«An Arrow Debreu Implementation of the Recovery Theorem»

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by

Tsogka Panagiota

MXRH1543

Supervisor: Lecturer Nikolaos Englezos

Committee

Professor Nikolaos Apergis

Lecturer Nikolaos Englezos

Lecturer Antonia Botsari

Department of Finance and Banking

University of Piraeus

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Abstract

Ross recovery is an option pricing model. The recovery theorem is a tool developed by Ross to determine the predictive content of market prices. It is an option pricing method similar but different to Black - Scholes. Ross recovery enables the investors to disentangle the future return distribution and pricing kernel from option prices which are in the Arrow Debreu environment and thus, option prices become state prices. Moreover, Arrow and Debreu model is the central model of General Equilibrium Theory where state prices use the sense of numeraire. The combination of these two models occurs under specific assumptions. More specifically, under the existence of a Markov process, a transition independence, an irreducible pricing matrix, no arbitrage, complete markets and discrete time.

In this thesis, firstly we present a historical review of the Arrow & Debreu model and its connections with the asset pricing theory. Secondly, we also present a historical review of the recovery theorem by Ross and criticize its extensions. As an empirical formulation of this theorem, we construct the transition state price matrix through a snapshot of option prices on the stock index FTSE/JSE Top 40, using least–squared minimization techniques subject to constraints in Matlab programming. According to Ross recovery theorem this matrix entails the knowledge of the real world probabilities matrix.

Key words: Arrow & Debreu, Recovery Theorem, pricing kernel, risk neutral, arbitrage free, transition independence, Markov process and Matlab optimization.

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Chapter 1

Introduction

1.1 Pricing financial derivatives with predictive ability and an economic reasoning.

There are plenty of types of derivatives and a well-developed theory of how to use their prices to extract the martingale[i] prices and risk neutral[ii] probabilities, *Ross (2013)*. However, it not easy for anyone to use the prices of their derivatives in order to make forecasts. Returns in the risk neutral world are natural returns that have been risk adjusted. In the risk neutral world the expected return on all assets is the risk free rate, the riskless rate of banks, because the risk neutral measure is the natural measure with the risk premium subtracted out. The risk premium is a function both of risk and the market's aversion. In models with a representative agent knowing the risk adjustment is equivalent to knowing the agent's utility function. The last one is not directly observable.

Financial markets' securities have prices with payoffs extending out in time. So, there is always the hope that these prices can be used to make forecasts about the future. This effort has attracted both scholars and practitioners. Nevertheless, the studies of the term structure of interest rates are connected to the efforts for forecasting. On the other hand, the foreign exchange markets and future derivatives markets have not been developed such others.

It is well known that derivative prices have not predictive ability. They do not contain useful predictive information, which is related to the distribution of future financial variables under the real world namely the risk neutral measure. On the contrary, derivatives are forward looking in nature.

« One may ask, whether the liquid derivative prices can be useful to create inferences about the future? », Ross (2013)

The natural expected return depends on the risk premium. It has long proved that any kind of examinations for the efficiency of the markets are at the same time tests for a particular asset pricing model and for the efficient market hypothesis. If we knew the pricing kernel, we could estimate how changeable the risk premium is. Also, a border on the variability of the pricing kernel would determine how foreseeable a model for returns could be without violating the efficient markets hypothesis.

1.2 Historical review. Derivative pricing with the knowledge of the General Equilibrium Theory and Ross Model

It is a historical review approaching pricing of assets. From the time of *Walras 1874* who firstly was interested in general equilibrium with the matter of consumption to today with the case of derivatives. How the *Arrow & Debreu* model *1954* connected the general economic equilibrium to pricing assets such as options.

1.2.1 Time-state preference model

L.Warlas[iii] created the marginal theory of value and he developed the General Equilibrium Theory. He is believed to be the father of the General Equilibrium Theory. He was the first who studies this theory assuming that there is always equilibrium in the market. He studied these themes in *1874* and in *1877* he published his paper work. In principle, Warlas putted forward the state of the economy at any point of time as the expression of the existence of a unique solution of a system of equations expressing the demand for goods by consumers, the supply of goods by producers and the equilibrium between these two partners. Unfortunately, he did not conclude to this solution, he did not give to his publication the solution!

$$\text{Demand} = \text{Supply}, \quad \text{for every market}$$

Another important paper is this of ***Samuelson P***. in *1937*, which the commodities are defined so as to create the desirable equilibrium in the economic system. The paper explains the importance of the allocation of these commodities between the two partners in the market and finally, it births the commodities with the properties which became later Arrow-Debreu commodities, *Arrow & Debreu (1954)* and *Geanakoplos (2004).*

It is always assumed that every consumer cares to maximize his utility, every producer cares to maximize his profit and the perfect competition wins independently of everyone choice. There is disadvantage in Walras study. Walras missed to conclude. But Arrow and Debreu' model did it! Also, the *Arrow & Debreu* proof is better because it is more coherent with the modern theory.

The ***Arrow-Debreu Model***, known as A-D model or A-D, we will use the expression A-D for now on, is a model of a static, multi-good, multi-partner or multi-agent economy. The A-D model is the central model in the General Equilibrium Theory and it uses state prices proving the existence of a unique solution, namely a unique general equilibrium. In financial economics, a state-price security is an Arrow–Debreu security and also, it can be called pure security or primitive security. These state prices use the sense of the numeraire. Numeraire is a basic standard, a constant by which value is calculated. In mathematical economics, numeraire is a tradeable entity. In monetary economy the numeraire is one of the function of money. Well, in this thesis numeraire will refer to a particular good as a benchmark and all other prices will be adjusted to the price of that good, just as general equilibrium theory does, *Arrow-Debreu (1954).*

State Price Security Pays:

- 1$ of numeraire if this certain state occurs in a certain time in the future
- 0$ of the numeraire in other states.

The A-D prices may be represented by a vector. We will use the help of the matrix to show the state prices as a vector. The state price vector is not only the vector of a particular state, but also the state prices for all states. So, any derivative contract whose settlement value is a function of an underlying asset, such as an option on a stock with an uncertain value at contract date can be analyzed as a linear combination of its Arrow-Debreu securities. Thus, any financial derivative contract can be expressed as a weighted sum of its state prices.

Basic Assumptions of the A-D model, *Arrow-Debreu (1954)*

1. Convex Preferences. With extra assumptions in this we conclude in a unique equilibrium in contrast to many equilibria in general level.

2. Perfect Competition

3. Demand Independence

There must be a set of prices which holds in every commodity in the economy.

Aggregate Demands = Aggregate Supplies

A-D model is one of the most general models of competitive economy and is crucial for the General Equilibrium Theory.

Breeden and Litzenberg in *1978,* mainly talked in their paper about the prices in A-D model. They examined and introduces a multi-period economy and they started to connect the prices of A-D commodities with probabilities and the prices of derivatives more specifically with the options. They used the example of the price of a European call option. In this paper, expressed a single-good model which can be extended to a multi-good if the assumption that all individual on the economy have the same price index holds. Finally, they conclude to the equilibrium between risk neutral probabilities and state contingent prices.

Furthermore, ***McKenzie*** in his paperwork in *1981* introduced many improvements to A-D model. He kept the assumptions of finiteness and the convexity for the preferences. But he did not adopt the assumption of transitive preferences like in A-D model, using a mapping of social demand. Also, he had a critical position in the role pf firms in the equilibrium. Finally, he conclude that any consumer can be without to trade.

Moreover, a totally different approach to the state price density was the paper of ***Aït-Sahalia and Lo*** in *1998*. A financial asset pricing method with including arbitrage free method and without including investment under uncertainty. It is a non-parametric[iv] estimation of the state price densities of option prices using the Monte Carlo analysis containing the assumptions of Black-Scholes pricing method. In this Monte Carlo analysis the experiments examine the practical performance and extract the state price densities or SPDs. They chose a non-parametric approach because so, they ended up to a more flexible and a more broaden variety of derivative securities and asset price dynamics. Also, they found that if we know at least two of the below we can result in the third. Namely,

1. Representative agents' preferences
2. Asset price dynamics
3. The state price density (SPD)

Dybvig and Ross, *2003.* A usual neoclassical financial model with a single-period context. They are interested in asset pricing. In this paper is expressed the pricing rule representation theorem or in other words, risk neutral representation or martingale representation. They adopted positive linear pricing rules which can be expressed as single state prices, risk neutral expectation and state price density. They also were interested in the portfolio choice problem. With the absence of arbitrage and free preferences they conclude that anyone can result in equilibrium prices when the preferences of each individual, each agent in the economy are known.

Also an important paper which is connected more to the A-D model is the article of ***Geanakoplos*** in his publication in *2004*. This paper did not provide newly information about the central of general equilibrium theory, the A-D. He mainly explains in a more comprehensive way the A-D model fully analyzing it and also, providing comprehensive examples. Instead of that, in the last pages he gives cases in which the A-D model is not enough. Important issues that present that A-D need some improvements to be more connected to the modern economy.

Some of the most important cases are the below

1. Bankruptcy is not acceptable in A-D world. That derives from that all agents in A-D model must obey to their budget restrictions.
2. There is no asymmetric information in A-D. There are only rational agents without taking into account what other agents in the economy believe and choose to do. Otherwise, the optimal equilibrium will be lost.

The Pareto[v] optimal allocation is tolerable as an A-D equilibrium.

Last but not least, the paper of ***Ludwig*** in ***2015.*** This article is interested in option markets and uses the robust estimation of state price densities (SPDs). It examines how to specify a well behaved SPDs in a state space from only a picture at a certain time of option prices. This picture of the certain time is very important in order to select the appropriate model and choose correctly the parameters. Finally, he conclude in providing information of current market sentiment and so, allow the researchers and readers to understand the investors' expectations about option prices and their perception about risk.

1.2.2 Recovery Theorem

It is an asset pricing model. A model which prices assets, stocks, financial derivatives, options etc. In this thesis we will occupy mostly on derivatives as options. The Recovery Theorem separates the risk aversion and the natural, namely the normal, probability distribution and derive their level from market prices. Over and above, it calculates the market's predictions for returns and the market's risk aversion from state prices alone. Risk neutral prices, A-D prices, are the product of risk aversion and the natural probability distribution, *Ross (2013).* This separation is very important because it gives us the opportunity to evaluate the market risk premium, the probability of a catastrophe and to contract model free tests for the efficient market hypothesis.

In the framework in the theorem there is a basic assumption. The martingale measure is observable. This means that not only the current state prices are observed, but also the martingale transition probabilities are observed.

In the Recovery Theorem from options move to the distribution of prices in a discrete time, discrete state space model. The Ross model ends up with the conclusion that if someone acquaints the diagonal state matrix with the undiscounted kernel on the diagonal of the matrix, then he will can define the exactly amount of the natural measure.

Another two important assumptions which there are in the recovery theorem in order to end up in the unique solution to the problem finding the probability matrix, finding the discount factor (*Ross* prefers the symbolism with the Greek letter δ for the discount factor) and the pricing kernel.

1. The underlying process is Markov. Then, the state space is discrete.
2. The pricing kernel is transition independent. This depends only on the current state.

So, markets subjective distribution of future returns does not include the historical distribution nor the parametric preferences.

The Recovery Theorem, *Ross (2013)*

In a world with a typical agent for any given set of state prices there is one and only one equivalent natural degree, and then, there is only a unique pricing kernel.

Let's define the probability, the martingale probability and the basic hypothesis for the Recovery Theorem.

- **P** is the frequency with which the market believes that future states happen. This frequency ends up in market prices, which also reveal investor attitudes towards risk. P does not need to be the true or the real probability.

- **Q** is the martingale probability measure. It is equivalent to risk neutral measure.
- The markets must be complete ⟶ Q is unique
- The utility function of the representative investor is an independent state and intertemporal additively separable with a constant rate of time preference
- **X** is a single state variable. Under Q it is a time – homogeneous Markov chain[vi] with a finite number of states. Thus, under P, X is also a finite – state time-homogeneous Markov chain and we can recover the transition probability matrix P of X from the assumed known risk-neutral transition probability matrix Q.

So, for any given set of state prices there is a unique natural measure and a unique pricing kernel, as we said in the previous subchapter in A-D model.

1.2.3 Ross Theorem's extensions.

There plenty of articles which examine the assumptions and the boundaries of the theorem and finally, propose how to extend the Ross theorem.

A first example of the kind of an extended recovery theorem is in the paper of ***Carr & Yu*** in *2012*. This paper searches the assumptions of Ross' approach. It suggests an alternative diffusions on a bounded state space, with limitations on the shape and dynamics of the numeraire portfolio without including agents' preferences. This article holds the theorem if the limitation on independence of utilities hold too, instead of the numeraire portfolio and its dynamics.

A second example**, *Dubynskiy & Goldstein*** in their paper in *2013*, examine consequences of bounds on the state space which are in Ross theorem. This approach has similar points with the above to *Carr & Yu*, *2012*. They believe that Ross restrictions on state vector dynamics are not plausible in reality. They stand that bounding allow recovery. Besides, derivatives prices do not supply further clues about actual trends and conditions corresponding to the risk aversion of the representative agent. Also, another contribution of this paper is that recovery of trends or drifts as *Dubynskiy & Goldstein* prefer to present, and preferences variables is might from cross sectional data only without the need of bounding state vector dynamics.

Moreover***, Martin & Ross,*** *2013,* studied the behavior of a long zero coupon bond which settles in the long future. Options on long bonds[vii] reveal the forward looking expected return on the underlying asset and the last's volatility movement. Also, the yield on these long bonds exposes the time preference level of a pseudo representative agent now. Another result from this article is that the yield curve must have slope on the average. Their paper is under two assumptions which are likely to the basic Recovery Theorem.

These are:

1. The market is complete. The fixed income derivatives are the best example of this kind of investment.
2. The state vector follows a Markov chain movement. So, it is stationary.

All above under the reasoning of no arbitrage market.

Furthermore, an extended recovery is that of ***Walden*** in *2013,* which explores the difficult of recovering the pricing kernel and the real likelihood from options values when the state parameter is a without bounds diffusion. As the paper of *Carr & Yu (2012*) who showed that for limited diffusion procedure, the recovery is likely. The article of Walden extends the methodology to the unbounded condition. Even if the space is limited, it continuous to exist a problem because of observable asset prices. Finally, he concluded that when recovery exists without worrying about the observable prices, then the kernel probability distribution may be approached well without enforcing extra limitations.

Another extended recovery approach is this of ***Audrino, Huitema and Ludwig*** (*2015).* In this approach they established a non-parametric valuation approach for the Ross' recovery, *Ross (2015).* They used options on the S&P 500 in the period from 2000 to 2012, investigating whether or not recovery yields have predictive ability regardless of what anything can be observed from risk neutral densities. As in the paper of *Breeden & Litzenberg (1978)* here in this article also lets us to calculate the prices for A-D eventual on the present state of the underlying asset.

Moreover, they used out of money options because they are more liquid, but they transformed these by using the *Aït-Sahalia & Lo (1998)* implied volatility way from put call parity, and so, they ended up with in the money options.

Over and above, there in an extended recovery approach from ***Borovicka, Hansen*** and ***Scheinkman*** in *2015*. They state asset prices include important clues about the probability distribution of future states and the stochastic discount factor. Asset prices are forward looking in nature as all investments. Invenstors beliefs are included in these prices. So, literature and policy makers follow the movements in financial market data and so, they try to elicit the view, beliefs of the private sector and the whole macro-economy mood. *Borovicka, Hansen and Scheinkman* insist on supplementary limits to data on asset prices because the last are not enough without the stochastic factor and transition likelihoods. More specifically these restrictions are.

1. Time series signal on process of the Markov state.

Or

2. Information on the marketplace defined stochastic discount factors.

Last but not least, is the extended recovery from ***Backwell*** in *2015*. He points out that Ross derivation, *Ross 2015*, is unceremonious and unfinished. Backwell makes some remarks in order to improve some of the boundaries of the recovery theorem. In his empirical part uses a snapshot of data on 18 September 2013 on FTSE/JSE Top40 applying Ross recovery.

He accepted these assumptions of the primitive model such as:

1. The market is complete and no arbitrage as Ross Recovery, *Ross 2015*.
2. There is a representative agent with expected utility function, *Samuelson 1937*.
3. There is unique balance on consumption.
4. A discrete and bounded state variable which follows the Markovian chain.

He made some remarks in order to be more comprehensive his work pointing the problem of Ross recovery assumptions and limitations. These six are:

- Remark 1: The boundaries of the primitive basic Ross model are wide enough to not worry about. On the contrary of other approaches as *Dubynskiy & Goldstein (2013)* did.

- Remark 2: The preference relation is incomplete because it does not succeed to captivate any interaction between two time points. The whole theory of expected utility has disadvantages on empirical base. So, he discussed preferences as meta-structural preferences.

- Remark 3: Consumption is imposed only by the current state not by the time or previous state.

- Remark 4: The Ross Theorem is an expansion to the class of preferences for which recovery is possible.

- Remark 5: State independent bond prices mathematically push the marginal utilities to be equal. Then, prices are simple with probability rather than the classically complicated interaction with the entire range of states. Also, literature illustrates the risk neutrality by linear utility functions, and so, the whole system in a matrix equation in a diagonal matrix is a scaled identity matrix.

- Remark 6: In his applications he is interested in including volatility in the state space. However, a discretized volatility might cause problems on estimations. His idea for a various states space without changing the basic assumptions is to include the former state in the state picture. The former state is a clever representative for volatility. High volatility state creates a better estimate Markov chain where the time homogeneity [vi] technique is more robust.

In his empirical section Backwell met two obstacles

1. The state space needs explicit definition. A solution to this is to keep the level of the index FTSE/JSE Top40.
2. However, financial prices do not follow Markovian process. Then the second problem raises which is the state price. Finally he used the time-homogeneity equations for some periods supposing high volatility.

Thus, Backwell concludes in a more relaxed state independent preferences recovery. He adopts the neutral technique to transform traded prices to state prices.

1.2.4: Empirical approaches to Ross Recovery Theorem

Three empirical approaches which are used in our empirical part, represent, analyze and make conclusions through Ross Theorem, *Ross (2013).* These approaches are of ***Audrino, Huitema*** *&* ***Ludwig*** *(2015),* ***Kiriu*** *&* ***Hibiki*** *(2015*) and ***Spears*** *(2013).* These papers access more the empirical part of the theorem explaining the construction of the crucial matrices such as S, P, Q and the mathematical reasoning which is needed for this process even some useful points for the final algorithms. All of them used the Matlab program in order to recover the option prices. They were based on the S&P 500 index option in the specific date as *Ross (2013)* uses which is 27 April 2011.

1.2.5: Pricing Kernel. A discount factor is a generalized method of moments[viii] with an only one assumption. Which is that the investor is free to think between a small investment or a small disinvestment. Stochastic Discount Factor (SDF) is a basic part in financial economics and mathematical finance. SDF is crucial in asset pricing. Now, where we know the pricing kernel all financial assets such as stocks, bonds, options etc., are pricing using the same methodology. Also, another important advantage to pricing kernel is that is simply and preferable to anyone than other methods of pricing financial assets.

There is the stochastic discount factor when the market is complete. It is a range of contingent prices in other words state prices, *Arrow & Debreu, (1954)*, and *Breeden & Litzenberg, (1978).* These prices are classified by probabilities which belong to risk neutral world without risk adjustments.

The name "stochastic discount factor" explains completely that the price of an asset or a derivative can be calculated by "discounting" the future cash flow **Xi** by the stochastic factor **m** and then, resulting in the expectation **E**, *Cohrane (2005).*

If {
- 1,….., n assets
- $p_{1,\ldots,}$ p_n with initial prices at the beginning of a period
- $x_{1.,.}$ .., x_n payoffs at the end of the period

} **then**, the SPD is any random variable m

$$\mathbf{E\ (mx_i) = p_i,\quad for\ every\ i}$$

«But what does this expressions mean? Can someone always find such a discount factor? Can we use this convenient representation without implicitly assuming all the structure of the investors, utility functions, complete markets etc.? » Cohrane (2005).

1.3 Description of the thesis

The primary goal of the thesis is to understand the work of Ross and to piece together his vital econometric process. This paper addresses the implementation of the Ross Recovery Theorem.
In chapter 1 thesis introduces its theme namely the combination of two models, Ross model which is a solution to the problem of determination the market's future returns by option prices alone. And *Arrow & Debreu model* (*1954)* which is the central model of the General Equilibrium theory.
In chapter 2 the *Arrow & Debreu* model (*1954)* is presented, discussed, pointing out crucial information on pricing derivatives.
In chapter 3 the Recovery Theorem is discussed, analyzed and formalized combining with the Arrow-Debreu environment mainly through mathematical way. Furthermore, the thesis further searches the theorem as the guided paper suggests *Backwell (2015)* and advices. Also, many extension and even critical approaches to recovery theorem are presented to thesis in this chapter.
In chapter 4 is the empirical application part with the use of the Matlab program in order to be constructed the state space which involves the levels of the stock index, the transition state prices matrix and the real world probabilities matrix.
Last but not least, the final chapter 5 involves conclusions to the whole work both mathematical discussion and Matlab applications.

Chapter 2

The Arrow-Debreu model and the SPD

2.1 Arrow-Debreu Model (A-D)

A significant issue in the economics of investment under uncertainty is the time-state preference model of Arrow-Debreu, *Arrow & Debreu (1954) and Aït-Sahalia & Lo (1998).* In A-D model is introduced elementary securities each paying 1$ in a certain state of nature and nothing in any other state. Now, known as Arrow-Debreu securities, which develop much of our contemporary understanding of economic equilibrium in an uncertain state of nature. In a continuum of states, the prices of Arrow-Debreu securities are defined by the state-price density (SPD), which gives for each state x the price of a security paying $1, if the state is between x and dx, *Arrow and Debreu (1954*).

If the state of nature $\in [x, dx]$ then, the price is 1$, otherwise is 0.

2.1.1 The Model

The most important and central part of General Equilibrium Theory is the A-D model of a static, multi-good, multi-agent economy.

The model needs to specify three fundamental matters:

1 An Economic Environment, which is described as:

- A set of agents. There are only two kind of partners in the economy. The consumers and the producers, who can play either or both of these roles at different times. When we say producers, we usually mean the firms.
- A commodity space

An economic characterization of each agent in terms of:

-A set of admissible actions. In order to be rational the consumers.

-A binary preference relation on the set of admissible actions

-An initial resource endowment

-An information system together with a system of beliefs about the state of the world

2 A Resource Allocation Mechanism, which is the competitive market

3 A System of Property Rights, such as that agents in the economy own all resources and all factors of productions.

2.1.2 Commodities in A-D model

Let there be L commodities, **l=1…., L**. The amount of a commodity is described by a real number. A list of all commodities is given by a vector in $\mathbf{R}^{\mathbf{L}}$. The sense of commodity is the primitive idea in economic theory. Each commodity is assumed to have an objective, quantifiable and universally agreed upon description. The differences between production and consumption are defined in terms of transformations of commodities that they cause. The set of commodities is the minimum pool of objects which is necessary to define production and consumption. Other objects, such as financial assets which may be traded, but they are not commodities.

General Equilibrium Theory is interested in the allocation of commodities with many ways like nations, individuals, time, etc. The individual is called «home economicus» in *Samuelson's paper (1937)*. The individual or home economicus is very important to be specified. His scale is required to be in certain ideal conditions where his observable performance will make open to unambiguous inference the shape of the function which is captured by maximization. The A-D model studies those allocations of commodities. It can be done with the help of the exchange of commodities at one moment in time. It is crucial to the agents in an economy to have precise physical description of commodities. So, agents know everything about the commodities. For example, they want to know all the clues of placing an order for a particular grade of steel or oil. The less untouched the classification of commodities becomes, the more space for agents to trade and the greater is the set of imaginable allocation, *Samuelson (1937).* When the descriptions for the commodities are so precise and so, further improvements cannot create imaginable allocations which raise the individual satisfaction of the agents. Then, these fully specified commodities are called **Arrow – Debreu commodities.**

An example: How an action sometimes depends on others.

If a field is more fertile and productive than another, it depends on how much rain has fallen on it. It depends on how much rain has fallen on other fields too. This demonstrates the paradoxical practicality of including in the description of an A-D commodity features of the economy.

Moreover, there are features which with a first glance is not important for the commodity but later we understand the connection between them. For instance, the commodities' geographic location, their state of nature and perhaps, even the name of their final consumer do not seem connected in the first place with the object itself, but they are in principle observable. This point of view for the sense of the commodity helped A-D to be connected with uncertainty, *Arrow & Debreu (1954)* and *Geanakoplos (2004).*

In reality is very difficult to not have connections and so, to have a pure market. The information about the commodities determine the amount of buyers and sellers. The amount raises when the description is for the start well known to all without unexpected connections. It is often than many sets of Arrow-Debreu commodities are traded together in different times. Nevertheless, this understanding of the limitations of real markets based on the idea of the A-D commodity is one of the most important issue in systematic accounting available to the General Equilibrium Theory. The A-D model with its idealization of a single market for each A-D commodity. When all hold simultaneously the calculation of the real economy can be done.

2.1.3 Consumers on an equilibrium base

In this base there should be H consumers**, h=1…... H**. Each consumer h can have plans and goals for his consumption which hold $x \in R^l$ in a set of consumption X^h which is a closed subset in R^l. Also, each consumer h has already a well determined set of preferences $\geq$ h which hold in every couple of $(x,y) \in X^h$.

Where $x \geq y$ ⟶ x is at least as derisible as y

Assuming that the relation $\geq$ is a complete, transitive, endless arrangement. This represents the neoclassical idea of rational consumers.

In the general equilibrium theory consumers have options not only between individual commodities, but also in the whole range of consumption plans. A sole product is important to consumer in his decision making process only in relation to the other commodities he has already consumed or plans to consume in the future. Thus, with the assumption above about the properties of relation $\geq$ we have the neoclassical idea of rational consumer as in *Geanakoplos*' paper (*2004*) is pointed out.

So, once utility is a function not of momentary but of the whole consumption strategy, then rational choice is equivalent to utility maximization.

Assumptions as A-D model for preferences, *Arrow & Debreu (1954)*

1. No satiety. This is in accordance with human nature.
2. Convexity. This infers that commodities are endlessly divisible. But when commodities are finely dated and so they must be though as flows, then this assumption is out of place. In any circumstance, if every agent is relative to the market, thereafter the non-convexities in preferences are quite useless.

- So, for each $x \in X^h$, there is a $y \in X^h$, with $y > x$

X^h: is a convex set and $\geq$: is convex

- If $y > x$ and $0<t<1$, then, $[\, ty + (l\text{-}t)\, x \,] > x$

Also, every agent-consumer h is considered to a vector of primary properties, which in the A-D model are called «endowments», e^h.

$$e^h \in X^h \in R^l \,, \qquad \text{for every } h=1\ldots,H$$

The «endowment» vector e^h as it is defined in the above expression, expresses the requirements which the consumer can have on all commodities without the necessity to own them. The relation $e^h \in X^h$ means that the consumer-agent in the economy can continue to exist even if he does not have any opportunity trade. Doubtless, this meaning is not so strictly as a fact in A-D model. Every consumer h has an ownership share of every firms in the market. Firms are expressed as j and j=1…..J. The firm in A-D is defined by its original distribution of possessors and by its technological efficiency. Production strategy is symbolized as y with $y \in R^l$.

Inputs	=	negative components of y
Outputs	=	positive components of y

So, if $y \in Y$ with $y \in R^l$ then any production plan is accomplished.

The classical theory is inseparable by its adoption of the second assumption above. The economy includes a limited amount of consumers who commerce in a single-good marketplace under circumstances of certainty, *McKenzie (1981).* The goods in this economy of certainty are finite and as also, the horizon is finite.

Production is demonstrated in a linear way or as convex input (negative components) and output (positive components), *as Arrow & Debreu (1954)* which belong to a finite number of firms in the economy. Consumption and relations on preference are convex as the 2^{nd} assumption on A-D model and jointly irrespective, *McKenzie* (*1981).*

3. Another assumption on consumers' base which is made in the A-D model is that they are independent of disposal. If l=1,…,L is any commodity and v_i is the unit vector in R^l with one in the l-th coordinate and zero elsewhere, then,

for all commodities l=1…L and k>0 , $kv_i \in Y_i$ for some firms j=1…J

Although it is bizarre to take into account that any commodity in the economy can be traded without charge.

4. In the empirical base now. The most important assumption is that for every j the Y_i is a closed convex set including the value of zero. This convexity assumption for Y_i guidelines agents in production, raises returns to scale, benefits through specialization, etc. If individuals of production are small, not important to the whole economy, then the results are not much affected. However, when individuals are large or when there are noteworthy raised returns to scale, then the model of competitive equilibrium cannot be applied. Nevertheless, convexity is consistent with reducing and stable returns to scale.

5. The level of productive activity has got limits. Even if the productive part captures all the options, all the sources of the consuming sector.

6. The economy does not diminish. So, for every two agents h and h´ the «endowment» e^h of agent h is positive in a commodity l, where agent h' might want to use and use it in order to make himself in a better position. It surely sounds rational that every agent's labor power can be used to make another agent wealthier. Surely, the potentials of production are taken into account.

7. The commodities are not classified by whom firm are produced or who wastes them. According to the competitive equilibrium this infers there are no externalities to production or consumption. On the other hand, form mathematically point of view this has no meaning. This assumption is essential for the matter of rationality in the theory and decision making behavior.

In the prototype and unconventional A-D model, *Arrow & Debreu (1954)* the firms have no right of owning initial resources. In the A-D equilibrium consumers might not pay same prices for H commodities. Hereafter, the discrepancy payment principle has be talked about in *Samuelson*' paper in *1937*. He points it out because he wanted to highlight the existence of a qualitative discrepancy between public and private goods economies.

McKenzie innovations on Arrow & Debreu model

Nevertheless it is important to mention the paper of McKenzie (1981) and his innovations on A-D model mainly on its assumptions. Following years of the primitive A-D model in 1954 create a new existence of this model in a multi different directions. The most important regarding to the article of *McKenzie (1981)* is the inference on the assumption about finiteness and convexity. Also, a crucial innovation is that preferences do not have to be transitive or complete. The hypothesis that consumers can continue to exist without trade is inessential for an undiminished market. *McKenzie, (1978)* takes into account the demand functions.

New innovated assumptions by *McKenzie (1981)* on A-D framework

X_i: a set of possible trades, a consumption set. Which can be done in the economy of i=1,….., m consumers
Y: firms in the economy

1. X_i is convex, closed and having limitations. Namely, these possible trades are convex implies that the relation among preferences follow this statement $\geq_i$. In other words, if $\mathbf{x} >_i \mathbf{x}'$ then, $\mathbf{x}'' >_i \mathbf{x}'$. So, there is a ξi, such that, $\mathbf{x} > \xi_i$, that retains for each x which $x \in X_i$

2. X_i is strictly classified by a convex and closed preference relation.

3. Y is a closed convex cone. This points out the crucial role of constant returns to scale by way of a presupposition to be the market ultimately competitive economy. Maybe, this can approximately exist when efficient firms have small size. This type of error about the small size is equal to the error of including the hypothesis about convex preferences on undivided commodities.

4. $Y \cap R_n^+ = \{ 0 \}$. This restriction does not actual exist because does not taking into account that commodities are available in any wanted quantity without cost.

5. $X_i \cap Y \neq \emptyset$. There is a common point of view between Y and X. Firstly, any agent-consumer in the economy may exist without the obligation to trade. Secondly, rational agents always select the price- space so as any price **p** will have $px < 0$ for some $x \in X$. Thus, if p is in accordance with the general equilibrium in production. Then, there will be a trade for consumers who have negative value. For example, a consumer can have income without limitations on his consumption.

6. Having m consumers in the economy let's define I_1 and I_2 sets of indicators for consumers.

$$I_1 \cap I_2 = \emptyset$$

And

$$I_1 \cup I_2 = \{1, \ldots, m\}$$

So, I_1 and I_2 sets can be selected whether $x_{I1} = y - x_{I2}$, with $x_{I1} \in X_{I1}$, $y \in Y$ and $X_{I2} \in X$. The goal of this 6th hypothesis is to guarantee that every individual in economy has got income. Thus, if a consumer has income at any level of price that maintenances the production set Y at y and exactly the same at the sets of consumption at the points x_i then, selecting I_1 consumers will end with an income $px_i' > 0$.

2.1.4 Prices and the General Equilibrium

Price is the central point in the A-D model, *Arrow & Debreu (1954)*. As commodity the price is something which can be exactly calculated and quantifiable. The A-D model has stated the crucial part which mathematics have in economies which is partially possessing to the quantifiable sense of these two original concepts the commodities and the prices. Thus, the wealthy mathematical relationship of double vector spaces can easily order the groups of price values and commodity quantities. The general equilibrium with the definition by A-D model states that it is adequate to offer L_l of these amounts of prices and commodities and all the rest can be specified.

