variables $\mathscr{E}_i$, $i \in \{1, 2, ..., n\}$, that are defined in a standard probability space (Ω,$\mathscr{F}$,P)[9]. Therefore, $\sum_{i=1}^{n} \mathscr{E}_i = \mathscr{E}$ is denoted to be the market endowment (aggregate endowment). Each agent's preferences are characterized by M-V utility function, specifically:

$$U_i[X] = E[X] - \gamma_i Var[X] \tag{2.1}$$

The interpretation of (2.1) is, that X stands for the random payoff, such that $X \in L^2(\Omega, \mathscr{F}, P)$, $E[.]$ and $Var[.]$ stand for the expectation and the variance, while γ_i denotes the agent's risk aversion coefficient. Moreover, we result to an optimization problem, which is known from the investment portfolio theory[10] and thus, we look for the maximum of the function (2.1) in order to solve it. With a slight abuse of notation, the Var[.] is associated with variance-covariance matrix, since we refer to vector spaces.

2.2 The pricing of the optimal contract in the incomplete market setting

When we refer to the incomplete market setting, we emphasize that, the risk sharing is completed via a given vector of standardized financial trade-able securities. We assume that these financial trade-able securities can be any structured financial derivative. Hereafter, we simply refer to them as contracts. Each agent can search whatever of these contracts (securities) and take the appropriate hedging position, towards to the needs of her endowment. The main question that we set in this problem is the following: *"Which is the best price, that the agent is going to pay on the contract according the position that she will take?"* The answer is that, the price of it, depends on the demand of the contract related to the individual's endowment. Thus, we consider the following proposition, which sets the appropriate optimization problem, in order to find the agent's demand function. Nevertheless, we mark, that trading a given vector

[9] It is postulated from the probability theory, that Ω illustrates the sample space, $\mathscr{F}$ the σ-algebra in order to encode the information and P is called the the subjective probability measure, which is common for every agent.

[10] Supposed that each agent is rational, she tries to maximize her expected payoff E[X] and simultaneously minimize her risk which is denoted by the term $\gamma_i Var[X]$.

of securities is not a Pareto optimal trade, while it concludes to improve the position of each player individually.

Supposing that C is the given vector of the contracts such that $C = (C_1, C_2, ..., C_k) \in (L^2)^k$, $k \in N$, which stands for their payoff, and $a \in R^k$ denotes the position that an agent takes on the contract. In addition, the vector $p \in R^k$, indicates the price of the contract. The answer in the aforementioned question is the solution of the next optimization problem:

$$sup_{a\in R^k} U_i(\mathscr{E}_i + a \cdot C - a \cdot p) = sup_{a\in R^k}\{U_i(\mathscr{E}_i + a \cdot C) - a \cdot p\} \tag{2.2}$$

Since function (2.1) is quasi concave (quadratic functions), its first derivative is a linear function, which means that we have a unique set of vectors that maximizes the function (2.2). Thus, the maximum is the solution of the optimization problem that gives the agent's demand function.

Proposition 1. *$\forall$ price vector $p \in R^k$ on the contracts C, the demand function of each agent is* $\zeta_i(p) := \left[\dfrac{E(C) - p}{2\gamma_i} - Cov(C, \mathscr{E}_i)\right] \cdot Var^{-1}[C]$

Proof. Given that p is a price vector on the contracts C we define that:

$$\zeta_i(p) := \arg\max_{a\in R^k} \{U_i(\mathscr{E}_i + a \cdot C) - a \cdot p\}$$

We have that

$$U_i = U_i(\mathscr{E}_i + a \cdot C - a \cdot p) = E(\mathscr{E}_i + a \cdot C - a \cdot P) - \gamma_i Var(\mathscr{E}_i + a \cdot C) \Rightarrow$$
$$\Rightarrow U_i = E(\mathscr{E}_i) + aE(C) - a \cdot p - \gamma_i Var(\mathscr{E}_i) - \gamma_i a \cdot Var(C) \cdot a^\intercal - 2aCov(C, \mathscr{E}) \Rightarrow$$
$$\Rightarrow U_i = U_i(\mathscr{E}_i) + aE(C) - a \cdot p - \gamma_i a \cdot Var(C) \cdot a^\intercal - 2aCov(C, \mathscr{E})$$

The quasi concavity of the agents preferences guarantees the negative definite Hessian matrix, thus the maximum is given by the zeroing of the first order partial derivative. Hence, the necessary condition of the gradient must be:

$$\nabla U_i(\mathscr{E}_i + a \cdot C - a \cdot p) = 0 \Rightarrow \frac{\partial U_i}{\partial a} = 0 \Rightarrow E(C) - 2\gamma_i a \cdot Var(C) - 2Cov(C, \mathscr{E}) - p = 0 \Rightarrow$$
$$\Rightarrow a = \left[\frac{E(C) - p}{2\gamma_i} - Cov(C, \mathscr{E}_i)\right] \cdot Var^{-1}[C]$$

or equivalently

$$\zeta_i(p) := \left[\frac{E(C) - p}{2\gamma_i} - Cov(C, \mathscr{E}_i)\right] \cdot Var^{-1}[C] \tag{2.3}$$

□

Hence, the demand function, for any price of vector p, is unique for each agent following her quadratic preferences. In the sequel, we discuss the interpretation of the model.

Based on (2.3), we are going to interpret thoroughly the demand function. We observe that: $\zeta_i(p) : R^k \to R^k$ is a linear function of the price vector $p \in R^k$ on the contracts C, under the M-V preferences of the agent. We illustrate, that $E(C)$ stands for the vector of the expected payoff of the contracts $(E(C_1), E(C_2), ..., E(C_k)) \in R^k$ and for every payoff $X \in L^2$, $Cov(C, X)$ denotes the vector $(Cov(C_j, X))_{j=1}^k \in R^k$.

It is assumed that $\det[Var(C)] \neq 0$, namely the contracts are uncorrelated with each other, while for every considered vector of C the variance-covariance matrix is non-singular. Therefore, the equation (2.3) is divided in two parts:

(a) the part $\dfrac{E(C) - p}{2\gamma_i} \cdot Var^{-1}[C]$, which is referred to the risk premium

(b) and the part $Cov(C, \mathscr{E}_i)Var^{-1}[C]$,which illustrates the correlation between the trade-able securities C and the agent's endowment.

The correlation between the contract and the endowment determines the position that an institutional investor (agent) takes. If the $Cov(C, \mathscr{E}_i)$ is negative the agent takes a long position to the contract, otherwise she takes short on the contract. In addition, we should highlight that the endowment appears only in the intercept point of the demand function and not in the slope. This is a crucial difference of this model in relation to the existent literature, because the most of the models in the literature are based on the slope of the agent's demand function.

In completion, setting the already mentioned model as a benchmark, we explain the way, that the risk sharing allocation is completed among the agents. Assuming that the set of matrices $A_{n\times k} \subset R^n \times R^k$ expresses the allocation of risk sharing among the n-agents with k-contracts. Moreover, we simply denote as $a_{i,j}$, where i$\in \{1, 2, ..., n\}$ defines the row and j$\in \{1, 2, ..., k\}$ denotes the column of an allocation matrix A $\in$ $A_{n\times k}$ respectively. If the term $a_{i,j}$ is negative this means that agent-i should take a short position on the contact-j, otherwise she will take the long one ($a_{i,j}> 0$). It is assumed, that the net supply of trade-able contracts is zero, which implies that $\sum_{i=1}^n a_{i,j} = 0$, for every j$\in \{1, 2, ..., k\}$. Hence, we sketch with a strict mathematical definition what we call as a competitive price-allocation equilibrium.

Definition 1. *We will call as a competitive price-allocation equilibrium of a given vector of contracts* $C \in (L^2)^k$*, the pair* $(p^*, A^*) \in R^k \times A_{n\times k}$ *if* $\zeta_i(p^*) = a^*$ *for each i*$\in$ $\{1, 2, ..., n\}$.

Note that, that the equilibrium allocation, in the incomplete market structure, coincides with the CAPM (see Magill and Quinzii [(1996)]). Now we are ready to give the answer to the question we mentioned in order to calculate the price, the position of the contract and the optimal contract payoff. Considering the above, the competitive price-allocation equilibrium is unique, because of the agent's demand function linearity.

Proposition 2. *The unique pair* $(p^*, A^*) \in R^k \times A_{n\times k}$*, which derives from the CAPM, given a vector of contracts C is:*

$$p^* = E(C) - 2\gamma Cov(C, \mathscr{E}) \tag{2.4}$$

and

$$a_i^* = Cov(C, C_i^o) Var^{-1}[C] \tag{2.5}$$

where the optimal contract is

$$C_i^o := \lambda_i \mathscr{E}_{-i} - (1-\lambda_i)\mathscr{E}_i, \tag{2.6}$$

$\forall i \in \{1, 2, ..., n\}$.

Proof. It is already known, that $\sum_{i=1}^n \zeta_i(p^*) = \sum_{i=1}^n a_{i,j}^* = 0$, $\forall j \in 1, 2, ..., k$, thus:

$$\left[\frac{E(C) - p^*}{2\gamma_i} - Cov(C, \mathscr{E}_i)\right] \cdot Var^{-1}[C] = 0 \Rightarrow \frac{E(C) - p^*}{2\gamma_i} - Cov(C, \mathscr{E}_i) = 0 \Rightarrow$$
$$\Rightarrow p^* = E(C) - 2\gamma Cov(C, \mathscr{E})$$

By definition we have, that $a_i^* = \zeta_i(p^*)$, hence we have the following:

$$a_i^* = \left[\frac{E(C) - p^*}{2\gamma_i} - Cov(C, \mathscr{E}_i)\right] \cdot Var^{-1}[C] = \left[\frac{\gamma}{\gamma_i} Cov(C, \mathscr{E}) - Cov(C, \mathscr{E}_i)\right] \cdot Var^{-1}[C] \Rightarrow$$
$$\Rightarrow a_i^* = \left[\lambda_i Cov(C, \mathscr{E}) - Cov(C, \mathscr{E}_i)\right] Var^{-1}[C] \Rightarrow$$
$$\Rightarrow a_i^* = \left[\lambda_i Cov(C, \mathscr{E}) - \lambda_i Cov(C, \mathscr{E}) + Cov(C, C_i^o)\right] \cdot Var^{-1}[C] = Cov(C, C_i^o) \cdot Var^{-1}[C]$$

By combining the equation (2.6) with $\mathscr{E} = \mathscr{E}_i + \mathscr{E}_{-i}$ which is the total portfolio of the market gives:

$$C_i^o = \lambda_i \mathscr{E}_{-i} - (1-\lambda_i)\mathscr{E}_i \Rightarrow C_i^o = \lambda_i(\mathscr{E} - \mathscr{E}_{-i}) - (1-\lambda_i)\mathscr{E}_i \Rightarrow$$
$$\Rightarrow C_i^o = \lambda_i \mathscr{E} - \mathscr{E}_i \Rightarrow \mathscr{E}_i = \lambda_i \mathscr{E} - C_i^o$$

Finally, we have that: $Cov(C, \mathscr{E}_i) = Cov(C, \lambda_i \mathscr{E} - C_i^o) = \lambda_i Cov(C, \mathscr{E}) - Cov(C, C_i^o)$.
□

Recall that, $\mathscr{E} = \mathscr{E}_i + \mathscr{E}_{-i}$, where $\mathscr{E}$ depicts the aggregate market endowment and $\mathscr{E}_{-i}$ to be the aggregate endowment of the rest of the agents and γ stands for the aggregate risk aversion coefficient, such that $\frac{1}{\gamma} := \sum_{i=1}^{n} \frac{1}{\gamma_i}$. Also, the relative risk tolerance coefficient is $\lambda_i = \frac{\gamma}{\gamma_i}$, while the rest of the players have $\lambda_{-i} = 1 - \lambda_i$.

We have shown that the equations (2.4) and (2.5) give us the unique price of the contract and a_i^*, which is the i-th row of matrix A^* stands for the position the agent takes on the contract.

Finally, the equations (2.4) and (2.5) clearly reflect, that the prices depend on the expected price of the vector of contracts minus the covariance of the contract related to the total market portfolio $\mathscr{E}$. The position on the contract depends on the covariance of the vector of securities with the optimal contract. Finally, the optimal contract, which is given by the equation (2.6) depicts, the difference of each agent's tolerance on the submitted endowment, that they report individually.

2.3 The complete of the market & the optimal risk sharing

In order to complete the market, (assuming no exogenous constrains or additional transaction costs such as brokerage fees or commissions) the agents can trade each other through financial tradeable contracts, which they co-design. The risk sharing allocation in such a case, is consistent with the Pareto optimal rules and so the market is completed. The display of Pareto optimal concept responds to the question: *"How the allocation of resources is going to be accomplished?"* Assuming that a social planner governs with absolute purpose to make the optimal allocation of goods and financial services for the institutional investors according to their needs, the unique answer is the following: *"We will find out the allocation of resources that is optimal for the community"*. By referring to the optimal allocation, it does not mean that this allocation is fair. Pareto optimality means the optimal allocation, given the needs of each agent's investment position. This governor aims to match the preferences of the agents based on their risk exposure by making the optimal risk sharing allocation.

In addition to this, irrespective of the allocation is about, there are specific possibilities for the risk sharing. More precisely, Pareto optimal risk sharing equilibrium assumes no market power and every agent submit her true risk exposure for the sharing. Each agent is going to purchase what the other agents offer and in this way, they co-design the contracts, which are summing up to zero aggregately (zero sum of contracts implies that the transaction is cleared out and there is no extra risk in the market). Consequently an allocation of risk sharing will be called Pareto optimal, if no contract is better than the existing one (constant contracts).

Therefore, in the case where no use of market power exists the sum of the contracts equals to zero. Thus, we define the set of all the possible risk sharing contracts to be $\mathscr{A} := \{C = (C_1, C_2, ..., C_n) \in (L^2)^n | \sum_{i=1}^{n} C_i = 0\}$. The term C reflects the payoff of the contract, which the agent receives. At this point we define Pareto optimal allocation with its strict mathematical definition.

Definition 2. *A vector of contracts $C^o \in \mathscr{A}$ is a Pareto optimal risk sharing if $\forall$ $C \in \mathscr{A}$, the following implication holds:*

If for some i, $U_i(\mathscr{E}_i + C_i) > U_i(\mathscr{E}_i + C_i^o)$, then $\exists$ j$\neq$ i such that $U_j(\mathscr{E}_j + C_j) < U_j(\mathscr{E}_j + C_j^o)$.

Theoretically, when the transferred risk attains a Pareto allocation means, no other risk sharing exists that some agent can increase her wealth without causing a reduction in the wealth of another agent. Therefore, no better contract exists than the Pareto optimal contract. If such a contract exists, the Pareto optimal risk sharing does not hold anymore. Equivalently, we can claim that the utility for some agent is better than the Pareto optimal, then there would be at least another agent, who suffers a greater loss of utility than the Pareto case.

With regard to the complete market setting, where the M-V preferences hold again, the risk sharing allocation problem is more restricted in order to find the solution of the optimal price allocation of the vector of endowments. We assume that the agents' vector of endowments is $E = (\mathscr{E}_1, \mathscr{E}_2, ..., \mathscr{E}_n) \in (L^2)^n$. We recall the functions (2.4) and (2.5) of the unique competitive equilibrium of E as a pair $(p^o, A^o) \in R^n \times A_{n\times n}$, which implies that the price is $p^o := E(E) - 2\gamma 1_n \cdot Var[E]$, where $1_n := (1, 1, ..., 1)$. It is also implied that the elements of $\mathscr{A}^o$ are $a^o_{i,i} = -\lambda_{-i}$ $a^o_{i,j} = \lambda_i$, for j$\neq$ i. Notice that the exponent ^{o} refers to the Pareto optimal risk sharing, while the exponent * refers to the incomplete market setting. The strict mathematical interpretation of the above are incorporated in the next proposition.

Proposition 3. *Let $(p^o, A^o) \in R^n \times A_{n\times n}$ be the competitive price-allocation equilibrium of the vector of securities E. Then, the vector of securities $\mathscr{A}^o(E - p^o) \in \mathscr{A}$ is the unique Pareto-optimal risk sharing*[11].

Proof. We clarify that the pricing of the aggregate payoff (compensation), is the one dimension of the problem that we study (the second dimension is the pricing of the contract payoff), which an agent gets is estimated to be $a^o_i \cdot E - a^o_i \cdot p^o$, where a^o_i stands for the vector $(a^o_{i,1}, a^o_{i,2}, ..., a^o_{i,n}) \in R^n$. It is induced by the *Proposition 1.* that $a^o_i E = C^o_i$, where C^o_i is the contract that agent-i gets and pays the price: $\pi^o_i = a^o_i \cdot p^o = a^o_i \cdot [E[E] - 2\gamma \cdot 1_n \cdot Var[E]] = a^o_i \cdot E[E] - 2a^o_i \cdot \gamma \cdot 1_n \cdot Var[E] = E(C^o_i) - 2\gamma Cov(C^o_i, \mathscr{E})$ to obtain it. The covariance of $E \in (L^2)^n$ and the market portfolio $\mathscr{E}$ is $Cov(E, \mathscr{E}) = 1_n \cdot Var[E]$, since E denotes the vector of the agents' endowments and $\mathscr{E}$ the aggregate endowment. Finally, the price is:

$$\pi^o_i := E(C^o_i) - 2\gamma Cov(C^o_i, \mathscr{E}) \tag{2.7}$$

[11] We highlight that Pareto optimality does contain the prices of the financial trade-able securities.

□

Taking into account the whole discussion of both markets, what we have to keep in mind is: *"Agent-i is going to exchange (short position) a part of her true endowment, that she submits in the hedging transaction, with equal part (long position) of the markets endowment, that the other agents posses."* This conclusion is robust under the aforementioned assumption with no use of market power.