The A-D economy is an array. We symbolize it as **E**:

$$\mathbf{E} = \{L, H, J(X^h, e^h, \geq), (Y^j), h=1\ldots,H, j=1\ldots,J\},$$

This equation satisfies the three basic assumptions we have already talked about in the subchapter 2.1.3: Consumers on an equilibrium base. Which are the 5th about the boundaries on the economy, the 6th about the undiminished nature of the economy and the 7th about no externalities.
So, for all firms j=1…, L, and for all consumers h=1…, H, we have the following.

$$\text{\underline{For all firms}: } y \in \arg\max\{ \sum \frac{py}{(y=y_1 \dots y_L)} \in Y_j \}$$

$$\text{\underline{For all consumers}: } X^h \in B^h(p)$$

$$\text{Where, } B^h(p) = \{x \in \frac{x^h}{\sum p_i x_i} \leq \sum p_i e_i^h + \sum d \sum p_i y_i\}$$

And

$$\text{if } X^h \in B^h(p), \longrightarrow \text{ then does not } x > x^h$$

$$\text{\underline{It holds}: } \sum x^h_i = \sum e^h_i + \sum y_i$$

The general equilibrium is symmetrical. All agents have access to the model separately, independently and motivated by their self-interest alone. Thus, no agent performances before any other does. In A-D equilibrium there is no cause to be a united rate of revenue. Finally, while the commodities are likely to contain physical goods which are dated over several periods there is only one budget restriction in an A-D equilibrium. The revenue that could be gained from the sale of an «endowed» product which is dated from the latter time is already available from the first period.

2.1.5 From stock prices to option prices

One of the major issues in economics is that when the markets hold in properly then prices carry major evidence which can be used to help the decision maker to take rational and beneficial decisions for him and for the whole economy. For instance, in finance using the information of prices of current bonds can help someone to conclude with future spot interest rates forecasts of inflations or even with the expectation of turns in economic cycles.

The effectiveness of the conclusions above depends on four characteristics of the model and the market, *Arrow & Debreu (1954*) and *Geanakoplos* (*2004*).

1. A compensatory model which combines prices with expected information.
2. A model which can be applied by appropriate time and low-cost techniques.
3. A model which calculates correctly the exogenous inputs.

4. A <u>market</u> which is <u>efficiency</u>.

In financial economics a major role perform financial derivatives and more specifically options because the latter are used to generate probabilistic information. However, in the markets are trades more American than European options and so, these American options do not generate easily conclusions about risk neutral probabilities.

The absence of a natural agreement on the set of state definition provokes important difficulties in examining information about observable financial assets from the point of view of underlying decisions on sequences of time state. In *Breeden & Litzenberg (1978)* signalize that the state preference approach is more general than the mean-variance approach. The first one, the state preference approach offers a not so difficult framework for examining theories. The time-state preference approach to general equilibrium, as analyzed in the paper of *Arrow & Debreu (1954)* is one of the most universal frameworks congenial to finance under uncertainty. The value of unknown future cash flows can be easily calculated by prices of initial securities. On the other hand, the implementation of the time-state preference approach on economic issues such as capital budgeting has not bounced. Unfortunately, it is difficult to end up with empirical part in the examination process as *Breeden & Litzenberg (1978)* have signalized.

«What is happening at multi-period economy? », Breeden & Litzenberg (1978)

«What is happening by delivering the prices of primitive securities from the prices of European call options on aggregate consumption expenditures at each date? », Breeden & Litzenberg (1978)

These prices in multi-period economy at equilibrium allow the evaluation of assets with uncertain payoffs at many future dates. For any given portfolio the price of a $1.00 is established at a time in the near future. If the portfolio's value stands between two given points at that time then, it is resulting from a second partial derivative of its call option pricing payoff. A capital asset pricing model which holds in every nature of time is inferred from payoffs whose distribution is joint log-normal with the total consumption of the economy. Pricing option methods with a stochastic interest rate stays an inextricable economic problem. *Breeden & Litzenberg (1978)* are interested in a single-good model which can be prolonged to a multi-good model if all partners in the economy belong to the same price index. The meaning of a price index that does not relate to individual's wealth demands that all income elasticities of demand must not be separate. *Breeden & Litzenberg (1978)* showed if individuals are time-dependent and state-independent utility function for every level of consumption and of potential consumption with the regard of probabilities of states of world, then every individual's ideal consumption level at that date can be stated as a function of total consumption at the same date. Therefore, any Pareto ideal differentiation of time-state eventual among individuals can be done in securities markets consisting only of the type of European not American call options. In more analytical approach we will discuss the existence of Pareto conditions in the economy in following sub-chapter 2.1.6. Prices of initial securities can be phrase from prices of rudimental total consumption. From this definition of initial security prices an estimation calculation can be done for all present securities in the market and capital budgeting projects always regarding to prices of European call option on total consumption.

The price of any unknown cash flows in the future like flows of capital budgeting project depend only on the uncertainty of the underlying asset value in the future. This value of any asset is not necessarily linear or with joint normal distribution or multivariate normal distribution.

Moreover, a serious issue which is taking into account in the paper of *Ludwig (2015)* is that about state price densities, SPDs. These densities hold the risk neutral probabilities which market signalizes in every state of the underlying asset until options expire.

« How can an individual in financial economy estimate well-behaved state price densities (SPDs) in a uniform space of states of nature using a snapshot of option prices? », Ludwig (2015).

« An answer: This question is difficult to be answered», *Ludwig (2015).*

Ludwig (2015) concluded in SPDs which are real only under certain conditions, and having forward-looking sense. These SPDs allow us to research how risk neutral measures exist down through the years and also, these densities offer a base for the recovery of real world probabilities namely risk neutral probabilities. In his estimation in his paper *Ludwig (2015)* uses parametric and non-parametric measures, the last one as *Aït-Sahalia & Lo (1998)* preferred on their estimation. Parametric techniques depend upon certain presuppositions on the function which combine inputs (negative components of production strategies) and outputs (positive components of production strategies), as we already have talks about above in the subchapter 2.1.3. On the other hand, non-parametric techniques depend only upon that the unknown function displays smoothness and these techniques are based to data. This smoothness means that the economy exists as in neoclassical case with an idealized form of economy without transaction cost, asymmetric information to agents, taxation or indivisibilities

From *Ludwig's (2015)* point of view, the estimation of parameters and the selection of models depend on the appropriate choice of a population of solutions which will be evaluated in the future by their properties and their standard errors. His approach used a snapshot of option prices and it is robust enough to deal with the problems of selecting the appropriate model and classifying the parameters. He used the in-sample and out-of-sample technique for model's estimators for the period of twelve years. *Ludwig (2015)* preferred this kind of estimation because these estimated SPDs surfaces offered a more understanding snapshot of current market sentiment. Also, this estimation allowed researchers to understand and follow the evolution of investors' expectations and their risk position in a period of time, here in 12 years.

Moreover, another crucial approach is this of *Dybvig & Ross* in *(2003)*. This combine the option prices to the portfolio case. General equilibrium theory always wanted the success in generality but nowadays, in financial economics it is preferred models with strict assumptions and restrictions which can be examined and be putted in practice. In their paper in *2003* they used a single period way approximating the asset pricing fundamental theorem with three important existences.

- <u>No arbitrage</u>
- <u>A positive linear pricing rule</u>
- <u>Agents want more to less</u> (differently no agent will succeed his optimum)

} These three generate two theorems in pricing as we will see.

They pointed out a very important relation between pricing rule and state contents. Namely, a positive liner pricing rule can also be expressed using state prices, risk-neutral expectations or a state-price density. They talked about the portfolio problem and how someone can make the best choices in order to accept the best results. A variety of applications in practice use the first order conditions on that portfolio that so taking the desirable results. The first order conditions state that marginal utility on every state is related to a state price density. If markets are complete then, this state price density is defined only taking into account investment chances and must be equal to all agents in the economy.

<u>Thus, simplifying the choice problem we take the following.</u>

Solving first order condition	**gives us**
For quantities	optimal portfolio choices
For prices	asset pricing models
For utilities	preferences
For probabilities	beliefs

An alternative estimation of SPDs using option prices by Aït-Sahalia & Lo (1998)

An alternative approach is this of non-parametric estimation by *Aït-Sahalia & Lo (1998)*. They estimated with a non-parametric way and extracted the state price densities (SPDs) using the actual prices from S&P 500 index on call options. They made plenty of Monte Carlo experiments in order to test if the Black, Scholes and Merton formula holds under the non-parametric SPD estimator. The basic assumption in their tests is an arbitrage free economy.

Implicit in the prices of traded financial assets ⟶ Arrow-Debreu prices

With continuous states ⟶ the state-price density (SPD)

This alternative non-parametric SPD estimator offers a technique with no arbitrage pricing new, complicated, or non-monetary assets while taking into account these characteristics of the relevant data under an asset-pricing point of view. Characteristics like negative skewness and exaggerated kurtosis in returns and volatility «smiles» for option prices. Combining the A-D model, *Arrow & Debreu (1954)*, Sahalia and Lo, *Sahalia & Lo (1998)* examine the economy under the condition of uncertainty. In A-D model the time state preferences represent basic assets which are called A-D securities as we have already seen in the sub chapter 2.1. Investors extract much of present comprehensive economic equilibrium under uncertainty using the knowledge of these A-D securities. In a continuous condition the prices of A-D securities are determined by the state price density (SPD).

Arrow-Debreu securities	Paying 1$ in one specific state of nature and nothing in any other state.
Prices of A-D securities (in a continuous of states)	Giving for each state x. The price of a security paying 1$ if the state falls between x and x + dx

In the equilibrium the state price density, SPD can be represented in terms of a stochastic discount factor or pricing kernel, which thesis will analytically discuss in the following chapters basically to *Cohrane' book (2005).* That asset prices are martingales[ix] under the real distribution of total consumption in the economy. Sahalia & Lo are based to the assumption of no arbitrage, *Aït-Sahalia Lo (1998*). So, the SPD can be called risk neutral density too. All investors are risk neutral and so, all assets in the financial market have to have an expected return equivalent to the interest risk free.

<u>Usually are used the following three parametric methods for the SPD estimations</u>

- In the <u>first method</u> are taken into account strictly assumptions on the underlying asset pricing dynamics for the SPD to get it in closed form. As for instance, when the movement of asset prices belong to geometric Brownian motion and the risk free rate does not change through time, then the distribution of the SPD is log-normal not normal. In a more advanced level of the stochastic process make the SPD unable to be calculated in closed form.

- In the <u>second method</u>, the SPD is obligated to be calculated in a parametric way.

- In the <u>third method</u> is taken a defined previous parametric distribution as a nominee SPD. Thereafter, the distance between the SPD estimation and the previous parametric set of derivatives are minimized.

However, in this alternative paper the method differs. The SPD is estimated in a non-parametric way, with no parametric boundaries on the underlying prices dynamics nor on the group of distributions where the SPD is. The estimators all use discrete data and require no discrete approximation even though the estimated model is not in discrete but in continuous time. Although, parametric approaches are preferable when the underlying asset's prices process is known to satisfy particular parametric assumptions, nowadays, empirical results create uncertainty and doubts on the parametric methods. This alternative non-parametric method to estimate SPD can provide information with great importance in the following four cases.

1. In this paper *Aït-Sahalia & Lo (1998)* is taken part the assumption of the complete market. As a risk manager may consider, *Sahalia & Lo* offer information with great importance in order to be comprehended the sense of the fat tails of asset return distributions estimated by options prices. This paper uses call options on the S&P 500. A serious problem is volatility and the use of it. Volatility cannot be accepted as just a summary statistic for the distribution. This is because when return series represent facts that are three standard deviations from the mean in short term approximately one year then, volatility cannot be use it in the whole

distribution. This estimation gives the complete return distribution system where single points like value at risk can easily end with solutions.

2. This alternative non-parametric estimation contains important characteristics. For example, data which are the most significant in an asset pricing and must be incorporated into an effective parametric model. Also, it is very helpful for the researcher to be sure what characteristics are missed because of strictly parametric models. On the contrary, the non-parametric SPD estimator contains another important characteristic of the data the volatility «smile». Also, the non-parametric SPD estimator shows persistent negative skewness and excess kurtosis. Besides these statistic characteristics are characteristics of the data too. Also, in a non-parametric analysis its estimators rarely miss important features of the data and so this non-parametric can be a precondition to the construction of a parametric model. On the other hand, a specific case in parametric models, parametric stochastic volatility models face difficulties in eliminating biases in short term option prices. This happens because of not having enough kurtosis. Jump models face difficulties with longer term options prices because of reverting too fast to prices as the maturity date raises.

3. This case is the most significant to all other three. The non-parametric estimator is robust to the classical joint hypothesis problem. The joint hypothesis problem highlights that test for market efficiency is doubtful, unbearable and even not possible. Joint hypothesis problem infers market efficiency cannot be tested. The non-parametric SPD estimator does not worry about the joint hypothesis. This non-parametric estimator is free of this hypothesis on asset price dynamics and risk premia on parametric arbitrage models or on preferences *Arrow & Debreu (1954*) and *Geanakoplos (2004*) in the equilibrium way of derivative pricing. Non-parametric estimations do not have strictly assumptions as parametric models do. Finally, non-parametric estimations are rarely broken in practice.

4. Last but not least, including the supplementary assumption that underlying prices follow a diffusion processes the non-parametric estimator of the SPD can be used in turn to estimate a snapshot of the non-parametric volatility function of the underlying.

Thus, there is a connection between state price density equilibrium models and financial derivative securities. In the case of a dynamic equilibrium model any price a financial security can be stated as the expected net present value of its future payoffs. The net present value is computed with the regard to the risk free rate and the expectation is regard to the marginal rate of substitution weighted probability density function (pdf) of the payments. This probability density function, pdf is the state price density, SPD or a risk neutral pdf or an equal martingale measure. The dynamic equilibrium case demonstrates the huge information that SPD can contain and that SPD can make reduction. For instance, whether parametric boundaries are enforced to the data producing process of asset prices, the SPD estimator will be useful to infer the representative agents' preferences in an equilibrium model of asset prices. Alterative, whether particular preferences are enforced like logarithmic utility then, the SPD will be able to conclude to the data which produce the process of asset prices.

- Representative agent's preferences
- Asset price dynamics
- The SPD

} Any two of these end to the third.

In pricing, state price densities (SPDs) are adequate statistics with economic reasoning. SPDs brief all relative contents on preferences and on businesses for purposes of pricing assets. On the other hand, the SPD cannot be easily evaluated by the time series of payoffs because it is also affected by preferences the marginal rate of replacement. Fortunately, the SPD can be evaluated by the time series of prices because prices stand for the combination of payoffs and preferences in an equilibrium base. The SPD is the second derivative which is normalized in order to be undivided of a call option pricing function with the regard to its strike price.

In *Sahalia & Lo's (1998*) paper, is presented a method of regression which is the non-parametric kernel regression. Non-parametric kernel regression concludes in an estimator of an expectation under a certain condition. This conditional expectation of **H** is without the requirement of the function **H(.)** which is parametrized by a limited amount of parameters. Kernel regression needs few hypothesises to be accepted apart from the smoothness of the estimated function. It is important to be accepted assumptions which help the estimation of the regularity of the data and the function to be robust to any misspecification of any result of call pricing function with parametric way. However, kernel regression intend to have data strenuous. Kernel' method naturally easily ends up to applications with economic and financial reasoning. It is a natural way out because previous basic parametric assumptions have already been rejected by high quality and large amount data. Parametric assumptions like normality and geometric Brownian movement.

«Market prices estimate an option pricing formula H (.) non-parametrically. But how we obtain non- parametrically? », Aït-Sahalia & Lo (1998).

Nevertheless, apart from kernel regression there are plenty of methods in the financial and mathematical literature on pricing financial derivatives. A variety of approaches which are interested in derivative prices without the non-parametric kernel sense. For instance, there are approaches using methods such as learning networks or implied binomial trees. The *Sahalia & Lo's (1998)* paper has got a momentous advantage. If the options which they used, call option prices on the index S&P 500, were estimated by another formula then, the non-parametric tactic of *Sahalia & Lo (1998)* must can approximately touch the parametric tactic. By definition, their estimation does not depend upon any parametric description for the underlying asset's price process, here the stock index. Thus, Monte Carlo simulation experiments such as *Sahalia & Lo (1998)* can also be dome for different option pricing models. But *Sahalia & Lo (1998)* preferred simulation experiments under the Black, Scholes and Merton hypothesises as most approaches preferred for estimations option prices.

Black and Scholes assumptions as *Hull's book* (2015) are presented and Sahalia & Lo (1998) accepted

1. Constant volatility. The most important assumption of others is this! A degree of how much a stock can be anticipated to change in the short-term is constant. A measure of the estimation of the future variability for the asset underlying the option contract. While volatility cannot never constant in longer term.

2. Efficient markets. This assumption mean that the market follows a random walk and so, it is not predictable. Random walk denotes that at any moment in time the price of the underlying assets as stock might raise or decline with the same probability. The price of a stock in future does not depend on the price of this stock in the present.

Price at time t+1 is independent from the price at time t

3. No dividends. The underlying stock does not give dividends until the option is exercised. In the real economy this assumptions does not hold. Many companies pay dividends. A usual and technique of adjusting the dividends in Black-Scholes model for is to remove the present value of a future dividend from the stock price.

4. Interest rates constant and known. The model uses as an interest rate the risk-free rate. In the real economy, risk free does not exist in pricing but traders may use the U.S. Government Treasury Bills short-term usually thirty day's rate. Besides the U.S. Government Treasury Bill is credit enough reliable. However, these treasury rates are not always constant in times.

5. Log-normally distributed returns on the underlying stock. This assumption holds in the real economy.

6. Only European not American options. European options can only be exercised on the expiration date not earlier as American which may be exercised at any time during the life of the contract. American options become more precious because of their pliability.

7. No commissions and transaction costs. There are no fees for buying or selling contracts and stocks. So, there are no limits in trading.

8. Liquidity. Markets apart from efficient are also perfectly liquid. So, it is possible to purchase and vend any quantity of stocks and options even their fractions at any given time.

So, *Sahalia & Lo (1998*) based in the above assumption recommended a non-parametric way on estimating price densities which are related to state price densities and option prices. They conclude in many statistical estimator's properties like pointwise asymptotic distributions, specification test statistics and a stability test in sub-samples. Despite the fact that, this *Sahalia & Lo* approach is data strenuous usually it requires more than a few data-points for a sensible level of accuracy. This approach offers an effective different to typical parametric pricing models when parametric boundaries are violated. It is often to be preferred parametric formulas to non-parametric when the underlying asset's price dynamics are comprehended. But this rarely happens in practice. Because in practice they do not depend on strict parametric hypothesises. Assumptions as log-normality and non-parametric approaches which are robust to the standard errors that overflow on parametric approaches.

Thus, a non-parametric model is notably precious in pricing applications because the usual parametric boundaries failed dramatically. Of course, there are parametric extensions to the Black & Sholes formula. If they perceive stochastic volatility, jumps, or even multi-parameter extensions, they should then, interest only in conceiving that empirical events. Furthermore, non-parametric models can be adjusted when it is necessary and can construct changes in the data generating function with the way of parametric models cannot. Finally, non-parametric models are pliant enough to contain plenty of financial derivative and asset price dynamics. Yet, a not-parametric way is simple, easily and effective calculated as *Sahalia & Lo (1998)* highlight in their article.

2.1.6 Portfolio approach

In this analysis in this chapter we taking into account the paper of *Dybvig & Ross* (*2003).* The portfolio theory operates on two points of time where between them nothing can happen. These points are time equal to 0, present, and time equal to 1, future. At time 0 the agent or in other words consumer in the economy is making decisions affecting the allocation of his consumption now and in the future.

Consumption: c
c_0: consumption at time 0

c_ω: consumption at time 1, with $\omega = 0,1,\ldots,\Omega$

Time = 0	**Time = 1**
c0	cω
Non-random consumption	Random consumption

Either at time 0 or in every state at time 1 there is a unique, single good for consumption. So, the consumption for this good at every time it will be a real number. This acceptance about the unique single good for consumption in the economy is for simplicity. In reality does not hold, where we have a multi good economy but in theoretical basis there is no problem for this acceptance.

«How to interpret our simple model? », Dybvig & Ross (2003).

There are two practices with important use when the economy is a single good model. Firstly, the case of single good uses nominal values and then evaluates the consumption in dollars \$. Secondly, in the same case but using real values now evaluates the consumption in adjusted dollars \$ by the level of the inflation in the economy. In *Dybvig & Ross (2003*) article the consumption units will be accepted as the numeraire. The numeraire is a benchmark as we have already see in A-D model, *Arrow & Debreu (1954*) and we will see in the following chapters discussing about the recovery theorem. So, all prices in the economy are adjusted to numeraire, this specific price of this single good. Thus, combining the above with the general equilibrium theory where there are units of consumption at different times, at different states of nature and at different goods we take a typical consumption vector symbolizing it as **C** in time 0 and **c** in time 1.

$C = \{c_0,\ldots,c_\Omega\}$, where the real number $c_0 \longrightarrow$ non-random consumption of the single good at 0

$c = \{c_1,\ldots,c_\Omega\}$, where the real numbers $c_1,\ldots,c_\Omega \longrightarrow$ random consumption of the single good in every state of nature at 1.

States of nature = $\{1,\ldots,\Omega\}$

In portfolio theory there are general results apart from the empirical conclusions. These results are preference free in the sense that they only depend on the thought that agents prefer more to less and are arbitrage free. So, these results as we have seen in the previous sub-chapter end up to two important theorems.

- Fundamental Theorem of Asset Pricing
- Pricing Rule Representation Theorem

The Fundamental Theorem of Asset Pricing, *Dybvig & Ross (2003*) and *Cohrane (2005*) say that the following are equivalent.

No arbitrage = Being a consequent positive linear pricing rule = Being the best representative agent who prefers more to less

The Pricing Rule Representation Theorem, *Dybvig & Ross (2003)* states different issues for the linear rule than the Fundamental theorem of asset pricing. It contains the following.

1. State prices
2. Risk neutral probabilities using martingale evaluation. Martingale pricing is a pricing method which is built on the sense of martingale and risk neutrality. The martingale pricing approach is a crucial to current quantitative finance and can be applied to a plenty of financial derivatives contracts. More specifically, in probability theory a martingale is a case of a fair-minded game where information or knowledge of previous events never are needed to forecast the mean of the future profits. A martingale is a sequence of random parameters such as the stochastic process at a certain time and the expectation of the future value in the sequence is equal to the value now even given awareness of all previous values.

   On the other hand, if the sequence is not martingale it does not change that the expected value at one time is equal to the expected value of the process at future. However, knowledge of the previous conclusions create a difference. This knowledge can diminish the uncertainty of future results and so, the expected value of the next result offer information to the present and all previous are higher than the present result.

3. State price density or stochastic discount factor or pricing kernel are an abstract positive linear sequence

The importance of having the risk neutral or martingale representation is that the price is the anticipated discounted value calculated by a riskless rate equal to the actual risk free rate if there are factitious risk neutral probabilities which determine positive probabilities to the same states as the true actual probabilities do. In risk neutrality all investments are fair games.

The efficiency condition is something that is always important to be considered. There are plenty of types of efficiency in economics. Three basic conditions are Pareto efficiency, market efficiency and portfolio efficiency. A usual fallacy is to believe that one condition of efficiency necessarily infers another.

- Pareto efficiency: Pareto efficiency or optimality as it is called is a condition of distribution of economy's resources. There is no likelihood to improve the positions of any individual in the whole economy without worsening the position of at least another one, *Arrow & Debreu (1954), Geanakoplos (2004).* A distribution or allocation of economy's resources is specified as «Pareto efficient or optimal» when there are no Pareto improvements which can be made.

- Market efficiency: The efficient in the market is determined by the prices in the economy. Asset prices testify all economy's information. No investor in an efficient market can exaggerate in his profit. The only technique can someone earn higher profit from his investments is just by good luck or by taking riskier positions. There three levels of market efficiency. (*Source: www.wikipedia.org)*

Weak	Semi-strong	Strong
Prices replicate all past information.	Prices replicate all past information and that prices instantaneously change to replicate new data.	Prices momently replicate even shadow information.

- Portfolio efficiency: From all risk averse investors point of view, a portfolio can make progress by gradually transporting funds from one

> asset to another or one sub set of assets to another sub set. Every risk averse investor must always have the goal to maximize his expected value of a raising utility function. So, the portfolio B is preferred to the portfolio A when the portfolio B' return leaders over to A. then, the portfolio A is inefficient because is not the optimal portfolio for anyone (*Source: www.wikipedia.org).*

In A-D model the market portfolio is always efficient! The reason is the array across states which are protected when investors make portfolio choices to compose the market portfolio.

The main themes in Finance:

- Time
- Uncertainty (strongly linked to the risk)
- Information

As principle, anyone is able to derive prices in equilibrium base when the preferences of each agent are acknowledged.

2.2 The stochastic discount factor

In the equilibrium basis the state price density (SPD) is combined to pricing kernel. Asset prices are martingale under distribution of entire consumption in the economy after multiplication by the stochastic discount factor. This factor is symbolized as with the Greek letter δ in *Ross (2013)* and with m in *Cohrane (2005).*

2.2.1 Asset pricing theory

Cohrane (2005) in his book explains with an analytically way the asset pricing theory and the role of the stochastic discount factor in pricing. Prices are equal to expected payoff which is discounted by the factor. There are two categories, two approaches in asset pricing using a discount factor as *Cohrane (2005)* shows. These categories are called the absolute pricing and relative pricing. The first one is a usual category as the example of Capital Asset Pricing Model (CAPM) in finance for stocks. But the second one, is more appropriate and often be used in pricing assets as derivatives.

More analytically these two categories of pricing assets.

- In absolute pricing, a trader or a researcher gives price to every asset with the regard to its exposure to fundamental economic resources of macro-economic risk. Models which are based in consumption or belong to the general equilibrium theory are the most primitive and purest examples of this category. An example is the Capital Asset Pricing Model (CAPM). This is a model that shows the relationship between systematic risk and the expected return of assets which trader is interested in. Assets which are priced through this approach are usually stocks. The overview ideal in CAPM is that trader should be compensated by time value risk represented by the risk free rate (the yield on government bonds as U.S. treasuries) and risk.

- In relative pricing traders are interested in more complicated assets as derivatives, options and then it is very important the help of Black and Scholes formula. The price of an asset here can be known using the knowledge of the prices of other assets in the market. There is no worry from where the prices of other assets originate. Also, little information about fundamental risk factors in this relative approach in pricing is taking into account.

The stochastic discount factor **m** as *Cohrane (2005)*

A universal method of moments[x] fits the asset pricing theory and is related to process.

Asset pricing by two equations:

$$p_t = E\,(m_{t+1}\, x_{t+1}) \qquad (1)$$

$$m_{t+1} = f\,(\text{data, parameters}) \qquad (2)$$

Where: p_t = asset price, x_{t+1} = asset payoff, m_{t+1} = stochastic discount factor.
These symbolisms permit equations to include many dissimilar asset pricing matters. With these equations can be priced assets such as stocks, bonds and derivatives like options. So, there is just one theory for all types of asset pricing.

<u>The only assumption</u>:

1. **1.** The trader can examine a short marginal investment or disinvestment. The last one refers to the use of an economic planned boycott in order to stress a government or a company to alter its policy or in the governments' case even a regime transformation. According to the principle of organizing as *Cohrane (2005)* explains everything can expressed through the basic pricing equation, $p = E(mx)$.

Well, the stochastic discount factor m, as *Cohrane* in his book *(2005*) highlights, has got a major advantage in its use. This advantage is its simplicity and generality. Without the use of the m there are three independent theories for stocks, bonds and options. But with the help of m these theories become to one with special cases as the assets differ. Also, this method of pricing with the stochastic discount factor m permits the separation of the step of determining economic hypothesises of the model, as in the second equation above, from the step of resolving.

Now, it is high time for us to choose a model with the figure f ( ) as a function. So, the first equation from above is able to guide to forecasts through the expressions of returns, price dividend ratios, expected return betas, moment conditions, applications on continuous or discrete time etc. From the discount factor point of view, everything is easier than as in terms of portfolios. Besides, it is easier and faster to persevere there is a positive m, than to examine every possible portfolio and then examine which one is the largest in terms of m. Furthermore, the stochastic discount factor rapprochement is connected to state spaces, as we have seen in *Arrow & Debreu (1954*) model and its analytically view of *Geanakoplos (2004*). Thus, from academic research to high technology implementations the discount factor and the state space are common.

On the other hand, there are a few obstacles in the comprehension of the asset pricing theory and not so mathematical issues. For instance, an asset with a high payoff variance is known that it will have a large connection to risk. However, if the payoff variance is uncorrelated with the discount factor, then, the asset does not have connection to risk. So, this riskless asset has got payments and expected returns equal to the risk free rate.

<u>In equations:</u>

$$\text{If} \quad cov(mx) = 0, \quad \text{then} \quad p = \frac{E(x)}{R^f} ,$$

where **x** is the payoff, **m** is the stochastic discount factor, **E(x)** is the expected returns of the payoffs, $\mathbf{R^f}$ is the risk free rate and cov is the covariance between discount factor and payoff.

These forecasts hold even in the case of high volatility in x where investors are risk averse. Namely, if someone buys a bit further of such a volatile asset, then he will not have first order result on the variance of his consumption. Continuous time if more often used than discrete time in asset pricing using differential equations. Besides, applications are more convenient and facile in continuous time than in discrete.

«Asset returns and consumption. Which is the chicken and which is the egg? », Cohrane (2005).

«Which variance is exogenous and which is endogenous? »

The answer is, it does not matter. If someone knows E (mx) then, he can determine prices, p. If he knows now prices, p, then he can also use them to determine consumption and savings decisions» Cohrane (2005).

The today's consumption can be defined by asset prices and asset payoffs not the opposite. Let us think an asset such a security like a contingent claim. This contingent claim has payment one dollar or one unit of the consumption good in one state only tomorrow. Where **s** is a state of nature with s ϵ S, where **S** are all possible states of nature. *Cohrane* (2005) in order to determine the today price of the contingent claim writes **p**, **c** for the claim and s to declare in which state of nature the security settles.

x(s) is the definition of an asset's payoff in state s.

So,

x(1) ⟶ s_1 x(2) ⟶ s_2

The price of the asset mast be equal to the value of the contingent claims.

$$p(x) = \sum p\, c(s)\, x(s)$$

Taking expectations rather than sum over states we have,

$$p(x) = \sum \pi(s) \left[\frac{pc(s)}{\pi(s)}\right] x(s)$$

Where π(s) is the probability that state s exist

Then, m is expressed as the ratio of contingent claim price to probability

$$m(s) = \frac{pc(s)}{\pi(s)}$$

The equation are presented as expectations including the symbolism ***** as *Cohrane (2005)* and others use.

$$p = \sum \pi^*(s)\, m(s)\, x(s) = E^*(mx)$$

To sum up, with a market completeness the stochastic discount factor m holds the equation p = E* (mx). It is just a group of contingent claims prices which are classified by probabilities. So, the combination of the discount factor and probabilities is often called state price densities (SPDs). Another important connection is this of m to the risk neutrality. *Cohrane* (*2005)* also shows the discount factor m as a transformation of risk neutral probabilities.

So, $$p = \frac{E*(x)}{R^f}$$

Another common transformation of p including risk neutral probabilities is the following.

$$\pi^*(s) = R^f m(s)\pi(s) = R^f pc(s)$$

Where R^f *is* $R^f = \frac{1}{\sum p\, c(s)} = \frac{1}{E*(m)}$

$0 \leq \pi(s) \leq 1$ ⟶ so, it is a combination of probabilities.

Then, we can transform the asset pricing function as,

$$p(x) = \sum p\, c(s)x(s) = \frac{1}{R^f}\pi^*(s)x(s) = \frac{E^*(x)}{R^f}$$

The symbolism ***** in E and in π is used because it is important to not be forgotten the expectation E contains the risk neutral probabilities π* and not the real probabilities π. So, this means that agents are all risk neutral with probabilities π* in the place of the true probabilities π. The probabilities π* offer more heaviness to states with higher utility than these states with the average utility! The risk aversion is here! People who offer high subjective probabilities of disagreeable facts, as the extreme scenario of airplane crashes, probably do not have irrational expectations. They might merely refer to the risk neutral probabilities or to the multiplication m x π. Above all, this multiplication takes the most important part in decision making under the information light.

The metamorphosis from actual probabilities to risk neutral is as following,

$$\pi^*(s) = \left[\frac{m(s)}{E(s)}\right]\pi(s)$$

The risk neutral probability performance is alike to asset pricing. More specifically, in derivative pricing where the results are independent of risk adjustments these two practice look the same. The risk neutral role is more preferred in continuous time diffusion processes than in discrete kind. That is because in continuous time it can adjust only the means without the covariances. In discrete time, changing the probabilities typically changes first and second moments as *Cohrane (2005)* has pointed out in his book.

2.2.2 Discount factor m

We will analytically present the stochastic discount factor basically to *Cohrane's* book *(2005)* and as Cohrane points it out and symbolizes it, m. Well, there is a connection between the state price density (SPD), which we have already talked about in previous sub-chapter using the knowledge of *Arrow & Debreu (1954*), *Aït-Sahalia & Lo (1998*) and *Ludwig (2015*) papers, and the characteristics of assets in economy. Let's think that incur n assets in the market with respectively primitive prices $p_1,\ldots..,p_n$ at the beginning of the time. Also, these assets have got respectively payoffs $x_1,\ldots..,x_n$ at the end of the duration. Then, SDF is any random variable m. The m is a random variable which is able to create prices taking respectively payoffs. Also, the m can conclude in one payoff's price using prices of other payoffs. A typical example of this in the case of Black, Scholes and Merton option pricing model, *Hull's book (2015)* and *Ross (2013)*. In the latter case of B-S model, the option payoff is capable to be reproduced by a combination of stocks and a bond in a portfolio. Here, any discount factor which evaluates these stock and bond can result to the price of this option. This is crucial in asset pricing!

The above in equation form.

$$E\ (mx_i) = p_i, \qquad \text{for every possible value of i}$$

Or in simpler expression of the above equation.

$$p = E\ (mx)$$

The title stochastic discount factor explains the characteristic of asset prices. They can be calculated by discounting cash flows x_i in the future using m. And so, the expectation can be expressed. This definition is crucial in asset pricing!