2.4 Risk sharing inefficiency measure

A basic milestone to capture is associated with the extra utility, which an agent gains at different equilibria. Namely, the measuring of the so-called utility surplus, in order to succeed in making comparisons between them. For each agent individually, we assume that the utility or equivalently her wealth (here is characterized by the random endowments) is given by:

$$u_i(p; C) = U_i(\mathscr{E}_i + \zeta_i(p) \cdot C) - \zeta_i(p) \cdot p \tag{2.8}$$

at any price p, ∀i∈ $\{1, 2, ..., n\}$.

Hence, the difference $u_i(p; C) - U_i(\mathscr{E}_i)$ is defined to be the agents' utility surplus, which gives the agent's additional wealth compared to the utility of the preceding equilibrium-condition. Namely, this measure is determined by identifying the surplus, given the contract demand at price p. Moreover, for some agent the utility is expressed at standard monetary unions. Following the literature, we set the agent's-i optimal utility in the situation, where strategical behavior does not exist, which is varying according to the market setting. In particular, u_i^* and u_i^o, indicate the incomplete market optimal utility and the complete one respectively.

Corollary 1. *The utility of agent-i at the Pareto optimal risk sharing is given by:*

$$u_i^o(p^o; E) = \gamma_i Var[C_i^o] + U_i(\mathscr{E}_i), \tag{2.9}$$

in contrast to this the utility at the competitive price-allocation equilibrium of a contract vector C is

$$u_i(p^*; C) = \gamma_i a_i^* \cdot Var[C] \cdot a_i^* + U_i(\mathscr{E}_i) = \gamma_i Cov(C, C_i^o) \cdot Var^{-1}[C] \cdot Cov(C, C_i^o) + U_i(\mathscr{E}_i)^{12} \tag{2.10}$$

[12]The utility gain can not be less than the utility of the initial position, if this were the case the agent will not participate in the market.

Proof. We have that:

$$u_i^o(p^o; E) = U_i(\mathscr{E}_i + C_i^o - \pi_i^o) = E(\mathscr{E}_i + C_i^o) - \pi_i^o - \gamma_i Var(\mathscr{E}_i + C_i^o) =$$
$$= U_i(\mathscr{E}_i) + 2\gamma Cov(C_i^o, \mathscr{E}) - 2\gamma_i Cov(C_i^o, \mathscr{E}_i) - \gamma_i Var(C_i^o)$$

Given that, $\mathscr{E}_i + C_i^o = \frac{\gamma}{\gamma_i}\mathscr{E}$, then:

$$Cov(C_i^o, \mathscr{E}) = Cov\left[C_i^o, \frac{\gamma_i}{\gamma}(\mathscr{E}_i + C_i^o)\right] = \frac{\gamma_i}{\gamma}\left[Cov(C_i^o, \mathscr{E}) + Var(C_i^o)\right]$$

Thus, we have

$$U_i(\mathscr{E}_i + C_i^o - \pi_i^o) = U_i(\mathscr{E}_i) + 2\gamma\frac{\gamma_i}{\gamma}\left[Cov(C_i^o, \mathscr{E}) + Var(C_i^o)\right] - 2\gamma_i Cov(C_i^o, \mathscr{E}_i) - \gamma_i Var(C_i^o) =$$
$$= U_i(\mathscr{E}_i) + \gamma_i Var(C_i^o)$$

Similar is the case of the incomplete market setting. Recalling the *Definition 1* we have that:

$$u_i(p^*; C) = U_i(\mathscr{E}_i + \zeta_i(p^*) \cdot C) - \zeta_i(p^*) \cdot C = U_i(\mathscr{E}_i + a^* \cdot C) - a^* \cdot C =$$
$$= U_i(\mathscr{E}_i) + a^* \cdot E(C) - \gamma_i a^* \cdot Var(C) \cdot a^* - 2\gamma_i a^* \cdot Cov(C, \mathscr{E}_i) - a^* \cdot p$$

Provided that, *Definition 1.* holds, the demand function (2.3) gives:

$$2\gamma_i a^* \cdot Var(C) \cdot a^* + a^* \cdot p = a^* \cdot E(C) - 2\gamma_i \cdot a^* \cdot Cov(C, \mathscr{E}_i)$$

Thus, we have that:

$$u_i(p^*; C) = U_i(\mathscr{E}_i) + 2\gamma_i a^* \cdot Var(C) \cdot a^* + a^* \cdot p - \gamma_i a^* \cdot Var(C) \cdot a^* - a^* \cdot p \Rightarrow$$
$$\Rightarrow u_i(p^*; C) = U_i(\mathscr{E}_i) + \gamma_i a^* \cdot Var(C) \cdot a^*$$

□

Easily, from this point onward we can determine, which is the utility "loss" of the individuals. Given that, trading a vector of contract is not consistent with Pareto efficient allocations (incomplete market setting), the agents suffer a loss of utility, that we can estimate it with the measure that we quote below. This measure is indicated by the difference $u_i^o(p^o; E) - U_i(p^*; C)$, which is compared to the complete market and non-negative. At last, under the quadratic preferences, the risk sharing inefficiency is accounted by the difference of the aggregate utility in the optimal equilibrium minus the aggregate utility of the sub-optimal equilibrium or the realized aggregate utility. Thus, it is given by:

$$\begin{pmatrix} Risk\ sharing \\ inefficiency \end{pmatrix} = \begin{pmatrix} Optimal\ Aggregate \\ Utility \end{pmatrix} - \begin{pmatrix} Realized\ Aggregate \\ Utility \end{pmatrix} \tag{2.11}$$

In the sections that follow this measure of inefficiency is defined more generally. The idea is based on the utility that an agent gains, which differs according to different equilibria. In this study they are thoroughly probed comparisons between competitive and Nash equilibria. Albeit, in a thin market, each agent, also, may suffer a "loss", which is indicated by a similar non-negative difference, this can not be taken as a general rule. We notice, that the strategic behavior may lead to a higher utility surplus, individually, in a Nash equilibrium, although the aggregate utility of the market diminishes.

2.5 Agents' incentives and the best response problem

The notion of the competitive equilibrium is catalyzed from the moment that agents use their market power. The thinness of the market, i.e. the modeling of an oligopoly with a finite number of participants, gives rise to the market power. Thus, agents have incentives to behave strategically and use their market power to gain extra profits (measured with the utility surplus individually) and improve their investment positions. Our interest is the pricing of financial tread-able securities (contracts) in an oligopoly, that are traded by the agents towards to their incentives. Although, in *Propositions 3* is clear that the risk sharing allocation can be transferred among the finite number of the market participants optimally, the market power, thus the strategic behavior results to allocation inefficiency.

The crucial query, that is labeled as the investor's best response is quoted with the following question: *''How does she react, given that she knows the market endowment?"* Namely, each agent individually inquires for the strategy, which will respond to the strategy of other agents (the so called dominant strategy). The main idea is based on the contract which will be nominated by the investor, giving the explanation of how the strategic behavior unfolds according to each market setting. We point out, that the best response endowment is used as an intermediary step to explain the Nash games.

2.5.1 Pricing according to the market structure - complete & incomplete market settings

Our interest is to study the price of the contract, on which the agents are going to trade. The key point is, that agent-i is considered to know the market endowment, but how will she respond to it under the information asymmetry benefit. Namely, we wonder if it is a good choice to submit her true endowment on the agreement or to use the information asymmetry benefit, to empower her investment position. By doing so, she will gain a better payoff than the others, who do not have the same information with her. In particular, agent-i has to take into account the risk, that she is going to share with the other participants of the market (complete setting) trying to take advantage from their hedging needs. Thus, for further analysis on the game theoretic approach, the agent will submit a different endowment than the real one that she owns, in order to exert the *best response endowment*, given that she has the knowledge of the real market portfolio or equivalently the needs for hedging of the whole market. Imagine, that she has a list of the best answers in the needs of each other agent due to the influence, that she can have on the market (market power).

Before we end up with the strict mathematical frame so as to define the *best response endowment*, we explain how a game evolves with a simple example. Let us consider a thin oligopolistic market which consists of only three insurance companies (hereafter IC). We assume that each IC holds some insurance endowment and more precisely, IC1 holds only life insurances, IC2 holds insurance calamities (i.e. natural disasters) and IC3 holds only auto insurance. Every IC is exposed to the (random) risk of the endowment, that she owns and due to no diversification exists to their portfolios, each undesirable event, in the sector which the individual IC has grown its investment position, may lead to bankruptcy. Consequently, if IC1 for instance has the knowledge of the endowments that IC2 and IC3 posses (the risk exposure of the other participants in the market) she can exercise the best response strategy to take advantage of their hedging needs (acts as a predator). Therefore, the contract that she will co-design with the rest of the participants to make the risk sharing allocation will maximize her own payoff. Hence, the IC1 will succeed in accomplishing the so called dominant strategy in the market. Obviously, the equilibria that we conclude are not competitive anymore. We mark here that the best response is, as always, the intermediary step before the presentation of the Nash equilibrium. By doing this step initially, we ensure the existence of the solution of the problem.

Not only the IC but also each institutional investor-agent will behave strategically in a thin market (imperfect competition from this step onward). We remind from the introduction that a thin market setting is similar to the Cournot market structure (see again in *Remark 1*), while the game differs in the strategies. The question that is quoted here is: *"What are the incentives, for the agent who holds the market power, to use it in order to make the best response?"* The answer is, that: Given that, she knows the aggregate endowment $\mathscr{E}_{-i}$ of the market she can be grouped between the two following categories:

(a) either she will submit a proportion of her true endowment on the agreement

(b) either she will submit exposure to the risk that all the other agents undertake (the response to the endowment $\mathscr{E}_{-i}$). She may act as a predator or a speculator, especially in the case where her risk aversion tends to risk neutrality.

Both of them aim to take advantage of the hedging needs that the rest of the agents face.

Supposing, that agent-i reports as her true endowment some random variable [13], say $\mathbf{B} \in L^2$, then the contract, that she buys, has the next payoff:

$$C_i^o(\mathbf{B}) := \lambda_i \mathscr{E}_{-i} - \lambda_{-i}\mathbf{B} \tag{2.12}$$

while the accumulated cash that she has to pay so as to obtain it, which is resulting by the combination of equations (2.7) and (.12) respectively, determines the price of the contract (compensation):

$$\begin{aligned} \pi_i^o(\mathbf{B}) &:= \lambda_i(E[\mathscr{E}_{-i}] - 2\gamma Cov(\mathscr{E}_{-i}, \mathscr{E}_{-i} + \mathbf{B})) - \lambda_{-i}(E[\mathbf{B}] - 2\gamma Cov(\mathbf{B}, \mathscr{E}_{-i} + \mathbf{B})) \Rightarrow \\ \pi_i^o(\mathbf{B}) &:= \frac{\gamma}{\gamma_i}(E[\mathscr{E}_{-i}] - 2\gamma Cov(\mathscr{E}_{-i}, \mathscr{E}_{-i} + \mathbf{B})) - (1 - \frac{\gamma}{\gamma_i})(E[\mathbf{B}] - 2\gamma Cov(\mathbf{B}, \mathscr{E}_{-i} + \mathbf{B})). \end{aligned} \tag{2.13}$$

Notice again that $\lambda_i = \frac{\gamma}{\gamma_i}$ denotes the relative risk tolerance of the investor.

The utility that an agent gain after the transaction is:

$$G_i(\mathbf{B}; \mathscr{E}_{-i}) := U_i(\mathscr{E}_i + C_i^o(\mathbf{B}) - \pi_i^o(\mathbf{B})) = E[\mathscr{E}_i + C_i^o(\mathbf{B})] - \gamma_i Var[\mathscr{E}_i + C_i^o(\mathbf{B})] - \pi_i^o(\mathbf{B}) \tag{2.14}$$

Apparently, we figure out, that the contract payoff, which the player will get and the price, that she pays as a compensation are determined by the next optimization problem. More precisely, the solution of this problem, which is unique due to the linearity of the function is given by:

$$\mathbf{B}_i^{res} := \arg\max_{\mathbf{B} \in L^2} \{G_i(\mathbf{B}; \mathscr{E}_{-i})\} \tag{2.15}$$

It is very important to mention, that the set of strategic choices is equal to $L^2(\Omega, \mathscr{F}, \mathrm{P})$, thus the endowment, that investors report is measurable with respect to the information which is generated by the true endowments. Namely, all investors encounter the same σ-algebra. Moreover, the solution $\mathbf{B}_i^{res}$ which defines the *best endowment response*, given that she knows the true market portfolio, is unique and gives the payoff that agent-i enjoys and the price that she pays at the equilibrium (not competitive anymore). Specifically, we define the contract of her preferences to be $C_i^r = C_i^o(\mathbf{B}_i^{res})$ (mapped by the solution of (2.12)) and the price that she will pay, which is accounted to be $\pi_i^r = \pi_i^o(\mathbf{B}_i^{res})$ in dollars (mapped by the solution of (2.13)).

[13] The random variable **B** stands for the term best based on the game theory literature.

The solution of the equation (2.15) is given by the next mathematical proposition:

Proposition 4. *The unique (up to constants) best endowment response for agent-i, $\forall i \in \{1, 2, ..., n\}$, given that she knows the submitted aggregate endowment of the other players $\mathscr{E}_{-i}$ is:*

$$\boldsymbol{B}_i^{res} = \frac{1}{1+\lambda_i}\mathscr{E}_i + \frac{\lambda_i^2}{1-\lambda_i^2}\mathscr{E}_{-i} \tag{2.16}$$

If we observe carefully the equation (2.16), $\mathbf{B}_i^{res}$is indeed a combination between the agent's-i endowment $\mathscr{E}_i$ and the rest of the market endowment $\mathscr{E}_{-i}$. In particular, this reflects the endowment, that she reports in the potential agreement with the rest of the agents (differs from the $\mathscr{E}_i$). Therefore, at the equilibrium the position that she takes is $C_i^r - \pi_i^r$ which also differs from the optimal equilibrium position $C_i^o - \pi_i^o$. More precisely, we will have that

$$C_i^r - \pi_i^r = \frac{C_i^o}{1+\lambda_i} - \frac{\pi_i^o}{1+\lambda_i} + 2\gamma\frac{\lambda_i Var(C_i^o)}{\lambda_{-i}(1+\lambda_i^2)} = \frac{1}{1+\lambda_i}[C_i^o - \pi_i^o] + 2\gamma\frac{\lambda_i Var(C_i^o)}{\lambda_{-i}(1+\lambda_i^2)}$$

which indicates, that the net payoff in this case is an increasing function of the variance of optimal contract C_i^o. Though, it seems to be lower than the Pareto optimal net payoff, in the case of agent who is assumed to be a speculator,[14] this interpretation may be of great significance, while at the moment this is not the basic issue of our study. Referring to the speculator case, a speculator is defined to be someone with a risk aversion coefficient γ_i and a constant endowment, which is very low and will submit only the same risk with her counterparty (i.e. the risk which her counterparty is endowed).

Also, ti is induced by (2.16) that since agent-i has the knowledge of the submitted aggregate endowment $\mathscr{E}_{-i}$ of the other agents, the act of the *best endowment response* is to share a proportion of her true endowment and report exposure to the risk that the other investors face. Thus, she will try to take advantage of her counterparties, because of their hedging needs, whereas the strategy that we mentioned will drive the prices in the way she wants. Specifically, she increases the demand of the contract, that she wants to sell during the transaction, which leads to a more tempting payoff, that she will get. Namely, agent-i with this strategic application simultaneously takes advantage of the other participants' hedging needs to cover hers and gets a higher price of the contract, that she sells. We recall the simplest case of the single predator existence,[15] who chooses his optimal trading strategy (which is the same with *best response endowment*), given that she knows the needs for liquidation, that the rest of the traders incur. A crucial difference in the predatory model is, that a predator,

[14] We define as a speculator the agent, that neither she has investment risk nor hedging needs and she wants to undertake risk in order to take cash.

[15] See in Markus K. Brunnermeier & Lasse Heje Pedersen [(2005)], in section III, the predatory phase, for more information.

who is referred as a strategic trader, is assumed to be risk neutral in relation to her prey, while initially the agents in our model are risk averse and homogeneous (with the same risk aversion). If we unravel the thread of Ariadne, we could say that the predator chases the whole pie, the exploitation of the prey and the liquidation of her own investment position as it was before the strategic application.

Another question that we should answer is: *"What happens with the agent's-i utility after the reporting of the best endowment response strategy?"* We conclude that the utility after the strategy application is:

$$G_i(\mathbf{B}_i^{res}; \mathscr{E}_{-i}) = E[\mathscr{E}_i + C_i^o(\mathbf{B}_i^{res})] - \gamma_i Var[\mathscr{E}_i + C_i^o(\mathbf{B}_i^{res})] - \pi_i^o(\mathbf{B}_i^{res}) \Rightarrow$$
$$\Rightarrow G_i(\mathbf{B}_i^{res}; \mathscr{E}_{-i}) = U_i(\mathscr{E}_i) + [\frac{2\gamma_i}{1+\lambda_i} + \frac{2\gamma\lambda_i}{\lambda_{-i}(1+\lambda_i)^2} - \frac{\gamma_i}{(1+\lambda_i)^2}]Var(C_i^o) \Rightarrow$$
$$\Rightarrow G_i(\mathbf{B}_i^{res}; \mathscr{E}_{-i}) = U_i(\mathscr{E}_i) + \frac{\gamma_i Var(C_i^o)}{1-\lambda_i^2} \tag{2.17}$$

Thus, we take the difference between the new utility minus the Pareto optimal one $G_i(\mathbf{B}_i^{res}; \mathscr{E}_{-i}) - u_i^o(p^o; E) = \dfrac{\lambda_i^2}{1-\lambda_i^2} Var(C_i^o)$ which depicts that in the case of the strategic application agent-i takes a surplus of utility compared to the Pareto optimal transaction. Note that, the percentage of the utility increase is an increasing function of λ_i.

The last but not least at this point has to do with the matching between $\mathbf{B}_i^{res}$and the $\mathscr{E}_i$ which agent-i holds. If for some case $\mathbf{B}_i^{res}$ becomes equal (up to constants) to $\mathscr{E}_i$ agent-i has no extra gains from the trading than the Pareto optimal with the rest of the participants in the market. For this reason, she may act by reporting $\mathbf{B}_i^{res} \neq \mathscr{E}_i$ due to the incentives we have already mentioned in (a) and (b). We emphasize, that it is the number of participants that reduces the difference $\mathbf{B}_i^{res} - \mathscr{E}_i$ or equivalently the market power use is less effective as much as the $\mathbf{B}_i^{res} - \mathscr{E}_i$ vanishes.

To sum up, we have shown that the connection between the agent's-i utility surpluses and is given by:

$$\begin{pmatrix} Utility\ surplus\ from\ the\ best \\ endowment\ response \end{pmatrix} = \frac{\lambda_i^2}{1-\lambda_i^2} \times \begin{pmatrix} Utility\ surplus\ in \\ competitive\ equilibrium \end{pmatrix} \tag{2.18}$$

Remark 3. *The game, when we refer to the incomplete market structure, is played through the agent's demand function. So, the parameterization is reflected in the part of the function, which contains the hedging needs of the trader's endowment. Namely, the term $Cov(C, \mathscr{E}_i)$ and the applied strategy is called the demand best response. Even though, the two games seem to be different, their outcomes are identical due to the specific preferences of the agents. Apart from this, the agent reveals only a part of her true demand for her endowment hedging needs, which also seems to be the same* *strategy approximation with the complete market structure, where the agent reports only a part of her true endowment. Referring to the basic paper, the effect of market power on* risk-sharing *of Anthropelos which generates this model, it is shown, that it is never optimal for an agent to submit her true demand function on the agreement (as it is never to optimal to submit her true endowment in the complete setting).*

However, in this market structure agent-i reveals the demand function, that clears out the market and maximizes her utility. The way that the game evolves responds to the next question: "What is the price impact in the market by the agent's use of market power, given that she knows the demand of the other participants?" More precisely, she exploits the information asymmetry given that she knows the hedging needs of the market (the rest of the agents). Hence, the solution of the optimization problem in such case is given by:

$$p_i^{res} := \arg\max_{a \in R^k} \{\phi_i(p; \zeta_{-i}(p))\} \tag{2.19}$$

where:

(a) $\zeta_{-i}(p)$*is the demand function of the market except the agent-i who exploits the information asymmetry and since* $a = \zeta_i(p)$ *then* $a + \zeta_{-i}(p) = 0 \Rightarrow \zeta_{-i}(p) = -a$*,*

(b) $\phi_i(p; \zeta_{-i}(p)) := U_i(\mathscr{E}_i - a \cdot C + a \cdot p)$ *for* $a, p \in R^k$*.*

Remark 4. *The best response strategy is independent of the market structure as the outcomes are identical due to the specific preferences of the agents. Therefore, from this point onward, we will focus our study in the complete market setting.*

2.5.2 The Nash equilibrium - One period model

Provided that, in the previous subsection we had a thorough conversation of the incentives, that agents have in order to apply the best response, we notice that they are still the same in Nash. What is more in Nash, has to do with the behavior of the whole market participants. Intuitively, assuming that we have a market with finite number of participants who all have the market power, each one will adopt the strategic behavior, that we marked in the best response strategy. Analyzing the game, the information symmetry and the assumption that they all have the market power, each agent will submit the same query. *"How much risk should she share on the trade, given that she knows the rest of the agents needs?"* More precisely, the equilibrium in which an agent submits a fraction of her true endowment, the rest of the agents will do the same. To respond exactly, the Nash equilibrium is given by:

$$g_i := \arg\max_{\mathbf{B} \in (L^2)^N} (U_i(\mathscr{E}_i) + C_i^o(\mathbf{B}) - \pi_i^o(\mathbf{B}); \mathbf{B}_{-i})$$

where $\mathbf{B}$ is the submitted endowment that agent-i declares, given that she knows what the rest of the agents have submitted $\mathbf{B}_{-i} = \sum_{j \neq i} \mathbf{B}_j$.

In this point, we introduce the Nash contract payoff and the price without proving them (you can see in Athropelos [(2017)]), since in the next sections our focus in the extended model refers to them. Specifically, the Nash contract payoff, for the case of two agents is:

$$C_2 = \frac{\lambda_1 \cdot \mathscr{E}_2}{2} - \frac{\lambda_2 \cdot \mathscr{E}_1}{2} = \frac{C_1^o}{2}$$

The price of the contract (compensation) in this case is given by:

$$\pi_i = E(C_i) - 2\gamma Cov(C_i, \mathbf{B}^\diamond)$$

where C_i is the Nash contract and $\mathbf{B}^\diamond = \sum_{i=1}^n \mathbf{B}_i$ the aggregate submitted endowment (it differs from $\mathscr{E}$).

A factor of major significance is the market risk sharing inefficiency, which is adjusted properly here, based on the logic of the equation (2.11). We present this, for the forthcoming challenged results of the next sections, and it is given by:

$$u_i^o(\pi_i) := U_i\left(\mathscr{E}_i + C_i - \pi_i\right) = \gamma_i \cdot \frac{\gamma_i + \gamma}{\gamma_i - \gamma} \cdot Var(C_i) + U_i(\mathscr{E}_i)$$

where C_i is set to be the Nash contract and π_i its price in order to obtain it.

Given that, the appropriate outcomes of the static model are thoroughly stated in the basic paper, this study will provide only comparisons between the static model and the extended one, that it is introduced in the next section. We aim to build a new finite horizon model (discrete time dynamic model), where the trades will be completed in infinitely many steps in the first phase. By doing so, we will take some crucial conclusions for the contract payoff and the compensation that an agent pays to her counterparty in order to obtain it.

3 A discrete time dynamic model of Re-trading in Thin Markets

The expansion of our model is based on the idea of re-trading in thin markets. According to the existent literature of re-trading, we notice some crucial similarities with the model of *Ghosal and Morelli* [(2004)]. They present a non-cooperative model of imperfect competitive markets, with finite number of agents who can affect prices, which is close to the thin markets setting and our model. Albeit, the concept of re-trading is based in the Edgeworth's idea of re-contracting, in the *Ghosal and Morelli* model this idea is presented in the domain of imperfect competition. The problem, which is set in a very challenging way, is about to respond in the query, whether an allocation can approximate the Pareto frontier. The answer is determinate and is confirmed also by our model, that agents can attain Pareto set, in such markets, in infinitely many steps, but the more challenging that we wonder is how? The complete reply is when agents behave myopic and re-trade in infinitely many steps, they can obtain the Pareto optimal allocation either this allocation is in commodities and goods or in assets/financial contracts.

In *Ghosal and Morelli* [(2004)] the results of their model, under the assumptions that they set, are expanded when traders behave non-myopically or they act with the so-called far-sighted behavior. In this case, they expect that future rounds of trade will take place between them and they act diversely. In general analysis, the equivalent concept is that agents must re-trade since the obtained allocation is not the optimal from the previous transactions. We recall that in a thin market, the price impact affects negatively the whole market, even those who hold the market power. Thus, we extent the notion of re-trading in order to avoid the price impact and succeed in pricing financial products, which is the holy grail of this search. Before we investigate the two cases of re-trading we introduce the definitions of myopic and far-sighted behavior.

a. A trader is set to behave myopically, when she plays her best response as the current round of trade is the last or equivalently she does not anticipate that future trades will occur [16]. This definition is broaden in this study, due to the fact that the game is played in the prices of tradeable financial securities (contracts) rather than the quantities of commodities [17] in the *Ghosal and Morelli's paper*. Agents, submit a strategic schedule of the reported risk, which they want to declare for the transaction between the contracts and cash that they want to achieve.

b. The agent, who behaves far-sighted or non-myopic, is assumed to be someone, who expects that there will be future rounds of trades. Provided that, each player behaves strategically, they anticipate that new trades will evolve between them in the future. Equivalently, they take into account not only the current transaction, but also the next-future trading round. This is the exact definition to set the non-myopic behavior by Ghosal and Morelli [(2004)] and we adopt it here.

The myopic process is marked as a naive process, because agents, truly, will behave in a forward looking way. Despite this fact, with this naive process we postulate,

[16]In the *Ghosal and Morelli* [(2004)], the definition of agents who behave myopically is the same.

[17]The non-cooperative game of the *re-trading in market games by Ghosal and Morelli* uses the Shapley-Shubik game, where the agents are allowed to make the price affection by sending quantitative signals. Contrarily, in the thin market game of our model, the price impact arises endogenously from the market power and we use the order break-up strategy (it is incorporated in the equilibrium strategy) to avoid it. We quote, that myopic re-trading is equivalent to the order break-up strategy, since the transactions are completed in many steps to avoid the price impact.

that even though a thin market is a market of imperfect competition and suffers of specific problems of inefficiency due to the price impact, agents transaction can reach theoretically the Pareto frontier if they behave myopic under specific cases.

We could claim, that far-sighted behavior can result in the Pareto optimality, in finite steps of re-trading, according to *Ghosal and Morelli*, which constitutes their basic argument, although this is not happening in reality. In particular, the agents have incentives not to conclude in the optimal situation, despite the fact that there is a way to access it. The agents varying towards their behavior, from hedgers and arbitrageurs to predators in consistency with *Brunnermeier and Pedersen* and speculators. The predator case is analyzed in preceding sections, while the latter we recall that is refereed to agents who have low risk averse coefficient γ_i and submit risk only to their counterparty's endowment. In the view of far-sighted behavior, it does not yield that both agents will result in greater gains, due to these incentives.

3.1 The extended model & the myopic re-trading behavior

The main purpose of this study is the pricing of securities in thin markets. Aiming to succeed in this purpose, we set specific mean-variance preferences in order to model the agents in the market by recalling (2.1). By doing so, we respond to the crucial queries of the pricing, which are clustered around the deviation of the Preto optimal risk sharing allocation. Due to the imperfect market competition and the strategic behavior of the agents, it is perfectly reasonable for such a departure to exist. Thus, the price of the contract which is co-designed by the agents may result to a different price than the optimal contract. This is how we conclude to a Nash (strategic) equilibrium for the price of it.

It is worth to recap the meaning of strategic actions. In first place, we refer to the necessity that some agent has in order to offset the risk, on which her endowment is exposed, given her investment position. More precisely, all investors do not have the same incentives to make transactions with some other. Some of them, may have hedging needs, but some others may want to take advantage of their counterparty hedging needs. Also, some agent may act in both strategic applications, to exploit the needs of her counterparty and simultaneously hedge her investment position. Thus, all these incentives, i.e. the strategic approach of the one party of the agreement may lead the price of the contract far enough of its optimal equilibrium price. With this discrete time dynamic model for re-trading in thin markets, we estimate this departure of the contract price.

We focus to the examination of bilateral re-trading as it is usually the case. Unfortunately, the problems that occur in transactions in thin markets are a little more complicated. The major problem is related with the price impact, which is moderated via the common slow trading strategy, the so-called order break-up equilibrium strategy. It is also claimed in *Rostek and Weretka [(2008)]*, *Mitchell, Pedersen and Pulvino [(2007)],* and in our introduction for thin markets. However, in our model where we assume, that agents re-trade in infinitely many steps, the Nash contract breaks-up in individual contracts for every round of transactions and thus we can estimate the final price of the sum of contracts at the end of the procedure. Thus, we claim that the re-trading procedure incorporates also this moderation of price impact in thin markets.

On the merits of our analysis, the examination of the risk aversion coefficient influence in the repetitive process of trades is of major importance. It is divided in two cases, in the first case agents are called homogeneous (i.e. agents with similar risk aversion), while in the latter they are called heterogeneous (i.e. agents with dissimilar risk aversion). Agents, not only break-up their transactions in infinitely many rounds, but also, they behave myopic. As we have already quoted how myopia is defined, we shall show, in the sequel, that it leads in different results in relation to the agents' risk aversion coefficient. Due to the fact that, the issue of the deviation from the Pareto optimal allocation is at the very center of our study, the answer varies when agents are homogeneous and behave myopically by contrast to the heterogeneous and myopic agents. In the first case we illustrate, under specific assumptions that we set in subsection 1.1, that agents attain the Pareto optimality while in the second one the agents with the lower risk aversion coefficient (i.e. the higher risk tolerance) will dominate the bilateral re-trading procedure. Independently of the incentives, it is obvious that strategic equilibria could lead some agents in higher utility gains, than the optimal risk sharing allocation.

We recall that the games between the agents are symmetric. Also, analyzing the Nash equilibrium in which an agent submits a fraction of her true endowment (or exposure too risk), her counterparty will act in the same way.

3.2 Setting the game conditions and the strategy

We set the conditions of the Nash game in the complete market structure. We set j to be the trading rounds that are carried out in the moments $\{t_j, j \in \{1, 2, ..., N\}\}$, where $\left\{t_1 = 1/N, t_2 = 2/N, ..., t_j = \frac{j}{N}, ..., t_{N-1} = \frac{(N-1)T}{N}, t_N = T\right\}$, N is the partition of the distance period between the rounds, which is zero when $N \to \infty$. Let $\mathscr{E}_{i,j}$ to be the $i-th$ agent's endowment after the $j-th$ round of trades. Moreover, we assume that:

i. The interest rate r is negligible during the time of re-trading, thus we set $r = 0$, in order not to affect the transactions.

ii. No reveals of new information occur between each round. This means that, the

$\sigma - algebra$ of the standard probability space, which embodies the information, remains unchanged.

iii. There are no transaction costs, such as brokerage fees, commissions or order costs.

In essence, agents re-trade in infinitely many steps by adjusting their whole strategic plan from the first ($j = 1$) to the last round ($j = N \to \infty$).Namely, her strategic approach, which is set from the beginning in a strategic schedule, it responds optimally in the "trivial process", where she submits some proportion of her counterparty's endowment in the transaction. Intuitively, since there is no information asymmetry (symmetric games), she has a picture of her counterparty's needs and she responses optimally in each round, but this holds for her counterparty.

In deterministic time, during transactions it is logic for new information to reveal and thus, every agent will change her strategy given the condition, which these information incorporate. We highlight, that we follow this naive process aiming to simply set the pricing in thin markets and investigate the circumstance of the improvement of agents' investment positions, under the aforementioned assumptions. In such way, the pricing is only a matter of the risk aversions (or risk tolerance) individually. It is shown that the agents, in thin markets where the one shot transactions are not optimal, can be very close to Pareto optimality, under the homogeneity condition by re-trading in infinitely many steps. Subsequently, myopic re-trading will give some challenging results in comparison with the competitive (optimal) equilibria.

3.3 Two agents re-trading - Economic perspective

Following the notation, the contract payoff of the agent for the trading round j is defined as $C_{i,j}$ and the compensation that she pays for it as $\pi_{i,j}$ respectively. Also, the aggregate endowment is defined as $\mathbf{B}_{j-1}$, whereas in the trading round $j = N$ the agents re-trade with the declared aggregate endowment $\mathbf{B}_{N-1}$, which is obtained in the previous round $N - 1$. This is consistent with the *Ghosal and Morelli* [(2004)]. Due to the focus on the bilateral trading we will make some technical claims to ease the explanation of the game. Specifically, we can easily observe that $C_{-i,j} = -C_{i,j}$ and for this reason hereafter we donate the contract as C_j. Similarly, holds the same for the cash compensation $\pi_{i,j}$, namely $\pi_{-i,j} = -\pi_{i,j}$ and hence we depict it as π_j. We mention again that, there is a dependence of j on N.

In the sequel, we present the outcomes of the trading rounds and interpret their economic perspective. Table 1 presents the important points on which we focus to explain theoretically, when the partition N of trading rounds tends to infinity ($N \to \infty$). Furthermore, in the first round, the Nash contract is the half of the Pareto optimal one, but as the trading rounds carry on, the sum of Nash contracts converges to the Pareto optimal, no matter the risk aversion of the agents.