«But what does this expressions mean? Can someone always find such a discount factor? Can we use this convenient representation without implicitly assuming all the structure of the investors, utility functions, complete markets etc.? », Cohrane (2005).

Cohrane (2005) discussed two theorems to stand the m in certain boundaries. The law of one price[xi] says that two portfolios with common payoffs in each state of nature will certainly have got and the same price. An offense of this law of one price will stand up an arbitrage profit. This will happen because one can sell the expensive part and buy the cheap one of the same portfolio.

The first theorem about the discount factor supports that there is an **m** which evaluates all the payoffs through the equation. Namely,

$$p = E\ (mx) \overset{\text{if and only if}}{\Longleftrightarrow} \text{this law of one price holds!}$$

Under financial economics point of view, absence of arbitrage is secured for a stronger idea. This is that if payoff A is all the time at least as good as payoff B and sometimes payoff A is better than B, then the price of A must has greater pricing. Namely,

If payoff A ≥ payoff B at any time, then price of A ≥ price of B

The second theorem says that exists a positive m can evaluate all the payoffs through,

$$p = E\ (mx) \overset{\text{if and only if}}{\Longleftrightarrow} \text{no arbitrage oppurtunities exist!}$$

These two theorems have important use of explaining how the stochastic discount factor can be used without including the assumptions of utilities, aggregation, complete markets and etc. Thus, the only thing someone needs to know about investors is that they will not support any kind of violations of the law of one price or the case of arbitrage. These two theorems are able to explain ideas of a payoff space like law of one price, absence of arbitrage including boundaries on the discount factor as it seems above in arrows.

Chapter 3

The Ross Model: The Recovery Theorem

3.1 The basic framework of Ross Model and the definition of kernel

The basic framework, *Ross (2013)* of the model expresses and connects with a great analytical way the natural probabilities to the risk neutral probabilities. Also, this framework presents the proof of recovery theorem. Using only the knowledge of state prices we can find the natural measure and pricing kernel. This happens with the help of this basic *Ross'* framework *(2013).*

Assumptions

1. Markov process is followed by the underlying asset
2. Transition independence of kernel function
3. No arbitrage, complete markets and discrete time
4. Irreducible state prices matrix P

It is meaningful to be said that the risk free rate here, *Ross (2013),* has the characteristic to be state dependent. If we try to transform this dependence across states, unfortunately, we will end up with a degenerated model. Also, the pricing kernel here is determined as the price per unit of probability. An important notice is there is no need to have state dependent utility. Pricing kernel relies only on the martingale ratio of substitution between the future consumption and the present.

The above frame is a typical root which supplies the fortune expectation that the solution of this will be in discrete space. And so, not an arbitrary large linear space. The recovery theorem controls this presentiment. Using option prices someone can end up with the distribution of state prices.

Risk aversion (pricing kernel) * natural probability distribution = state prices

Moreover, the recovery theorem is able to define the market's forecast of returns and the market's risk aversion from state prices. The theorem allows the recovering of pricing kernel, the market risk premium, the probability of a catastrophe and the examination of the model using free tests of the null hypothesis of market efficiency. The very low probability level of a catastrophe event is a serious problem of uncertainty. And further, changes in that observed probability effect on asset prices.

In *Ross' (2015)* paper the recovery theorem transforms to a multinomial theorem. This multinomial recovery supplies a different approach to recover the natural distribution for binomial and multinomial procedures.

3.1.1 The basic framework as *Ross (2013)* defined

In a discrete time with asset payoffs $g(\theta)$ at time T on the realization of a state of nature $\theta \epsilon \Omega$. As in the fundamental theorem of asset pricing without arbitrage can exist positive state space prices.

$P(\theta)$ = the function of the price distribution

$p(\theta)$ = state prices

If the market is complete ⟺ then these state prices are unique!

If there is not arbitrage in the market ⟺ then these state prices are positive!

The today price (or value) of an asset which pays $g(\theta)$ in one period is able to be found through the following equality.

$$p_g = \int g(\theta) \, d\, P(\theta) \qquad (1)$$

The aggregate of the state prices is the today value of a dollar (1$) in the near future with no risk. The interest rate is the risk free rate.

$r(\theta^0)$: The risk free rate or the riskless rate r as a function of the present state of nature θ^0 . So, with this r the equality (1) becomes,

$$p_g = \int g(\theta)\, dP(\theta) = \int dP(\theta) \int \frac{dP(\theta)}{dP(\theta)} = e^{-r(\theta^0)T} = \int g(\theta) d\pi * (\theta) =$$

$$e^{-r(\theta^0)T} E^* \left[g(\theta)\varphi(\theta)\right] = E[\, g(\theta)\varphi(\theta)]$$

Where, asterisk ***** highlights the expectation in the martingale measure and the pricing kernel.

The Radon-Nikodym Theorem[xii] is another useful and significant theorem which extends the concept of the probability theory from probability multitudes and probability which are determined by over real numbers to probability measures determined by sets. Namely, this theorem explains how we can be transferred from one probability measure to another. Well, the Radon-Nikodym derivative, *f* as we will see later, is the probability density function (pdf) of a random variable with the regard of some base measure. With other words, this theorem is useful in order to have the proof of the existence of expectation under certain conditions for probability meters. This is a key idea because in probability theory the conditional probability is a particular case. Also, financial mathematics utilizes the recovery theorem widely in pricing of derivatives and more specifically when actual probabilities alter to risk neutral probabilities. A risk neutral measure or an equilibrium measure or an equivalent martingale measure is exactly the same measure. This is a probability meter which every stock price equals to the discounted expectation of this price under the risk neutral meter in.

From the mathematics point of view, the recovery theorem is an outcome of meter theory. Namely,

(X,Σ): a measurable space

ν: a σ-finite measure on measurable space

μ: a σ-finite measure on measurable space

- ν taking into account a given (X,Σ) is also unconditionally continuous with respect to μ.

So, the measurable function of the above.

$f: X \longrightarrow [0,\infty]$ in any measurable subset $A \subset X$ then $\nu(A) = \int_A f \, d\mu$

The function *f* is called the Radon-Nikodym derivative and denoted by $\frac{d\nu}{d\mu}$.

Thus, the state price / probability $\varphi(\theta)$ is the Radon Nikodym derivative of $P(\theta)$ with respect to the natural meter, $F(\theta)$.

With continuous distributions: $\varphi(\theta) = \frac{p(\theta)}{f(\theta)}$ where, $f(\theta)$ is the natural probability.

θ_i : the current state which is a state one period forward.

θ_j : following states in T periods.

Assumption

This is a total report of the state of nature containing the stock price and other information relevant to the future movement of the stock market index. So, we end up with the following equation with the crucial point that time interval plays important role to this function but calendar time does not, *Ross (2013).*

$$Q(\theta_i, \theta_j, T) = \int Q(\theta_i, \theta_j, t)\, Q(d\theta, \theta_j, T-t) \qquad (2)$$

where $Q(\theta_i, \theta_j, T)$ is the forward martingale probability transition function and where the completion is over and above the middle state θ at time t, θ_t .

To sum up, above we are discussing about the basic framework in recovery theorem. This is a universal framework and so, permits the researcher to understand and comment the issues which are examined as he desires. Namely, if the distribution of martingale returns is defined only by the volatility thereafter the transformation can be expressed as a movement from the current state θ_i to θ_j.

$\theta_i=(S,\sigma)$

$\theta_j = (S(1+R), \sigma')$, where σ and σ' are dissimilar volatilities

R: the rate of return

$$\text{So, } Q(\theta_i, \theta_j, t) = Q((S, \sigma)(S(1+R),\sigma'),t) \qquad (3)$$

Simplifying and using state prices rather than martingale probabilities the equation (3) becomes,

$$Q(\theta_i, \theta_j, t, T) = e^{-r(\theta_i)(T-t)} Q(\theta_i, \theta_j, T-t) \qquad (4)$$

So, there is no need to continually correct the equation for the interest parameters.

Another assumption

A time homogeneous procedure where calendar time is not relevant for the transformation from any time t to time t+1, *Ross (2013).*

So, we have the following equation,

$$P(\theta_i, \theta_j) = e^{-r(\theta_i)(T-t)} Q(\theta_i, \theta j, T\text{-}t) \qquad (5)$$

The kernel φ in this basic framework is determined as we can see as the price p per unit of probability measure *f* in continuous state spaces.

$$\varphi(\theta i, \theta j) = \frac{p(\theta_i, \theta_j)}{f(\theta_i, \theta_j)} \qquad (6)$$

So, when a positive kernel exists there is no arbitrage opportunities!

3.1.2 Definition about pricing kernel in Ross model

φ: kernel

δ: stochastic discount factor

h: function of the states of nature θ

h(θ) = U′(c(θ)): the representative agent formula (we will need it more, later in Recovery Theorem)

Kernel is a transition which does not depend on whether there is a positive function of the states or a positive constant discount factor δ, as *Ross (2013)* symbolizes the stochastic discount factor such as m in *Cohrane's book (2005*). So, for any transition from current state θ_i to others as θ_j the kernel has the following equation.

$$\varphi(\theta i, \theta j) = \delta \frac{h(\theta_j)}{h(\theta_i)}, \qquad \text{pricing kernel} \qquad (7)$$

A usual comprehensive example which gives a transition without the need of kernel is the perpetual additive utility function.

Now, including the independence of transition the equation (6) from above can be written as well.

$$p(\theta i, \theta j) = \varphi(\theta_i, \theta_j) f(\theta_i, \theta_j) = \delta \frac{h(\theta_j)}{h(\theta_i)} f(\theta_i, \theta_j) \qquad (8)$$

Unfortunately, in equation (8) there are three unknowns. More unknowns than equations!

Which unknowns?
- The natural probability transition function $f(\theta_i, \theta_j)$
- The kernel $\varphi(\theta_i, \theta_j) = \delta \frac{h(\theta_j)}{h(\theta_i)}$
- The stochastic discount factor δ

So, transition independence is needed for defining with a separate way the kernel or the natural distribution. This means that we could not define them only using the information of p (θ_i, θ_j).

3.2 The Recovery Theorem

Ross (2013) takes the hypothesis that we are on discrete time. This is important for empirical analysis as we will see later. When it is necessary and possible we will include the representative agent formulation to our analysis. So, this equation

$$\mathbf{p}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j) = \boldsymbol{\delta} \frac{h(\theta_j)}{h(\theta_i)} \boldsymbol{f}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j)$$

can be written as well, including the representative agent U′ using the formula with states of nature.

$$U'_i \, \mathbf{p}_{ij} = \boldsymbol{\delta} \, U'_j \, \mathbf{f}_{ij} \qquad (9)$$

where $U'_i = U'(c(\theta_i))$ including so, the states of nature θ_i. U′ is any positive function of the state. So, the equation (7) pricing kernel can be written including now the states of nature. Let denote the current state θ_i as state i=1 and then, we have θ_1.

$$\varphi(\theta_1, \theta_j) = \delta \, (U'_j / U'_1) \qquad (10)$$

A connection between states of nature and kernel to assets values like stock, S(θ_i)

An important connection is the states of nature to stock values. We in our thesis as states of nature take into account the value of the underlying asset and more specifically, of the index FTSE/JSE Top 40. Kernel φ is the core of the state space determined by the refining asset prices. Also, marginal utility is decreasing monotonous in consumption but it is not necessary to have the same monotony in the asset value S (θ_i).

In a matrix form the equation (8) can be written

P: the mxm matrix of state contingent prices p_{ij}. *Arrow & Debreu (1954)*

F: the mxm matrix of real probabilities f_{ij}

D: the diagonal matrix. Its diagonal have got the undiscounted kernel.

$$D = \left(\frac{1}{U'_1}\right)\begin{bmatrix} U'_1 & 0 & 0 \\ 0 & U'_i & 0 \\ 0 & 0 & U'_m \end{bmatrix} = \begin{bmatrix} \varphi_1 & 0 & 0 \\ 0 & \varphi_i & 0 \\ 0 & 0 & \varphi_m \end{bmatrix}\left(\frac{1}{\delta}\right)$$

$$\mathbf{p}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j) = \boldsymbol{\delta}\frac{h(\theta_j)}{h(\theta_i)}\boldsymbol{f}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j)$$

$$D\mathbf{P} = \boldsymbol{\delta}\mathbf{F}D \qquad (11)$$

- Including the assumption of no arbitrage in the market

In a model with exogenous consumption this absence is an outcome of the equilibrium with positive state prices. These prices confirm that the carrying cost net of the dividend which is paid by stocks to investors equal to any position that try to earn from the increase of the lowest asset value or from the decrease of the highest asset value.

It is wised to always remember that the state prices P and the natural measure of probabilities F are connected so from one we can result to another. F can be written as so is the dependent variable of variable P. Namely, the equation (11) becomes,

$$F = \left(\frac{1}{\delta}\right) DPD^{-1} \qquad (12)$$

where D^{-1} is the inverse matrix of D matrix.

Thus, if we know the matrix variable D then we can have price to F! Probabilities which belong to F are natural in the risk neutral world as we have already said, *Ross (2013)* and *Arrow & Debreu (1954).* Unfortunately, the risk neutral measure is known but there is no knowledge on the marginal rates of substitution of states. For instance, the risk adjustment in the equation does not give solutions as the natural measure F. However, F is a matrix whose rows are transition probabilities. This F matrix is a stochastic matrix. A positive matrix whose rows aggregate to one with an additional group of m restrictions. So, the matrix F is equal to the vector e.

e: the vector with the number 1 in all entries. The unit vector

$$Fe = e \qquad (13)$$

So, combining the equation (13) to equation (12) the latter becomes,

$$Fe = \left(\frac{1}{\delta}\right) DPD^{-1} e = e \qquad (14)$$

- Including the assumption of the undiminished transition matrix P

Thereafter, all states are possible to happen from all other states in n movements. If the matrix P has positive value, then this matrix cannot be decreased. Even though there is a zero price in the node ij it is likely to get to j with two movements by the transition from l to k. And then, from the point k we continue with n steps. P cannot be diminished if there is always a route where every state i can reach every state j. It is very important to keep in mind that if P is irreducible and so F will be.

Theorem 3.2.1: The Recovery Theorem, *Ross (2013)*

- If there is no arbitrage in the complete market } There is a

- If the pricing matrix is undiminished unique solution**!**
- If it is generated by a transition independent kernel

Namely, there is one and only positive solution to the matter of discovery the **F**, the **δ**, and the **φ** as we have already discussed in sub-chapters 3.1.2 and 3.2. Thus, for every group of state prices, *Arrow & Debreu (1954)* there is one and only pursuant natural measure and one and only pricing kernel.

<u>The Proof of the Recovery Theorem</u>

z: the unique positive characteristic vector with characteristic root λ. z it is also called eigenvector. $z := D^{-1}e$

The matter of solving for the probabilistic matrix F is the same to finding the characteristic roots, called eigenvalues δ and characteristic vectors z of the matrix P. Consequently, if we know the stochastic discount factor δ and the characteristic vector z, we will generate the following equation (15).

$$Pz = \delta z \qquad (15)$$

«What does it mean this equation?

<u>Answer</u>: This is a characteristic root problem and offers some hope that the solution set will be discrete and not an arbitrary cone », Ross (2013)

Ross (2013) explains that all non-negative undiminished matrices have one and only positive characteristic vector z and a connected to that positive characteristic root λ with the crucial equality λ = δ. The characteristic root is equal to the stochastic discount factor. Combining this to equations (8), (9) and (10) form above we result in the following.

$$f_{ij} = \frac{1}{\delta}\frac{\varphi_i}{\varphi_j} = \frac{1}{\delta}\frac{U'_i}{U'_j}p_{ij} \qquad (16)$$

$$f_{ij} = \frac{1}{\delta}\frac{\varphi_i}{\varphi_j} = \frac{1}{\delta}\frac{U'_i}{U'_j}p_{ij} = \frac{1}{\lambda}\frac{z_j}{z_i}p_{ij} \qquad (17)$$

Moreover, the probability matrix can be separated from kernel as in the proof of theorem above, whether kernel is an independent transition. Also, we do not know whether the kernel will be monotone as in the case of stock values.

Corollary 3.2.1 of the basic recovery theorem of *Ross (2013)*

- δ, the subjective discount rate, has limitations by the biggest interest factor.

More analytically, the stochastic discount rate δ is the largest characteristic root of matrix P. This root has limitations by the highest row sum of matrix P. This highest row includes the interest factors. Besides, these entries of P are the undiluted contingent events of state prices, *Arrow & Debreu (1954*). The highest the maximum row sum is also the supreme interest factor. This interest factor *Ross (2013)* symbolizes it with the Greek letter γ.

Important Notice: The interest rate in the following equations is the risk free rate.

risk free rate $\longrightarrow$ risk neutrality.

Theorem 3.2.2: **The Recovery Theorem with neutrality**, *Ross (2013)*

The exclusive natural density refers to a descriptive group of prices which belong to risk neutral world. Natural density or asymptotic density or arithmetic density as someone can meet this density to be called is one of the likely ways to meter the size of a set of natural numbers. The natural density is accurate. So, if the risk free rate is state independent, thereafter the exclusive natural density is the martingale. With other words, pricing is risk neutral. We are in the risk neutral world! References do not include in the theorem! This can be proved by the mean and the variance of the model.

Proof of the theorem 2

Let us remember the definition of the symbolisms from the framework and the theorem above as *Ross (2013)* preferred.

Q: the forward martingale probability transition function

P: the matrix of state prices

F: the matrix of natural probabilities

φ: kernel

e: the unit vector

δ: the stochastic discount factor

z: the eigenvector. The unique positive characteristic vector

λ: the eigenvalue. The characteristic root of z

γ: interest factor

So, the previous (15) equation **P**z = δz becomes now as follows,

$$\mathbf{P}e = \gamma e \qquad (1)$$

Namely, only the P is the same and the others change. The unit vector takes the position of the eigenvector and the interest factor replaces the stochastic discount factor. It is also important to highlight that the forward martingale distribution is equal to the same equation to F, as in (24) equation from the recovery theorem.

$$F = Q = \left(\frac{1}{\gamma}\right) P$$

There are a variety of approaches to this base such as the extension to a multinomial models but we will also analyze another approach to the recovery theorem who Ross shows in his article, *Ross (2013)*. The following theorem.

Theorem 3.2.3: An extended approach by *Ross (2013)*

There is a combination between two densities on consumption the risk neutral as *Aït-Sahalia & Lo (1998)* and the natural. The risk neutral density for consumption and the natural density for consumption have the «single crossing property», *Ross (2013)*.

«What does single crossing property mean? »

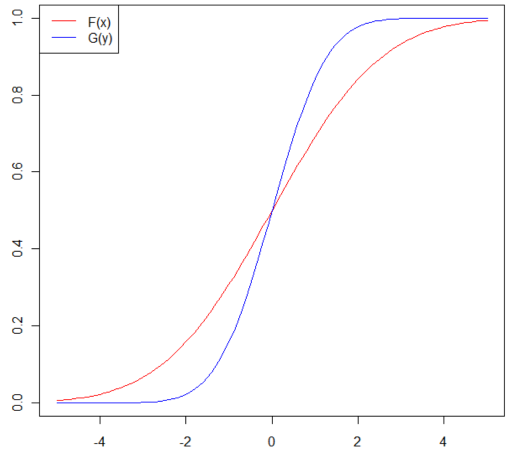
Answer: The single crossing condition or single crossing property refers to how the probability distribution of outcomes changes as a function of an input and a parameter.

F(x) , G(x) : two cumulative distribution function as in our case with the risk neutral density and the natural one.

These two cdf have the single crossing property for every x and y when:

$\overset{x \geq y}{\Longrightarrow} F(x) \geq G(x)$ and $\overset{x \leq y}{\Longrightarrow} F(x) \leq G(x)$

A graph of that relationship



Source: *www. wikipedia.org*

Also, the natural density indicates with a stochastic way the risk neutral density. And the exactly the opposite in a one period world. The natural density indicates with a stochastic way the risk neutral density with its turn.

Proof of theorem 3

$$\varphi(\theta i, \theta j) = \frac{p(\theta i, \theta j)}{f(\theta i, \theta j)} = \delta \frac{U'(c(\theta j))}{U'(c(\theta i))} \quad (1)$$

This proof includes items and equations which hold from the framework of the recovery theorem.

θ: states of nature

θi: current state of nature in one period forward, θi = (S, σ)

θj: following states of nature, θj = (S(1+R), σ′)

R: rate of return

σ, σ′: different volatilities

h(θ) = U′(c(θ)): the representative agent in the market, where c is the contingent claims

pij: state prices

*f*ij: a function with the equality as in the equation (17) above. The Radon-Nikodym derivative

v: a σ-finite meter on a measurable space (X, Σ)

Kernel φ diminishes in c(θj). As we have already seen in the explanation about the de*f*inition single crossing property we have the following.

Kernel φ overdoes discount factor δ when c(θj) < c(θi). This happens when also holds the equality δU′(v) = u′ (c(θi)). So, we have the inequalities below.

$$p>f \quad \text{for} \quad c<v \qquad \text{and} \qquad p<f \quad \text{for} \quad c>v$$

This particular property confirms that *f* with stochastic way determine the state prices p, *Ross (2013)*, *Arrow & Debreu (1954).*

Corollary 3.2.2 of the theorem 3 of *Ross (2013)*

In a one period model as here these two parts wealth and consumption will be treated as the same. Consumption concurs with the value of the market.

In a one period world as here in this model the market reports a risk premium which is the expected return on the asset such as stocks. This expected return has got higher value than the risk free rate.

R: rate of return. The expectation. The natural

r: risk free rate, R > r

Z: non-negative

ε: the mean zero conditional on R-Z

E: symbolizes the expectation

In every future in the period T the return R including risk neutrality follows the relationship below.

$$R^* \sim R - Z + \varepsilon \qquad (1)$$

Combining with the expectations now, we have the equation (1) with E as below.

$$E\,[R] = r + E\,[Z], \qquad \text{its value is} > r \qquad (2)$$

Final comments on recovery theorem

Thus, the recovery theorem becomes sufficient and ready for the empirical part of the analysis. The recovery belongs to the state space. *Ross (2013)* includes the general equilibrium approach and more specifically the *Arrow & Debreu* model (*1954)* state condition in his theorem. Conditions such as the existence of no arbitrage opportunities refer to the presence of positive A-D prices. A risk neutral adjustment in measurement which seems from the risk free rate from expectations provides a positive pricing kernel to derivative pricing. So, assets pricing happens with the expectations of the underlying assets payoffs which are balanced with the kernel.

To highlight some non-parametric conditions which are taken part in recovery theorem.

1. The underlying process such as a stock's movement follows the Markov chain. The model exists in discrete time and so, the state variables are discretized.
2. Kernel is characterized by independency. It is a function in the last state which depends upon the contemporary state as normal. It is similar to what is happening the circumstance in the marginal rate of substitution over time to a typical agent in the economy. This type of agent has got a timeless additively separable utility function.

3. The determination of the kernel, the stochastic discount rate, future values and the underlying natural probability of returns is based only to the transition state prices.

4. Neither the historical distribution of returns nor independent parametric hypotheses are included on preferences in the model with final goal the market's subjective distribution of future returns.

To conclude, this pricing holds with the probabilities of risk neutrality which are the product of an unknown kernel, more specifically the risk neutral, and natural probabilities. These two categories of probabilities can exist separately!

3.3 Extensions of Recovery Theorem by others

There still is a variety of areas which can be also searched and still allowing the recovery theorem as *Ross (2013)* explains. For instance, *Ross (2013)* points that the following can be changed as the researcher demands and so explores the new, extended recovery,

- Bounding the assumed kernel or not
- Bounding the underlying process or not
- Having continuous or bounded process
- Weakening some of the basic assumptions of the recovery theorem
- Implied volatility. This is respective to the state variable. A problem appears because of the assumptions of the theorem that the state is able to be concluded by the current level of an index.

Ross (2013) said that his research has only scratched the very beginning of a significant area which will be searched further in future with great interesting. Many approaches from other mathematicians, economists etc. approximate to result in an extended recovery theorem. Some of the most important approaches of them are the following sub chapters as *Backwell (2015)* pointed out.

3.3.1 Options on bonds as *Martin & Ross (2014)* suggest

We study the behavior of the long bond, a zero-coupon bond that pays off in the far-distant future, under the following assumptions.

1. The fixed income market is complete.

And

2. The state vector follows a Markov chain. This guides interest rates.

The transition independence in pricing kernel means that there is an investor whose utility preferences are separate. So, the yield curve must have a slope up and results to a form expressing the expected return on the long bond. This return is in terms of the prices of options on long bonds. In this paperwork, *Martin & Ross, (2013)* present several theoretical conclusions on the properties of the long end of the yield curve.

Combining the Recovery Theorem, this article highlights a problem. *Ross* showed in his paper *(2015)* that when a matrix of A-D prices *Arrow & Debreu (1954)* is given, then, it is also possible to conclude to the objective state transition probabilities and implied marginal utilities. Unfortunately, the above can happen in rich asset price data with risk neutral probabilities, Q but cannot happen for the objective or real world probabilities, F. Furthermore, it is difficult to construct the matrix of A-D prices.

This paper of *Martin & Ross (2013)* shows that the yield on the infinitely long zero coupon bond simultaneously expresses the time preference rate of a pseudo-representative agent. A pseudo-representative agent is uniquely defined. If there is a representative agent as in typical part, then a pseudo-representative agent will exist and be the same as the representative one. Besides, the time series of returns on the long bond disclose the pseudo-representative agent's marginal utilities. These marginal utilities can be a useful guide but now in this analysis there still is the assumption of arbitrage free market.

Martin & Ross (2013) point out two issues.

1. On average, the yield curve must slope up. This is regular to the empirical part.
2. A mathematic form for the expected return on the long bond is expressed in terms of the prices of long bond options.

They end up in several results basic to the pseudo-representative agent and the yield curve. Five results in the theoretical part of their approach and other two in application process.

Results from theoretical analysis

Result 1

For random asset prices a unique decomposition matrix D exists. The matrix D consists only of positive scalar multipliers. This points out the uniqueness of a pseudo-representative agent and his probability measure. So, the asset prices are rationally measured.

Result 2

The returns on the long bond are patterned by the v eigenvector. This explains that there is a way that the kernel is able to be watched empirically through the long bond's returns. Having linked the long bond to the v eigenvector, we can also link the first to the φ eigenvalue. Well, the long rate does not depend to the current state but it is equal to the conditional expected log return of the bond and so, it presents that both are computed with the help of φ eigenvalue. They are defined by φ which is the biggest eigenvalue pricing data.

Result 3

However, as *Martin & Ross (2013)* pointed out, there is an issue when the bonds have finite maturity.

Result 4

The T-period yield is not able to stand off the long yield. Meaning that whether the index's risk estimations for pricing Q is small, then the long end of the yield curve has to be flat and the yield volatility low. On the other hand, whether there is a crucial variation in long dated yields either to maturities or to states, even if the yield curve has got a significant slope at the long end or long dated yields are volatile, then the matrix Q has risk positions important for a fixed income pricing.

Result 5

Including the category of fixed income assets the long bond has a great growth. This occurs in each state that the long and the peak expected log return of all fixed income assets.

Results from applications process

Result 6

The yield curve neither always leans down nor leans up while the curve should lean up on average.

Result 7

The case now of options on long bonds reveals its conditional expected excess return. Besides, option prices disclose the conditional moments of the return on the long bond. If options' payoff on long bonds can be priced by a static no arbitrage argument, as *Backwell (2015)* suggests in his application part and we adopt it in our empirical part. This outcome links the forward looking in nature bond's expected return to its volatility surface. As *Spears (2013)* explains volatility surface is a plot with three dimension. In x axis appears the time to maturity, in y axis appears the strike prices and in z axis appears the current market implied volatility.

«Why Martin & Ross (2013) preferred to focus on fixed income markets? », Martin & Ross (2013).

Because

1. The assumption that the state variable is Markovian means that this variable is stationary. Namely, it does not be influenced by any shock in the financial market. This is in accordance with the properties of the yield curve but is not with the asset pricing which have raising cash flows.

2. In fixed income markets the assumption of market completeness is safe.

3.3.2 An empirical approach. Suggestions for extensions by *Audrino, Huitema and Ludwig (2015)*

In this paper they focus on how to construct robust state price surfaces of option prices using as an underlying asset the index S&P 500. This empirical approach is in accordance to *Aït-Sahalia' (1998*) paper which gives to *Audrino, Huitema and Ludwig (2015)* the non-parametric reasoning in order to be created by them the estimation strategy for Ross' recovery, *Ross (2013).* Also, this approach is in accordance to *Backwell' (2015*) paper which refers to the case of a snapshot of option prices as they three prefer. *Backwell (2015)* and we use a snapshot of option prices of data on 18 September 2013 on the index FTSE/JSE Top 40.

Audrino, Huitema & Ludwig (2015) research the possibility that recovery yields forecasting information according to risk neutral densities. They use a period of 13 years, more specifically from 2000 to 2012. They conclude market timing strategies are supported by recovered moments drastically overdraw those which are based on corresponding risk neutral.

In this paper is taken into account closing prices of only out of money (OTM) call and put options on the stock index S&P 500 for each Wednesday between 5 January of 2000 and 26 December 2012 with a daily frequency. They preferred out of the money options due to this type of derivatives are more liquid.

« Why there are more liquid? »

Because: *Out of the money options are made up of time and volatility values. The time value (theta) is non-linear and the volatility is able to alter in a minute note. Thus, the value is elusive! OTM options earn profit as you shift closer to the strike price.*

Also, they left out options which violated the no arbitrage limitation. As in the paper of *Aït-Sahalia & Lo (1998)* is used the put call parity to create the implied forward from close to at the money (ATM). Call and put pairs are given the implied forward. We back out the implied dividend yield via the spot-forward parity and translate out the money puts into in the money calls. Finally, combining the paper of *Ludwig (2015) Audrino, Huitema & Ludwig (2015)* used as a guide the Black, Merton & Scholes model, *Hull (2015)*, to map prices to implied volatilities.

An important notice is that the market's risk aversion combining the pricing kernel is able to make use only of a snapshot of current option prices. Knowledge of the pricing kernel lets the researcher to recover real world probabilities using the risk neutral densities. The last ones get the forward looking information nature to be involved to options market prices those options which are straight available for applications. Applications like management or portfolio optimization even trading strategies.

In contrast to above risk, the recovery theorem neither depends on historical returns nor limitations on form of the pricing kernel. Furthermore, a careful reader will notice that while option implied volatility has long been chosen to determine the market's sensitivity of risk, on the other hand option prices are thought to be inactive in case of forecasting the average return. Let alone, the whole natural distribution.

In this article some points of *Ross (2013)* continue to hold:

1. State prices follow Markov chain movement, *Arrow & Debreu (1954).*
2. The structure of risk neutral analysis belongs to Black, Scholes and Merton as *Hull (2015)* explains.
3. Time-homogeneity. This creates the basis to exist a non-negative undiminished matrix which with its turn has got a one and only one

positive eigenvalue and also, has got corresponding unique and now severely positive eigenvectors.

4. Pricing kernel is path self-governing. This means that kernel is based only on the marginal rate of substitution between the future and current consumption.

Audrino, Huitema & Ludwig (2015) result among others in the state price, A-D securities, transition matrix $P_{i,j}$ which is analogous to the prices of single period.

$$\mathbf{P_{i,j} = \delta \; M_{i,j} \; F_{i,j}}$$

δ → Market's average discount rate

$M_{i,j}$ → Pricing kernel

$F_{i,j}$ → Real world transition matrix

These make the start of the research if recovery yield ends with a common sense result. More specifically, the research if the above result meliorates the forecasting information which is contained to risk neutral densities.

They were based on linear models without neglecting the risk neutrality. Neutral lattices behave as a linear grouping of features instead of fixed non-linear modifications. So, the basic shape of expansions is revealed simultaneously with the coefficients of the linear model.

Real world: Ross' key ideas

Other studies with option pricing are usually focused on determining the stochastic procedure of an asset and producing option prices according to imposed dynamics and no arbitrage condition. Equating the price of a contingent claim to that of a dynamic trading plan combining only the underlying asset and a cash account, then the pricing is promising even in the case of lacking the knowledge of the expected return on the underlying. Thus, the only important matter now is the determination of the stochastic procedure in the shape of matching the market option prices.

Raising the liquidity of markets for contingent claims-demands options become assets. These options as assets have market prices that determine the model parameters. The mathematical model's quality is often evaluated in the space of implied volatilities among unlike strikes and maturities. The usual disposition traditional models are normally not capable for capturing all the different patterns in market prices. This issue piloted to an approach with option at different strikes and maturities.

Over the last years, a great interest has been created in the forecasting information that the market traded option prices contain. Also, the movement from risk neutral to natural or real probabilities has been studied with a great concern. Finally, the stochastic discount factor or pricing kernel which is determined as the ratio between the two meters presented onto return states.

From Option Prices to State Prices: Steps to Ross recovery

The conclusions of *Breeden & Litzenberger (1978)* allow *Audrino, Huitema & Ludwig (2015)* to construct the A-D prices for claims contingent on the current state of the underlying asset, here the stock index S&P 500. The core of recovery theorem is the fulfillment of the transition matrix P with the state prices and the real world probabilities matrix F as we do in the chapter 4, our empirical part.