In each round, the agents trade only a proportion of their new endowment which, is shaped in relation to their previous transactions. For instance, assuming that agent-1, initially, owns the random endowment $\mathscr{E}_{1,0}$, after the first trading round her new endowment will be $\mathscr{E}_{1,1} := \mathscr{E}_{1,0} + C_1$. In short, for agent-2 will be accounted that $\mathscr{E}_{2,1} := \mathscr{E}_{2,0} - C_1$. Thus, endowments $\mathscr{E}_{1,1}$ and $\mathscr{E}_{2,1}$ are under set for the second round and so on. On the subject of the compensations, we calculate them aggregately in the end of the trading rounds, as long as this does not affects our results. Below we have the following:

Proposition 5. *The sum of Nash contacts $\sum_{j=1}^{N} C_j$, when $N \to \infty$, converges to the Pareto optimal contract C_1^o.*

Proof. When agents enter in the $j-th$ trading round, the declaring aggregate endowment is $\mathbf{B}_{j-1}$, thus for the contract in each round we have the following:

1. in the round $j = 1$: $C_1 = \dfrac{\gamma_2 \mathscr{E}_{2,0} - \gamma_1 \mathscr{E}_{1,0}}{2(\gamma_1 + \gamma_2)} = \dfrac{C_1^o}{2}$, as it is seen the Pareto optimal contract is combined with the Nash contract.
2. in the round $j = 2$: $C_2 = \dfrac{\gamma_2 \mathscr{E}_{2,0} - \gamma_1 \mathscr{E}_{1,0}}{2(\gamma_1 + \gamma_2)} - \dfrac{\gamma_1 + \gamma_2}{2(\gamma_1 + \gamma_2)} C_1 = C_1 - \dfrac{C_1}{2} = \dfrac{C_1}{2}$
3. in the round $j = 3$: $C_3 = \dfrac{\gamma_2 \mathscr{E}_{2,0} - \gamma_1 \mathscr{E}_{1,0}}{2(\gamma_1 + \gamma_2)} - \dfrac{\gamma_1 + \gamma_2}{2(\gamma_1 + \gamma_2)} C_1 - \dfrac{\gamma_1 + \gamma_2}{2(\gamma_1 + \gamma_2)} C_2 = C_1 - \dfrac{C_1}{2} - \dfrac{C_2}{2} = C_1 - \dfrac{C_1}{2} - \dfrac{C_1}{4} = \dfrac{C_1}{4}$

Thus, for round $j = N$, we imply that $C_N = \dfrac{C_1}{2^{N-1}}$ and we will show that the aggregate contract converges to Pareto optimal one as the partition N tends to infinity. We have that:

$$\sum_{j=1}^{N} C_j = C_1 + C_2 + C_3 + ... + C_N = C_1 + \frac{C_1}{2} + \frac{C_1}{4} + ... + \frac{C_1}{2^{N-1}} = C_1 \sum_{j=0}^{N-1} \frac{1}{2^j} \Rightarrow$$

$$\Rightarrow \sum_{j=1}^{N} C_j = C_1 \left(2 - \frac{1}{2^{N-1}}\right) = C_1^o \left(1 - \frac{1}{2^N}\right)$$

Hence, when $N \to \infty$ we presume the challenging result:

$$\lim_{N\to\infty} \sum_{j=1}^{N} C_j = C_1^o \lim_{N\to\infty} \left(1 - \frac{1}{2^N}\right) = C_1^o$$

□

The meaning of the above result, is that agents can reach the Pareto optimality through infinitely many steps of trades. It is clear, that the Nash contract convergence to the Pareto one, is independent of the agents' risk aversion.

In this part, we emphasize the pricing of the compensations, which is of major importance. We illustrate that the aggregate pie of compensations $\sum_{j=1}^{N} \pi_j$, that an agent pays to her counterparty, grows in relation to the static model as the frequency of trading rounds increases unlimited. We remind that the Nash compensation is given for each round by $\pi_{i,j} = E(C_{i,j}) - 2\gamma Cov(C_{i,j}, \mathbf{B}_{i,j-1})$, where i stands for the agent and j for the trading rounds. Because we refer to bilateral re-trading procedure, it is implied that $\pi_j = E(C_j) - 2\gamma Cov(C_j, \mathbf{B}_{j-1})$. In contrast to the contract payoff ,the aggregate compensations depend on the agents risk aversions. Thus, we examine the two cases of compensations pricing, when the agents are homogeneous and heterogeneous. Specifically, in this subsection we mainly argue the event of agents, that are homogeneous. In order to succeed this, we will make a comparison between the compensations at the limit, $\lim_{N\to\infty} \sum_{j=1}^{N} \pi_j (= \pi_{lim})$, and the optimal one π_1^o. Also, we denote hereafter the variance of the optimal contract to be $\Sigma_1^o = Var(C_1^o)$. At last, we make the following proposition, when the agent are homogeneous.

Proposition 6. *The aggregate compensation on the limit of the re-trading procedure is:* $\pi_{lim} = \pi_1^o + \dfrac{2(\gamma_1 - \gamma_2)}{3} \cdot \Sigma_1^o$.

Proof. The aggregate submitted endowment in each round is formed as shown below

1. for the round $j = 1$ the agg. submitted endowment is: $\mathbf{B}_0 = \dfrac{\gamma_1 \mathscr{E}_{1,0} + \gamma_2 \mathscr{E}_{2,0}}{2\gamma}$

2. similarly for the round $j = 2$: $\mathbf{B}_1 = \dfrac{\gamma_1 \mathscr{E}_{1,1} + \gamma_2 \mathscr{E}_{2,1}}{2\gamma} = \dfrac{\gamma_1 \mathscr{E}_{1,0} + \gamma_2 \mathscr{E}_{2,0}}{2\gamma} + \dfrac{\gamma_1 - \gamma_2}{2\gamma} C_1 = \mathbf{B}_0 + uC_1$, where $u = \dfrac{\gamma_1 - \gamma_2}{2\gamma}$

3. for the round $j = 3$: $\mathbf{B}_2 = \dfrac{\gamma_1 \mathscr{E}_{1,2} + \gamma_2 \mathscr{E}_{2,2}}{2\gamma} = \dfrac{\gamma_1 \mathscr{E}_{1,0} + \gamma_2 \mathscr{E}_{2,0}}{2\gamma} + \dfrac{\gamma_1 - \gamma_2}{2\gamma} C_1 + \dfrac{\gamma_1 - \gamma_2}{2\gamma} C_2 = \mathbf{B}_0 + u(C_1 + C_2)$

So, in the round $j = N$ we end up with $\mathbf{B}_{N-1} = \mathbf{B}_0 + u(C_1 + C_2 + ... + C_{N-1}) = \mathbf{B}_0 + u \sum_{j=1}^{N-1} C_j = \mathbf{B}_0 + u[2 - \dfrac{1}{2^{N-2}}]C_1 = \mathbf{B}_0 + u[1 - \dfrac{1}{2^{N-1}}]C_1^o$, where $u = \dfrac{\gamma_1 - \gamma_2}{2\gamma}$.

We examine where the agent's aggregate compensation results from the re-trading procedure. Substantially, it is accounted by:

$$\sum_{j=1}^{N} \pi_j = \sum_{j=1}^{N} E(C_j) - 2\gamma \sum_{j=1}^{N} Cov(C_j, \mathbf{B}_{j-1}) \tag{3.1}$$

Obviously, the equation is divided in two parts

1. $\sum_{j=1}^{N} E(C_j)$

2. $\sum_{j=1}^{N} Cov(C_j, \mathbf{B}_{j-1})$

Firstly, we calculate the second term 2. as it is quoted below. The proof is separated in two parts, initially we calculate each factor of the sum individually and then we calculate the sum. In order to make easier the plot of the mathematical calculations, hereafter we depict the variance of the Nash contract $Var(C_1)$ as Σ_1.

We have the next:

- $Cov(C_2, \mathbf{B}_1) = \frac{1}{2}Cov(C_1, \mathbf{B}_0) + \frac{u}{2}\Sigma_1$
- $Cov(C_3, \mathbf{B}_2) = \frac{1}{4}Cov(C_1\mathbf{B}_0) + \frac{3 \cdot u}{8}\Sigma_1$
- $Cov(C_N, \mathbf{B}_{N-1}) = \frac{1}{2^{N-1}}Cov(C_1, \mathbf{B}_0) + \frac{2^j - 1}{2^{2j-1}}u\Sigma_1$

From the aforementioned equations we can take the next sums:

- $\left(1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{N-1}}\right) Cov(C_1, \mathbf{B}_0) = \sum_{j=0}^{N-1} \frac{1}{2^{N-1}}Cov(C_1, \mathbf{B}_0)$
- $\left(\frac{1}{2} + \frac{3}{8} + ... + \frac{2^j - 1}{2^{2j-1}}\right) u\Sigma_1 = u\Sigma_1 \sum_{j=1}^{N-1} \frac{2^j - 1}{2^{2j-1}}$

We readily can show that

$$\sum_{j=1}^{N-1} \frac{2^j - 1}{2^{2j-1}} = \sum_{j=1}^{N-1} \frac{1}{2^{j-1}} - \sum_{j=1}^{N-1} \frac{1}{2^{2j-1}} = \left(2 - \frac{1}{2^{N-2}}\right) - \frac{2}{3}\left(1 - \frac{1}{4^{N-1}}\right) \Rightarrow$$

$$\Rightarrow \sum_{j=1}^{N-1} \frac{2^j - 1}{2^{2j-1}} = \frac{4}{3} - \frac{2}{2^{N-1}} + \frac{2}{3}\frac{1}{4^{N-1}}$$

Therefore the 2. sum is accounted to be:

$$\begin{aligned}\sum_{j=1}^{N} Cov(C_j, \mathbf{B}_{j-1}) &= Cov(C_1, \mathbf{B}_0) + Cov(C_2, \mathbf{B}_1) + Cov(C_3, \mathbf{B}_2) + ... + Cov(C_N, \mathbf{B}_{N-1}) \\ &= \left(1 - \frac{1}{2^N}\right) Cov(C_1^o, \mathbf{B}_0) + \frac{\gamma_1 - \gamma_2}{2\gamma}(\frac{4}{3} - \frac{2}{2^{N-1}} + \frac{2}{3}\frac{1}{4^{N-1}})\frac{\Sigma_1^o}{4} \\ &= a_N \cdot Cov(C_1^o, \mathbf{B}_0) + b_N \cdot \frac{\gamma_1 - \gamma_2}{2\gamma}\frac{\Sigma_1^o}{4}\end{aligned}$$

where $a_N, N \in 1,2,3...$ and $b_N, N \in 1,2,3...$ are sequences that we denote below and $\Sigma_1 = \frac{\Sigma_1^o}{4}$. Below we cite the sequences that all of them converge to a real number.

- $a_N = 1 - \frac{1}{2^N}$ which converges to 1 while $N \to \infty$
- $b_N = \frac{4}{3} - \frac{2}{2^{N-1}} + \frac{2}{3} \cdot \frac{1}{4^{N-1}}$ which converges to $\frac{4}{3}$ while $N \to \infty$

In respect of the first term 1. we have that:

$$\sum_{j=1}^{N} E(C_j) = E\left(\sum_{j=1}^{N} C_j\right) = E\left(\frac{C_1^o}{2}\left(2 - \frac{1}{2^{N-1}}\right)\right) = \left(1 - \frac{1}{2^N}\right) \cdot E(C_1^o) = a_N \cdot E(C_1^o)$$

Thus, (3.1) transforms to the next equation:

$$\begin{aligned}\sum_{j=1}^{N} \pi_j &= a_N \cdot E(C_1^o) - 2\gamma \cdot \left(a_N \cdot Cov(C_1^o, \mathbf{B}_0) + u \cdot \frac{\Sigma_1^o}{4} \cdot \textstyle\sum_{j=1}^{N-1} \frac{2^j - 1}{2^{2j-1}}\right) = \\ &= a_N \cdot E(C_1^o) - 2\gamma \cdot a_N \cdot Cov(C_1^o, \mathbf{B}_0) - \frac{\gamma_1 - \gamma_2}{4} \cdot \Sigma_1^o \cdot b_N\end{aligned}$$

Hereafter, we set the compensations at the limit as $\pi_{lim} = \lim_{N \to \infty} \sum_{j=1}^{N} \pi_j$ and express them in relation to the optimal compensations. As it is known from the (2.7) we have that:

$$\pi_1^o = E(C_1^o) - 2\gamma Cov(C_1^o, \mathscr{E}) \Rightarrow E(C_1^o) = \pi_1^o + 2\gamma Cov(C_1^o, \mathscr{E})$$

Combining the above equation with (3.2) we have that:

$$\begin{aligned}\sum_{j=1}^{N} \pi_j &= a_N \cdot (\pi_1^o + 2\gamma Cov(C_1^o, \mathscr{E})) - 2\gamma \cdot a_N \cdot Cov(C_1^o, \mathbf{B}_0) - \frac{\gamma_1 - \gamma_2}{4} \cdot \Sigma_1^o \cdot b_N = \\ &= a_N \cdot \pi_1^o + 2 \cdot a_N \cdot Cov[C_1^o, \gamma \cdot (\mathscr{E} - \mathbf{B}_0)] - \frac{\gamma_1 - \gamma_2}{4} \cdot \Sigma_1^o \cdot b_N = \\ &= a_N \cdot \pi_1^o + a_N \cdot (\gamma_1 - \gamma_2) \cdot \Sigma_1^o - \frac{\gamma_1 - \gamma_2}{4} \cdot \Sigma_1^o \cdot b_N = \\ &= \left(1 - \frac{1}{2^N}\right) \pi_1^o + \left(1 - \frac{1}{4^N}\right) \frac{2\,(\gamma_1 - \gamma_2)}{3} \Sigma_1^o \end{aligned} \tag{3.2}$$

Also we have that:

$$\begin{aligned}\gamma \cdot (\mathscr{E} - \mathbf{B}_0) &= \gamma \mathscr{E}_{1,0} + \gamma \mathscr{E}_{2,0} - \frac{\gamma_1 \mathscr{E}_{1,0} + \gamma_2 \mathscr{E}_{2,0}}{2} = (\gamma - \frac{\gamma_1}{2}) \cdot \mathscr{E}_{1,0} + (\gamma - \frac{\gamma_2}{2}) \cdot \mathscr{E}_{2,0} \\ &= \frac{-\gamma_1(\gamma_1 - \gamma_2)}{2(\gamma_1 + \gamma_2)} \cdot \mathscr{E}_{1,0} + \frac{\gamma_2(\gamma_1 - \gamma_2)}{2(\gamma_1 + \gamma_2)} \cdot \mathscr{E}_{2,0} = \frac{\gamma_1 - \gamma_2}{2} \cdot \frac{\gamma_2 \mathscr{E}_{2,0} - \gamma_1 \mathscr{E}_{1,0}}{(\gamma_1 + \gamma_2)} \\ &= \frac{(\gamma_1 - \gamma_2) \cdot C_1^o}{2}\end{aligned}$$

Hence the limiting compensation is:

$$\begin{aligned}\pi_{lim} &= \lim_{N \to \infty} \textstyle\sum_{j=1}^{N} \pi_j = \lim_{N \to \infty} \left(\left(1 - \frac{1}{2^N}\right) \pi_1^o + \left(1 - \frac{1}{4^N}\right) \frac{2\,(\gamma_1 - \gamma_2)}{3} \cdot \Sigma_1^o\right) = \\ &= \pi_1^o + \frac{2(\gamma_1 - \gamma_2)}{3} \cdot \Sigma_1^o\end{aligned} \tag{3.3}$$

□

(⇒) Obviously, the pie of utility surplus increases as the rounds increase unlimited, compared with the static model. Based on the equation (3.3), it shows that the aggregate price π_{lim} of transactions are equivalent to the optimal transactions π_1^o plus the term $\frac{2(\gamma_1 - \gamma_2)}{3}\Sigma_1^o$. The latter term, which is expressed in relation to the risk aversion of each agent and the risk on the contract, that the agents undertake in order to begin the re-trading procedure, needs some thorough interpretation. To be more specific, if the agents are homogeneous, the term $\frac{2(\gamma_1 - \gamma_2)}{3}\Sigma_1^o = 0$ and the submitted aggregate endowment of the agents $\mathbf{B}_0$ is equal with their true agg. endowment $\mathscr{E} = \mathscr{E}_{1,0} + \mathscr{E}_{2,0}$. In this way, the aggregate compensation after the infinitely many steps of transactions, is equal to the optimal compensation, namely $\sum_{j=1}^{N} \pi_j = \pi_1^o$, as well the sum of Nash contracts is $\sum_{j=1}^{N} C_j = C_i^o$. This gives an explicit message, when $\gamma_1 = \gamma_2$. When the trades are unlimited, the condition to succeed the Pareto optimal risk sharing allocation (both agents improve their position by receiving the optimal allocation individually) is exactly the same to succeed infinite trades to the already mentioned prices. After all, their contract and compensations pricing converge to the Pareto optimal while $N \to \infty$.

(⇐)Vice versa, from the equation (3.3) the $\pi_{lim} = \pi_1^o$ if the term $\frac{2(\gamma_1 - \gamma_2)}{3}\Sigma_1^o = 0$ and thus $\gamma_1 = \gamma_2$ or $\Sigma_1^o = 0$. Regarding to the contract, if it is constant it is implied that $\Sigma_1^o = 0$ or equivalently the agents are already in a Pareto optimal transaction ($\gamma_2 \mathscr{E}_{2,0} = \gamma_1 \mathscr{E}_{1,0}$).

Corollary 2. *Hence, taking into account Proposition 5 and 6, the re-trading procedure results in Pareto optimal, through infinitely many steps of trades. Both the contract and the prices coincide with the optimal, if $\gamma_1 = \gamma_2$ or the agents are already in Pareto optimal position ($\Sigma_1^o = 0$).*

Remark 5. *It is shown, that the agents will never attain the Pareto optimal risk sharing allocation, in finite steps with myopic re-trading. Through the myopic re-trading process, the agents achieve an allocation, where the assets or financial tradeable securities, that are shared in each round, are inherited by the previous rounds of trades. Recall that, these financial tradeable securities are contracts, which can be any financial derivative. In addition to this, every trading round consists a static Nash Equilibrium, in relation to the inherited risk sharing allocation of the former rounds. Hence, the agents re-trading procedure approximates the competitive equilibrium price in infinitely many steps, if they are homogeneous and behave myopically.*

The conclusion is more challenging when agents are heterogeneous. Let us assume that $\gamma_1 < \gamma_2$. Then (3.3) indicates that, although agent-2 (after the infinite trading rounds) gains more cash from the compensations than the one shot transaction case, her counterparty gets a discount (the lower risk averse agent-1) to obtain the contract, from the effect of the term $\frac{2(\gamma_1-\gamma_2)}{3}\Sigma_1^o$. The clear message is that the lower risk averse agent gains more profits, not only from the sum of Nash contracts payoff convergence to the Pareto optimal one, but also from her higher risk tolerance, which yields an extra gain by declining the cash that she will pay for the contract in the limit. In re-trading, the compensation gives a discount on the side of the agent with lower risk aversion.