In reality, there are not continuous trading strikes. So, *Audrino, Huitema & Ludwig (2015)* firstly estimated option prices on a dense grid in order to conclude to the transition matrix. They need a current snapshot of the whole option pricing surface linking various maturities over a uniform state space, the underlying asset prices. These necessities created problem to this article in its application section. So they decided to deal only with single option maturities or to pass over the issue of extrapolation [ix]. Paper’s target is to completely captivate the information priced into options through supply and demand.

Their empirical analysis of the recovery theorem results in the four important following conclusions.

1. Without having any limitations on pricing kernel, this paper acquires a positive equity risk premium and a negative variance risk premium.
2. This work find that recovered skewness and kurtosis have remarkably a greater stationary shape than their corresponding risk neutral.
3. The recovered pricing kernel refers to state prices and clarifies the likelihood of presudo recovery.
4. The recovery theorem offers economic value.

3.3.3 An extended recovery approach by *Borovicka, Hansen & Scheinkman (2015)*

Here, in this paper the researchers *Borovicka, Hansen & Scheinkman (2015)* focus on the predictive ability of assets which exists in information they carry. More specifically, asset prices which can be used as underlying assets enclose clues as information about the probability distribution of future states and the stochastic discounting of those states. This issue is really interesting to investors and their beliefs about risk.

The above three researchers choose to separate a positive martingale component of the stochastic discount factor as *Cohrane' book (2005*) procedure. They do that in order to better comprehend the provocations in separating investors' beliefs from risk adjusted discounting. This separation recovers a probability meter that engrosses long-term risk adjustments. If the martingale is not degenerate then the recovered probability will mistakenly contain investors' beliefs. Namely, recovered probability will distort the conclusions risk and return trading relationships. In several models of asset prices their empirical martingale components have stochastic discount factors too.

Furthermore, asset prices are forward looking investment kind and this enclosed information that they contain about investors' beliefs leads the literature and economic policy makers to advise financial market data. This happens in order to literature and makers comprehend what the market thinks about the future. This is a serious macro-economic issue. Combining the thoughts of *Arrow & Debreu (1954)* about assets prices, this paper points out the estimation of risk from the combination of investors' risk aversion and the probability distribution. In the case of dynamic models, investors' risk aversion is shown by stochastic discount factors and the A-D prices are shown through investors' beliefs.

Unfortunately, using only the data on asset prices is not enough to clarify both the stochastic discount factor and transition probabilities. There is a need of supplementary limitations. This supplement can be the example of time series evidence on the development of the Markov state. Another example can be the information on the market which with its turn settles on stochastic discount factors.

Finally, this paper highlights that long-term valuation is just a concept of systematic study of pricing intimations over different trading horizons.

3.3.4 Extensions on recovery theorem for financial derivatives by *Dubinsky & Goldstein (2013)*

This paper points out some implausible limitations of Ross recovery theorem, *Ross (2013).* Although the great importance of recovery theorem has, from a more theoretical point of view, it infers economically fantastic limitations. This article provides an alternative unbounded conditions.

«Limitations, where?

Answer:

Implausible bounding restrictions on state vector dynamics», Dubynskiy & Goldstein (2013).

In this paper the innovative issue is that the explanation they give in bounding permits recovery. As in general economies happen derivative prices are results to partial different equations stated in terms of risk-neutral «drifts», as *Dubynskiy & Goldstein (2013)* named the expected change. These risk-neutral drifts have two parts.

1. Actual drifts
2. Relevant terms of risk aversion of the representative agent

The enforced by Ross frontier conditions give the permition of the above two parts of drift to separately be determined. In general, derivative prices do not provide enough information to disentangle these two separate components. More notably, this article displays that many models which belong to the literature recovery of drifts and preference parameters are likely without the state of having bounds on state vector dynamics. This is the main difference of this paper of *Dubynskiy & Goldstein (2013*) and *Ross'* paper *(2015).*

Under asset pricing point of view, no one is able to define expected returns of assets only by observing a cross section of derivative prices, derivatives which are written on those assets. The most common example of this is the case of Black, Sholes and Merton option pricing model, *Hull (2015).* This option pricing formula depend on that option prices are independent of the expected return on the underlying asset which are usually stocks. True transition densities can be recovered having certain constraints, *Dubynskiy & Goldstein (2013)*, *Ross (2013).* Also, *Ross (2013)* states that a researcher can recover the levels of risk aversion of the representative agent using derivative prices such as option prices.

Ross (2013) in his recovery make two vital restrictions.

1. The underlying state vector have definitely to be limited.
2. The ratio of marginal utilities, which the representative agent have among two dates, is a function based only upon future and current consumption, *Carr & Yu (2012)* and *Dubynskiy & Goldstein (2013).* This «path independence», as *Ross (2013)* prefers the expressions is a key element to his analysis of recovery theorem. This «path independence» creates an

environment without utility functions which are forced by additional state variables. State variables as for example, habit formation or preference shocks.

The recovery theorem trails the first order conditions of the representative agent. The ratio of A-D prices, *Arrow & Debreu (1954),* per unit probability equals to the ratio of marginal utilities. So, when the utility function is «path independent» these first order conditions are able to be formed as a matrix equation. In this matrix will still exist the property that each row of the transition density have to sum to one having the probabilities properties. Thus, pricing eigenvector of the A-D matrix defines the marginal utility for every value of the state vector which means that risk aversion and actual drifts can be recognized. Without loss of generality, this paper is focusing on one single period A-D transition matrix. This happens since the multi-period transition density is exclusively defined by the one period density.

However, *Dubynskiy & Goldstein (2013*) disagree with Ross' bounding limitations on consumption. Also, they believe that stocks dynamics which are demanded for the implementation of the implement the recovery theorem economically doubtful to exist. So, the new clue here in this paper is that they are taking a broad view of Ross' the perceptions by releasing the above limitations.

Artificially enforcing restrictions in application of the recovery theorem provokes huge changes to the solution which causes a unique estimate for the parameter vector even if a continuous of parameter vectors cause with their turn identical derivative prices. This is true even if parameters' estimates become sensitive to the position of the reflecting boundaries when these are boundlessness. Thus, *Dubynskiy & Goldstein (2013*) decide to neglect Ross' boundaries.

Because of the above economic doubt on Ross' restrictions this article investigates other general equilibrium models that allow drifts and levels of risk aversion to continue to be separately determined from derivative prices including the case of the unbounded state vector. Those state variables that are not straight noticeable such as drifts and jump-intensities are substituted to those which are noticeable. For example, the risk free rate and risk-neutral intensities can straight estimate every day from a cross section of securities with fixed income and derivative as option prices. Therefore, there is no need for boundaries on the state vector.

This paper points out that there is a natural way to recognize levels of risk aversion and expected returns. This does not happen through options on the stock index S&P 500 as *Ross (2013)* and a variety of approaches choose. But it does happen with the help of the term structures of interest rates and dividends.

Uniquely the risk free rate is the coefficient of relative risk aversion.

<u>This paper is advantageous for many reasons. Mainly for the three are following.</u>

- Firstly, its application exist even when preferences are «path dependent». The paper shows that state variables which guide preferences can truly be helpful in determination. Every state variable that influences derivative prices infers the presence of one supplementary noticeable state variable whose risk-neutral drift provides crucial information for the parameter vector. For instance, *Dubynskiy & Goldstein (2013)* adopt a simple example of a representative agent who deals with shocks in his preferences. These shocks are autonomous of the whole macro-economy. This autonomy makes the model less complex. Also, models as habit formation have the levels of habit and consumption correlated. So, then, models such these could drive to parallel conclusions.

- Secondly, the positions of the limitations are treated as parameters. This adoption can be determined through the maximum likelihood using data of derivative prices.

- The last and most important object is that the paper clarifies that the bounding of the state variable procedure is the cause why derivative prices are able to unravel drifts and risk aversion. Without boundaries A-D prices will be functions based only on risk-neutral drifts. So, these A-D prices will be possible separately determined by expected returns and risk aversion.

<u>A boundless state vector</u>

When someone numerically estimates a model with a boundless state vector usually artificially enforces restricted conditions at some finite limits and then studies how this solution pushes those limits further to infinity. If the influence is trivial then the boundary conditions do not influence the final solution. Namely, this article investigates the case of the economy with an unbounded log-endowment procedure which trails a discrete binomial time at every period Δt either increases by Δx or decreases by the same amount, where x is the log-endowment. This log-endowment is the single state variable which explains the economy. It is equal to log-consumption in general equilibrium. The representative agent of economy should consumes this log-endowment of the economy.

Three routes for recovery of drifts and preference parameters by *Dubynskiy & Goldstein (2013)*

1. Boundaries are demanded for the state vector. This route or «path» as this paper and *Ross (2013)* name, is used even by Ross' recovery.

However, in this paper there is the doubt that consumption has got upper and lower limits. It believes that these boundaries will be avoided in the future even by Ross.

2. Limitations are demanded for the parametrization of drifts and risk aversion.

3. Multiple state variables lead the economy.

Let us notice, that the spot rate value and the risk neutral parameters are all recognizable by bond prices using the database of bloomberg. Furthermore, the sensitivity of the slope to interest rate changes determines the coefficient of the risk aversion in the model of *Dubynskiy & Goldstein (2013)*. This application of these observations is the term structure of interest rates and dividends might give a better determination of drifts and preference parameters than the case of option prices. So, here, appears a huge difference between this paper and this to *Ross (2013)*. They preferred observations of term structure of interest and Ross chose historical options prices on the stock index S&P 500.

3.3.5 A critical view of recovery theorem by *Carr & Yu (2012)*

Carry & Yu (2012) have a more critical point of view than other approaches to recovery theorem in the financial literature. They suggest and explain an alternative preference-free way in order to export the same result as Ross did. Also, they present that the separation beliefs to preferences can be done with the help of the separation of variables. As in the above paper of *Dubynskiy & Goldstein (2013)* recovery theorem holds if the «path independence» limitation on the utility is instead imposed on the numeraire portfolio and its dynamics.

This paper is based on two asset pricing environment as all papers adopt and we do too.

- No arbitrage in the market. Namely, there is not a portfolio which can lose or earn under P.

- Markets are complete. So, information is perfect, there is no transaction costs and there is a price for every asset in every state of the world, *Cohrane (2005), Arrow & Debreu (1954*) and *Geakoplos (2004).* Namely, in the complete financial market future payoffs of any state contingent claim or state claim depend on future states of the world (as the example with the fair coin) as state of world in this study is the levels of stock prices.

A market is complete *if and only if* the equivalent martingale measure Q is unique!

And so, later in the analysis we will see that if there is a unique Q then will be a unique P!

Carr & Yu (2012) analyzing the recovery theorem point out that there are two types of probability measures. As *Ross (2013)* symbolizes P the probability measure which testifies the frequency that the market believes for future states and Q the risk neutral probability measure.

More mathematically the probability measures

P: holds in an arbitrage free market condition involving a money market account with its price $S_{0t} > 0$ in every $t \geq 0$.

Q: A probability measure which is equivalent to P as *Carr & Yu (2012)* characterize. Every asset's spot price is expressed as S_{it} with i=0,1,…..,n. This S_{it} is the relative price of the underlying asset, here stocks. Then, the Q measure will be a martingale and it will be expressed as following and it is a risk neutral measure.

$$Q = \frac{S_{it}}{S_{0t}}$$

A numeraire is a self-financing portfolio with the value always positive, *Carr & Yu (2012)* and *Arrow & Debreu (1954).* The money market account usually is used as a numeraire. Generally speaking, if the benchmark numeraire, as *Dybvig*

& Ross (2003) preferred the expression, changes containing the probability measure fixed, then the drift of a relevant price will change too. However, there is no just single one numeraire when pricing options.

Let's assume as *Carr & Yu (2012)* suggest that the probability measure P is not known to us but we do know market prices. From a combination of these prices and a set of assumptions the probability Q can be revealed. So, we end up with the known Q and unknown P.

«Well, knowing Q does imply that we will exactly know P and how we will do that? Are there more than one way to recover P from Q?», Carr & Yu (2012)

Answer:

«There is a disagreement between Ross and Carr & Yu on determining P from Q containing the Markov chain setting» Carr & Yu (2012).

Ross recovery assumptions as *Carr & &Yu (2012)* explain

1. Markets are complete
2. The representative agent's utility function is both state-independent and over time separate to a constant rate of time preference.
3. There is a unique state variable X that according to Q is a time-homogeneous Markov chain with a restricted number of states.

Adopting the above assumptions they end up that there is a way to the recover the transition probability matrix P of X using the supposed known risk-neutral transition probability matrix Q. In practice, it is used models in continuous state space.

«Is a way that P can be learnt from Q when the state variable X is a diffusion under Q? », Carr & Yu (2012)

Furthermore, this paper is interested in whether is essential the state variable X to lead the price of each asset in the market. *Ross* in his paper *(2015)* supposes a representative agent who will lead the prices. However, many combinations of financial assets in the market are highly correlated. So, there is a need to clarify that we mainly search in a subset of the market and no in the whole economy where all prices are leaded by a unique state variable X.

«Could there be more than one way to recover P from Q? », Carr & Yu (2012)

This article presents that exists a preference free relationship to extract P from Q when the state variable X, as we have already talked about, is a time homogeneous bounded diffusion.

«What does bounded diffusion process mean?»

Answer:

In probability theory and statistics a diffusion process as above is a solution to a stochastic differential equation. This process is Markov process with continuous time and continuous sample paths. A major example of this case of process is the Brownian motion. The characterization bounded explain the limitations up and down. There is a frame in process.

However, *Carr & Yu (2012)* ended up with an alternative example of differential process which holds the result of P from Q without being bounded. Finally, they conclude that the interest rate and the asset prices must not be bounded. On the other hand, the state variable X must be.

Numeraire: A self-investing portfolio

S_{0t}: The spot price of the money market account.

Assumptions

1. n risky assets with spot prices S_i for i=1,…,n
2. No arbitrage between n+1 assets. Namely, someone can always create a portfolio that every asset's relative price Si / L is a P martingale.
3. There is a portfolio with L value. L > 0 for all times u and t with $u \geq t \geq 0$:

$$\frac{Siu}{Lu}\bigg|_{\cdot Ft} = \frac{Sit}{Lt}$$, i=0,1,…..,n.

.

More assumptions for simplicity

4. Discrete in time.

5. The interest rate equals to 0.

Using the investor the help of a loan can conclude to the real amount ($S_{i,n+1} / S_{i,n}$) -1 as the realized gain. Also, this gain is the net return per $ which is invested in asset i at time n and it is realized at time n+1.

$$\frac{Si,n+1}{Si,n} - 1 = \frac{Si,n+1 - Si,n}{Si,n}$$

This gain / net return with its turn can be invested in the money market account and so, to be unchanged to the future. Namely, this can be expressed as following.

$$\frac{Si,n+1 - Si,n}{Si,n}$$

«Can we also gave a financial explanation to a backward return?», Carr & Yu (2012).

Answer:

The trader buys 1 share of asset i at time n. He funds the cost by borrowing $S_{i,n}$ with r =0. At n+1 the investor sells the share and refunds the loan. So, he realizes the gain of $S_{i,n+1} - S_{i,n}$. Another tactic is to invest his gain in the money market account. Thus, the backward return of this investment would be the number of shares that he purchased at time n+1. Having no dividends this backward return is time unchanged.

$$\frac{Si,n+1 - Si,n}{Si,n+1} \qquad \in \mathbb{R}$$

Backward Return **vs** Forward Return as *Carr & Yu (2012)* briefly referred

Carr & Yu's (2012) 3 trading strategies:

- Buy 1 share, borrow and realize into money market account
- Buy $1 worth of shares, borrow and realize into money market account
- Buy 1 share, borrow and realize into the i-th risky money market account

Using a numeraire portfolio with L value avoiding Ross' constrains to the shape of preferences really helped the application of this paper. More specifically, this article assumes that a specified set of assets' prices are determined by a univariate time-homogeneous bounded diffusion process. This process is a state variable and is symbolized as X as we have already seen. So, the value L is just a function of X and t. Well, L is a bi-variate time-homogenous diffusion process.

3.3.6 *Walden' (2012)* approach on extensions on Ross' theorem

Problem which is analyzed: Recovering the pricing kernel and real probability

When: The state variable is an unbounded diffusion

Well, this *Walden's* paper *(2012)* examining the recovering problem of pricing kernel and real probability distribution deriving from noticeable option market prices. This examinations occurs in the condition of an unbounded diffusion process. Namely, it occurs in an unbounded state variable. In practice, this paper belongs to a continuous time which is approximated on a restricted or discrete space without defining exactly the conditions of limits. Thus, *Walden (2014)* concludes that recovery is possible to be occurred in several diffusion processes on unbounded fields.

Walden's (2014) eye on Ross Recovery Theorem

- There are noticeable prices among states of nature as pricing kernel does. This is accordance to transition independence.
- Pricing kernel positive sign adds serious limitations
- State space finiteness

« When can the recovery work in an unbounded state space?», Walden (2014).

A rational approach does not depend upon whether the state space is bounded or unbounded as *Walden (2014)* expresses. The state space can be restricted too far so every rare events will not affect the results in each case in boundaries. Similarly to paper of *Carr & Yu (2012)* is presented that under suitable exogenously specified boundary conditions recovery is possible. Ross focuses on unbounded diffusion process but even he mentions the paper of *Carr & Yu (2012)* in order to refer to the alternative bounded condition.

Even if the true space is bounded the problem of truncation might continue to exist due to a restricted number of noticeable asset prices even in case of unknown bounds. For instance, in our real world with finite resources it is necessary to have an upper bound on the value of the stock market or countries GDP etc. However, to a scale as thousands, millions etc. the above looks implausible.

Rare events are important to the recovery because they relate to many fragility endings in equilibrium asset pricing models which are searched by the financial field in recent years. *Walden (2014)* adopting as all paper do based on *Ross (2013)* theorem a representative agent in the economy. State of nature grows in continuous time according to a time homogeneous diffusion process on an unbounded domain as *Ross (2013)* states.

Walden's (2012) major contribution to the recovery theorem

1. He exports properties of the diffusion process which control whether recovery is possible. So, the shape of the pricing kernel does not play an important role. Notice that a sufficient but not necessary condition is that the diffusion process should revert to mean after every shock. In addition,

when it is demanded the marginal utility to have up and down limits then, the drift of the diffusion process should be constrained to just one direction.

Furthermore, he proves that state prices are not able to define the recovery's possibility by themselves because information about the underlying asset process is vital. This paper points out that recovery is likely to occur for a wide range of different diffusion processes but there are cases where recovery cannot occur.

2. He emphasizes the relationship between recovery conditions for the unbounded circumstances and discrete state spaces. If option prices are familiar only in the bounded case then the recovery is possible on this bounded circumstance *if and only if* complete recovery is possible on the opposite unbounded circumstance. More specifically, in the last case of circumstance a near pricing kernel can be built using noticeable option prices on a bounded interval. As this bounded length raises, the approximation tries to meet the true kernel. Thus, it is important this approximation method to not have boundaries. In practice, it is really useful.

   Furthermore, *Walden (2014)* highlights that the solution, which belonging to a discrete approximation of the continuous case, in recovery problem is fragile to minor changes. Consequently, apart from the fact that a unique solution always holds in the discrete case this solution is probably wrong in conditions where continuous recovery fails.

A great notice by *Walden (2014)* is the importance of recovering depends μ and σ for large state variable $|x|$ in recovery. The state variable grows according to a one-dimensional time-homogeneous diffusion process as following and other approaches showed as *Ross (2013)* and *Carr & Yu (2012).*

$$dX_t = \mu(X_t)dt + \sigma(X_t)d\omega \qquad t \geq 0$$

Where,

X: the state variable

μ: will be equal to r in risk neutrality avoiding the traders' preferences always >0 and assuming here to be always constant.

σ: is the volatility always > 0 and assuming again to be always a constant.

«Specifically, in an economy where recovery is not possible given an X process, could it be that by defining $Y_t = G\ (X_t)$ for some smooth, strictly increasing transformation, G, with the whole real line as range to keep the process unbounded, recovery is possible under Y process ?», Walden (2012).

Answer:

No. He denies it, because the demand for recovery does not change in such alterations and therefore, it is not able to create again the recovery property.

A very interesting example by *Walden (2012*) is with a representative agent with **u** as the utility power on **c** for consumption as follows.

Assumption

υ(x): takes the form $e^{-\gamma x}$ for a parameter γ,

⟶ Recover **γ, ρ** and the diffusion process for **X**.

$$u'(c) = c^{-\gamma},$$

$$g(t,X) = e^{\alpha t + X},$$

$$\alpha = -\alpha\gamma \qquad , \alpha > 0$$

$$\upsilon(x) = e^{-\gamma x}$$

$\mathbf{e^{\alpha t}}$: a long-term growth component

α: a parameter which is assumed to be known

$\mathbf{X_t}$: the deviation from this long-term growth, and its dynamic is not a priori known

Although, recovery in the case as above with the representative agent is able to be covered from option prices there is need to take powerful restrictions on the kernel.

«So, altogether, whether does this leave us with respect to recovery? », Walden (2012)

Thus, from *Walden's (2012*) paper it is clear that recovering with these setting re not in accordance with the classical Black-Scholes economy. Well, recovery will come up to nothing in any model with stochastic growth. In stochastic growth models have got the long-term growth rate strictly positive and unknown. In the case of the black-Scholes model it is necessary to be taken powerful restrictions as marginal utilities which have boundaries strictly away from zero.

To conclude, this paper presents a general characterization explaining when recovery of the pricing kernel and real probability distribution is able to happen in the case of models with a time homogeneous diffusion process without boundaries. So, when recovery has results on unbounded conditions then prices can only be noticeable on a bounded space which is a sub unbounded space. However, recovery is likelihood to occur for many interesting occasions but this cannot work in economies which are «*too close» as Walden (2012)* chose the expression, to the condition with a positive and long-term growth and to the marginal utility without having boundaries..

3.3.7 When *Backwell (2015)* proves the Recovery Theorem

Backwell (2015) discusses and analyses *Ross (2013)* recovery theorem. He criticizes it for the boundaries Ross chose and the existence of Markov chain which is in doubt in Ross recovery, combining so papers such as *Dubynskiy & Goldstein (2013*) and *Carr & Yu (2012).* He combines the *Arrow & Debreu (1954)* model with *Ross (2013)* recovery as we do. In his application section he chose to use a snapshot of option prices on the stock index FTSE/JSE instead of historical data of option prices on the stock index S&P 500 as Ross did. We in out thesis follow the *Backwell (2015)* decision as it is our guidance paper. His application work is analyzed in the following chapter where we provide our empirical approach. Through that paper *Backwell (2015)* concludes to a variety of remarks.

Remark 1

The combination of the discrete state space with the assumed Markov property [x] creates a model which is more familiar to researcher. The one period nature helps the practitioner and so he suggests to focus on a specific time horizon. Though, this simplicity creates problems to the preference aspect of the model. For this *Backwell (2015)* believes that the decision making by the investor point of view becomes «myopic», as he uses the expression. Furthermore, the boundaries in recovery are wide enough for the state space, here is the levels of the stock index, to not annoy the agent's utility function. Also, the application is robust to minor ups downs in the boundaries.

Remark 2

Expected Utility Theory (EUT), as *Samuelson (1937)* talked about that is a dominant method for risk decision making. However, *Backwell (2015)* believes that the EUT has major empirical faults. So, he remarks that in economic theory investors do not concern about the value of their investments or the state of nature but they do concern about how it disturbs the consumption. Also, the time additive extension which *Ross (2013)* suggests, has to be improved. It fails to capture any interaction between the two time points. In Backwell's paper there is not a parametric utility function but there is a need that this utility function holds. Thus, the preference assumptions which are needed are meta-structural.

Remark 3

Consumption is assumed as *Cohrane (2005)* agrees with *Backwell (2015)* to be determined only by the state of nature and not by the time or even by a previous state. This condition of the state independent is criticize by *Carr & Yu (2012).* Someone could have a more risk averse utility function in worse states than in other cases. *Backwell (2015)* loosens this assumption by subscripting the utility function in accordance with the current state where they apply.

Remark 4

Backwell (2015) takes bond prices combining *Martin & Ross (2014*) paper in order to have more simplicity in his application part. He points out that if the implied riskless bond prices are state independent, then pricing kernel is risk neutral.

Remark 5

In continuity with the above, state independent bond prices mathematically must conclude to equal marginal utilities. The marginal utilities are the mean that the utility function can influence the price probability relationship. So, prices now have a simpler sense as *Backwell (2015)* highlights.

Remark 6

Backwell (2015) in his application section decided to include volatility in the state space in order to have a better approximation to a Markov process where the time homogeneity technique would be more robust. On the other hand, involving the discrete volatility states the price discretization might provoke obstacles. It might not be easy to apply the model over different states of nature. *Backwell (2015)* advices a wealthier state space where there is a model freeway using the previous state in the state description which is a good choice as a proxy for volatility and it could let the model to have a realistic Markovian concept.

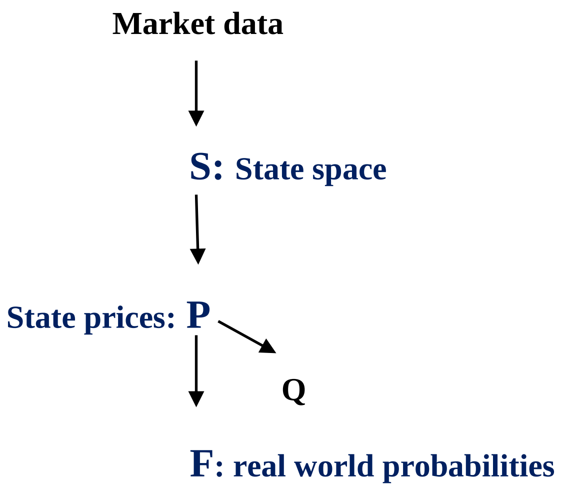
Chapter 4

Empirical Applications

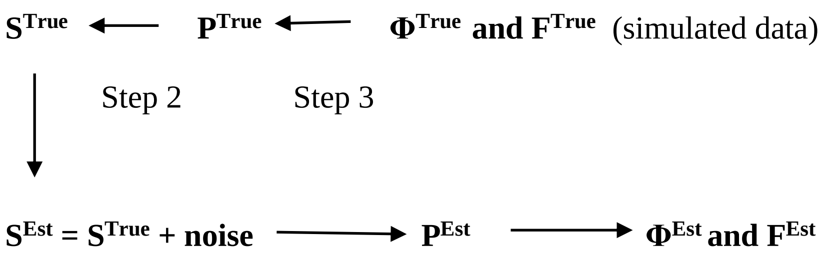
4.1: The core of the Ross recovery process

Ross (2013) represents through with paperwork his beliefs on a method for recovering the real world probability distribution as implied collecting market data from Bloomberg database. These market data correspond to S&P 500 index options with the today date of 27 April 2011 with a time horizon of 3 years. An index option as in our case is a type of derivative which provides the holder of the option the right and not the obligation to buy if we are talking about a call options or sell if we are talking about a put option the value of an underlying index. According to the stock markets index options have always a cash settlement. The historical range which is used is from April 2011 to April 2013 demanding only mild limitations on risk preferences. This mild approach *Backwell* (*2015)* characterizes it as meta-structural in his paper. The whole process includes five steps as *Ross (2013)* and other papers such the *Spears (2013)* approach.

1. Collect market data through Bloomberg as the risk free rate, the dividend yield from the current state etc.
2. Using them in order to estimate the matrix of states of nature S.
3. The matrix S guides to the matrix of state prices including the importance of *Arrow & Debreu (1954*) model.
4. The matrix P with its turn estimates the matrix Q which is the risk neutral transition matrix in order to finally estimate the real world transition matrix F.
5. The final algorithm concluding to a Monte Carlo simulation of option prices in order to conclude in single one call price for the whole transition state price matrix P.



- In a more detailed figure including the noise in the data from the market as *Kiriu & Hibiki (2015)* shows and *Spears (2013)* suggests.



On the other hand, we in our work will focus on the process of estimation of the matrix S (state space) and through that to be estimated the matrix P (state prices). We will reach out the state price transition matrix P based to a snapshot of FTSE/JSE Top 40 index in the market of South Africa as *Backwell (2015)*, our guided paper, suggests. *Bachwell (2015)* emphases that the state space matrix is the different levels of the underlying asset and more specifically of the FTSE/JSE Top 40 index and the state price matrix which include *Arrow & Debreu (1954)* reasoning for the prices. This matrix P consist of option implied state prices. It does not contain market option prices. We as Backwell (2015) will be based in the case of an in the money European call option and through the put call parity, as *Hull (2015)* shows, is easy to be determined the European put option price too.

<u>Put- Call - Parity</u>

$$c + K e^{-rT} = p + S_0$$

In this thesis, the application part is focusing on creating the S vector of the state space as *Backwell (2015)* explains and conclude to the (21x21) matrix P through Matlab codes. Finally, we conclude to the matrix F (21x21) with the real probabilities. In order to end up with the F we have to firstly calculate the eigenvector and the eigenvalue of P. So, we will recover the real probabilities and thus we will fulfill our goal!

4.2 Estimating the state space matrix S

State Space: This space is the different levels of the index FTSE/JSE Top 40 in the market of South Africa as *Backwell (2015)* our guided paper was based.

Assumptions

1. The state space is bounded as *Arrow & Debreu (1954)* model points out.
2. We are in discrete time as *Arrow & Debreu (1954).*

With the above two assumptions option prices become independent of risk preferences on options.

3. There is a representative agent in the economy as *Samuelson (1937*) and *Cohrane (2005)* explain.
4. No arbitrage as in articles of *Arrow & Debreu (1954)* and *Carr & Yu* (*2012).* And so then, a positive kernel exists as Radon-Nikodym derivative shows.
5. Market is complete.

There exists an equilibrium condition on consumption c for every possible value for i and j.

i: is the spatial step

j: is the time step

For j = 1: n $\longrightarrow$ **j = 1:21**

1st Obstacle

Unfortunately, this process in not a Markov chain because it has not got a particular probability space. So, we take as *Backwell (2015)* advises a new assumption for having a high volatility space. This is a suggestion to 1st obstacle in empirical part.

→ Solution: High volatility space

(nx1) vector S over a state space and more specifically with particular numbers as in the application section of *Backwell (2015)* (21x1) vector of states with a particular way to be fulfilled this vector. This way is, S1 = S11 /2 , S11 = the value of the index FTSE/JSE Top 40 in a snapshot of data on 18/09/2013, S21 = 3/2 * S11 and with the numbering distance between these states the prices of the index is 5% * S11.

From the database of DataStream we took the value of 38.832.74 ZAR but *Backwell (2015)* ended with the value 38.981.4 ZAR. This happened as *Backwell* explains (*2015)* because options were valued before the close of the day and then this value 38.981.4 is different from the published which is mine, 38.832.74 as it was in the stock market of South Africa. He faced the same inequality as we did and took into account only the value of 38.981.4 ZAR.

Taking into account the value of *Backwell (2015)* S11 = 38.981.4 ZAR

S2 = S1 * 5/100 * S11

S1 = S11/2	**19490,7**
	21439,77
	23388,84
	25337,91
	27286,98
	29236,05
	31185,12
	33134,19
	35083,26
	37032,33
S11	**38.981,4 ZAR**
	40930,47
	42879,54
	44828,61
	46777,68
	48726,75
	50675,82
	52624,89
	54573,96
	56523,03
S21=3/2*S11	**58472,1**

S20 = S19* 5/100 * S11

4.3 Estimating the state price matrix P

From option prices to state prices. The *Ross (2013)* recovery combining the *Arrow & Debreu (1954)* model.

Breeden & Litzensberg (1978) work in continuous time but we approximate the recovery in discrete time as our guided paper of *Backwell (2015).*

2nd Obstacle

The initial state links to a single row of the P matrix. It has to be estimated.

⟶ Solution: Low volatility space

Backwell (2015) refers to a t-bond for the 1st equality in order to include the Markov chain, to a future price for the 2nd equality to avoid the complications of dividends which caused problems to *Ross'* application part (*2015)* and an option on index to ends up in the last equality with the goal of pricing option under A-D reasoning.

Under these three constraints

$$\sum_{j=1}^{n} p_j^{(t)} = e^{-r(t)t} \qquad (1)$$

$$\sum_{j=1}^{n} p_j^{(t)} S_j = F^{(t)} e^{-r(t)t} \qquad (2)$$

$$\sum_{j=1}^{n} p_j^{(t)} \max(S_j - X_k, 0) = H_k^{(t)} \qquad (3)$$

where r, Sj, F, X_k, H_k are known from Bloomberg and t=3/12, 6/12, 9/12 and k=1,…..,9.

The linear equation (1) refers to a treasury bond t-bond price with continuous yield the risk free rate $r^{(t)}$ of this specific time 18/19/2013. The approach of using the t-bond we have already seen in *Martin & Ross (2013)* paper. The sum of the each row of matrix o is equal to the price of a riskless t-bond and as the whole setting of A-D securities give 1 unit in all states of the world, namely, the price levels of the index. Also, in the equation (2) *Backwell (2015)* preferred to use the future contract in order to avoid as he said, the complications of Ross' applications. Finally, the equation (3) refers to options payoffs in three different dates 3 months, 9 months and 9 months later to 18/09/2013 and so, the (3) ends up with 9 call option prices in the above three dates. This last equation does have to be discounted as *Backwell (2015)* highlighted that option prices H_k are already discounted because that take account of the fact that the payment is in the future.

Data from Bloomberg for a snapshot of 18 September 2013

Risk free rate r	2.000%	
The solution of 1st equation	t=3/12=T	**0.995**
	t=6/12=2T	**0.990**
	t=9/12=3T	**0.985**
	t=4T=12/12	**0.980**
	t=5T=15/12	**0.975**

Future price on FTSE/JSE	38,842.0 ZAR	
The solution of 2nd equation	t=3/12	**38648.27 ZAR**
	t=6/12	**38455.52 ZAR**
	t=9/12	**38263.72 ZAR**
	t=12/12	**46028.00 ZAR**
	t=15/12	**43473.00 ZAR**

3 month expiry t=T	
Strike price X for 9 European call options ZAR	9 European call prices ZAR the 3rd solution when t=3/12
X 1 = 38,832.74	**H1 = 1401.86**
X2 = 40774.38	**H2 = 520.05**
X3 = 42716.01	**H3 = 112.67**
X4 = 44657.65	**H4 = 15.68**
X5 = 46599.29	**H5= 2.62**
X6 = 48540.93	**H6 = 0.67**
X7= 50482.56	**H7= 0.22**
19X8= 52424.20	**H8= 0.07**
X9= 54365.84	**H9= 0.02**

The rest tables for nine call options at different time horizon are in Appendix B and more specifically in the corresponding tables from B.1 to B.4.

Then, we will continue with the method of using the remaining freedom in order to maximize the smoothness of the discrete distribution. This will be done with the help of the squared second differences. Namely, we will solve the following equality to result in twenty-one particular state prices p_j, j=1,…..,21 which the above three equation will hold with.

$$p^{(t)} = \underset{p \geq 0}{\operatorname{argmin}} \sum_{j=1}^{n} (p_{j-1} - 2p_j + p_{j+1})^2$$

- The equations from 1 to 3 can be also written as with b_{1j}=1 for the 1st , b_{2j} = S_j for the 2nd and b_{ij} = max(S_j-X_{i-2},0) for the 3rd.

$$b_{1,1}p_1 + b_{1,2}p_2 + \ldots + b_{1,21}\, p_{21} = c_1 \qquad (1)$$

$$b_{2,1}\ p_1 + b_{2,2}\ p_2 + \ldots + b_{2,21}\ p_{21} = c_2 \qquad (2)$$

$$\begin{cases} b_{3,1}\ p_1 + b_{3,2}\ p_2 + \ldots + b_{3,21}\ p_{21} = c_3 \\ \\ b_{11,1}\ p_1 + b_{11,2}\ p_2 + \ldots + b_{11,21}\ p_{21} = c_{11} \end{cases} \qquad (3)$$

These three equations can be written as well as matrices and vectors.

$$B*P = C$$

where the matrix B (11x21) consists of the values of the parameter b, the vector C (11x1) consists of the solutions from above equations, namely the data from Bloomberg and the vector which we are looking for P (21x1) consists of the state prices in *Arrow Debreu (1954)* environment.

- These first 21 state prices of p that will be given through the arg min equality will also be the first row of the transition state prices matrix P. We uses the mandate quadprog as the book of *Bandimate (2006)* explains in order to resolve this minimization argument and so, we continue to our final step.

  The *quadprog function* deals with quadratic programming problems such as our here.

$$\min_{p}: \frac{1}{2}\, p^{tr} H\, p + f\, p$$

$$s.t.\ A\, p \le b$$

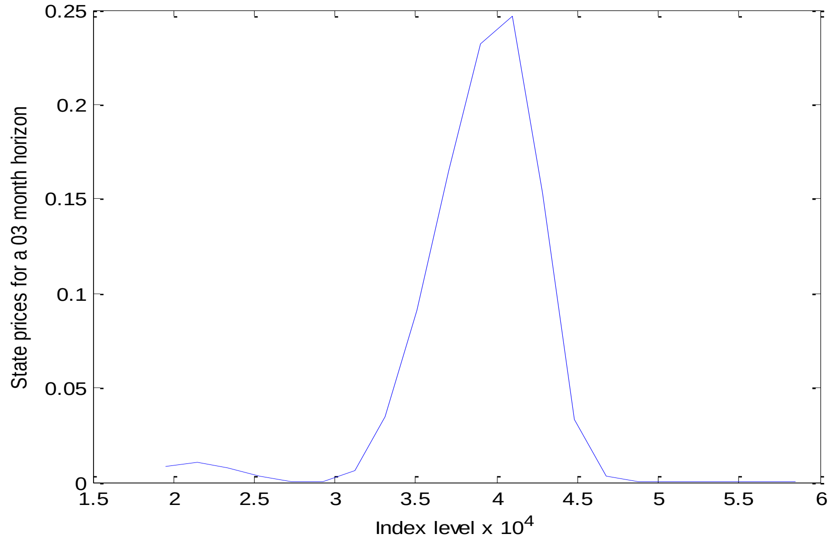
$$Aeq\ p = Beq$$

$$l \le p \le u$$

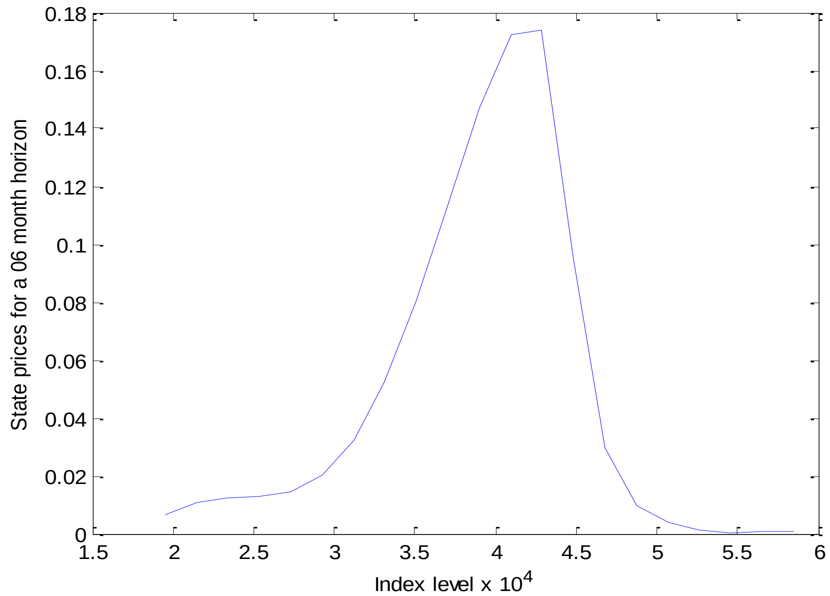
Where p is the state price vector we are looking for. H is the Hessian matrix which is calculated in Appendix using the Matlab program and its function is the A.1 algorithm. f is a zeros matrix for us, Aeq is the coefficient of the equations (1) (2) and (3) and beq their results. Finally, l is the lower bound which is 0.000 and u is the upper bound which is 1.000. These boundaries are involved because of the property of state prices to be probabilities. The corresponding functions for the quapdorg is in Appendix A and more specifically, the function A.2 refers to the t=3/12 and the function A.3 refers to the combination of different t. Also, the corresponding plots are in the Appendix B.

Indicatively, we have the following two plots which present the state prices in the y axis and the state space which is the stock index in the x axis. The plots for time horizon 3T, 4T and 5T are in Appendix B.

Plot B.1: State prices for t=3/12=T



Plot B.2: State prices for t=6/12=2T



<u>3rd Obstacle</u>

The matrix P with dimension 21x21 is difficult to be fulfilled

→ <u>Solution</u>: *Backwell (2015)* takes another assumption. The final matrix P is time-homogeneous.

This assumption is not powerful because of the non-Markovian issue which becomes important in long term maturities. But, having low volatility in the state we can approach in a better level the Markov process and not having problems in our estimated matrix P.

- The state prices in t=3/12=T consist the middle row which is the 11^{th} row of the following transition matrix P. It is a serious constraint.

The (nx1) vector of t-dated state prices $p^{(t)}$ becomes a matrix of dimension (21x21).

- Each row is a state price vector in its own right. We will focus on this point of view.

<u>Under the following constraints</u>

$$\sum_{j=1}^{21} p_{ij} \leq 1 \qquad (1)$$

$$\sum_{j=1}^{21} p_{ij}\, S_j = S_i\, e^{-q(T)T} \qquad (2)$$

$$(p^{(T)})^{tr} = (p^{(0)})^{tr} P \qquad (3)$$

where T=0.25 as Backwell (2015) defines in his paper with i=1,…..21 representatives the rows of the matrix P. The dividend yield q = 2.84% is from Bloomberg and S the state space as we have already calculated with the *Backwell's* way *(2015).* The (1) restriction here exist in order to help the state prices to exist even as probabilities. Also, he preferred to use here in the (2) equation the dividend yield to have a simpler restriction than before. The (3) equation is in accordance to *Ross (2013).* He presented the relationship $p^{t+1} = p^t$ P. Namely, *Ross (2013)* said that by looking at only m time periods we have the m^2 equations indispensable to give solution for the m^2 unknown $p_{i,j.}$ transition prices. A system of m^2 individual equations in the m^2 variables $p_{i,j..}$

The solution of 2nd equation with the prices of underlying assets as in the first table and with q=2.84%	**19352.81**
	21288.09
	23223.37
	25158.65
	27093.93
	29029.21
	30964.49
	32899.77
	34835.05
	36770.33
	38705.61
	40640.89
	42576.17
	44511.45
	46446.73
	48382.02
	50317.30
	52252.58
	54187.86
	56123.14
	58058.42

- Before continue to another arg min equality as we have above. Firstly, we have to determine the following function.

A smoothness function

$$S(P) := \sum_{i=1}^{21}\sum_{j=1}^{21}(p_{i,j-1} - 2p_{i,j} + p_{i,j+1})\hat{}2, \quad (1)$$

which is non-parametric and robust as *Spears (2013)* agrees. Also, as *Backwell (2015)* explains a smoothness function has got derivatives of all orders everywhere in its domain. It is a continuous function. The smoothness in mathematics is a property measured by the number of derivatives a function has.

A distance function

$$D(P) := \sum_{i=1}^{21}\sum_{j=1}^{21}(p^{((k+1)T} - p^{(kT)}\,P)^{tr}(p^{((k+1)T)} - p^{(kT)}P \quad (2)$$

which is also, non-parametric and robust as *Spears (2013)* agrees. As *Backwell (2015)* explains it is the function which minimizes the time distance between two rows of P. In other words, this distance function minimizes the difference between $P^{(0)}$ to $P^{(T)}$ and similarly the rest.

Then, we continue to the following to extract the matrix P.

$$P = \underset{p\geq 0}{\text{argmin}}\big(S(P) + D(P)\big)$$

In order to succeed that second argument minimization we should use as Matlab suggest the *fmincon function*. This is a function appropriate for nonlinear constrained programming solver.

$\min_{p} f(p)$: c(p)$\leq$ 0

ceq(p)=0

A*p ≤ b, A is a matrix and b is a vector

Aeq*p=beq, Aeq is a matrix and beq is a vector

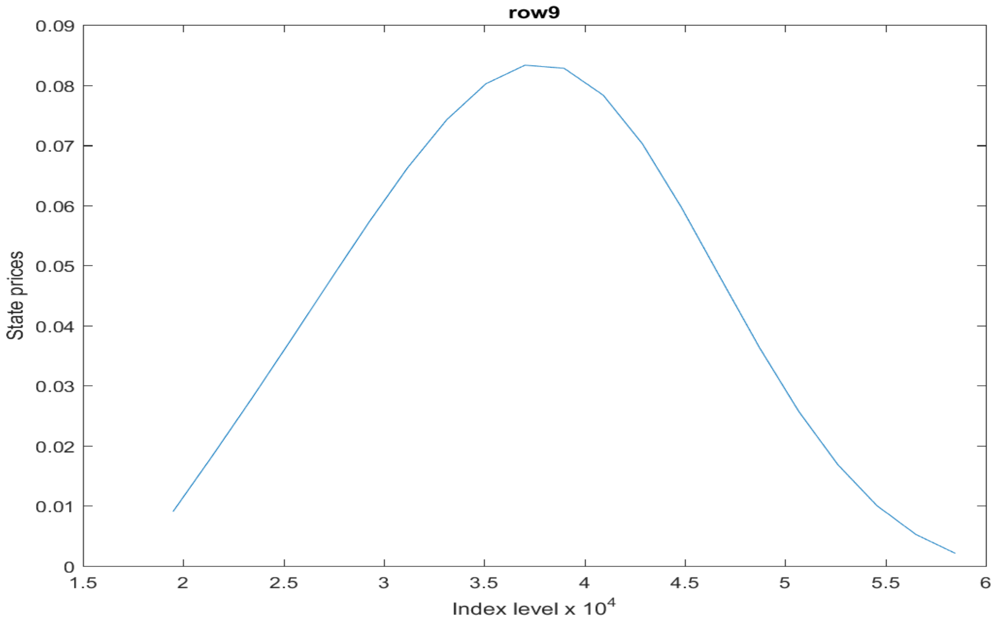
lb$\leq p \leq ub$

where, there is no Hessian matrix as before and we have to construct the function f which will be the sum of the functions S(P) and D(P). This specific algorithm is in Appendix A and it is the function A.4. Also, the algorithm which provides the whole transition matrix P is the function A.5 in Appendix A.

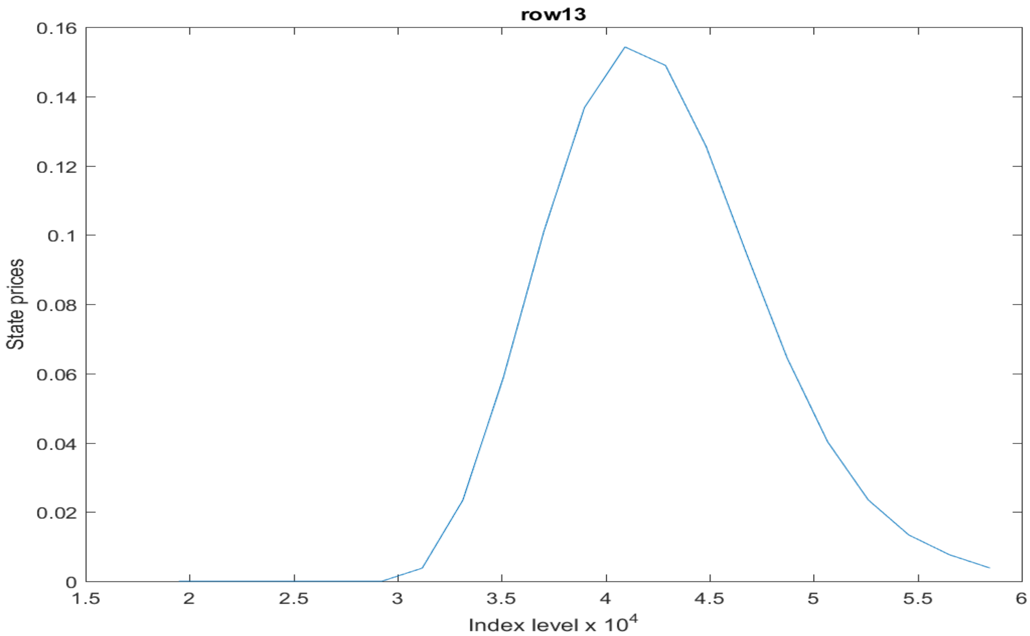
We materialize that for k=1 and k=2 and so, we end up with the table C at Appendix C which is the matrix P.

Indicatively, we present the two following plots and the rest of them there are in Appendix C. These plots presents different rows of the matrix P as the state price densities in the y axis and the state space in the x axis.

Plot C.9: State price distribution for the 9th row of the estimated state price matrix P

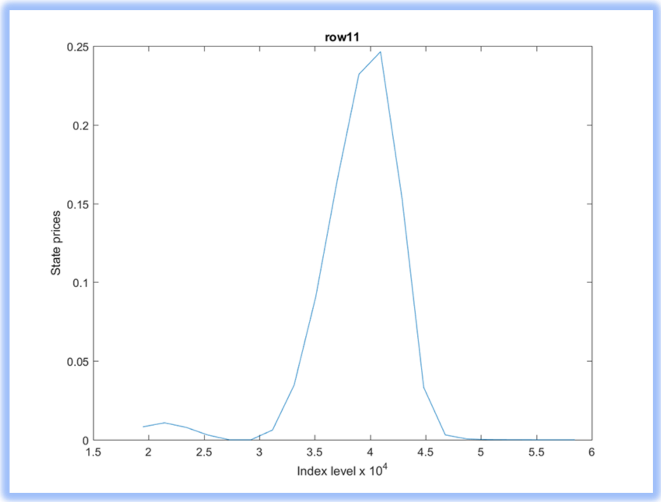


Plot C.13: State price distribution for the 13th row of the estimated state price matrix P

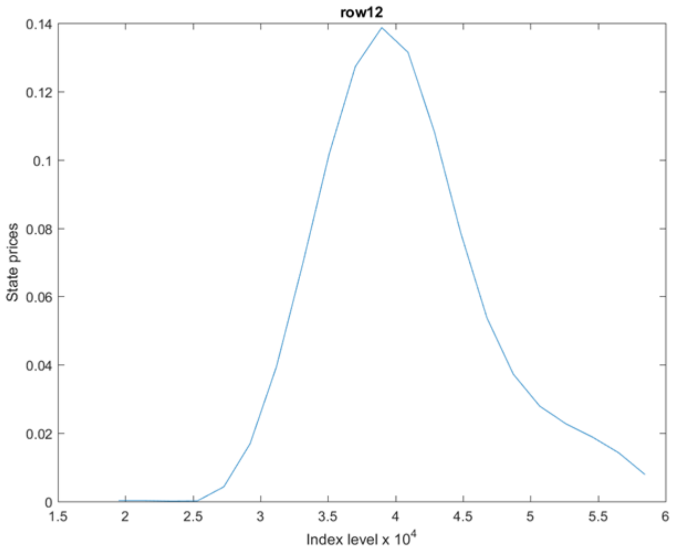


We materialize that for k=1:3 and so, we end up with the table D at Appendix D which is the new matrix P.

Plot D.11: State price distribution for the 11th row of the new matrix P



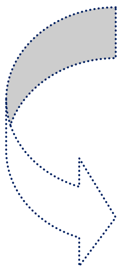
Plot D.12: State price distribution for the 12th row of the new estimated state price matrix P



Thus, we conclude to desired estimated state price matrix P at time t=T as *Backwell (2015)* did!

4.4 Estimating the real world probabilities matrix F

Furthermore, taking the knowledge of the transition matrix P we can as another deeper step recover the real probabilities matrix F, *Backwell (2015).* Knowing P is knowing F, *Ross (2013)*!



$$DP = \delta P\mathbf{F}$$

$$Pz = z\delta$$

$$z: = D^{-1} e$$

$$\mathbf{F}=1/\delta \, DPD^{-1}$$

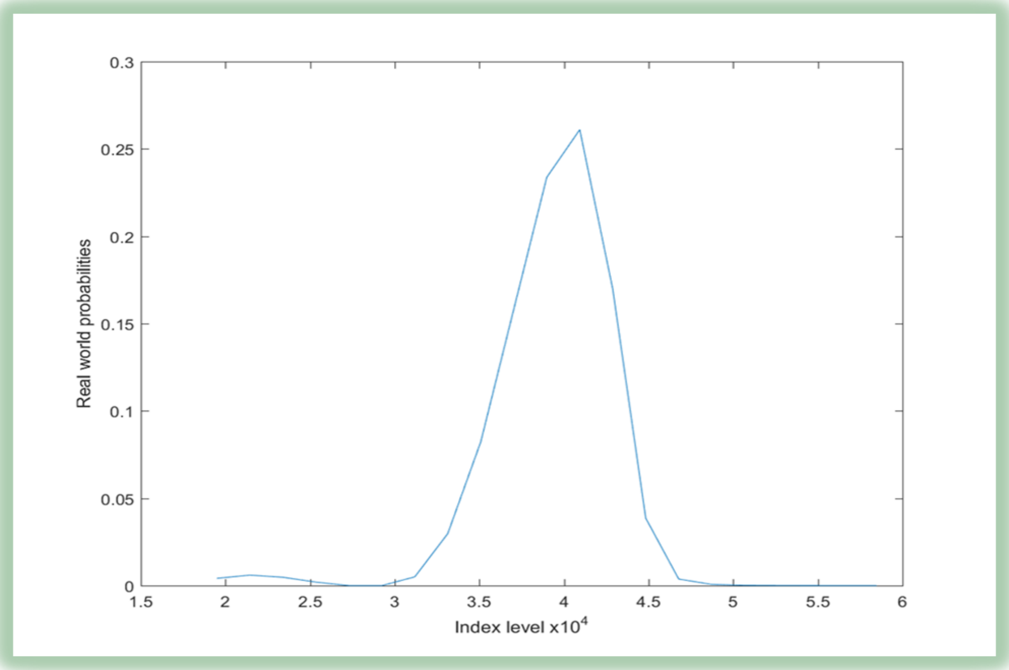
z: is the eigenvector of P

δ: is the eigenvalue of P. The characteristic root.

e: is the vector with 1 in all entries

D: is the diagonal matrix with the undiscounted kernel in its diagonal

From Perron - Frobenius Theorem holds the state that all of the positive eigenvectors, here are z, of P create an eigenspace which is one-dimensional and it is associated with a specific real and positive eigenvalue, here it is the discount factor δ. As in the above theorem δ exists and it is unique. We take these values using the corresponding mandates in Matlab which are available in Appendix A. Consequently, we conclude to the real world transition matrix F! The core of the Recovery Theorem! The whole matrix F is also available in the Appendix D as a table. The most important row of F is the 11^{th}. So, we present it as a diagraph from MatLab. We observe that this picture is very similar to the diagraph of the 11^{th} row of state prices P.



Real world probabilities
0.3
0.25
0.2
0.15
0.1
0.05
0
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
Index level x10^4

Chapter 5

Conclusions

If we can observe and estimate the transition of price matrix the Recovery Theorem allows us to recover the pricing Kernel. Armed with the market's risk aversion and the market's subjective assessment of returns, there is a huge variety of applications. It is a huge importance that a financial option can be now determined by the initial value of constructing a self-financing portfolio of assets replicating the option payoff in the maturity date. State prices under certain assumption are able to predict he probability and also, to price kernel uniquely.

Economists and investors are regularly asked to fill out surveys to determine some consensus estimate for the expected return on the stock market. It is really interesting because in fixed income markets forward rates help practitioners to infer the market's prediction of future spot rates. On the other hand, equity markets cannot resolve this apart from the solution of recovery theorem, state prices. Knowledge of both kernel (measurement of the degree of risk aversion in the market) and the natural probability distribution will also provide information to the controversy of whether the market is too volatile to be consistent with rational pricing models.

Ross research and its extensions are useful in passive equity management, risk management and asset allocation. Consultant and researchers can use the recovery theorem in order to make forecasts on market volatility, on the probability of upheavals and the long term equity risk premium for determining optimal asset allocation.

One may well ask about the extent to which Ross should change our world view. However, there is no a direct answer to this.

- The Ross recovery idea provides multiple interesting directions for future research. An important issue which should be tested is whether the real world transition matrix F has the ability to make forecasts hold up over time.

Appendix

List of algorithms, tables and plots used in Matlab simulations

A: List of algorithm

Function A.1: The Hessian matrix

```
function H=matrixH(H)
v=[ 10 12 12 12 12 12 12 12 12 12 12 12 12 12 12 12 12 12 12 12 10];
D =diag(v)
% Table with all 0 except for the diagonal
A=[ -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 -8 ]
D1 =diag (A,1)
D2=diag(A,-1)
B=[2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2]
D3=diag(B,2)
D4=diag (B,-2)
%H = sum of tables
>> H=D+D1+D2+D3+D4
H =

  10  -8   2   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
  -8  12  -8   2   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
   2  -8  12  -8   2   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
   0   2  -8  12  -8   2   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
   0   0   2  -8  12  -8   2   0   0   0   0   0   0   0   0   0   0   0   0   0   0
   0   0   0   2  -8  12  -8   2   0   0   0   0   0   0   0   0   0   0   0   0   0
   0   0   0   0   2  -8  12  -8   2   0   0   0   0   0   0   0   0   0   0   0   0
   0   0   0   0   0   2  -8  12  -8   2   0   0   0   0   0   0   0   0   0   0   0
   0   0   0   0   0   0   2  -8  12  -8   2   0   0   0   0   0   0   0   0   0   0
   0   0   0   0   0   0   0   2  -8  12  -8   2   0   0   0   0   0   0   0   0   0
   0   0   0   0   0   0   0   0   2  -8  12  -8   2   0   0   0   0   0   0   0   0
   0   0   0   0   0   0   0   0   0   2  -8  12  -8   2   0   0   0   0   0   0   0
   0   0   0   0   0   0   0   0   0   0   2  -8  12  -8   2   0   0   0   0   0   0
   0   0   0   0   0   0   0   0   0   0   0   2  -8  12  -8   2   0   0   0   0   0
   0   0   0   0   0   0   0   0   0   0   0   0   2  -8  12  -8   2   0   0   0   0
   0   0   0   0   0   0   0   0   0   0   0   0   0   2  -8  12  -8   2   0   0   0
   0   0   0   0   0   0   0   0   0   0   0   0   0   0   2  -8  12  -8   2   0   0
   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   2  -8  12  -8   2   0
```

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	-8	12	-8	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	-8	12	-8
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	-8	10

Function which are required for the quadprog programming (engineering toolbox for optimization)

Function A.2: The vector of state prices at t=3/12 for the 1st arg min

```
clear
% it take file with the values of S
load S.txt
% it takes the variables X for the first case and the Hessian matrix
load xh.txt
% From Strikes vector it takes 1st column
X=xh(:,1);
% From the Hessian matrix it takes 2nd column
H=xh(:,2);
% it takes the whole Hessian matrix with all dimension
load Hessian.txt
% We define the length of the vector S
n=length(S);
% We define the length of the vector X
k=length(X);
% This is the 1st column of the A matrix. The A matrix is a combination of the
first three restrictions for the first arg min. The coefficients to p at each restriction.
% 1st column of A
A(:,1)=ones(n,1);
% 2nd column of the vector S
A(:,2)=S;
for i=1:n,
   for j=1:k,
      SX(i,j)=max(S(i)-X(j),0);
   end
end
% it takes the combination S and X to A
A(:,3:k+2)=SX;
% We define the dimension to Beq which have the solutions at first case of time
Beq = [0.995 38648.27 H'];

% We must create the inverse matrices because we face problems at quadprog
A=A';
Beq=Beq';
%The bounds should take these expressions as quadprog advices in order to not
take plausible and negative call option state prices
% lb>0
lb=eps*10*ones(n,1);
```

```
ub=ones(n,1);
f=zeros(n,1);
% We are requiring the quadprog now
p = quadprog(Hessian,f,[],[],A,Beq,lb,ub);
% 21 p which are the optimum solution
disp('optimun solution')
```

Function A.3: It combines the case of t=6/12, t=9/12, t=12/12 and t=15/12 and creates the state price vector for the 1st arg min

```
clear
%
d=menu('Dedomena','3/12','6/12','9/12','12/12','15/12');
%exp(-r(t))
emr=[0.995 0.990 0.985 0.98 0.975];
% F* exp(-r(t))
F=[38648.27 38455.52 38263.72 46028 43473];
% All strikes for all t
Xmat=[38832.74   38832.74   38832.74    45903         45903
          40774.38     40774.38    40774.38   50034.27     50034.27
          42716.01     42716.01   42716.01    54165.54     54165.54
          44657.65     44657.65    44657.65   58296.81     58296.81
          46599.29     46599.29   46599.29    62428.08     62428.08
          48540.93     48540.93   48540.93    66559.35     66559.35
          50482.56     50482.56    50482.56   70690.62     70690.62
          52424.20     52424.20     52424.20   74821.89     74821.89
          54365.84     54365.84     54365.84   78953.16     78953.16];
%  All option market prices for all t
Hmat = [1401.86 2090.82  2688.79    1070.28     1752.22
              520.05     1120.47    1701.71    21.02       236.80
              112.67     480.20       942.95      0.04         4.64
              15.68       176.06      432.03       0              0.50
              2.62          69.30        172.14       0              0.08
              0.67          31.20        73.20         0              0.01
              0.22          15.69        36.61         0              0
              0.07          8.67          20.48         0              0
              0.02          5.09          12.32         0              0];
% The state space as above
S=[19490.7  21439.77    23388.84    25337.91    27286.98    ...
    29236.05   31185.12    33134.19    35083.26    37032.33    ...
    38981.4    40930.47    42879.54    44828.61    46777.68    ...
    48726.75   50675.82    52624.89    54573.96    56523.03    58472.1];
 S=S';
%
X=Xmat(:,d);
```

```
H=Hmat(:,d);
%
load Hessian.txt
%
n=length(S);
k=length(X);
% 1st column of A matrix
A(:,1)=ones(n,1);
% 2nd column of the vector S (nx1)
A(:,2)=S;
% The column from 3 to k+2 to the whole matrix is the sub-matrix is the max(Si-
Xk,0)
for i=1:n,
   for j=1:k,
      SX(i,j)=max(S(i)-X(j),0);
   end
end
% It takes the matrix SX to the matrix A
A(:,3:k+2)=SX;
% I define the dimension 1x(k+2)
Beq = [emr(d) F(d) H'];
% The quadprog here needs our inverse matrices
A=A';
Beq=Beq';
% the lower bound as above lb>0
% lb vector (nx1)
lb=eps*10*ones(n,1);
% vector nx1
ub=ones(n,1);
f=zeros(n,1);
% This is a warning from Matlab
options = optimset('LargeScale','off');
% I call the quadprog. The program advices the following expression
p = quadprog(Hessian,f,[],[],A,Beq,lb,ub,[],options);
% eps = zero of Matlab
Namely, as in the command window
>> eps
ans =   2.2204e-016
p=max(p,eps);
% p are the optimum solution
disp('optimum solution')
p'

% I want to create plot diagrams for all different t. I put ; Matlab to change row
titles=['03/12';'06/12';'09/12';'12/12';'15/12'];
figure(d),
title(titles(d,:))
% We divide with 10^e to take the same climax as Backwell did
plot(S/1e04,p)
% In order to plots have labels
```

```
xlabel('Index level x 10^{4}')
ylabel(['State prices for a ' titles(d,1:2) ' month horizon'])
```

Function which are required for the fmincon programming (engineering toolbox for optimization)

Firstly we do the following for k=1:2

Function A.4: Function f namely f=S(P)+D(P)

```
function f=fun12(P,pT,p2T,p3T)
% Backwell said that P11=pT
P(21*10+1:21*10+1+20)=pT;
% ph is the desired P matrix

ph=[];

for i=1:21,
   ph(i,:)=P(21*(i-1)+1:21*(i-1)+1+20)';
end
% Because Matlab has to start from somewhere. For sum we use 1.
SP=0.0;
for i=1:21,
   for j=2:20,
      SP=SP+(ph(i,j-1)-2*ph(i,j)+ph(i,j+1))^2;
   end
   SP=SP+(-2*ph(i,1)+ph(i,2))^2+(ph(i,20)-2*ph(i,21))^2;
end
% A is the auxiliary matrix that we need with dimension 21x1. It is a vector
A=p2T-ph*pT;
% B is the auxiliary matrix that we need with 21x1 dimension as a vector.
B=p3T-ph*p2T
% this is the required function D(P)
DP=A'*A+B'*B;
f=SP+DP;
% we use the absolute values of f to find appropriate values and succeed a better
arg min result values of P matrix
f=abs(f)
% return
```

Function A.5: Function for the matrix P 21x21

```
clear

% S is a vector with 1x21 dimensions
```

```
S=[19490.7  21439.77    23388.84    25337.91    27286.98    ...
    29236.05   31185.12    33134.19    35083.26    37032.33    ...
    38981.4    40930.47    42879.54    44828.61    46777.68    ...
    48726.75   50675.82    52624.89    54573.96    56523.03    58472.1];

% Seq is a vector with 1x21 dimension. It is the 2nd solution in the 2nd arg min
with q.
Seq=[19352.81   21288.09    23223.37    25158.65    27093.93    ...
         29029.21   30964.49    32899.77    34835.05    36770.33  ...
         38705.61   40640.89    42576.17    44511.45    46446.73  ...
         48382.02   50317.30    52252.58    54187.86    56123.14   58058.42];

% we take the previous results of the 1st arg min.
% for t=3/12=T
load run312;
% for t=6/12=2T
load run612;
% for t=9/12=3T
load run912;
% we create the coefficient with 1 for all values. We need it for the 1st constraint.
assoi is the vector of A which takes only 1.
assoi=ones(1,21);
A=zeros(21,21^2);
for i=1:21,
          % matrix A have 1 from 1 to 21 for the 1st row and later all 0. In the 2nd
row A has 0 until 21 and from 22 to 43 it takes 1 later all 0 etc. A matrix has
21x21^2 dimensions.
          A(i,21*(i-1)+1:21*(i-1)+1+20)=assoi;
          Aeq(i,21*(i-1)+1:21*(i-1)+1+20)=S;
end
b=ones(21,1);
% beq now is again for the 2nd equation restriction
beq=Seq';
% the bounds for the values of pij
lb=zeros(21^2,1);
ub=ones(21^2,1);
% fmincon explain how the options in this part should be expressed
options = optimset('LargeScale','off','TolX',1e-06,'TolFun',1e-6);
% the program cannot run without an initial value for P as Matlab and fmincon
explain. If we define P0=zeros (_) then the program ends up with only zero
values. This is not correct. So we use the rand mandate.
P0=rand(21^2,1);
% this the appropriate expression. Without @ the program cannot "run"!
P=fmincon(@fun12,P0,ones(1,21),ones(1,21),S,Seq,lb,ub,[],options,pT,p2T,p3T);
Is exactly what fmincon demands and what we have
P= fmincon(@fun12,P0,A,b,Aeq,beq,lb,ub,[],options,pT,p2T,p3T);
```