Below, we define a measure to examine the utility that agents get from the re-trading procedure. We set this measure as the Nash utility surplus, which measures the extra utility of the agent at the end of each trading round. It is set as the expected utility of the difference of her new endowment $\mathscr{E}_{i,j}$ minus the aggregate compensation $\sum_{j=1}^{N}\pi_j$ that she pays for each new round. Hereafter, $U_{1,j}^{Nash}$ will stand for surplus which also incorporates the agent's wealth before the re-trading $U_{1,0}(\mathscr{E}_{1,0})$.[18] Hence, we set the utility of agent-1, in the $j-th$ Nash equilibrium to be:

$$U_{1,j}^{Nash} = U_{1,j}\left(\sum_{i=1}^{N}\mathscr{E}_{1,j} - \sum_{j=1}^{N}\pi_j\right) = U_{1,j}\left(\mathscr{E}_{1,0} + \sum_{j=1}^{N}C_j - \sum_{j=1}^{N}\pi_j\right) \tag{3.4}$$

Our purpose is to analyze the evolutionary course of the utility of agent's aggregate position. We use it as a measure for the market inefficiency, which is carried by both agents, no matter their risk tolerance, in combination with the (2.11). This inefficiency is anticipated due to the fact that the aggregate submitted endowment is $\mathbf{B}_0 \neq \mathscr{E}$. In the sequel, we indicate the total utility surplus of the agent at the end of each trading round.

Proposition 7. *The utility surplus of agent-1, at the trading round N is:*

$$U_{1,N}^{Nash} - U_{1,0}(\mathscr{E}_{1,0}) = \left(1-\frac{1}{4^N}\right)\cdot\frac{\gamma_1+2\gamma_2}{3}\cdot\Sigma_1^o \tag{3.5}$$

Proof. The limiting Nash utility surplus is given by the difference of the Nash surplus in round N minus the position that the agent already holds before the re-trading procedure $U_1(\mathscr{E}_{1,0})$ and hence we set $U_{sur} = U_{1,N}^{Nash} - U_{1,0}(\mathscr{E}_{1,0})$:

[18] Following the same notation with the contract and the compensations, the $U_{i,j}^{Nash}$ illustrates the surplus of the $i-th$ agent in the $j-th$ round of trade. Due to the symmetric games we examine the surplus of agent-1, $U_{1,j}^{Nash}$, adjusting the results properly for agent-2.

$$
\begin{aligned}
U_{sur} &= U_N\left(\mathscr{E}_{1,N} - \sum_{j=1}^{N}\pi_j\right) - U_1(\mathscr{E}_{1,0}) = U_N\left(\mathscr{E}_{1,0} + \sum_{j=1}^{N}C_j - \sum_{j=1}^{N}\pi_j\right) - U_{1,0}(\mathscr{E}_{1,0}) \\
&= E\left(\sum_{j=1}^{N}C_j\right) - \sum_{j=1}^{N}\pi_j - \gamma_1 Var\left(\sum_{j=1}^{N}C_j\right) - 2\gamma_1 Cov\left(\sum_{j=1}^{N}C_j, \mathscr{E}_{1,0}\right) \\
&= 2\gamma\sum_{j=1}^{N}Cov(C_j, \mathbf{B}_{j-1}) - \gamma_1 Var\left(\sum_{j=1}^{N}C_j\right) - 2\gamma_1 Cov\left(\sum_{j=1}^{N}C_j, \mathscr{E}_{1,0}\right) \\
&= 2\cdot a_N\cdot[Cov(C_1^o, (\gamma\cdot\mathbf{B}_0 - \gamma_1\cdot\mathscr{E}_{1,0})] + \left(b_N\cdot\frac{\gamma_1-\gamma_2}{4} - (a_N)^2\cdot\gamma_1\right)\cdot\Sigma_1^o \\
&= \left(1 - \frac{1}{4^N}\right)\cdot\frac{\gamma_1+2\gamma_2}{3}\cdot\Sigma_1^o
\end{aligned}
$$

Since we have that

$$
\begin{aligned}
E\left(\sum_{j=1}^{N}C_j\right) - \sum_{j=1}^{N}\pi_j &= E\left(\sum_{j=1}^{N}C_j\right) - \sum_{j=1}^{N}E(C_j) + 2\gamma\sum_{j=1}^{N}Cov(C_j, \mathbf{B}_{j-1}) \\
&= 2\gamma\sum_{j=1}^{N}Cov(C_j, \mathbf{B}_{j-1})
\end{aligned}
$$

and

$$
\gamma\cdot\mathbf{B}_0 - \gamma_1\cdot\mathscr{E}_{1,0} = \frac{\gamma_1+\gamma_2}{2}\cdot C_1^o
$$

finally we can prove the required. □

Corollary 3. *The Nash utility surplus of agent-2 is given by:*

$$
U_{2,N}^{Nash} - U_{2,0}(\mathscr{E}_{2,0}) = \left(1 - \frac{1}{4^N}\right)\cdot\frac{2\gamma_1+\gamma_2}{3}\Sigma_1^o
$$

(symmetric games), we mark that:

$$
U_{2,j}^{Nash} = U_{2,j}\left(\sum_{i=1}^{N}\mathscr{E}_{2,j} + \sum_{j=1}^{N}\pi_j\right) = U_{2,j}\left(\mathscr{E}_{2,0} - \sum_{j=1}^{N}C_j + \sum_{j=1}^{N}\pi_j\right)
$$

When $N\to\infty$, we have that the utility surplus of agent-1 is:

$$
\lim_{N\to\infty}\left[U_{1,N}^{Nash} - U_{1,0}(\mathscr{E}_{1,0})\right] = \frac{\gamma_1+2\gamma_2}{3}\Sigma_1^o
$$

and due to the fact that the outcomes are symmetric for her counterparty is:

$$\lim_{N\to\infty}\left[U_{2,N}^{Nash} - U_{2,0}(\mathscr{E}_{2,0})\right] = \frac{2\gamma_1+\gamma_2}{3}\Sigma_1^o$$

The case of the utility surplus is similar to the aforementioned conversation. In particular, the Nash utility surplus converges to the optimal utility in the limit, when agents are homogeneous. Namely, $U_{1,N}^{Nash} = \gamma_1$ *and* $U_{2,N}^{Nash} = \gamma_2$ *and thus we demonstrate that the homogeneity case with myopic re-trading approximates the competitive equilibrium. The whole search is verified once again, reinforcing the Remark 6*

So far, the re-trading explicitly leads two homogeneous agents to the Pareto optimal allocation via unlimited continuum of transactions. The question is: *"How do these results change if the agents are heterogeneous?"*. For sure, they are even more fancy and they illustrate that the lower risk averse trader dominates re-trading. To respond better in this question, we first introduce the outcomes of re-trading for the first $j = 3$ rounds and then, we determine the exact neighborhoods, in which the aversion of the the lower risk averse trader belongs. A thorough debate is presented in the last subsection.

3.4 Step by step of bilateral re-trading evolution

Considering the above, in this subsection we present the evolutionary course of re-treading step by step ($\forall$ round) in the Table 1, for the $j = 3$ trading rounds. Notably, in order to present the idea how the table was formed, we refer to the meaning of each row. In particular, we present in the table the next outcomes, the aggregate submitted endowment (A.S.E), the purchased contract of agent-1 (P.C. of agent-1), the compensation that is paid in each round by agent-1 (Compensation), the utility surplus of agent-1 (U.S. of agent-1) and the market inefficiency (M.I).

We recall, that $\pi_1^o = E(C_1^o) - 2\gamma Cov(C_1^o, \mathscr{E})$ is the Pareto optimal compensation that agent-1 pays to agent-2, when they do not behave strategically. Furthermore, the outcomes of the first trading round coincide with those of the static model. The price of the contract in each round is given by:

$$\pi_j = \frac{\pi_1^o}{2^j} + \frac{\gamma_1-\gamma_2}{2}\cdot\frac{\Sigma_1^o}{4^{j-1}} \tag{3.6}$$

where $j \in \{1, 2, ..., N\}$.

Remark 6. *The Nash utility surplus is accounted by the equation* (3.5) *for every round of trade. According to the risk sharing inefficiency measure, which is set by* (2.11)*, we adjust it properly for the different equilibria, the Nash equilibrium and the Pareto* *optimal one. Thus, we generalize this measure and the aggregate market inefficiency is given by:*

$$\begin{pmatrix} Aggregate\ Market \\ Inefficiency \end{pmatrix} = (U_1^o + U_2^o) - \left(U_{1,N}^{Nash} + U_{2,N}^{Nash}\right) \Rightarrow$$
$$\Rightarrow \begin{pmatrix} Aggregate\ Market \\ Inefficiency \end{pmatrix} = (\gamma_1 + \gamma_2) \cdot \Sigma_1^o - \left(1 - \frac{1}{4^N}\right) \cdot (\gamma_1 + \gamma_2) \cdot \Sigma_1^o \Rightarrow$$
$$\Rightarrow \begin{pmatrix} Aggregate\ Market \\ Inefficiency \end{pmatrix} = \frac{\gamma_1 + \gamma_2}{4^N} \cdot \Sigma_1^o$$

Ultimately, we quote the Table 1 outcomes:

Table 1: Bilateral re-trading evolution in each round

aggregate submitted endowment (A.S.E), the purchased contract of agent-1 (P.C. of agent-1), the compensation that is paid in each round by agent-1 (Compensation), the utility surplus of agent-1 (U.S. of agent-1) and the market inefficiency (M.I), $u = \frac{\gamma_1 - \gamma_2}{2\gamma}$

	trade round 1	trade round 2	trade round 3
A.S.E.	$\mathbf{B}_0$	$\mathbf{B}_1 = \mathbf{B}_0 + \frac{u \cdot C_1^o}{2}$	$\mathbf{B}_2 = \mathbf{B}_0 + \frac{3u \cdot C_1^o}{4}$
P.C. of agent-1	$C_1 = \frac{C_1^o}{2}$	$C_2 = \frac{C_1^o}{4}$	$C_3 = \frac{C_1^o}{8}$
Compensation	$\frac{\pi_1^o}{2} + \frac{\gamma_1 - \gamma_2}{2}\Sigma_1^o$	$\frac{\pi_1^o}{4} + \frac{\gamma_1 - \gamma_2}{2}\frac{\Sigma_1^o}{4}$	$\frac{\pi_1^o}{8} + \frac{\gamma_1 - \gamma_2}{2}\frac{\Sigma_1^o}{16}$
U.S. of agent-1	$\frac{\gamma_1 + 2\gamma_2}{4}\Sigma_1^o$	$\frac{5\gamma_1 + 10\gamma_2}{16}\Sigma_1^o$	$\frac{21\gamma_1 + 42\gamma_2}{64}\Sigma_1^o$
M.I.	$\frac{\gamma_1 + \gamma_2}{4}\Sigma_1^o$	$\frac{\gamma_1 + \gamma_2}{16}\Sigma_1^o$	$\frac{\gamma_1 + \gamma_2}{64}\Sigma_1^o$

In the last part of our discussion, knowing all the steps of the Nash re-trading, our attention is focused in the much talked case of heterogeneous agents. Since the agents are $n = 2$, we will show the final inductions in steps, like in each part of our conversation. From this point onward, we expand our results arguing about the event that agents are heterogeneous. Initially, we will determine the “efficient neighborhoods” in which the lower risk averse agent dominates the bilateral trading. Below, we unfold the game plot from another aspect, making the main assumption that $\gamma_1 < \gamma_2$.

We indicate for the first $j = 3$ trading rounds the “efficient neighborhoods” . In the sequel, we endorse this event as re-trading continues indefinitely till the last Nash equilibrium.

1. In the first trading round the Nash contract payoff is the half of the optimal one, $C_1 = \frac{C_1^o}{2}$, and the compensation that she pays to her counterparty is given by the equation (3.6) (see in Table 1). Consequently, by comparing the compensation with the Pareto optimal one, it gives the next result:

$$\pi_1 - \frac{\pi_1^o}{2} = \frac{\gamma_1 - \gamma_2}{2} \cdot \Sigma_1^o \Rightarrow \pi_1 < \frac{\pi_1^o}{2}$$

   This means that in the beginning agent-1 pays a higher compensation than the half of the Pareto optimal one. In order to be accurate about the efficient neighborhood, which in fact is an open set of the subset $(0, +\infty) \subset R$, we determine them. Hence, the lower risk averse agent gains more in the transaction than her counterparty when particularly $\gamma_1 < \frac{2\gamma_2}{3}$ or

$$\gamma_1 \in A_1 = \left(0, \frac{2\gamma_2}{3}\right).$$

2. Similarly, in the second trading round the Nash contract payoff is also a fraction of the Pareto optimal one, namely $C_2 = \frac{C_1^o}{4}$, and the compensation that she pays to her counterparty is given by the equation (3.6) (see in Table 1). Consequently, by comparing the compensation with the Pareto optimal one, it gives the next result:

$$\pi_2 - \frac{\pi_1^o}{4} = \frac{\gamma_1 - \gamma_2}{8} \cdot \Sigma_1^o \Rightarrow \pi_1 < \frac{\pi_1^o}{4}$$

   This means that in the second round agent-1 pays a lower compensation than the quarter of the Pareto optimal one. More precisely, she acquires another part of the Pareto optimal contract, $\frac{C_1^o}{4}$, by discount, although the compensations pie increases as the transactions continue indefinitely. Thus, for agent-1 the compensation, that she pays, diminishes after making trades for another round if $\gamma_1 < \frac{10\gamma_2}{11}$. In brief, this indeed shows that the scales lean towards the lower risk averse agent. Hence, the efficient open set is:

$$\gamma_1 \in A_2 = \left(0, \frac{10\gamma_2}{11}\right).$$

3. In the third trading round the Nash contract payoff is $C_3 = \frac{C_1^o}{8}$, and the compensation that she pays to her counterparty is given by the equation (3.6) (see again in Table 1). Consequently, by comparing the compensation with the Pareto optimal

one, it gives the next result:

$$\pi_3 - \frac{\pi_1^o}{8} = \frac{\gamma_1 - \gamma_2}{32} \cdot \Sigma_1^o \Rightarrow \pi_1 < \frac{\pi_1^o}{8}$$

Similarly the development of the game gives in this round the same conclusions with the previous one. Thus, agent-1 obtains another part of the Pareto contract by paying, respectively, less than the part of the optimal compensation, if $\gamma_1 < \frac{42\gamma_2}{43}$ or

$$\gamma_1 \in A_3 = \left(0, \frac{42\gamma_2}{43}\right).$$

Corollary 4. *Therefore, to sum up the evolution course of the iterative process of trades, afterward the trading round $j = 2$ the lower risk averse trader gains even if she is slightly lower risk averse than her (higher risk averse) counterparty. Although, a fairly large neighborhood, in which agent-1 looses utility in relation to her counterparty, exists in the first trading round ($A_1^c = (\frac{2\gamma_2}{3}, \gamma_2)$), it diminishes in the second one and so on. It is remarkable that the open set grows from the right hand of γ_1 tending close to γ_2 but they can never be equal. Intuitively, the "efficient neighborhood" of agent's-1 risk aversion coefficient belongs to the open space:*

$$A_j = \left(0, \frac{2^{2\cdot j+1} - 2}{2^{2\cdot j+1} + 1} \cdot \gamma_2\right)$$

where $j \in \{1, 2, ..., N\}$.

Despite the fact that agent-1 gains more utility than her counterparty in the case of heterogeneous agents, in the limit it is illustrated that the aggregate utility, when traders behave strategically is equal to the aggregate Pareto optimal utility. Below, it is indeed shown, that the difference between aforementioned aggregate utilities is equal to zero. Moreover, by making the comparison of the agent's individual Nash utility in the limit minus the Pareto one, agent-1 is verified to be the dominant trader in the repeated trading process.

$$\begin{pmatrix} \textit{Aggregate utility} \\ \textit{on the limit} \end{pmatrix} - \begin{pmatrix} \textit{Aggregate utility} \\ \textit{at the Pareto} \end{pmatrix} = U_{1,lim}^{Nash} + U_{2,lim}^{Nash} - (U_1^o + U_2^o) = 0$$

We notice that each trader is affected by two different streams. The initial stream for each trader is originated by the utility surplus which converges to $U_{1,lim}^{Nash} = \frac{\gamma_1 + 2\gamma_2}{3} \cdot \Sigma_1^o$ and $U_{2,lim}^{Nash} = \frac{2\gamma_1 + \gamma_2}{3} \cdot \Sigma_1^o$ for each agent respectively, while the Pareto utilities are accounted to be $U_1^o = \gamma_1 \cdot \Sigma_1^o$ and $U_2^o = \gamma_2 \cdot \Sigma_1^o$ respectively.

In particular, for agent-1 we have that:

$$U_{1,lim}^{Nash} = \frac{\gamma_1 + 2\gamma_2}{3} \cdot \Sigma_1^o > U_1^o = \gamma_1 \cdot \Sigma_1^o \Rightarrow \gamma_2 > \gamma_1$$

while for agent-2

$$U_{2,lim}^{Nash} = \frac{2\gamma_1 + \gamma_2}{3} \cdot \Sigma_1^o < U_2^o = \gamma_2 \cdot \Sigma_1^o \Rightarrow \gamma_2 > \gamma_1$$

Therefore, for as long as $\gamma_1 < \gamma_2$, given that γ_1 belongs to the "efficient neighborhoods", at the last Nash equilibrium, agent-1 not only gains more utility than the Pareto optimal risk sharing allocation, since she gets the optimal contract payoff C_1^o with a discount (as it is shown below in *Proposition 8*).