Function A.6: New algorithm for the new fun12 for k=1:3

```
%
function f=fun12(P,pT,p2T,p3T,p4T)
P(21*10+1:21*10+1+20)=pT;
ph=[];
for i=1:21,
    ph(i,:)=P(21*(i-1)+1:21*(i-1)+1+20)';
end

SP=0.0;

for i=1:21,
    for j=2:20,
        SP=SP+(ph(i,j-1)-2*ph(i,j)+ph(i,j+1))^2;
    end
    SP=SP+(-2*ph(i,1)+ph(i,2))^2+(ph(i,20)-2*ph(i,21))^2;
end
% array 21x1. New P matrix new D(P) for k=1:3
A=p2T-ph*pT;
B=p3T-ph*p2T;
C=p4T-ph*p3T;
DP=A'*A+B'*B+C'*C;
f=SP+DP;
% they are probabilities
f=abs(f)
% return
```

Function A.6: New algorithm for the new P matrix for k=1:3

```
clear

% S is a vector 1x21. State Space
S=[19490.7    21439.77      23388.84      25337.91      27286.98      ...
    29236.05  31185.12      33134.19      35083.26      37032.33      ...
    38981.4   40930.47      42879.54      44828.61      46777.68      ...
    48726.75  50675.82      52624.89      54573.96      56523.03
          58472.1];
%
% Seq is a vector 1x21. fmincon
Seq=[19352.81         21288.09       23223.37       25158.65       27093.93
          ...
         29029.21         30964.49       32899.77       34835.05       36770.33
          ...
         38705.61         40640.89    42576.17          44511.45    46446.73
          ...
```

```
        48382.02        50317.30        52252.58        54187.86        56123.14
58058.42];
% from command window run2 t= 3/12
load run312; %
% run2 t= 6/12
load run612;
%  run2 t= 9/12
load run912;
%  run2 t= 12/12
load run412;
assoi=ones(1,21);
A=zeros(21,21^2);
for i=1:21,
   A(i,21*(i-1)+1:21*(i-1)+1+20)=assoi;
   Aeq(i,21*(i-1)+1:21*(i-1)+1+20)=S;
end
b=ones(21,1);
beq=Seq';
%
lb=zeros(21^2,1);
ub=ones(21^2,1);
options=optimset('LargeScale','off','TolX',1e-06,'TolFun',1e-
6,'MaxFunEvals',1000000);
P0=rand(21^2,1);
% P = the matlab defines
fmincon(@fun12,P0,ones(1,21),ones(1,21),S,Seq,lb,ub,[],options,pT,p2T,p3T);
P= fmincon(@fun12,P0,A,b,Aeq,beq,lb,ub,[],options,pT,p2T,p3T,p4T);
arrayP=[];
for i=1:21,
   arrayP(i,:)=(P(21*(i-1)+1:21*(i-1)+1+20))';
end
arrayP(11,:)=pT;
save dataP P arrayP
for i=1:21,
   figure(i);
   plot(S/1e04,arrayP(i,:));
   title(['row' int2str(i)])
   xlabel('Index level x 10^{4}')
   ylabel(['State prices'])
   save as(gcf, ['fig' int2str(i)], 'bmp')
end
```

Function A.7: Algorithm for the eigenvalue, eigenvector and the matrix F.

```
clear all
%  from the run3
load dataarrayP2121
```

```
P=arrayP;
%  P is 21x21
% r=rows = columns
r=size(P,1);
% z=eigenvector, d=eigenvalue
[z,d]=eig(P);
% Backwell and Ross defines e
e=ones(r,1);
% The process  to find positive eigenvalue and its corresponding eigenvector
i=1;
q=[1];
while i<=r & ~isempty(q),
   % positive eigenvalue
 if real(d(i,i))>0,
        % eigenvectors
        q=[];q=find(real(z(:,i)<0));
       % We found positive z and δ
             if isempty(q),
             delta = d(i,i);
            disp('a positive eigevalue assigned to a posotive eigenvector is found')
            delta
            % positive z
            zp=z(:,i);
        end
    end
    i=i+1;
end
%
D=zeros(21);
for i=1:r,
   % D =diagonal
 D(i,i)=1./zp(i);
end
% array F
F=[];
if abs(det(D))> 0.001  % criterion for bad condition of D-matrix
    % Ross
        F=1./delta*D*P*inv(D);
else
    disp('MAtrix D - bad condition')
end
% S is a vector 1x21
S=[19490.7   21439.77     23388.84     25337.91     27286.98     ...
     29236.05    31185.12     33134.19     35083.26     37032.33     ...
     38981.4     40930.47     42879.54     44828.61     46777.68     ...
     48726.75    50675.82     52624.89     54573.96     56523.03
            58472.1];
% figure 11th row of F
   figure(11),
   plot(S/1e04,F(11,:))
```

```
xlabel('Index level x10^4')
ylabel('Real world probabilities')
saveas(gcf, ['fig' int2str(11)], 'bmp')
```

B Tables and plots for the state price vector

Table B.1: Call option market prices for t=6/12

6 month expiry t=2T	
Strike price X for 9 European call options ZAR	9 European call prices ZAR the 3rd solution when t=6/12
X1= 38832.74	**H1= 2090.82**
X2= 40774.38	**H2= 1120.47**
X3= 42716.01	**H3= 480.20**
X4= 44657.65	**H4= 176.06**
X5= 46599.29	**H5= 69.30**
X6= 48540.93	**H6= 31.20**
X7= 50482.56	**H7= 15.69**
X8= 52424.20	**H8= 8.67**
X9= 54365.84	**H9= 5.09**

Table B.2: Call option market prices for t=912

9 month expiry t=3T	
Strike price X for 9 European call options ZAR	9 European call prices ZAR the 3rd solution when t=9/12
X1= 38832.74	**H1= 2688.79**

X2= 40774.38	**H2= 1701.71**
X3= 42716.01	**H3= 942.95**
X4= 44657.65	**H4= 432.03**
X5= 46599.29	**H5= 172.14**
X6= 48540.93	**H6= 73.20**
X7= 50482.56	**H7= 36.61**
X8= 52424.20	**H8= 20.48**
X9= 54365.84	**H9= 12.32**

Table B.3: Call option market prices for t=12/12

1 year expiry t=4T	
Strike price X for 9 European call options ZAR	9 European call prices ZAR the 3rd solution when t=12/12
X1 = 45903.0	**H1 = 1070.28**
X2 = 50034.27	**H2 = 21.02**
X3 = 54165.54	**H3= 0.04**
X4 = 58296.81	**H4 = 0.00**
X5 = 62428.08	**H5 = 0.00**
X6 = 66559.35	**H6 = 0.00**
X7 = 70690.62	**H7 = 0.00**
X8 = 74821.89	**H8 = 0.00**
X9 = 78953.16	**H9 = 0.00**

Table B.4: Call option market prices for t=15/12

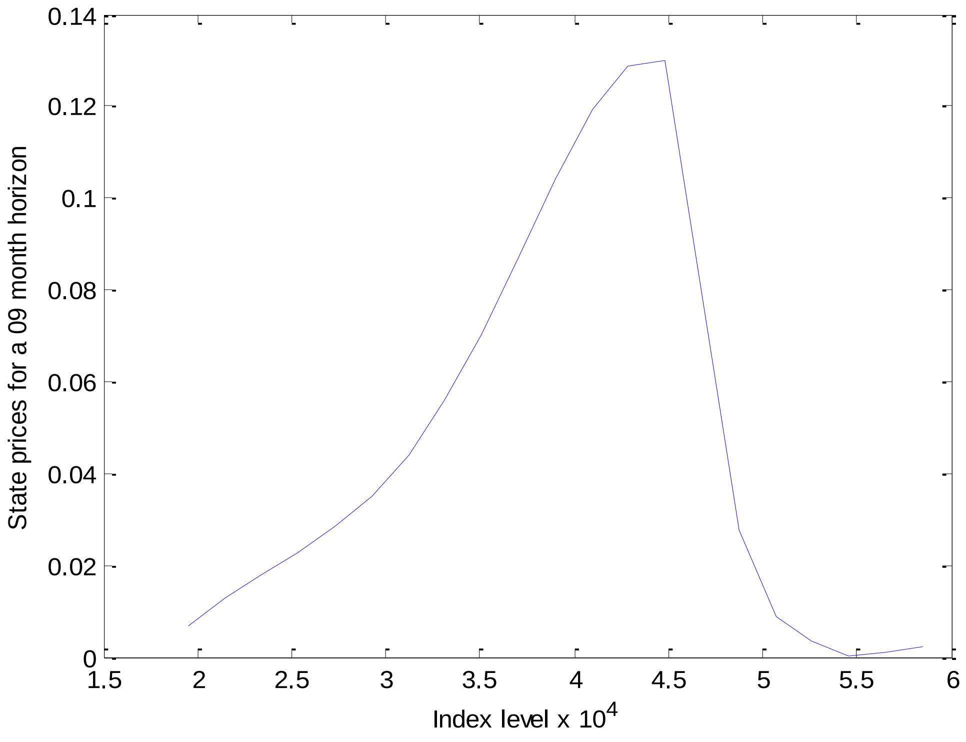
1 year and 3 months expiry t=5T	
Strike price X for 9 European call options ZAR	9 European call prices ZAR the 3rd solution when t=15/12
X1 = 45903.00	**H1 = 1752.22**
X2 = 50034.27	**H2 = 236.80**
X3 = 54165.54	**H3= 4.64**
X4 = 58296.81	**H4 = 0.50**
X5 = 62428.08	**H5 = 0.08**
X6 = 66559.35	**H6 = 0.01**
X7 = 70690.62	**H7 = 0.00**
X8 = 74821.89	**H8 = 0.00**
X9 = 78953.16	**H9 = 0.00**

Table B.5: The vector of the state prices at different time

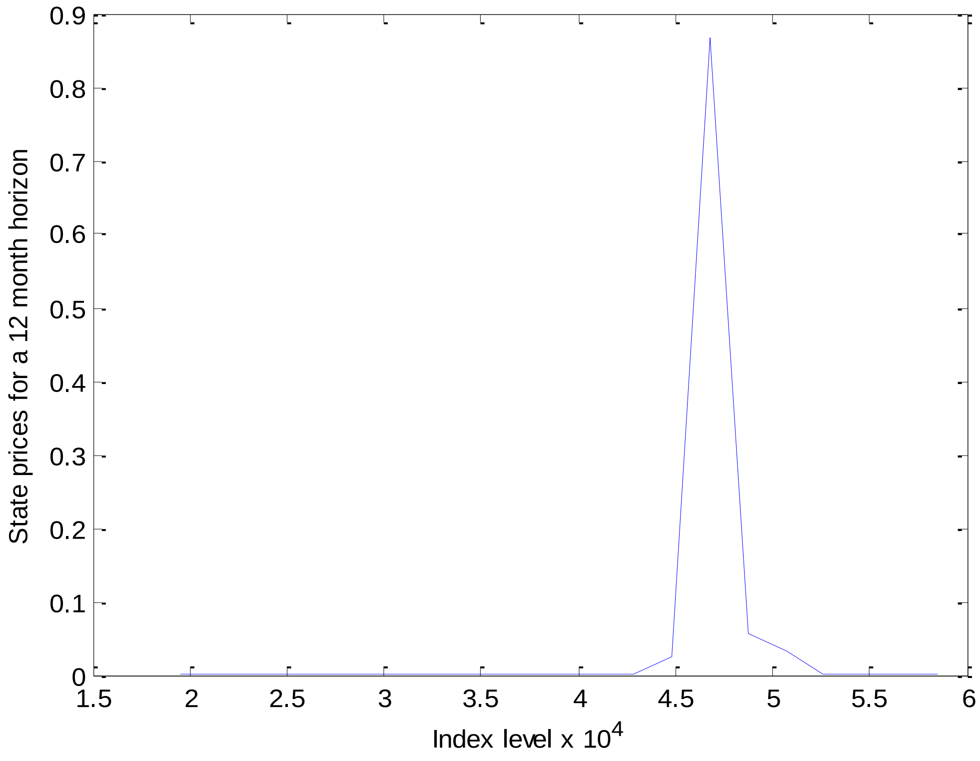
	t=3/12 we will use it as the 11th row of P (21x21) matrix	**t=6/12=2T**	**t=9/12=3T**	**t=4T**	**t=5T (zero prices of Matlab)**
p1	**0.0082497**	0.0064654	0.0067434	2.2204e-016	2.2204e-016
p2	**0.010779**	0.010579	0.012741	2.2204e-016	2.2204e-016
p3	**0.0078879**	0.012185	0.017959	2.2204e-016	2.2204e-016
p4	**0.0030377**	0.012701	0.022873	2.2204e-016	2.2204e-016
p5	**2.2286e-015**	0.014492	0.028273	2.2204e-016	2.2204e-016
p6	**2.2184e-015**	0.020248	0.03506	2.2204e-016	2.2204e-016
p7	**0.0061963**	0.032364	0.044047	2.2204e-016	2.2204e-016

p8	**0.034735**	0.052309	0.055762	2.2204e-016	2.2204e-016
p9	**0.09065**	0.08001	0.070244	2.2204e-016	2.2204e-016
p10	**0.16501**	0.11322	0.086847	2.2204e-016	2.2204e-016
p11	**0.23206**	0.14691	0.10404	2.2204e-016	2.2204e-016
p12	**0.24638**	0.17263	0.11921	2.2204e-016	2.2204e-016
p13	**0.15293**	0.17389	0.12892	2.2204e-016	2.2204e-016
p14	**0.033276**	0.095412	0.12986	0.025335	2.2204e-016
p15	**0.0030745**	0.029689	0.078689	0.86662	2.2204e-016
p16	**0.00054193**	0.0098494	0.027516	0.05543	2.2204e-016
p17	**0.00011399**	0.0038092	0.008778	0.032593	2.2204e-016
p18	**4.4786e-005**	0.0015531	0.0036035	2.2204e-016	2.2204e-016
p19	**1.4828e-005**	0.00019893	0.00027051	2.2204e-016	2.2204e-016
p20	**4.5828e-006**	0.00053692	0.0012073	1.6967e-005	2.2204e-016
p21	**1.7115e-006**	0.00094742	0.0023523	2.2204e-016	2.2204e-016

Plot B.3: State prices for t=9/12=3T



Plot B.4: State prices for t=12/12=4T



The last plot for state prices at t=5T is a straight line starting from the zero. There is no reasoning to picture it.

IF we change the stating value Matlab's plot takes this value and neglect the zero holding it as the unique state price for all different index levels.

C: The table of the state price matrix and its plots

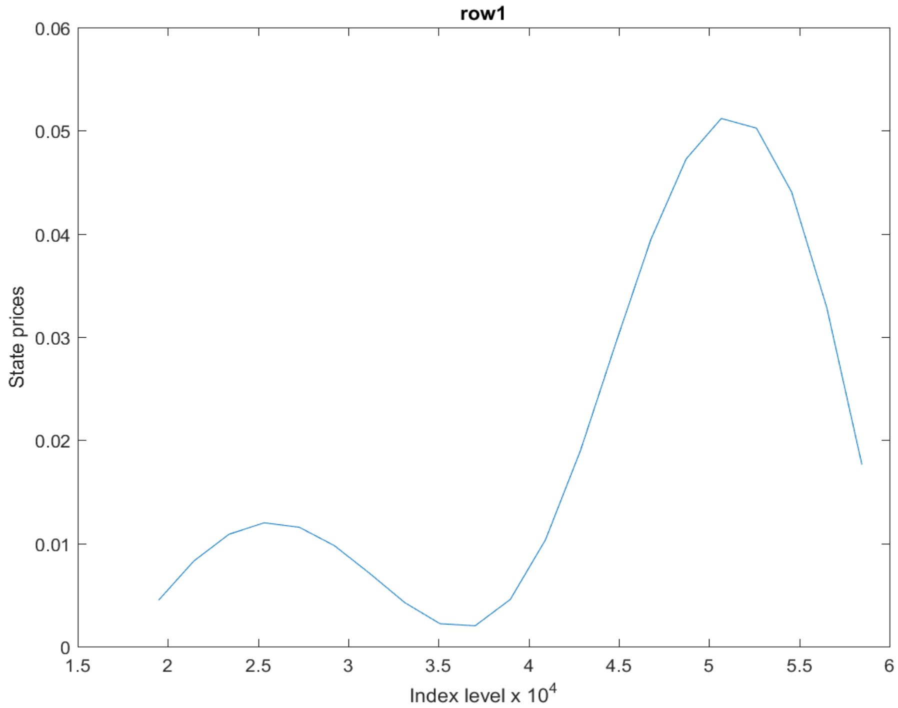
Table C: The transition matrix P (21x21) for k=1:2

i,j	1	2	3	4	5	6	7	8	9	10
1	0,074585	0,039541	0,072931	0,10721	0,10423	0,060134	0,029772	0,029892	0,066357	0,096965
2	0,046943	0,084224	0,019021	0,13809	0,068793	0,038631	0,03476	0,043174	0,06413	0,10302
3	0,037495	0,022881	0,095565	0,070977	0,071825	0,055538	0,044115	0,08164	0,055567	0,082064
4	0,0047382	0,039527	0,018284	0,043279	0,021111	0,047245	0,032933	0,038256	0,01874	0,074341
5	0,0066558	0,035616	0,084772	0,059963	0,044335	0,058685	0,04759	0,063452	0,078943	0,020472
6	0,047297	0,087485	0,062574	0,06382	0,021314	0,027656	0,03988	0,035178	0,046209	0,029981
7	0,016799	0,06043	0,040841	0,13248	0,073422	0,048815	0,026439	0,013156	0,045246	0,027441
8	0,088998	0,069715	0,12147	0,091295	0,045578	0,018292	0,07288	0,034309	0,042751	0,0089132
9	0,025136	0,017432	0,05753	0,080309	0,14864	0,083155	0,019098	0,079715	0,041358	0,051964
10	0,047797	0,063694	0,08382	0,10314	0,038961	0,071525	0,016573	0,037481	0,030301	0,041732
11	0,22847	0,0027338	0,0031571	0,0032167	0,00030784	0,0010041	0,0081967	0,0026329	0,1097	0,0062965
12	0,0061344	0,038036	0,038568	0,008353	0,0044196	0,0025672	0,0031056	0,028872	0,01806	0,087388
13	1,51E-17	9,59E-18	0,001639	0,023675	0,031818	0,043382	0,043345	0,03198	0,028679	0,09345
14	0,00020058	-1,97E-18	0,031556	0,0090707	0,029681	0,017206	0,033576	0,025574	0,021132	0,034051

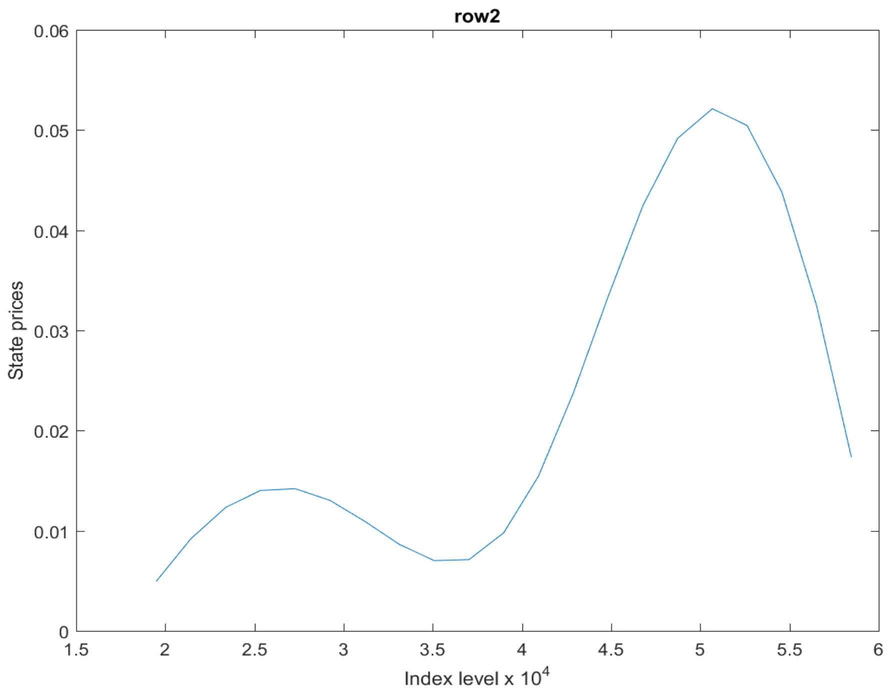
15	2,76E-18	5,11E-18	7,57E-05	2,09E-05	0,00013308	0,012774	0,046573	0,040284	0,023532	0,065202
16	1,32E-08	0,007285	0,010599	0,0042838	0,003895	0,01325	0,026072	0,013286	0,0065386	0,039557
17	6,41E-12	-5,02E-18	1,54E-18	2,57E-11	-9,93E-19	5,84E-10	-8,15E-18	3,58E-18	0,0036738	0,011751
18	1,23E-18	-4,50E-18	4,42E-18	-1,30E-17	7,03E-18	6,16E-19	-3,37E-18	7,09E-18	-2,00E-18	2,32E-18
19	-5,78E-18	-3,37E-18	-3,82E-18	3,09E-18	3,90E-18	1,46E-19	-5,40E-19	-1,03E-18	-7,90E-18	6,88E-18
20	6,61E-18	7,75E-18	3,08E-18	2,21E-18	1,67E-18	1,23E-18	1,43E-18	-5,03E-18	-7,43E-18	2,62E-18
21	-4,90E-18	-3,20E-18	-1,38E-17	-8,80E-18	1,39E-18	6,78E-18	-1,06E-17	4,27E-19	1,30E-18	-7,59E-18

11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**
0,0040412	0,00075933	8,48E-06	-3,87E-18	5,37E-19	1,45E-17	-2,19E-17	1,53E-17	-2,48E-18	-3,89E-18	4,05E-18
0,05321	0,014786	0,001358	0,00046265	7,66E-18	4,32E-18	3,37E-05	0,0052485	2,50E-05	1,52E-18	4,87E-18
0,082392	0,042643	0,0078532	5,25E-05	1,41E-17	7,81E-19	1,82E-09	9,22E-09	7,47E-07	0,00033946	1,62E-11
0,017132	0,035199	0,0093178	0,025888	0,06057	0,10131	0,066784	0,0039615	5,39E-09	2,12E-05	5,15E-18
0,036056	0,056957	0,034631	0,014262	0,029239	0,039607	0,020238	0,0063337	0,0088933	0,021085	0,0079364
0,079491	0,13267	0,09883	0,012744	0,019127	0,038923	0,008921	0,00064477	4,11E-05	4,58E-05	-2,42E-18
0,011609	0,015626	0,042342	0,046377	0,11371	0,056269	0,0084071	0,016385	0,024066	0,033697	0,003099
0,041403	0,02438	0,063193	0,073408	0,12519	0,012564	0,019825	0,010409	0,014134	0,0015421	0,0001126
0,023091	0,060481	0,089086	0,074639	0,032072	0,014661	0,021649	0,016508	0,0060416	0,038071	0,0049285
0,046941	0,067775	0,043874	0,010968	0,02303	0,032377	0,093949	0,035647	0,07064	0,019614	0,019589
0,005699	0,01197	0,11033	0,010059	0,17761	0,0011476	0,26295	0,013934	0,018549	0,0023036	0,0020054
0,15869	0,094847	0,075743	0,03152	0,097185	0,139	0,12225	0,012995	0,0066377	0,0020252	0,0011812
0,11099	0,081852	0,09292	0,082334	0,034871	0,031742	0,057798	0,073867	0,017492	0,048217	0,066651
0,057108	0,10589	0,095459	0,048502	0,072677	0,076711	0,13197	0,044319	0,046711	0,071689	0,046917
0,020311	0,060491	0,094792	0,057882	0,046056	0,13285	0,096848	0,099184	0,095681	0,070517	0,036773
0,056266	0,018605	0,050019	0,02821	0,056571	0,11771	0,16647	0,10154	0,081431	0,064098	0,134
0,031915	0,072423	0,033551	0,023533	0,052958	0,19725	0,15667	0,13418	0,088909	0,095455	0,097728
-2,46E-18	0,012729	0,024287	0,035997	0,096604	0,13055	0,11923	0,11491	0,17891	0,20051	0,085673
6,65E-18	4,16E-18	1,55E-18	0,0046461	0,067889	0,047387	0,14257	0,10977	0,19995	0,22181	0,20597
-5,24E-18	-8,44E-18	-1,09E-18	-9,07E-18	3,97E-18	2,75E-18	0,004756	0,073643	0,30751	0,35019	0,2639
-1,26E-18	8,67E-18	5,17E-20	9,07E-18	-1,19E-18	4,39E-18	1,74E-18	-1,09E-18	-7,71E-18	0,21224	0,78776

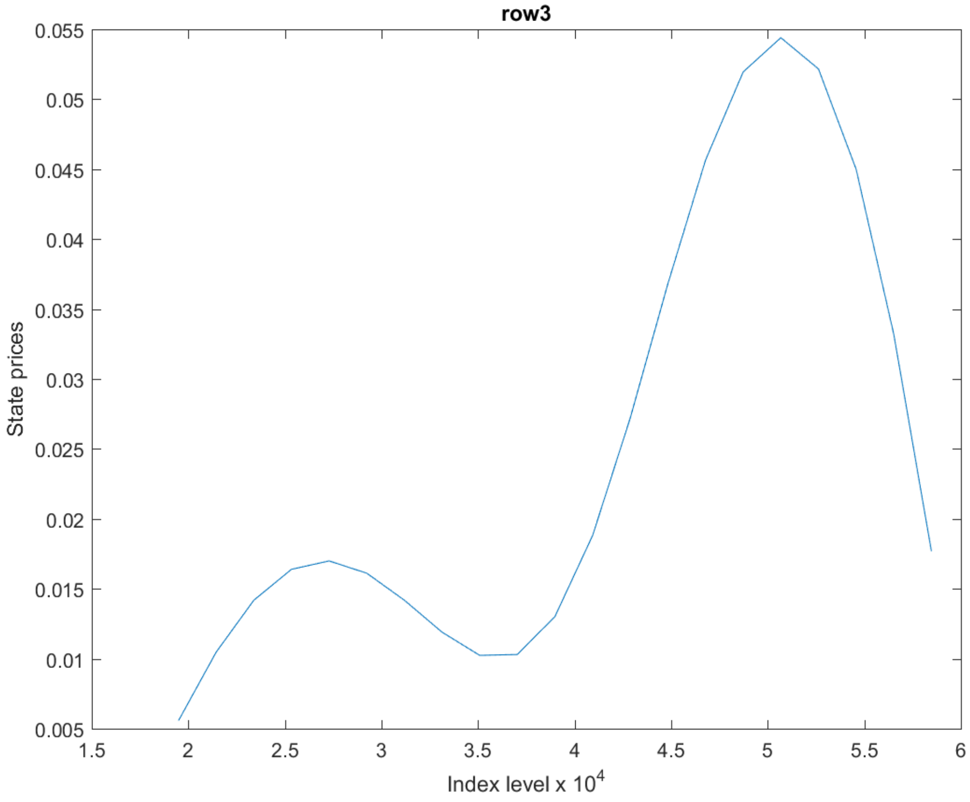
Plot C.1: State price distribution for the 1st row of the estimated state price matrix P



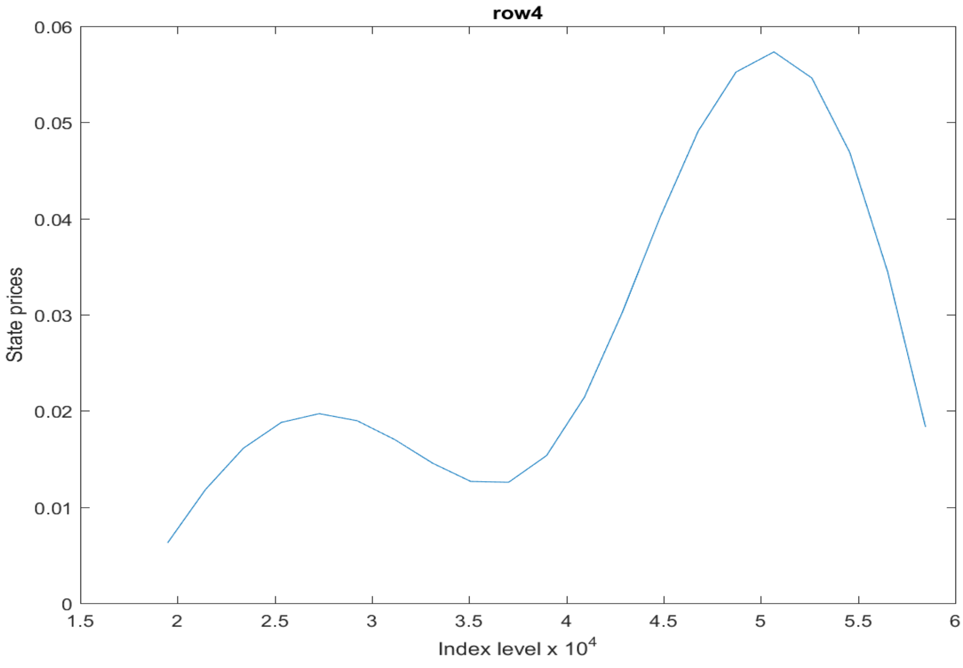
Plot C.2: State price distribution for the 2nd row of the estimated state price matrix P



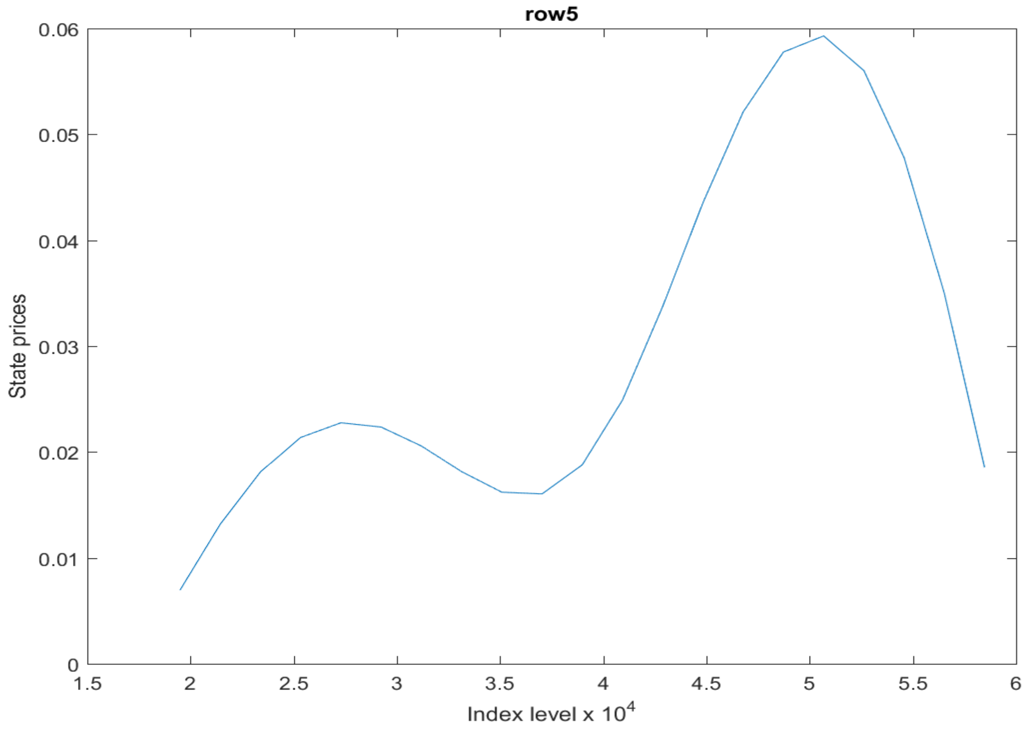
Plot C.3: State price distribution for the 3rd row of the estimated state price matrix P



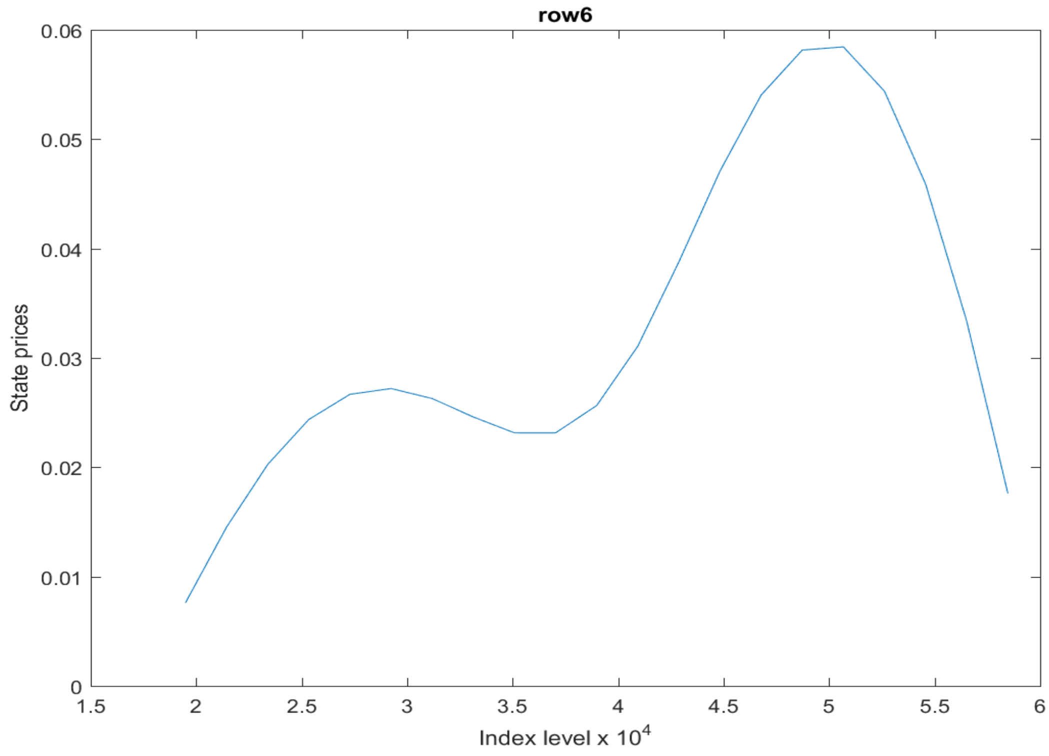
Plot C.4: State price distribution for the 4th row of the estimated state price matrix P



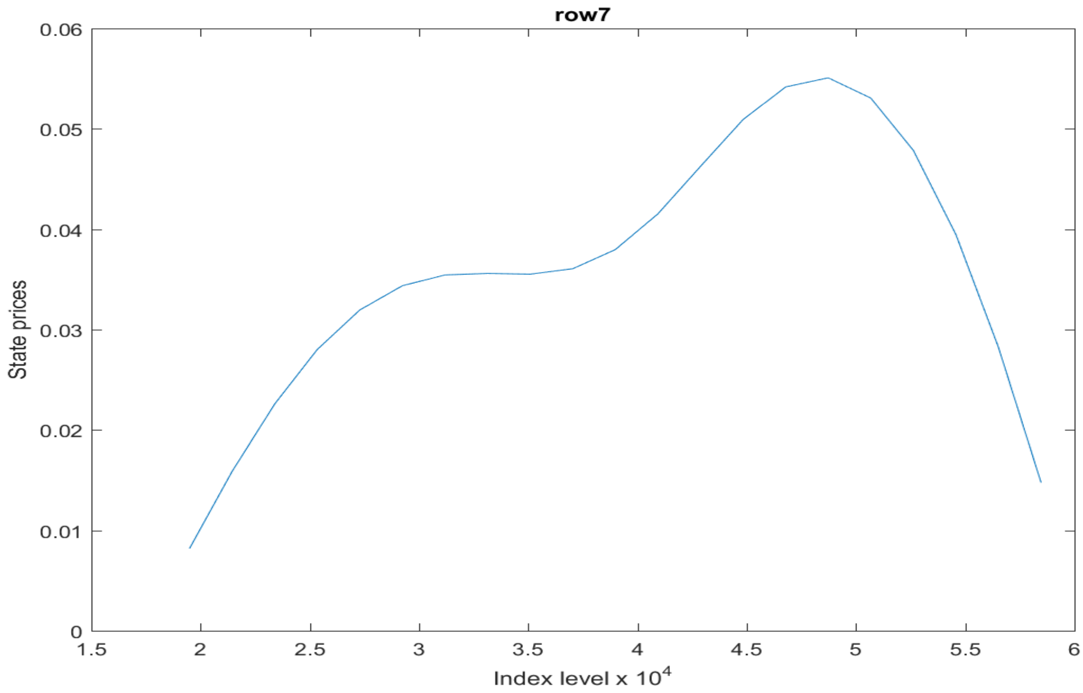
Plot C.5: State price distribution for the 5th row of the estimated state price matrix P



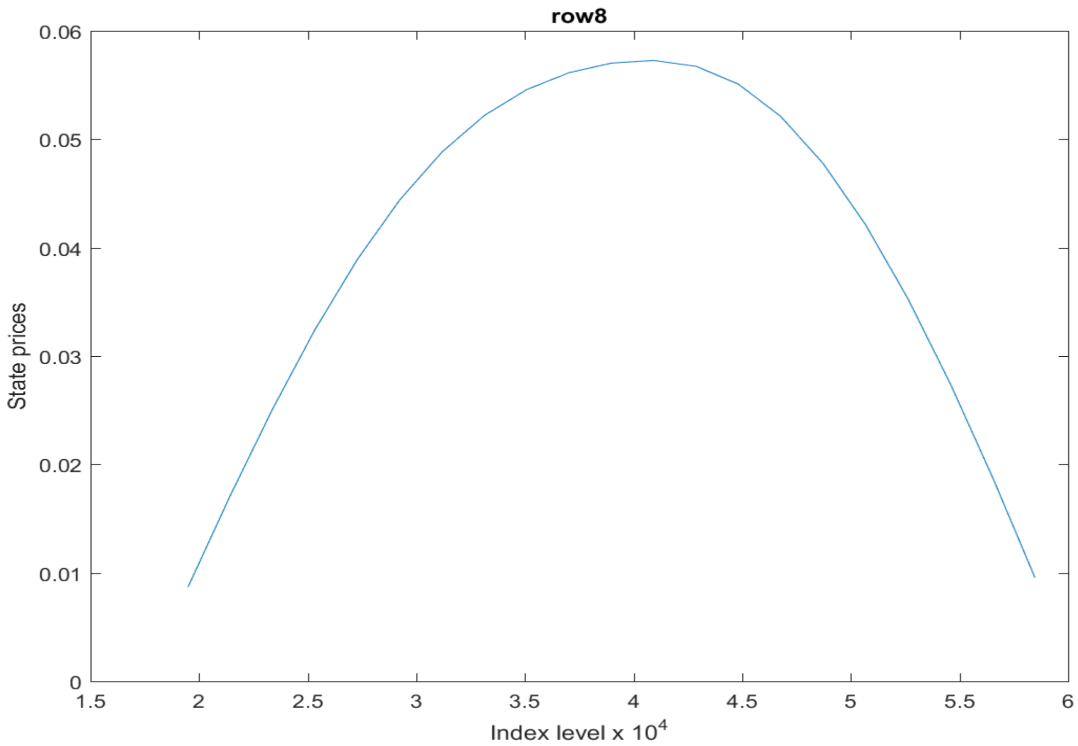
Plot C.6: State price distribution for the 6th row of the estimated state price matrix P



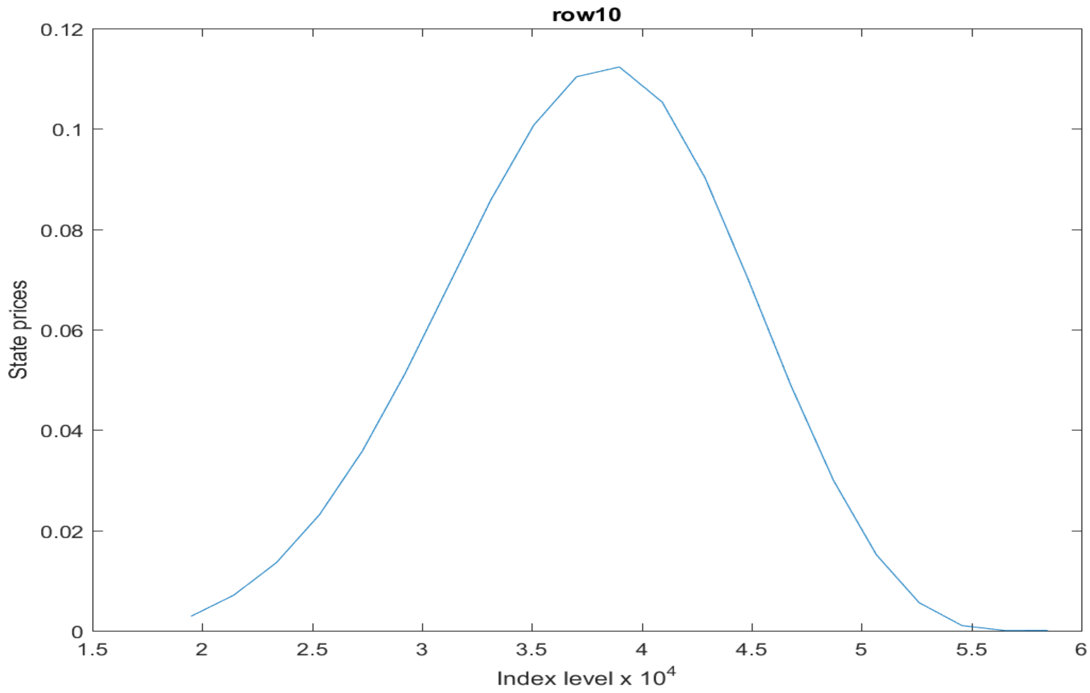
Plot C.7 State price distribution for the 7th row of the estimated state price matrix P



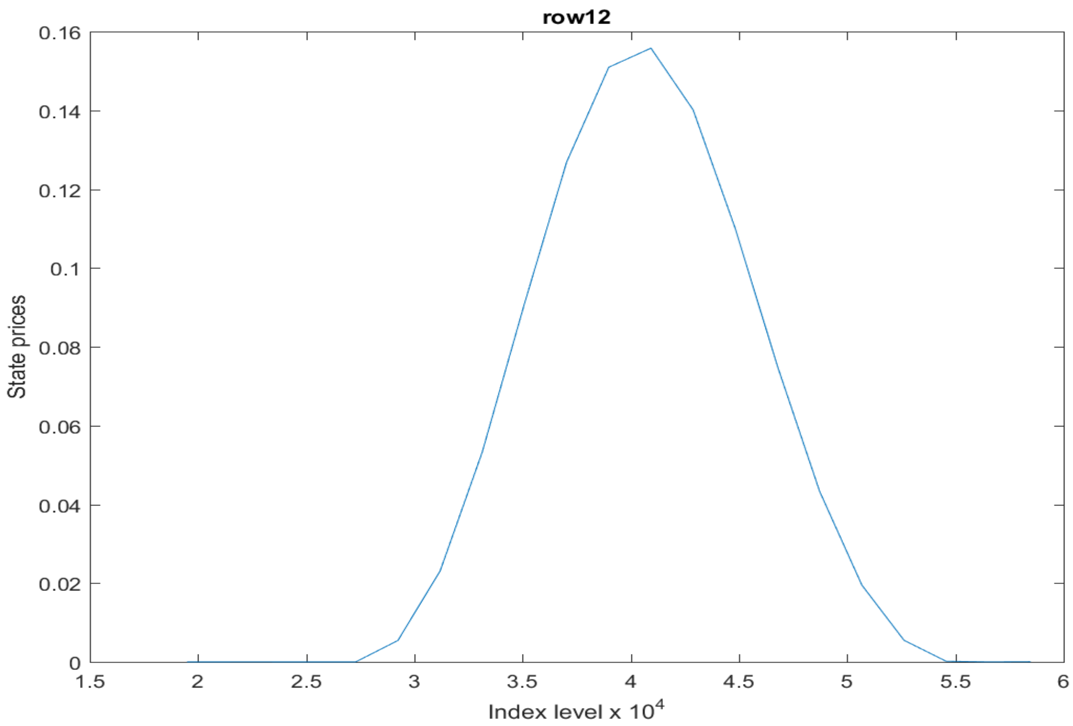
Plot C.8: State price distribution for the 8th row of the estimated state price matrix P



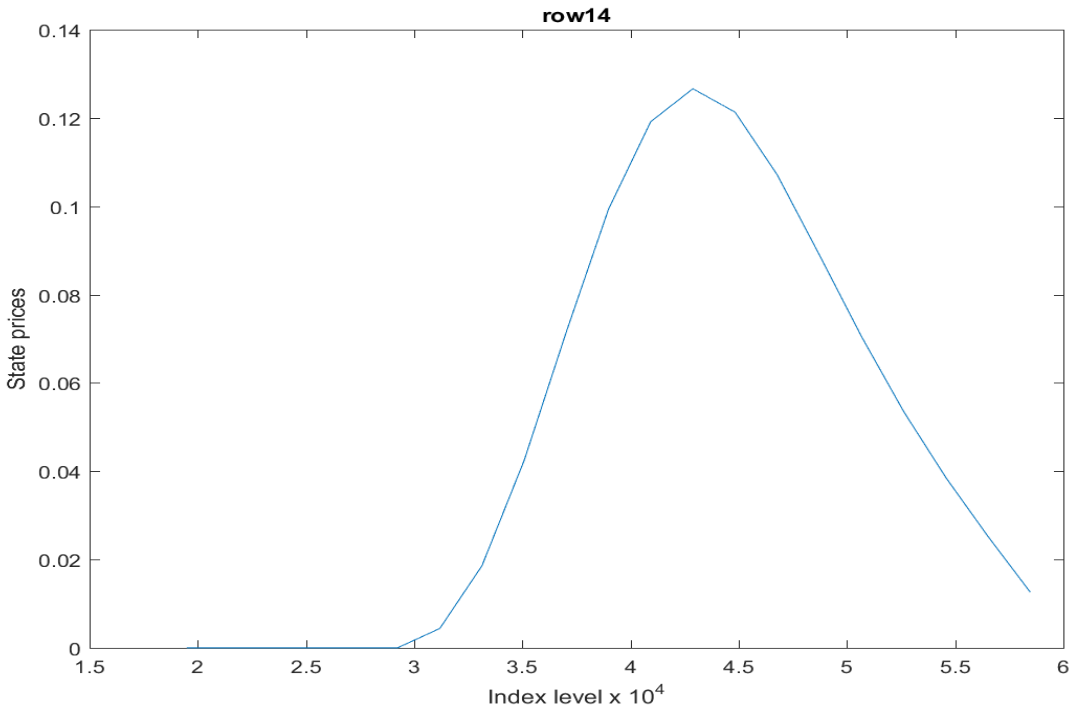
Plot C.10: State price distribution for the 10th row of the estimated state price matrix P



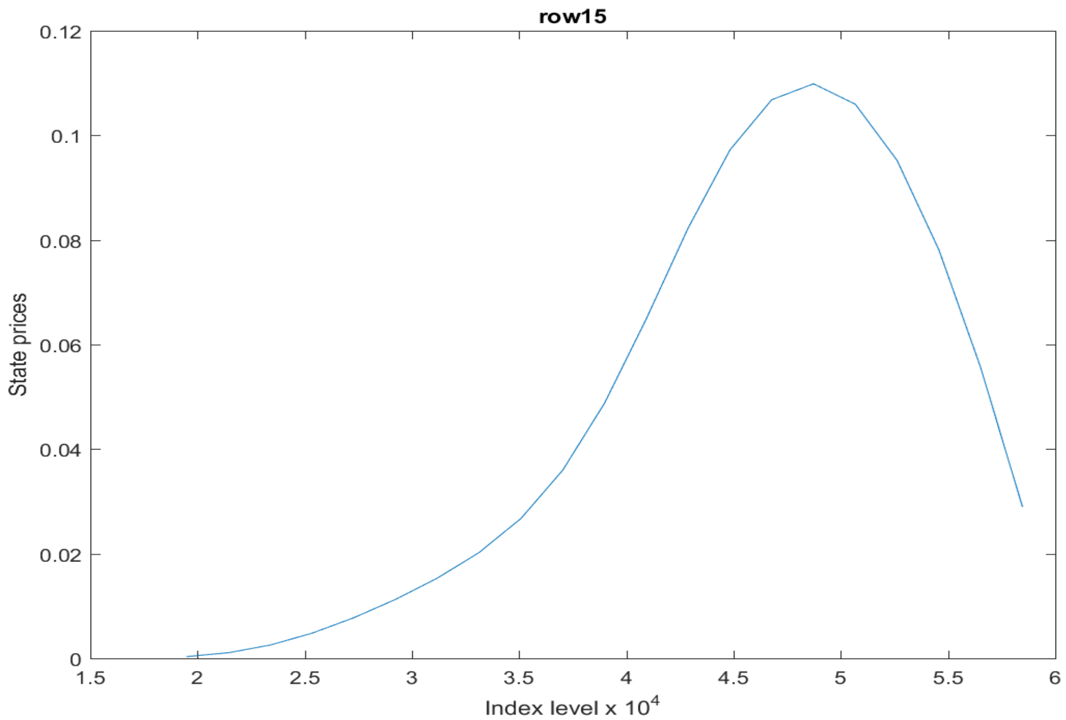
Plot C.12: State price distribution for the 6th row of the estimated state price matrix P



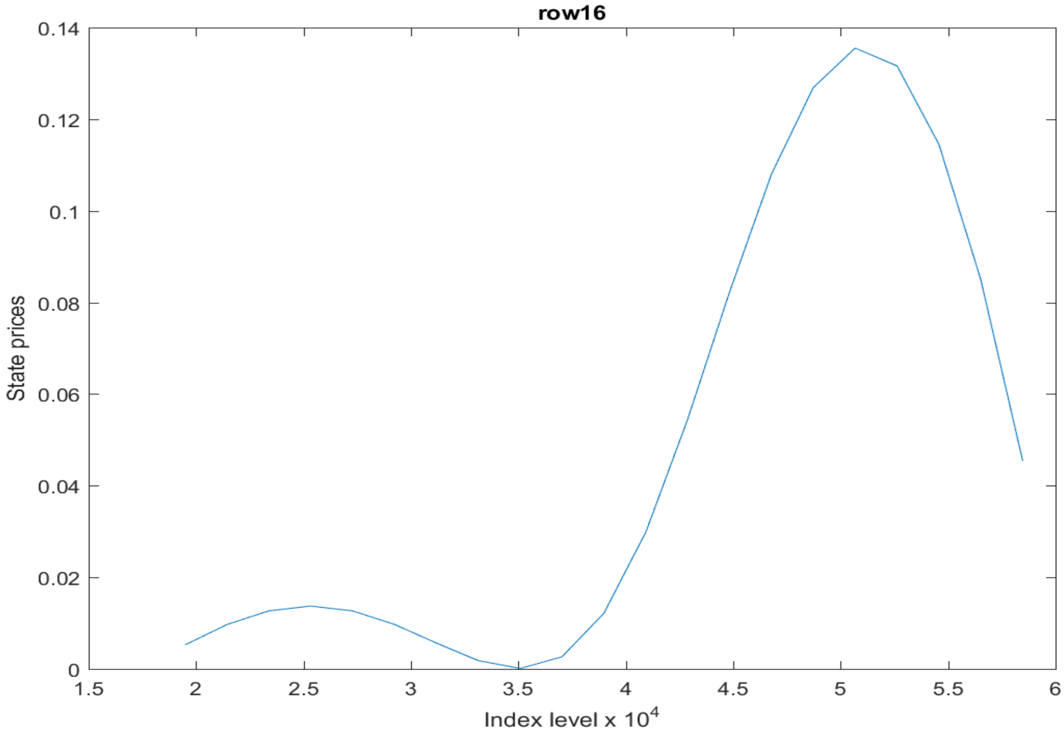
Plot C.14: State price distribution for the 6th row of the estimated state price matrix P



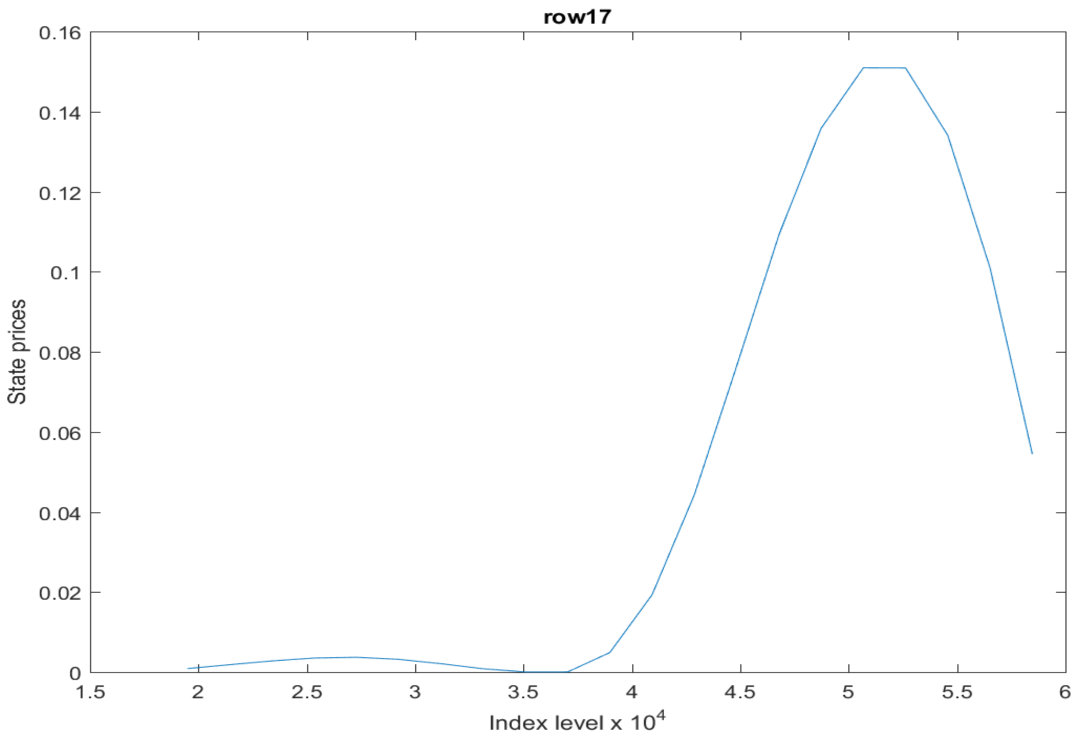
Plot C.15: State price distribution for the 6th row of the estimated state price matrix P



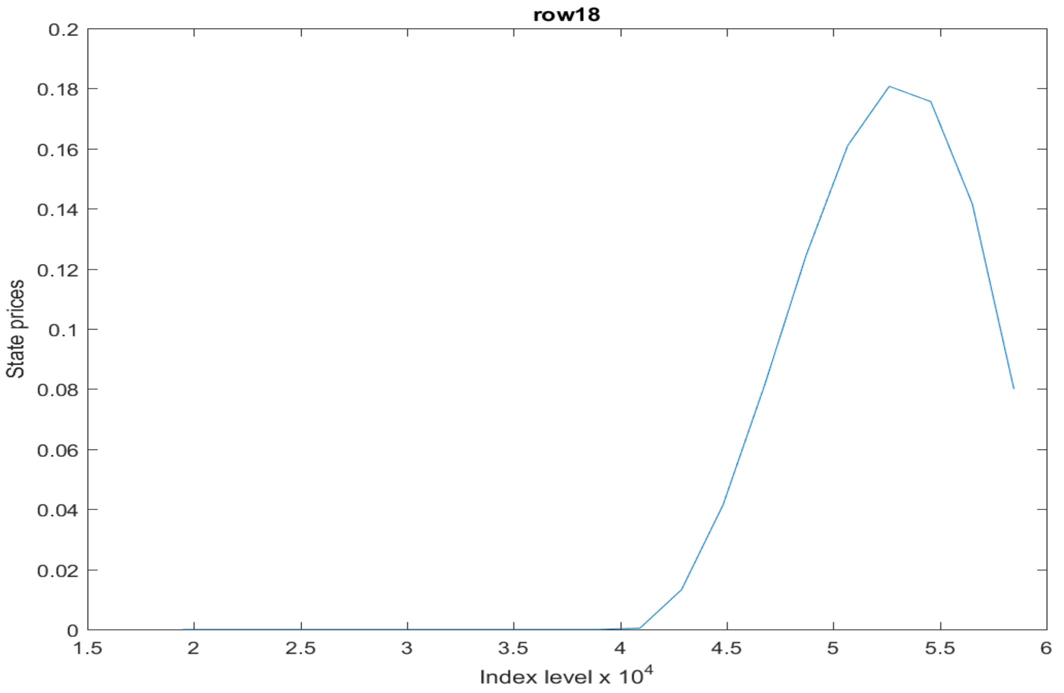
Plot C.16: State price distribution for the 6th row of the estimated state price matrix P



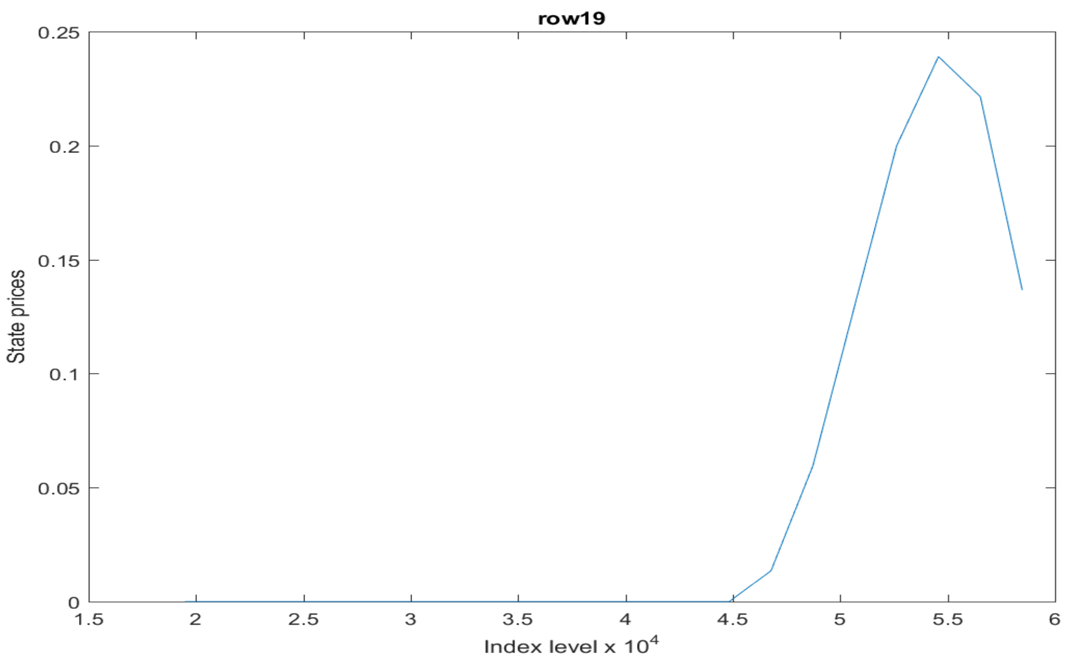
Plot C.17: State price distribution for the 6th row of the estimated state price matrix P



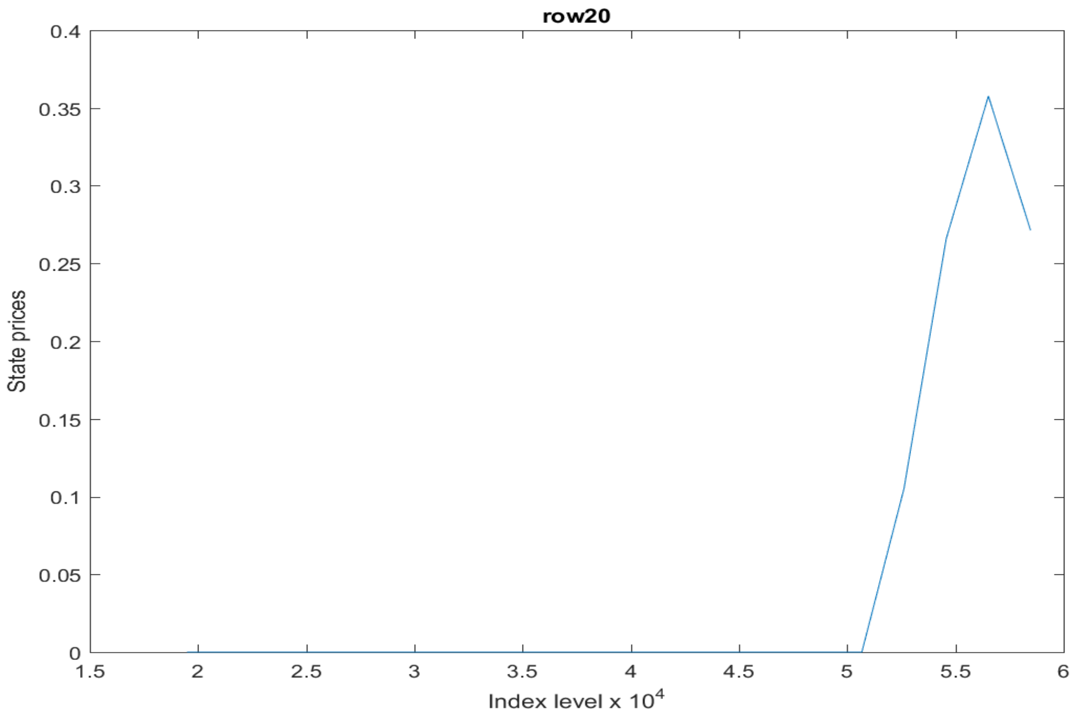
Plot C.18: State price distribution for the 6th row of the estimated state price matrix P



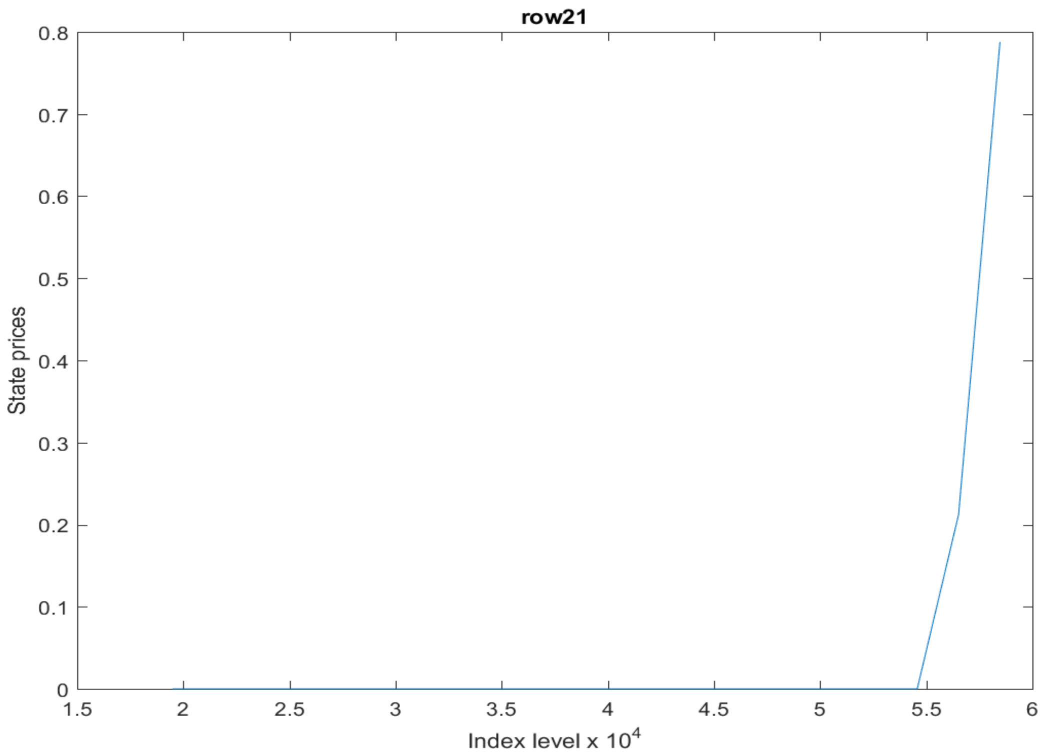
Plot C.19: State price distribution for the 6th row of the estimated state price matrix P



Plot C.20: State price distribution for the 6th row of the estimated state price matrix P



Plot C.21: State price distribution for the 6th row of the estimated state price matrix P



D: The new table of the state price matrix and its plots

Table D: The transition matrix P (21x21) for k=1:3

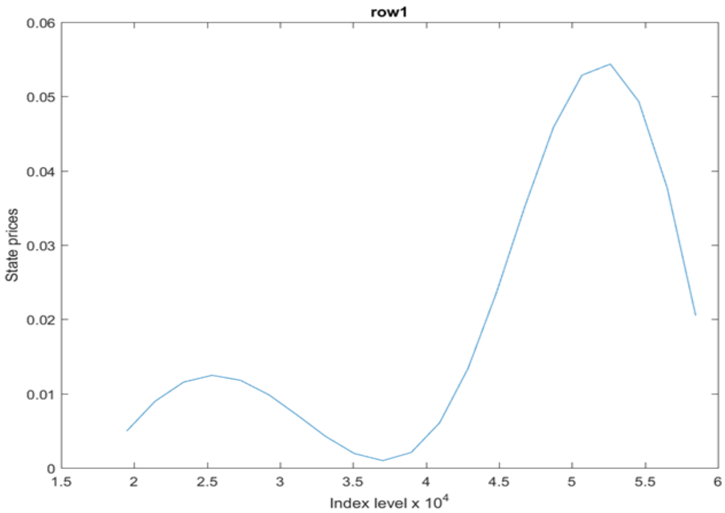
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.00496	0.00898	0.01155	0.01246	0.01179	0.00983	0.00709	0.00421	0.00193	0.00098
2	0.00462	0.00831	0.01062	0.01139	0.01075	0.00904	0.00678	0.00461	0.00324	0.00337
3	0.00463	0.00834	0.01066	0.01147	0.0109	0.00932	0.00727	0.00541	0.00445	0.00511
4	0.00477	0.00861	0.01102	0.01189	0.01137	0.00983	0.00783	0.00606	0.00528	0.00625
5	0.00483	0.00872	0.0112	0.01216	0.01178	0.01044	0.00872	0.00729	0.00692	0.00837
6	0.00461	0.00834	0.01079	0.01192	0.01197	0.01136	0.01062	0.01035	0.01115	0.01353
7	0.00405	0.00736	0.00969	0.01119	0.01219	0.01317	0.01457	0.01673	0.01979	0.02373
8	0.00315	0.00578	0.00796	0.01008	0.0127	0.01635	0.02128	0.02737	0.034	0.04019
9	0.00208	0.00392	0.00597	0.00901	0.0139	0.02123	0.03101	0.04246	0.05388	0.06295
10	0.00119	0.00237	0.00441	0.00862	0.01623	0.0279	0.04338	0.06109	0.07806	0.09032
11	0.00825	0.01078	0.00789	0.00304	2.23E-15	2.22E-15	0.0062	0.03473	0.09065	0.16501
12	0.00029	0.00029	0.00011	0.00018	0.0043	0.01694	0.03944	0.06959	0.10162	0.12733
13	8.61E-05	9.25E-05	9.00E-05	6.72E-05	4.56E-05	0.00229	0.01574	0.04232	0.07727	0.11102
14	0.00014	0.00016	0.00016	9.88E-05	0.00012	0.00288	0.01255	0.02991	0.05268	0.07641
15	3.08E-06	3.34E-06	3.64E-06	3.99E-06	4.43E-06	5.02E-06	5.86E-06	7.12E-06	6.05E-06	0.00025
16	0.00101	0.00226	0.00372	0.00499	0.0055	0.0049	0.00342	0.00219	0.00339	0.01003
17	0.00295	0.00528	0.0067	0.00707	0.00643	0.00496	0.00309	0.00136	0.0003	7.06E-05
18	2.61E-05	2.80E-05	3.01E-05	3.25E-05	3.54E-05	3.87E-05	4.25E-05	4.66E-05	5.10E-05	5.34E-05