The second stream, however equally important, is originated by the market inefficiency, which is carried by both traders, it is proved to be higher for agent-2 in the limit (comparing to the optimal situation). Consequently, with the measure (2.11) for each trader individually, the difference between the utility at the limit minus the Pareto one is accounted to be:

1. For agent-1 we have that:

$$\begin{pmatrix} \textit{Utility at} \\ \textit{the limit} \end{pmatrix} - \begin{pmatrix} \textit{Utility at} \\ \textit{the Pareto} \end{pmatrix} = U_{1,lim}^{Nash} - U_1^o \;= \left(\frac{\gamma_1 + 2\gamma_2}{3} - \gamma_1 \right) \Sigma_1^o = \frac{2(\gamma_2 - \gamma_1)}{3} \Sigma_1^o$$

2. Similarly for agent-2:

$$\begin{pmatrix} \textit{Utility at} \\ \textit{the limit} \end{pmatrix} - \begin{pmatrix} \textit{Utility at} \\ \textit{the Pareto} \end{pmatrix} = U_{2,lim}^{Nash} - U_2^o = \left(\frac{2\gamma_1 + \gamma_2}{3} - \gamma_2 \right) \Sigma_1^o = \frac{2(\gamma_1 - \gamma_2)}{3} \Sigma_1^o$$

Namely, the market inefficiency stream for trader-2 is equal to $\frac{2(\gamma_1 - \gamma_2)}{3} \Sigma_1^o$, which controls that she suffers more utility loss in the limit from re-trading. Albeit, her wealth is accounted to be $\frac{2\gamma_1 + \gamma_2}{3} \Sigma_1^o$, she is affected more negatively by her inefficiency stream individually. Agent-2 looses utility in the Nash occasion compared to the case that no strategic behavior exists. To be more accurate, agent-2 improves her position, but still is distant from the Pareto. Hence, it is confirmed that for the neighborhood where $\gamma_1 < \gamma_2$, trader-1 is the one that benefits in the limit (again). Apparently, the utility loss, of agent-2, is a matter not only of the trader's higher risk aversion, but also from the mutual carriage of the different risks, that is reported by the pair of the traders. From our scope, we follow this myopic-naive path of continuous trading, only as an intermediate step to simply show that, even though agents behave strategically, the pricing can result to the Pareto set, if they are homogeneous. In contrast to this, if agents are heterogeneous, the scales lean towards agent-1 even in myopic case.

The whole discussion is confirmed when the procedure continues to infinitely many steps of trades. The pie of compensation increases for agent-2 comparative to the one shot transactions, however it is lower than the Pareto optimal in the end of re-trading procedure. Considering the above mentioned discussion, when $N \to \infty$ we result to the following proposition.

Proposition 8. *In the limit, the lower risk averse agent gains both the optimal contract,* $\sum_{j=1}^{N} C_j \to C_1^o$ *and a discount in the aggregate compensations that she pays,* $\lim_{N\to\infty} \sum_{j=1}^{N} \pi_j < \pi_1^o$*, if and only if agents are heterogeneous (say* $\gamma_1 < \gamma_2$*).*

Proof. Say $\gamma_1 < \gamma_2$. As we have already shown:

$$\pi_{lim} = \pi_1^o + \frac{2(\gamma_1 - \gamma_2)}{3}\Sigma_1^o \Rightarrow \pi_{lim} - \pi_1^o = \frac{2(\gamma_1 - \gamma_2)}{3}\Sigma_1^o$$

($\Rightarrow$) If $\gamma_1 < \gamma_2$, apparently we have that the lower risk averse agent will pay less than the Pareto optimal compensation in the limit to obtain the optimal contract:

$$\lim_{N\to\infty} \sum_{j=1}^{N} \pi_j - \pi_1^o = \frac{2(\gamma_1 - \gamma_2)}{3}\Sigma_1^o \Rightarrow \lim_{N\to\infty} \sum_{j=1}^{N} \pi_j < \pi_1^o$$

($\Leftarrow$) Vice versa, if we have that:

$$\lim_{N\to\infty} \sum_{j=1}^{N} \pi_j - \pi_1^o = \frac{2(\gamma_1 - \gamma_2)}{3}\Sigma_1^o < 0 \Rightarrow \gamma_1 < \gamma_2$$

which means that agent-1 will pay less than the Pareto optimal compensation to get the contract, if her risk averse is lower than her counterparty. Finally, in the limit we have that $\lim_{N\to\infty} \sum_{j=1}^{N} \pi_j < \pi_1^o$, if and only if $\gamma_1 < \gamma_2$. □

Remark 7. *In conclusion, the pair of agents attains the Pareto allocation if they are homogeneous and behave myopically. Contrarily, in Proposition 8 is stated that, the agent with the lower risk aversion,* γ_1*, acquires the contract* C_1^o*, hence the optimal payoff, through infinite steps of trade with discount, as she will pay lower cash than* π_1^o*. As for her counterparty, the agent with the risk aversion* γ_2*, she seems to gain more as the pie increases, while the re-trading frequency is unlimited, but she will never attain* π_1^o*. Also, agent-2 carries the higher part of the inefficiency in the market. Thus, in the case of heterogeneous agents, the scales lean in favor of the agent-1, while agent-2 will never attain the optimal compensation, which is an issue that stems from both, her higher risk aversion and the the submitted risk that differs from the true one. Eventually, when agents behave strategically and they are heterogeneous, it is* *strengthened the result that agent-1 dominates the bilateral re-trading even though they are myopic. In this case, the risk share allocation can not reach the Pareto efficiency.*

4 The Non-myopic problem

The pricing, taking into account agents' strategic behavior, is probed thoroughly in this section, setting the problem in a forward looking way. Provided that, in the real world two agents will never attain a transaction within infinite time, we set the conditions for the non-myopic problem. In principal, we determine the risk, that the agents should share in the simple case of two trading rounds, by responding the question: *"How much risk should an agent share, given that she knows the aggregate submitted endowment of her counterparty in both trading rounds?* In deterministic time, we should say that the most precise answer is the following: *"The risk that an agent submits in the current trading round, depends not only on the risk that her counterparty submits in this round, but also in the next one (or future transaction)"*. We recall that, when agents behave non-myopic or far-sighted they expect that future transactions will be completed between them.

Each agent will declare a fraction of both endowments, the one she holds and the other that her counterpart owns, in the trade. These proportions are vectors of the Euclidean space, say $(\alpha_j, \beta_j) \in R^2 \times R^2$, where $j = 1, 2, ..., N$ (we have the same notation for $j-$dependence on time likewise in the previous section of the myopic case) and α_j reflects the declared proportion of risk sharing in accordance with the aggregate endowment for agent-1, while β_j reflects the same for agent-2 respectively.

Specifically, agent-1 will report in the first trade, $\alpha_1 =$($\alpha_{1,1}$ coefficient of $\mathscr{E}_{1,0}$, $\alpha_{1,2}$ coefficient of $\mathscr{E}_{2,0}$), whereas agent-2 will respond by reporting $\beta_1 =$($\beta_{1,1}$ coefficient of $\mathscr{E}_{1,0}$, $\beta_{1,2}$ coefficient of $\mathscr{E}_{2,0}$). The coefficients stand for the submitted risk of every endowment that they will report for the transaction. Similar is the occasion of the second trading round, where α_2 and β_2 vectors demonstrate the submitted risk sharing of agent-1 and agent-2 respectively. In the sequel, we present this simple case of two trading rounds bilateral transactions. Thus, we result to the next optimization problems from each one's scope.

Supposing that agent-1 has the knowledge of the β_1 and β_2 vectors, she will make the best endowment response by reporting α_1 and α_2 vectors, in the trade, respectively. Hence, the maximum utility that agent-1 will gain is given by the solution of the problem, which is defined below:

$$\max_{(\alpha_1,\alpha_2)\in R^2\times R^2} \left\{ U_{1,2}\left(e_1 \cdot \mathscr{E} + \sum_{j=1}^{2} C_j^o(\alpha_j, \beta_j) - \pi_1^o(\alpha_1, \beta_1) - \pi_2^o(\alpha_2, \beta_2); (\beta_1, \beta_2) \right) \right\} \tag{4.1}$$

Contrarily, given that agen-2 knows that agen-1 will submit α_1 and α_2 vectors in each trading round, the best endowment response of agent-2 is β_1 and β_2 respectively. The corresponding optimal problem for the second trader gives her individual utility maximization as it is quoted below:

$$\max_{(\beta_1,\beta_2)\in R^2\times R^2}\left\{U_{2,2}\left(e_2\cdot\mathscr{E}-\sum_{j=1}^{2}C_j^o(\alpha_j,\beta_j)+\pi_1^o(\alpha_1,\beta_1)+\pi_2^o(\alpha_2,\beta_2);(\alpha_1,\alpha_2)\right)\right\} \tag{4.2}$$

where $e_1 = (1,0)$, $e_2 = (0,1)$, $\mathscr{E} = (\mathscr{E}_{1,0}, \mathscr{E}_{2,0})$ the vector of the endowments, the contract payoff is $C_j := C_j^o(\alpha_j, \beta_j) \in (R^2)^N \times (R^2)^N$ and the price of the contract is $\pi_j := \pi_j^o(\alpha_j, \beta_j)$ with $j \in 1, 2, ..., N$. Each optimizer expression, gives the unique Nash equilibrium $\forall$ agent towards her own scope.

We have the following for the contract and the compensations in each round:

- The contract payoff in the first round is: $C_1^o(\alpha_1,\beta_1) = \lambda_1(\beta_1\cdot\mathscr{E}) - \lambda_2(\alpha_1\cdot\mathscr{E})$
- The contract payoff in the second round is: $C_2^o(\alpha_2,\beta_2) = \lambda_1(\beta_2\cdot\mathscr{E}) - \lambda_2(\alpha_2\cdot\mathscr{E})$
- The price in the first round is: $\pi_1^o(\alpha_1,\beta_1) = -2\gamma Cov\left[C_1^o(\alpha_1,\beta_1),(\alpha_1,\beta_1)\cdot\mathscr{E}\right]$
- The price in the second round is:
  $\pi_2^o(\alpha_2,\beta_2) = -2\gamma Cov\left[C_2^o(\alpha_2,\beta_2),(\alpha_1+\alpha_2,\beta_1+\beta_2)\cdot\mathscr{E}\right]$

  Notably, the aggregate endowment [19] is $\mathbf{B}_1 = (\alpha_1,\beta_1)\cdot\mathscr{E}$ in trading round 1 and $\mathbf{B}_2 = (\alpha_1+\alpha_2,\beta_1+\beta_2)\cdot\mathscr{E}$ in trading round 2. Without loss of generality we can assume that $E(\mathscr{E}) = E(\mathscr{E}_{1,0}) = E(\mathscr{E}_{2,0}) = 0$ and this induces that $E(\sum_{j=1}^{2} C_j^o(\alpha_j,\beta_j)) = 0$, given that:

$$\sum_{j=1}^{2} C_j^o(\alpha_j,\beta_j) = C_1^o(\alpha_1,\beta_1) + C_2^o(\alpha_2,\beta_2) = (\lambda_1(\beta_1+\beta_2) - \lambda_2(\alpha_1+\alpha_2))\cdot\mathscr{E}$$

By solving the optimization problems that are marked by the (4.1) and (4.2) we will result to the risk sharing between the agents, when they behave strategically and non-myopic. We remind that the M-V preferences are ticked off by the quadratic (2.1) equation and hence we denote as $U_{1,2}$ the utility function for agent-1 at the end of the second round and as $U_{2,2}$ the corresponding one for agent-2. Thus we have the following.

[19] For the $j-th$ round we can say that $\mathbf{B}_j = \sum_{j=1}^{N}(\alpha_j,\beta_j)\cdot\mathscr{E}$, where $j = 1, 2, ..., N$. Furthermore, the risk sharing allocation between the agents is completed on the inherited endowments by the previous round of trade.

The utility for agent-1 is accounted to be:

$$U_{1,2} = U_{1,2}\left(e_1 \cdot \mathscr{E} + \sum_{j=1}^{2} C_j^o(\alpha_j, \beta_j) - \sum_{j=1}^{2} \pi_j(\alpha_j, \beta_j)\right) \Rightarrow$$

$$\Rightarrow U_{1,2} = -\left(\sum_{j=1}^{2} \pi_j(\alpha_j, \beta_j)\right) - \gamma_1 \cdot Var\left(e_1 \cdot \mathscr{E} + \sum_{j=1}^{2} C_j^o(\alpha_j, \beta_j)\right) \tag{4.3}$$

The aggregate compensations for both rounds are given below:

$$\sum_{j=1}^{2} \pi_j(\alpha_j, \beta_j) = \pi_1^o(\alpha_1, \beta_1) + \pi_2^o(\alpha_2, \beta_2) = -2\gamma\left(Cov(C_1^o(\alpha_1, \beta_1), \mathbf{B}_1) + Cov(C_2^o(\alpha_2, \beta_2), \mathbf{B}_2)\right) =$$

$$= -2\gamma\left[(\alpha_1 + \beta_1) \cdot \Sigma \cdot (\lambda_1\beta_1 - \lambda_2\alpha_1)^\intercal + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \cdot \Sigma \cdot (\lambda_1\beta_2 - \lambda_2\alpha_2)^\intercal\right]$$

where $\Sigma = Var(\mathscr{E})$, namely the variance-covariance matrix of the endowments. We assume that the endowments are uncorrelated or equivalently the inverse matrix of Σ exists. We can do so, since there is no affection in the final outcomes of the game. Concerning with the variance between agent's-1 endowment and the sum of contracts, we end up with the following:

$$Var\left(e_1 \cdot \mathscr{E} + \sum_{j=1}^{2} C_j^o(\alpha_j, \beta_j)\right) = Var\left((e_1 + \lambda_1(\beta_1 + \beta_2) - \lambda_2(\alpha_1 + \alpha_2)) \cdot \mathscr{E}\right) =$$

$$= (e_1 + \lambda_1(\beta_1 + \beta_2) - \lambda_2(\alpha_1 + \alpha_2)) \cdot \Sigma \cdot (e_1 + \lambda_1(\beta_1 + \beta_2) - \lambda_2(\alpha_1 + \alpha_2))^\intercal$$

Equivalently, the utility for agent-2 is accounted to be:

$$U_{2,2} = U_{2,2}\left(e_2 \cdot \mathscr{E} - \sum_{j=1}^{2} C_j^o(\alpha_j, \beta_j) + \sum_{j=1}^{2} \pi_j(\alpha_j, \beta_j)\right) \Rightarrow$$

$$\Rightarrow U_{1,2} = \left(\sum_{j=1}^{2} \pi_j(\alpha_j, \beta_j)\right) - \gamma_2 \cdot Var\left(e_2 \cdot \mathscr{E} - \sum_{j=1}^{2} C_j^o(\alpha_j, \beta_j)\right) \tag{4.4}$$

Where the variance between agent's-2 endowment and the sum of contracts is the following:

$$Var\left(e_2 \cdot \mathscr{E} - \sum_{j=1}^{2} C_j^o(\alpha_j, \beta_j)\right) = Var\left((e_2 - \lambda_1(\beta_1 + \beta_2) + \lambda_2(\alpha_1 + \alpha_2)) \cdot \mathscr{E}\right) =$$

$$= (e_2 - \lambda_1(\beta_1 + \beta_2) + \lambda_2(\alpha_1 + \alpha_2)) \cdot \Sigma \cdot (e_2 - \lambda_1(\beta_1 + \beta_2) + \lambda_2(\alpha_1 + \alpha_2))^\intercal$$

4.1 The solution of non-myopic re-trading

In this point, we are going to clarify the non-myopic problem by solving the individual problems that are asserted by the equations (4.3) and (4.4) mutually. Accurately, we will apply the matching between the two problems, in order to succeed it. Particularly, the solution of the linear equations system, in which we end up from the combination of the necessary conditions is unique. We expected that because of the linearity of the partial derivatives (quadratic forms of M-V preferences).

Another fact to be mentioned is the relaxation of the assumptions that set in the non-myopic problem, especially the ii) and the iii). Albeit, we ought to and readily relax the second assumption so as to determine the model in a forward looking or far-sighted way, this is not obvious in the model at the moment. Furthermore, we have not incorporated the appropriate coefficient for the transaction costs or the penalty of the continuous trading. These issues can be left for further research, but firstly we need to solve the problem and then we can incorporate them. We need to find the solution and then we will adjust the model properly.

The necessary conditions to find the optimal risk sharing for each agent is quoted below: The gradient of the equation (4.3) must be:

$$\nabla U_{1,2}\left(e_1\cdot\mathscr{E}+\sum_{j=1}^{2}C_j^o(\alpha_j,\beta_j)-\pi_1^o(\alpha_1,\beta_1)-\pi_2^o(\alpha_2,\beta_2)\right)=0\Rightarrow(\frac{\partial U_{1,2}}{\partial\alpha_1},\frac{\partial U_{1,2}}{\partial\alpha_2})=(0,0)$$

The partial derivatives for agent's-1 utility are:

$$\frac{\partial U_{1,2}}{\partial\alpha_1}=\left[(2\gamma_1\lambda_2+2\gamma)\left(\lambda_1(\beta_1+\beta_2)-\lambda_2(\alpha_1+\alpha_2)\right)-2\gamma\lambda_2(\alpha_1+\beta_1)+2\gamma_1\lambda_2e_1\right]\cdot\Sigma$$

$$\frac{\partial U_{1,2}}{\partial\alpha_2}=$$

$$=\left[(2\gamma_1\lambda_2+2\gamma)(\lambda_1\beta_2-\lambda_2\alpha_2)-2\gamma\lambda_2(\alpha_1+\alpha_2+\beta_1+\beta_2)-2\gamma_1\lambda_2(\lambda_1\beta_1-\lambda_2\alpha_1+e_1)\right]\cdot\Sigma$$

The gradient of the equation (4.4) must be:

$$\nabla U_{2,2}\left(e_2\cdot\mathscr{E}-\sum_{j=1}^{2}C_j^o(\alpha_j,\beta_j)+\pi_1^o(\alpha_1,\beta_1)+\pi_2^o(\alpha_2,\beta_2)\right)=0\Rightarrow(\frac{\partial U_{2,2}}{\partial\beta_1},\frac{\partial U_{2,2}}{\partial\beta_2})=(0,0)$$

The partial derivatives for agent's-2 utility are:

$$\frac{\partial U_{2,2}}{\partial\beta_1}=\left[-(2\gamma_2\lambda_2+2\gamma)\left(\lambda_1(\beta_1+\beta_2)-\lambda_2(\alpha_1+\alpha_2)\right)-2\gamma\lambda_1(\alpha_1+\beta_1)+2\gamma_2\lambda_1e_2\right]\cdot\Sigma$$

$$\frac{\partial U_{2,2}}{\partial \beta_2} =$$
$$= -\left[(2\gamma_2\lambda_1 + 2\gamma)(\lambda_1\beta_2 - \lambda_2\alpha_2) + 2\gamma\lambda_1(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + 2\gamma_2\lambda_1(\lambda_1\beta_1 - \lambda_2\alpha_1 - e_2)\right] \cdot \Sigma$$

We remind from *section 2*, that the relative risk tolerance $\forall$ agent, is denoted as $\lambda_i = \frac{\gamma}{\gamma_i}$ where $i = 1, 2$ in our case. Therefore, it is $\lambda_1 + \lambda_2 = 1$. Hence, we conclude to the following linear system:

$$\left(\frac{\partial U_{1,2}}{\partial \alpha_1}, \frac{\partial U_{1,2}}{\partial \alpha_2}, \frac{\partial U_{2,2}}{\partial \beta_1}, \frac{\partial U_{2,2}}{\partial \beta_2}\right) = (0, 0, 0, 0)$$

which results to the next equations:

- $\lambda_2(1 + \lambda_1)\alpha_1 + \lambda_2\alpha_2 - \lambda_1^2\beta_1 - \lambda_1\beta_2 = \lambda_2 e_1$, (1)
- $\lambda_2 \cdot \alpha_1 + \lambda_2 \cdot (1 + \lambda_1) \cdot \alpha_2 - \lambda_1^2 \cdot \beta_2 = \lambda_2 e_1$, (2)
- $\lambda_2^2\alpha_2 + \lambda_2\alpha_1 - \lambda_1(1 + \lambda_2)\beta_1 - \lambda_1\beta_2 = -\lambda_1 e_2$, (3)
- $\lambda_2^2\alpha_2 - \lambda_1\beta_1 - \lambda_1(1 + \lambda_2)\beta_2 = -\lambda_1 e_2$, (4)

Considering and setting as a benchmark the above system of linear equations, we will have the following:

- By calculating
$$(1) - (2) \Rightarrow \alpha_1 = \alpha_2 + \frac{\lambda_1}{\lambda_2}\beta_1 + \beta_2 \tag{4.5}$$
- From the equations
$$(3) - (4) \Rightarrow \alpha_2 = -\frac{\lambda_2}{\lambda_1}\alpha_1 + \beta_1 - \beta_2 \tag{4.6}$$
- By combining the equations (4.5) & (4.6)
$$\alpha_2 = -\beta_2 \tag{4.7}$$
- Finally, we have, that by combining (4.5) with (4.7), then
$$\alpha_1 = \frac{\lambda_1}{\lambda_2}\beta_1 \tag{4.8}$$

The final step to take the solution of the system, by combining the equations (1), (2), (3), (4) an (4.5), (4.6), (4.7), (4.8), is quoted below:

-
$$\lambda_1\beta_1 - \beta_2 = \lambda_2 e_1$$

- $$\lambda_1 t\beta_1 + \beta_2 = \lambda_1 e_2$$
- $$\beta_1 = \frac{\lambda_2}{2\lambda_1}e_1 + \frac{\lambda_1}{2\lambda_1}e_2 \Rightarrow \beta_1 = (\frac{\lambda_2}{2\lambda_1}, \frac{1}{2})$$
- $$\beta_2 = (-\frac{\lambda_2}{2}, \frac{\lambda_1}{2})$$
- $$\alpha_2 = (\frac{\lambda_2}{2}, -\frac{\lambda_1}{2})$$
- $$\alpha_1 = (\frac{1}{2}, \frac{\lambda_1}{2\lambda_2})$$

The sufficient condition, in order to find the maximum, is a negative definite Hessian matrix. Since the variance-covariance matrix is positive definite, we can easily prove this for both problems, (4.1) and (4.2), that the sufficient condition holds. For our scope the Hessian matrix is omitted due to the easy way to check it and provided that we assumed uncorrelated endowments, the variance covariance matrix is positive definite.

4.2 The contract payoff and the compensations

We emphasize, that on the merits of the far sighted discussion, even if the re-trading window of opportunity exists, the agents prefer to trade in one shot process. Independently from the trading rounds, they respond in the last round of the potential ones, the submitted endowment, that they report in order to negotiate. The non-myopic case induces, that they want to gain instantly the optimal share of the pie based on the strategical approach, which they set. This explicit message is confirmed by the verification of the outcomes, that are originated on the matching of the two problems. Specifically, in the first round if agent-1 reports α_1 her counterparty will respond risk $\beta_1 = \frac{\lambda_2}{\lambda_1}\alpha_1$ (equation(4.8)). This will give as a result, that the contract payoff is $C_1(\alpha_1, \beta_1) = 0$ and the price also $\pi_1^o(\alpha_1, \beta_1) = 0$. In the second trading round, if agent-1 submits α_2 her counterparty will respond the exact opposite risk $\beta_2 = -\alpha_2$ (equation (4.7)). In this occasion, the contract payoff results to half of the Pareto optimal contract which coincides with the outcomes of the static model and the same holds for the prices. Namely, the outcomes of the non-myopic case in the second round coincide with the static Nash equilibrium of the first period transactions in *Table 1*.

The reverse conversation can be unfold, with the same outcomes, if we wonder how the agent-1 is going to respond, given that she knows the risk of each endowment, will be submitted by her counterparty.

Proposition 9. *The contract payoff, when agents behave far-sighted is* $C_2(\alpha_2, \beta_2) = \frac{C_1^o}{2}$ *and its price is* $\pi_2^o(\alpha_2, \beta_2) = \pi_1$ *or equivalently they coincide with the corresponding payoff and prices of the Nash equilibrium of the static model.*

Proof. The contract payoff and the prices in the first trading round, where α_1 and β_1 are known from equation (4.8), give us the following:

$$C_1(\alpha_1, \beta_1) = \lambda_1 \beta_1 \cdot \mathscr{E} - \lambda_2 \alpha_1 \cdot \mathscr{E} = \lambda_1 \left(\frac{\lambda_2}{2\lambda_1} \mathscr{E}_{1,0} + \frac{1}{2} \mathscr{E}_{2,0} \right) - \lambda_2 \left(\frac{1}{2} \mathscr{E}_{1,0} + \frac{\lambda_1}{2\lambda_2} \mathscr{E}_{2,0} \right) = 0$$

$$\pi_1^o(\alpha_1, \beta_1) = -2\gamma Cov \left[C_1^o(\alpha_1, \beta_1), (\alpha_1, \beta_1) \cdot \mathscr{E} \right] =$$
$$= -2\gamma \left[(\frac{\lambda_1}{\lambda_2} \beta_1 + \beta_1) \cdot \Sigma \cdot (\lambda_1 \beta_1 - \lambda_2 \frac{\lambda_1}{\lambda_2} \beta_1) \right] = -2\gamma (\frac{\beta_1}{\lambda_2}) \cdot \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \cdot (0, 0)^\intercal = 0$$

The contract payoff and the prices in the second trading round, where α_2 and β_2 are known from equation (4.7), are depicted below:

$$C_2(\alpha_2, \beta_2) = \lambda_1 \beta_2 \cdot \mathscr{E} - \lambda_2 \alpha_2 \cdot \mathscr{E} = (\lambda_1 + \lambda_2) \beta_2 \cdot \mathscr{E} = (-\frac{\lambda_2}{2}, \frac{\lambda_1}{2}) \cdot (\mathscr{E}_{1,0}, \mathscr{E}_{2,0}) = \frac{C_1^o}{2},$$

where C_1^o is the optimal contract.

$$\pi_2^o(\alpha_2, \beta_2) = -2\gamma Cov \left[C_2^o(\alpha_2, \beta_2), (\alpha_1 + \alpha_2, \beta_1 + \beta_2) \cdot \mathscr{E} \right] =$$
$$= -2\gamma \left[(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)) \cdot \Sigma \cdot (\lambda_1 \beta_2 - \lambda_2 \alpha_2) \right] = -2\gamma (\frac{\beta_1}{\lambda_2}) \cdot \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \cdot (-\frac{\lambda_2}{2}, \frac{\lambda_1}{2}^\intercal) =$$
$$= -2\gamma (-\frac{\lambda_2}{4\lambda_1} \sigma_1^2 + \frac{\lambda_1}{4\lambda_2} \sigma_2^2) = \frac{\gamma_1^2}{2(\gamma_1 + \gamma_2)} \sigma_1^2 - \frac{\gamma_2^2}{2(\gamma_1 + \gamma_2)} \sigma_2^2 = \pi_1$$

where $\sigma_1^2 = Var(\mathscr{E}_{1,0})$, $\sigma_2^2 = Var(\mathscr{E}_{2,0})$ and $\pi_1 = E(C_1) - 2\gamma Cov(C_1, \mathbf{B}_0)$. In completion, the price of the contract in the non-myopic problem coincides with the Nash equilibrium price of the one period static model. □

Remark 8. *We highlight, that Proposition 9 is equivalent with the statement: "The contract payoff and its price, when agents behave far-sighted, coincide with the corresponding payoff and the price of the static Nash equilibrium of the one period model." The query, that we can postulate in this point is, whether it is enough for a pair of traders, to attain the Pareto efficiency by behaving non-myopically. The precise answer is that: "The traders by being far-sighted, they will not be led in the Pareto optimality." Far-sighted behavior does not imply, that they will approximate the Pareto* *frontier. The reason is the incentives of the traders, which are ranged from hedgers or arbitragers to predators or speculators. These incentives in correlation with the thinness of the market, will lead them to trade in the last round of re-trading (the second round in this problem). Namely, if the rounds of trade were j, the traders would complete transactions, where the aggregate prices are zero for $j-1$ rounds (i.e. no trade until the last round, $\sum_{j=1}^{N} \pi_{j-1} = 0$), while in the $j-th$ round, they would attain the static Nash equilibrium of the one period model outcomes.*

5 The explicit costs analysis

The deviation of Pareto optimal risk sharing allocation, not only in thin markets, but also in the competitive ones, is related with the much talked issue of transaction costs, which can have any form such as commissions, brokerage fees, fares etc. Practically, if the intermediary fees are very low (i.e $\delta = 0$), there may be a chance for agents to attain the Pareto optimal risk sharing allocation, when no strategic behavior exists or after infinite steps of trades according to myopic re-trading procedure (strategic behavior). In order to set simply the problem of transaction costs, we introduce a discount factor, say δ, which constitutes the measure of these explicit costs. We stress that this factor $\delta \in (0,1)$, though it is postulated to be increasingly high under circumstances, where the price of the agreement is too high or equivalently agents share too much risk. Hence, they expect much higher gains.

Our study targets the determination of this factor δ, that makes some agent indifferent for the trade, given that the costs overwhelming the aggregate utility surplus during the evolving course of re-trading. Hence, we will result to an (upper) bound $\bar{\delta}$, which reflects, that no other trade will incur because it will be unprofitable for some trader. Agents loose some part of the expected wealth, which is anticipated after new rounds of transactions due to the explicit costs. Since these costs are ex-ante we examine the $\bar{\delta}_j$ in the $j-th$ round which is this upper bound or the terminal factor that makes them indifferent for a new agreement. Given that, the market structure, that we study, is a thin (oligopoly) complete market structure, where both agents co-design the contracts and have the power to influence the prices, the explicit costs can be shared equally to each one without loss of generality. Because of the market power, that agents possess the whole skeptical of this terminal factor application, in order not to trade, if the new round is too much costly for some trader, it can be assumed as an equal divide of the costs between them.

Suppose that, the price of the optimal contract is $\pi_1^o > 0 \Rightarrow Cov(C_1^o, \mathscr{E}) < 0$ and $\gamma_1 \leq \gamma_2$, our analysis focuses in the case of the $\bar{\delta}$ from the scope of agent-2 in each round of trade. We mention that the negative covariance between the optimal contract and the total market endowment depicts the state where agents gain the higher profits from re-trading process. Furthermore, we have shown that agent-1 is the dominant trader for as much as her risk aversion is lower than her counterparty's and belongs to the "efficient neighborhoods". Thus, the utility surplus with the incorporation of the discount factor δ is adjusted to be for every agent as below:

- The utility surplus of agent-1:

$$U_{1,j}^{\delta}\left(\mathscr{E}_{1,0}+\sum_{j=1}^{N}C_j-\sum_{j=1}^{N}\pi_j-\delta\cdot\sum_{j=1}^{N}\pi_j\right)-U_{1,0}\left(\mathscr{E}_{1,0}\right) \tag{5.1}$$

- The utility surplus of agent-2:

$$U_{2,j}^{\delta}\left(\mathscr{E}_{2,0}-\sum_{j=1}^{N}C_j+\sum_{j=1}^{N}\pi_j-\delta\cdot\sum_{j=1}^{N}\pi_j\right)-U_{2,0}\left(\mathscr{E}_{2,0}\right) \tag{5.2}$$

If agents are heterogeneous, the process of re-trading is presented step by step until the $N-th$ round of re-trading. Particularly, in the first trading round:

- Agent's-1 investment position is accounted to be: $\mathscr{E}_{1,0}+C_1-\pi_1\left(1+\delta\right)$, while her "new" utility surplus is shaped to be:

$$U_{1,1}^{\delta_1}-U_{1,0}\left(\mathscr{E}_{1,0}\right)=\frac{\gamma_1+2\gamma_2}{4}\cdot\Sigma_1^o-\delta\cdot\pi_1$$

- Similarly, agent's-2 investment position is accounted to be: $\mathscr{E}_{2,0}-C_1+\pi_1\left(1-\delta\right)$, while her "new" utility surplus is shaped to be:

$$U_{2,1}^{\delta_1}-U_{2,0}\left(\mathscr{E}_{1,0}\right)=\frac{2\gamma_1+\gamma_2}{4}\cdot\Sigma_1^o-\delta\cdot\pi_1$$

Therefore, the agent-2 will be indifferent to the transaction from the moment that her anticipated gains are zero or $U_{2,1}^{\delta_1}-U_{2,0}\left(\mathscr{E}_{2,0}\right)=0\Rightarrow\bar{\delta}_1=\dfrac{\left(2\gamma_1+\gamma_2\right)\cdot\Sigma_1^o}{4\cdot\pi_1}$.

From the second trading round,the economic discussion is gathered only in the utility of agent-2, as it is mentioned above, until the end of the re-trading procedure:

- Agent's-2 investment position is accounted to be: $\mathscr{E}_{2,0}-C_1+\left(\pi_1+\pi_2\right)\left(1-\delta\right)$, while her utility surplus is shaped to be:

$$U_{2,2}^{\delta_2}-U_{2,0}\left(\mathscr{E}_{1,0}\right)=\frac{10\gamma_1+5\gamma_2}{16}\cdot\Sigma_1^o-\delta\cdot\left(\pi_1+\pi_2\right)$$

As a consequence, agent-2 will be indifferent to the transaction from the moment that her anticipated gains are zero or $U_{2,2}^{\delta_2}-U_{2,0}\left(\mathscr{E}_{2,0}\right)=0\Rightarrow\bar{\delta}_2=\dfrac{\left(10\gamma_1+5\gamma_2\right)\cdot\Sigma_1^o}{16\cdot\left(\pi_1+\pi_2\right)}$.

In the third trading round:

- Similarly, agent's-2 position is accounted to be: $\mathscr{E}_{2,0} - C_1 + (\pi_1 + \pi_2 + \pi_2)(1-\delta)$, while her utility surplus is the following:

$$U_{2,3}^{\delta_3} - U_{2,0}(\mathscr{E}_{2,0}) = \frac{42\gamma_1 + 21\gamma_2}{64} \cdot \Sigma_1^o - \delta \cdot (\pi_1 + \pi_2 + \pi_3)$$

Agent-2 will be indifferent to the transaction when $U_{2,3}^{\delta_3} - U_{2,0}(\mathscr{E}_{2,0}) = 0 \Rightarrow \bar{\delta}_3 = \dfrac{(42\gamma_1 + 21\gamma_2) \cdot \Sigma_1^o}{64 \cdot (\pi_1 + \pi_2 + \pi_3)}$.

Finally, in the $N-th$ round of trades, given the outcomes of the transactions, the discount factor concludes to the following:

- Agent's-2 position is: $\mathscr{E}_{2,0} - C_1 + (1-\delta)\sum_{j=1}^{N} \pi_j$, whereas her utility surplus is the following:

$$U_{2,3}^{\delta_N} - U_{2,0}(\mathscr{E}_{2,0}) = \left(1 - \frac{1}{4^N}\right) \frac{2\gamma_1 + \gamma_2}{3} \cdot \Sigma_1^o - \delta \cdot \sum_{j=1}^{N} \pi_j$$

Agent-2 will be indifferent to the transaction when

$$U_{2,N}^{\delta_N} - U_{2,0}(\mathscr{E}_{2,0}) = 0 \Rightarrow \bar{\delta}_N = \frac{\left(1 - \frac{1}{4^N}\right) \frac{2\gamma_1 + \gamma_2}{3} \cdot \Sigma_1^o}{\sum_{j=1}^{N} \pi_j} \Rightarrow$$
$$\Rightarrow \bar{\delta}_N = \frac{\left[2\left(4^N - 1\right)\gamma_1 + \left(4^N - 1\right)\gamma_2\right]\Sigma_1^o}{3\left(4^N - 2^N\right)\pi_1^o + 2\left(4^N - 1\right)(\gamma_1 - \gamma_2)\Sigma_1^o}$$

Recall that $\sum_{j=1}^{N} \pi_j$ is the aggregate compensations that are known from the equation (3.2).

Proposition 10. *Assuming that $\pi_1^o > 0$, the sequence $\left(\bar{\delta}_j\right)_{j \in N}$ is a strictly decreasing, uniformly bounded from below and its limit is $\bar{\delta}^* = \dfrac{(2\gamma_1 + \gamma_2)\Sigma_1^o}{3\pi_1^o + 2(\gamma_1 - \gamma_2)\Sigma_1^o}$.*

Proof 1. *In order to show the monotony of the sequence, we will show that:*

$$\bar{\delta}_N < \bar{\delta}_{N-1}$$

$$\bar{\delta}_N < \bar{\delta}_{N-1} \Rightarrow \frac{\left(1-\frac{1}{4^N}\right)\frac{2\gamma_1+\gamma_2}{3}\cdot\Sigma_1^o}{\sum_{j=1}^{N}\pi_j} < \frac{\left(1-\frac{1}{4^{N-1}}\right)\frac{2\gamma_1+\gamma_2}{3}\cdot\Sigma_1^o}{\sum_{j=1}^{N-1}\pi_j} \Rightarrow$$

$$\Rightarrow \left(1-\frac{1}{4^N}\right)\sum_{j=1}^{N-1}\pi_j < \left(1-\frac{4}{4^N}\right)\left(\sum_{j=1}^{N-1}\pi_j+\pi_N\right) \Rightarrow \frac{3}{4^N}\sum_{j=1}^{N-1}\pi_j < \left(1-\frac{4}{4^N}\right)\pi_N \Rightarrow$$

$$\Rightarrow 3\left(1-\frac{2}{2^N}\right)\pi_1^o + 2\left(1-\frac{4}{4^N}\right)(\gamma_1-\gamma_2)\,\Sigma_1^o < \left(4^N-4\right)\left[\frac{\pi_1^o}{2^N}+\frac{2\,(\gamma_1-\gamma_2)}{4^N}\Sigma_1^o\right] \Rightarrow$$

$$\Rightarrow \pi_1^o < \frac{4^N-4}{3\,(2^N-2)}\pi_1^o \Rightarrow 1 < \frac{4^N-4}{3\,(2^N-2)}$$

which holds because we have that:

$$4^N-4 > 3\left(2^N-2\right) \Rightarrow (2^N)^2 - 3\cdot 2^N + 2 > 0 \Rightarrow \chi^2 - 3\cdot\chi + 2 > 0$$

$\forall\chi > 2$ *and thus* $\forall N > 1$ *(we set* $\chi = 2^N$*).*

Provided that, $\bar{\delta}_N$ *is strictly decreasing and uniformly bounded from below it converges to:*

$$\bar{\delta}^* = \lim_{N\to\infty}\bar{\delta}_N = \frac{(2\gamma_1+\gamma_2)\,\Sigma_1^o}{3\pi_1^o + 2\,(\gamma_1-\gamma_2)\,\Sigma_1^o}$$

by recalling the equation (3.2) *for the aggregate compensations.*

Hence the proposition is confirmed with regard to the reasoning above.

Corollary 5. *If* $\bar{\delta} < \bar{\delta}^*$*, they may continue to re-trade for infinite steps, while the traders will not attain the Pareto since they loose a fraction of their anticipated profits.*

As the iterative rounds of trades progress, the discount factor $\bar{\delta}$has a diminishing trend, due to the fact that the gains in each “new” round of transactions follow a decreasing course. As we have seen, the re-trading procedure, when agents face frictions such as transaction costs, stops in a very small number of trades and as a consequence the Pareto optimality is practically excluded. Either agents are myopic or non myopic the re-trading procedure will stop instantly since agent-2 anticipates greater loss than gains from the transaction. We remind that, even if agents act far-sighted, it does not imply that they can approximate Pareto optimality, let alone the occasion in which they incur transaction costs.

In this point we mark the homogeneous agents version, who share different risks, $Var(\mathscr{E}_{1,0}) \neq Var(\mathscr{E}_{2,0})$. We have shown, when agents are homogeneous, that after infinitely many steps they can attain the Pareto optimal risk share if they act myopically. No matter the noncompetitive structure of the thin markets, agents can approximate the optimal risk sharing allocation. If $\gamma_1 = \gamma_1 = g$, where g is defined to be the common risk aversion of the agents we show that agents fix jointly the same indifference point $\bar{\delta}_g$. Actually, *Proposition 10.* holds also for the new upper bound, i.e. $\bar{\delta}_{j,g} = \dfrac{(4^j - 1)\, g\Sigma_1^o}{2^j\,(2^j - 1)\,\pi_1^o}$ and due to the fact that $\left(\bar{\delta}_{j,g}\right)_{j\in N}$ is strictly decreasing and lower bounded the $\bar{\delta}_g^* = \dfrac{g\Sigma_1^o}{\pi_1^o}$.

Remark 9. *An important consideration, with regard to $\bar{\delta}$ is related with the variance of t the contract. It can be easily shown that $\bar{\delta}(\Sigma_1^o)$ is an increasing function of Σ_1^o in each case of assents' risk aversion. Provided that,*

$$\Sigma_1^o = \frac{\gamma_2^2 Var(\mathscr{E}_{2,0}) - 2\gamma_1\gamma_2 Cov(\mathscr{E}_{1,0}, \mathscr{E}_{2,0}) + \gamma_1^2 Var(\mathscr{E}_{1,0})}{(\gamma_2 + \gamma_1)^2}$$

if the $Cov(\mathscr{E}_{1,0}, \mathscr{E}_{2,0}) < 0$, the discount factor tends to be increasingly high, but the gains of the potential trade are expected to be quite higher than the explicit costs. In short, if they make the deal, they expect to pay a huge amount of transaction costs. In essence, the discount factor reflects the benefits of the transaction. When the covariance of the endowments is negative, we have the perfect matching between the risks that are transferred from the agents and thus the $\bar{\delta}$ is expected to be its highest levels. Even if, it is increasingly high, the investor is willing to pay the price in order to complete the trade. Conversely, if the $Cov(\mathscr{E}_{1,0}, \mathscr{E}_{2,0}) > 0$, then $\bar{\delta}(\Sigma_1^o)$ tends to its lowest levels.

Equally important to notice is the term $\dfrac{\Sigma_1^o}{\pi_1^o}$. If we re-define it, we conclude to the next induction:

$$\frac{\Sigma_1^o}{\pi_1^o} = \frac{1}{\dfrac{\pi_1^o}{\Sigma_1^o}} = \frac{1}{\dfrac{-2\gamma Cov(C_i^o, \mathscr{E})}{Var(C_i^o)}} = \frac{1}{2\gamma(-\hat{\beta})}$$

where $\hat{\beta}$ reflects the sensitivity of the price of the contract in the various fluctuations of the contract.

Equivalently, the term $\dfrac{\Sigma_1^o}{\pi_1^o}$ can be written as below:

$$\frac{\Sigma_1^o}{\pi_1^o} = \frac{\gamma_2^2 Var(\mathscr{E}_{2,0}) - 2\gamma_1\gamma_2 Cov(\mathscr{E}_{1,0}, \mathscr{E}_{2,0}) + \gamma_1^2 Var(\mathscr{E}_{1,0})}{-2\gamma\,(\gamma_1 + \gamma_2)\,(\gamma_2 Var(\mathscr{E}_{2,0}) - (\gamma_2 - \gamma_1) Cov(\mathscr{E}_{1,0}, \mathscr{E}_{2,0}) - \gamma_1 Var(\mathscr{E}_{1,0}))}$$

Corollary 6. *In the event of far-sighted agents, by combining the* Proposition 9. *and* Proposition 10. *the indifference point for agent-2 to make the transaction with her counterparty is $\bar{\delta}^F = \dfrac{(2\gamma_1 + \gamma_2)\cdot \Sigma_1^o}{4\cdot \pi_1}$*[20]*. Far-sighted agents prefer to trade in one shot process, i.e they will not trade until the last round of re-trading procedure, while the outcomes coincide with the static model.*

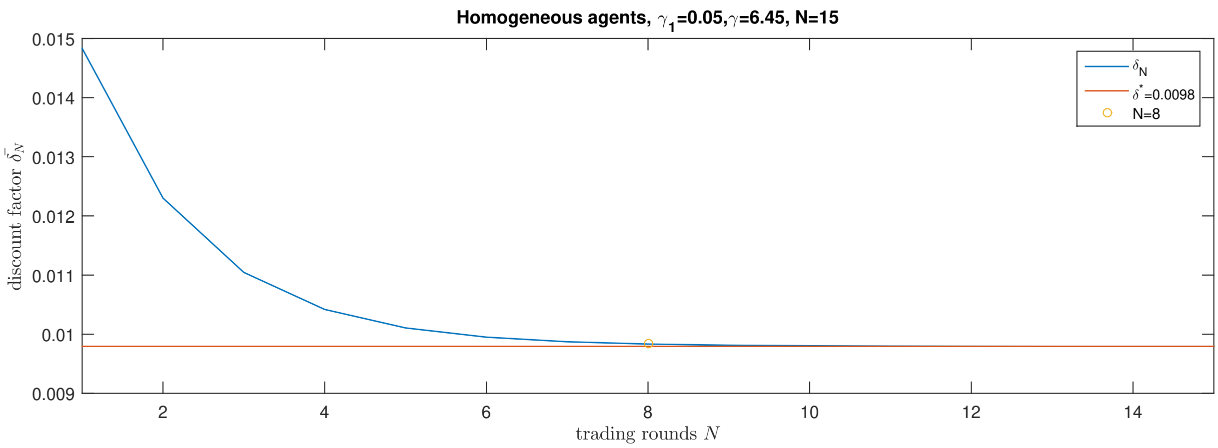
[20] The exponent F is used to define the upper bound of the far-sighted trader.

5.1 Comparative statics for δ

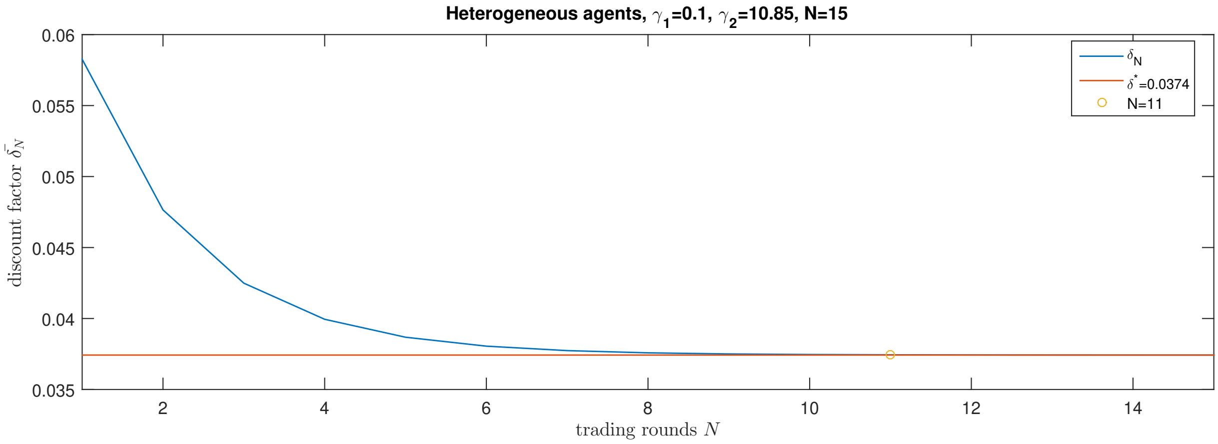
We analyze the state where the agents are homogeneous and heterogeneous, emphasizing the case of the negative covariance of the contract with the total market endowment, in which the profits of the trade are at their maxima. In the various fluctuations of δ, the discount factor moves to its lowest and highest values (as a percentage of the contract price), provided that δ is an increasing function of Σ_1^o.

Below we quote some special occasions of the discount factor fluctuations, where it is obvious that after the trading round $N = 10$ converges to its lower bound $\bar{\delta}^*$.

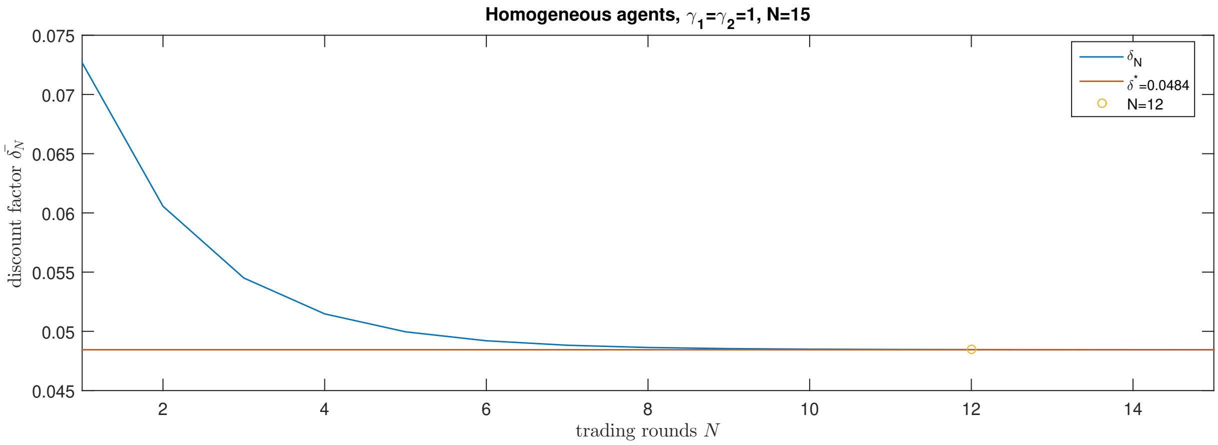
1. In the following diagrams we present the state where agents are heterogeneous in two different occasions. We mention that the covariance of the endowments is slightly higher that zero (i.e. $Cov(\mathscr{E}_{1,0}, \mathscr{E}_{2,0}) > 0$). As it is obvious agent-2 will trade with her counterparty since the $\bar{\delta}$ is at its lowest values as a percentage of the aggregate price on which is accounted the re-trading process. In the first diagram $\bar{\delta}_1 = 1.48\%$ and $\bar{\delta}^* = 0.98\%$, while in the second one $\bar{\delta}_1 = 5.83\%$ and $\bar{\delta}^* = 3.745\%$.



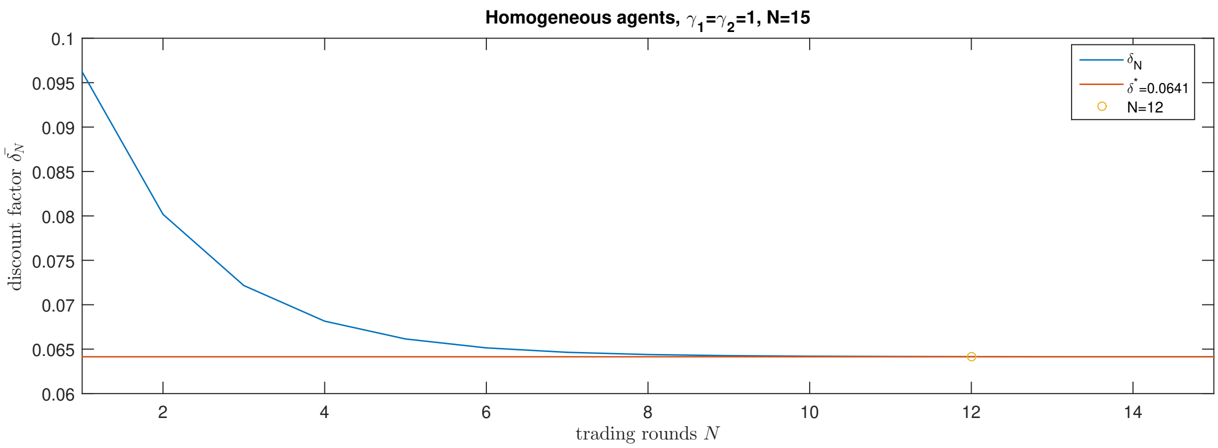
2. Here we present the second diagram of heterogeneous agents, where the $Cov(\mathscr{E}_{1,0}, \mathscr{E}_{2,0}) > 0$ and quite higher than zero. The two crucial differences are, firstly, the aforementioned domain of $\bar{\delta}$, which moves in higher values than the first diagram (it was anticipated due to the monotony of the discount factor towards Σ_1^o) and secondly the $\bar{\delta}$ converges to $\bar{\delta}^*$ after $N = 11$ rounds.



3. In the next diagrams, we present two different cases of homogeneous agents, where it is apparent that $\bar{\delta}$ is quite higher than the preceding case. As we can see in the third diagram $\bar{\delta}_1 = 7.27\%$ and $\bar{\delta}^* = 4.84\%$, while in the forth one $\bar{\delta}_1 = 9.62\%$ and $\bar{\delta}^* = 6.41\%$. In the third diagram the covariance of the endowments is slightly higher than zero.



4. The following diagram depict the case where the $Cov(\mathscr{E}_{1,0}, \mathscr{E}_{2,0}) > 0$ (quite higher that zero) and hence the discount factor as an increasing function of the Σ_1^o moves in fairly higher levels than the previous diagram.



The key question to answer here is, why the transaction costs are higher in the situation of homogeneous agents than the situation of heterogeneous. According to all the above, if δ is increasingly high, then agent-2 will pay some great value for the transaction. Given that, for the risk aversions holds the inequality $\gamma_1 < \gamma_2$, she does not have the will to do so, because the gains of the transaction are in favor of agent-1. Namely, the transaction will be made if the $\bar{\delta}$ is at its lowest values, which is justified by the first and the second diagram too. Diversely, in the case of homogeneity, since the profits are higher for agent-2, she has the will to pay a higher amount for the transaction costs. This is verified from the third and the fourth diagram.

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