19	3.66E-06	3.89E-06	4.16E-06	4.47E-06	4.82E-06	5.24E-06	5.73E-06	6.32E-06	7.05E-06	7.96E-06
20	4.05E-07	4.29E-07	4.56E-07	4.86E-07	5.22E-07	5.62E-07	6.10E-07	6.66E-07	7.34E-07	8.17E-07
21	6.65E-09	7.01E-09	7.41E-09	7.87E-09	8.38E-09	8.96E-09	9.63E-09	1.04E-08	1.13E-08	1.24E-08

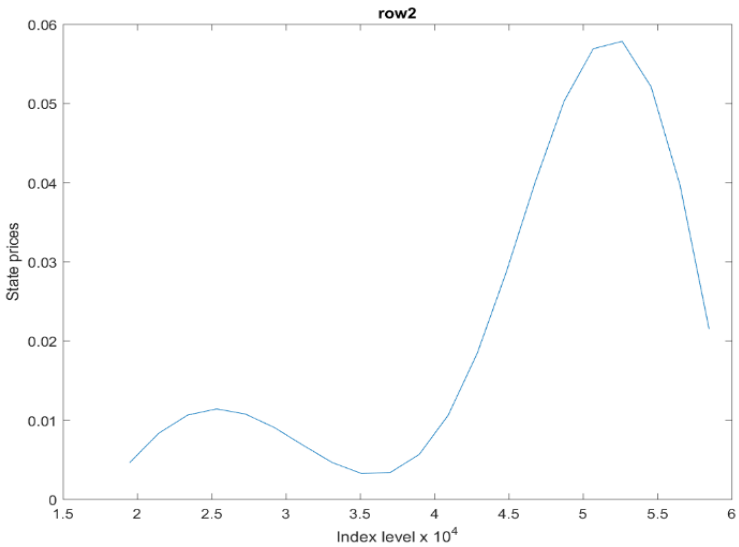
	11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**
1	0.00207	0.00607	0.0134	0.02362	0.03524	0.04584	0.05286	0.05436	0.04931	0.03768	0.02051
2	0.00567	0.01064	0.01845	0.02867	0.04005	0.0503	0.05688	0.05783	0.05208	0.03962	0.02151
3	0.00802	0.01362	0.02197	0.03259	0.04424	0.05464	0.06116	0.0618	0.05544	0.04207	0.02281
4	0.00966	0.01591	0.02498	0.03624	0.04842	0.05916	0.06577	0.06614	0.05915	0.04479	0.02426
5	0.01228	0.01901	0.02846	0.03999	0.05234	0.06318	0.06972	0.06981	0.06226	0.04706	0.02546
6	0.01793	0.02455	0.03337	0.04396	0.05544	0.06563	0.0717	0.07142	0.06352	0.04795	0.02593
7	0.02845	0.03393	0.04036	0.04805	0.05696	0.0653	0.07028	0.06954	0.06169	0.04653	0.02516
8	0.04498	0.04798	0.04983	0.05217	0.0564	0.06142	0.06453	0.06324	0.05595	0.04221	0.02284
9	0.06749	0.0666	0.06165	0.05615	0.05356	0.05379	0.05427	0.05236	0.04617	0.03488	0.01892
10	0.09426	0.0884	0.07496	0.05978	0.0488	0.04313	0.04044	0.03788	0.03322	0.0252	0.01376
11	0.23206	0.24638	0.15293	0.03328	0.00307	0.00054	0.00011	4.48E-05	1.48E-05	4.58E-06	1.71E-06
12	0.13869	0.13146	0.10822	0.07867	0.0537	0.03727	0.02793	0.02273	0.01888	0.01427	0.00791
13	0.13275	0.13517	0.11878	0.09278	0.06907	0.05281	0.04314	0.03714	0.03166	0.02417	0.01344
14	0.0959	0.10698	0.10811	0.1014	0.09169	0.08192	0.0723	0.06201	0.05005	0.03568	0.01876
15	0.03789	0.10526	0.16974	0.19649	0.17116	0.11686	0.06759	0.04073	0.03426	0.03451	0.02523
16	0.02467	0.04779	0.07643	0.10417	0.124	0.13271	0.13023	0.11766	0.09648	0.06846	0.03556
17	0.00125	0.0106	0.03141	0.06238	0.09813	0.13085	0.15287	0.15841	0.14438	0.11064	0.06034
18	2.99E-05	0.00046	0.01282	0.04085	0.08096	0.12469	0.1615	0.1812	0.17584	0.14145	0.0798

19	9.15E-06	1.07E-05	1.02E-05	1.45E-05	0.01345	0.05947	0.12986	0.19992	0.23906	0.22148	0.13666
20	9.21E-07	1.06E-06	1.24E-06	1.49E-06	1.88E-06	1.79E-06	7.58E-05	0.1052	0.26573	0.35766	0.27132
21	1.37E-08	1.54E-08	1.75E-08	2.02E-08	2.40E-08	2.94E-08	3.81E-08	5.32E-08	9.43E-08	0.21224	0.78776

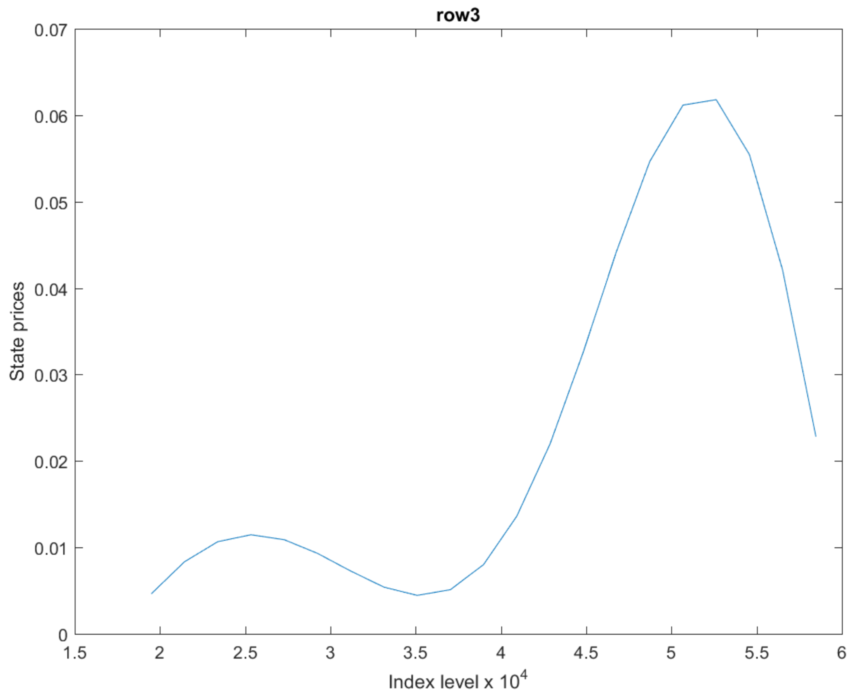
Plot D.1: State price distribution for the 1th row of the New estimated state price matrix P



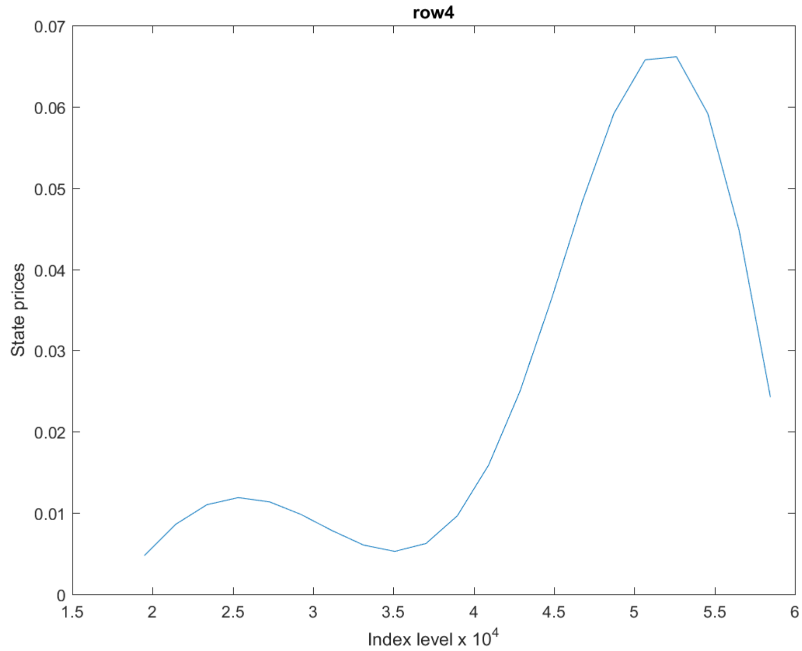
Plot D.2: State price distribution for the 2nd row of the New estimated state price matrix P



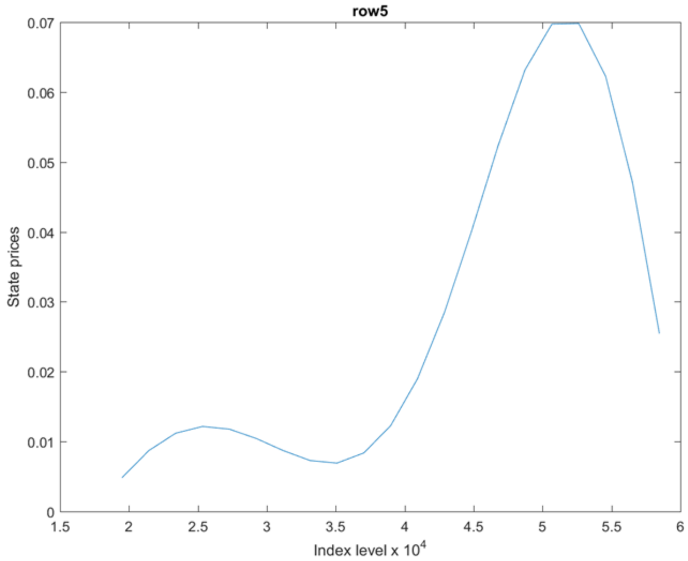
Plot D.3: State price distribution for the 3rd row of the estimated state price matrix P



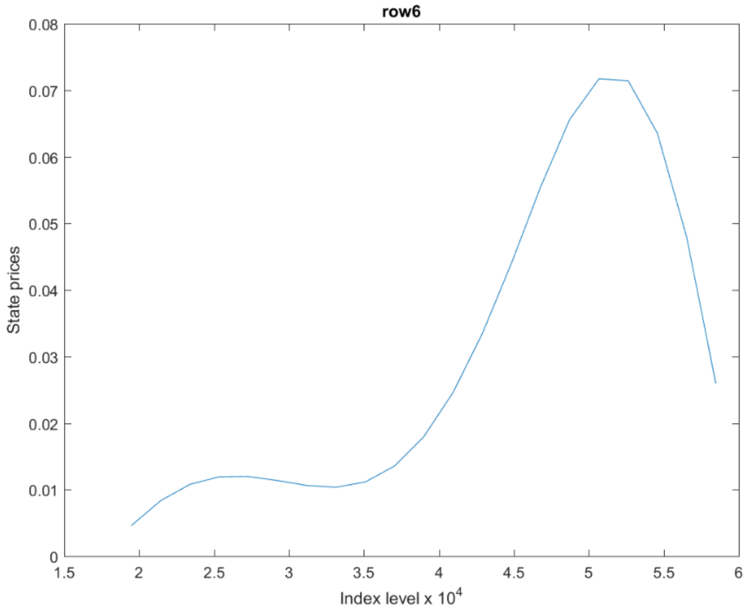
Plot D.4: State price distribution for the 4th row of the estimated state price matrix P



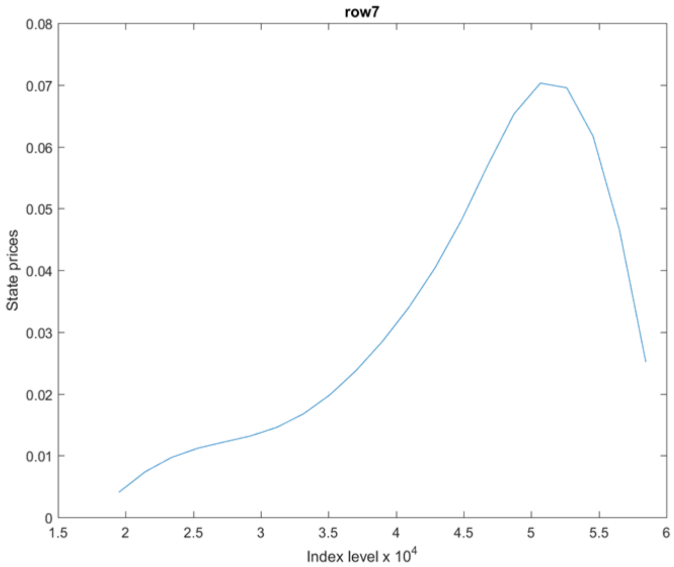
Plot D.5: State price distribution for the 5th row of the estimated state price matrix P



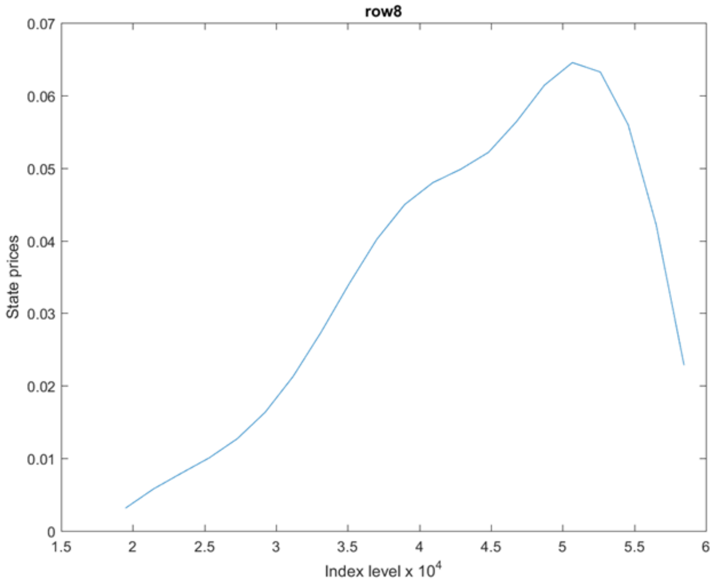
Plot D.6: State price distribution for the 6th row of the estimated state price matrix P



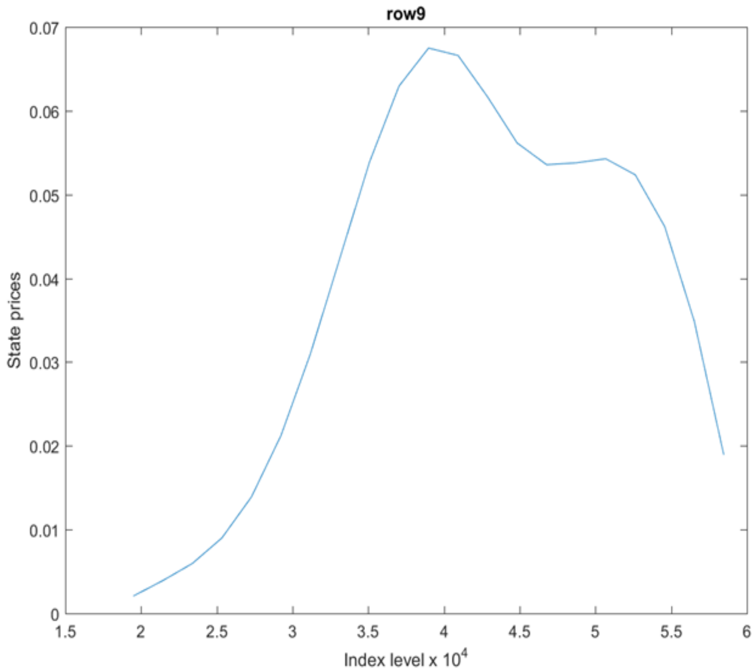
Plot D.7: State price distribution for the 7th row of the estimated state price matrix P



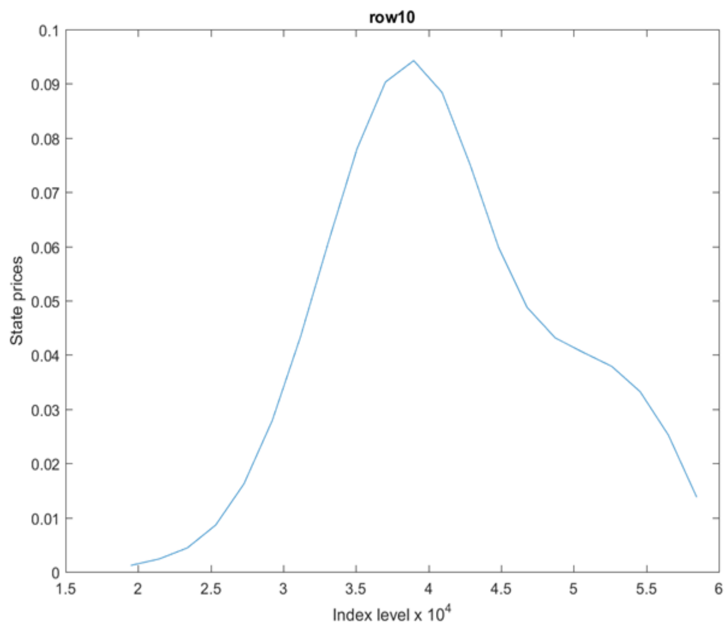
Plot D.8: State price distribution for the 8th row of the estimated state price matrix P



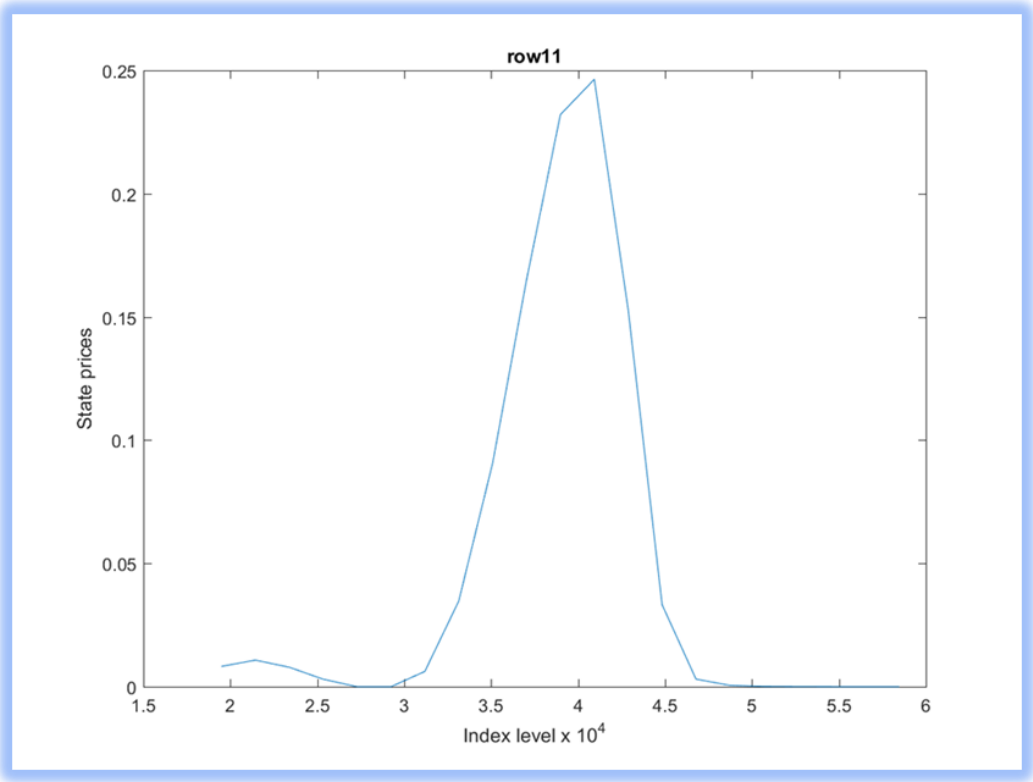
Plot D.9: State price distribution for the 9th row of the estimated state price matrix P



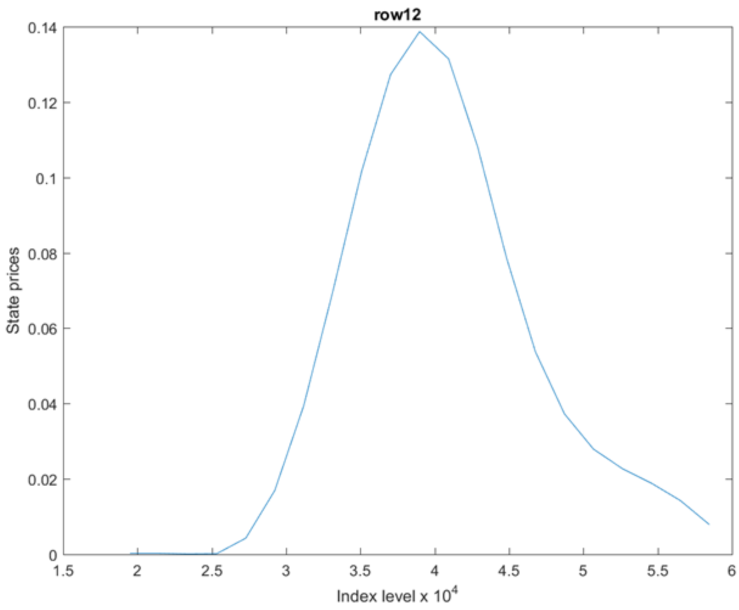
Plot D.10: State price distribution for the 10th row of the estimated state price matrix P



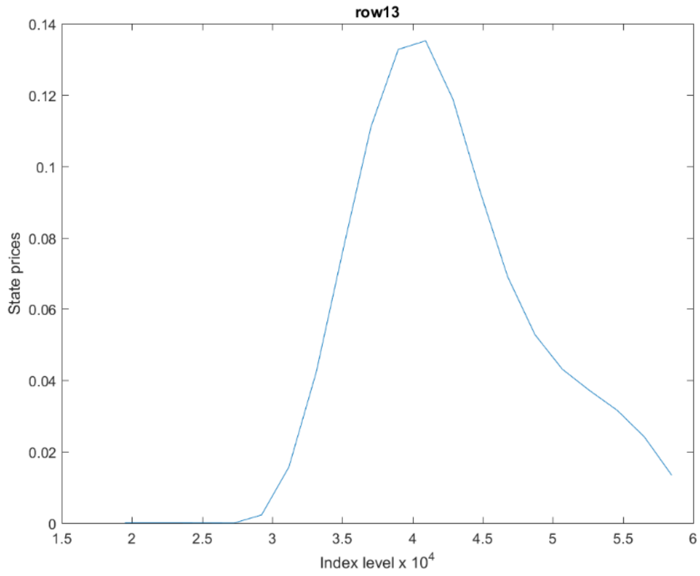
Plot D.11: State price distribution for the 11th row of the estimated state price matrix P



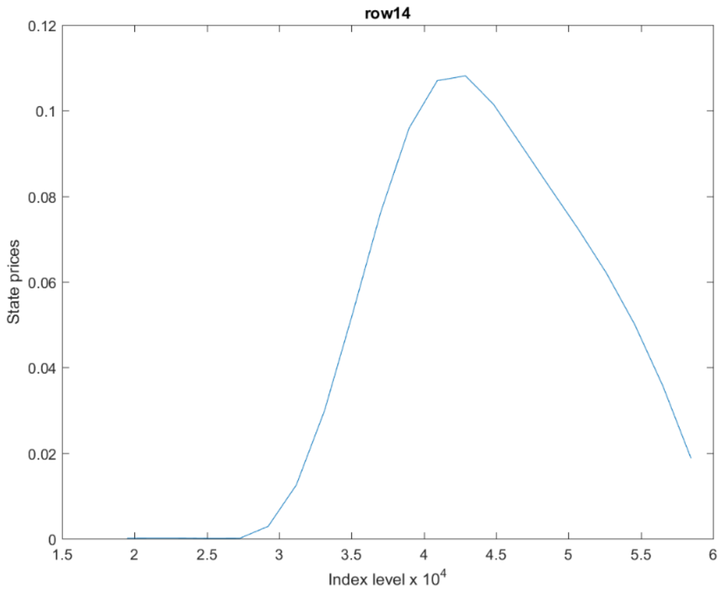
Plot D.12: State price distribution for the 12th row of the estimated state price matrix P



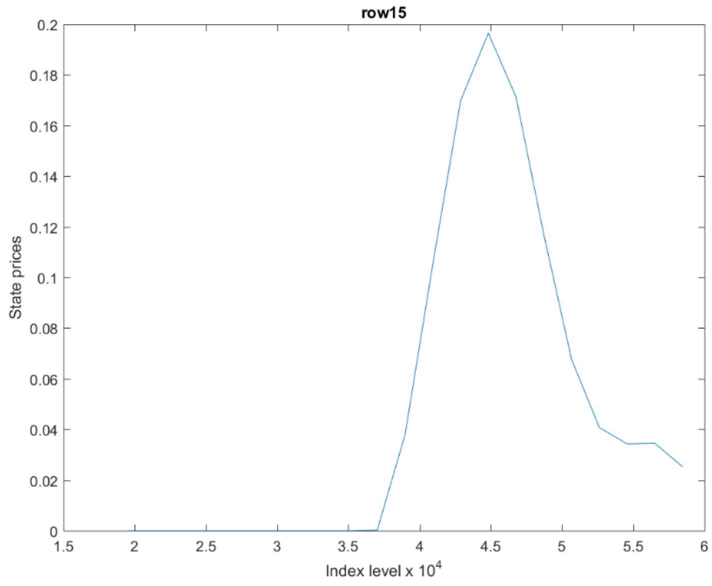
Plot D.13: State price distribution for the 13th row of the estimated state price matrix P



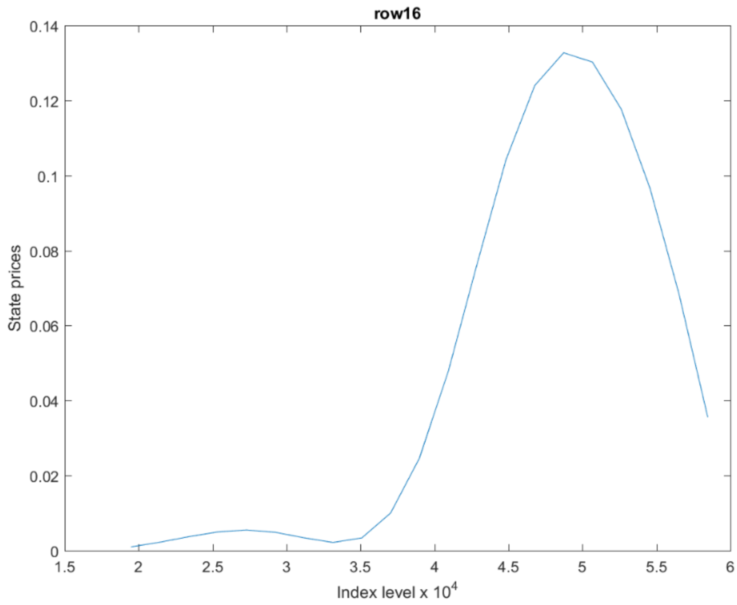
Plot D.14: State price distribution for the 14th row of the estimated state price matrix P



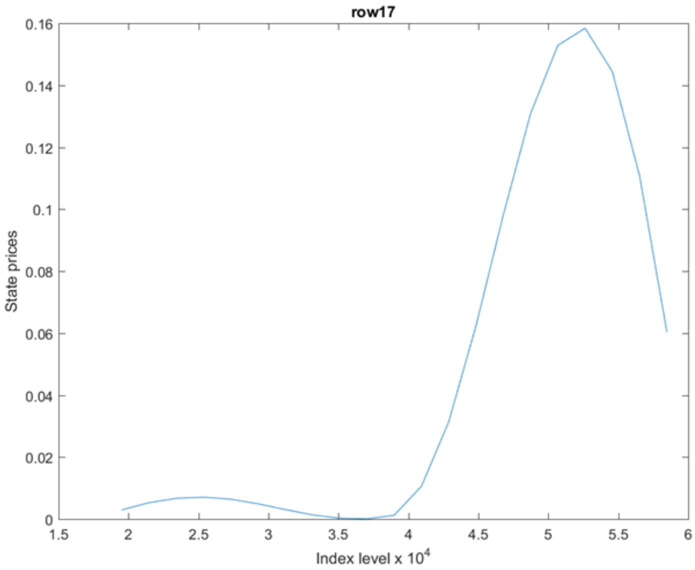
Plot D.15: State price distribution for the 15th row of the estimated state price matrix P



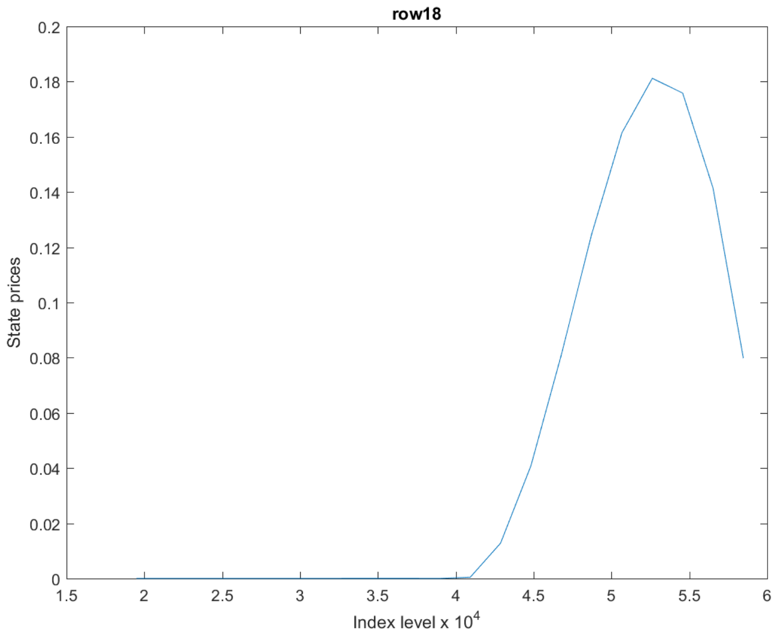
Plot D.16: State price distribution for the 16th row of the estimated state price matrix P



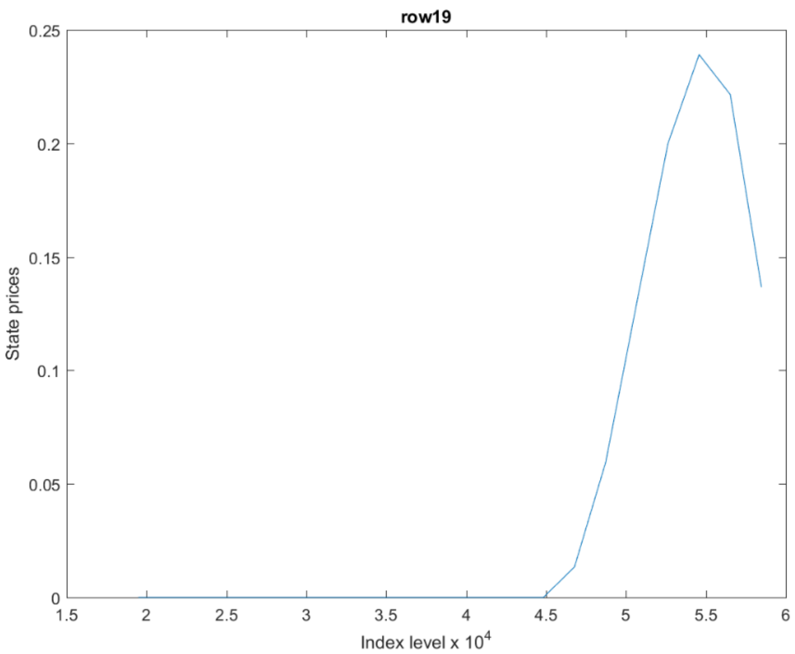
Plot D.17: State price distribution for the 17th row of the estimated state price matrix P



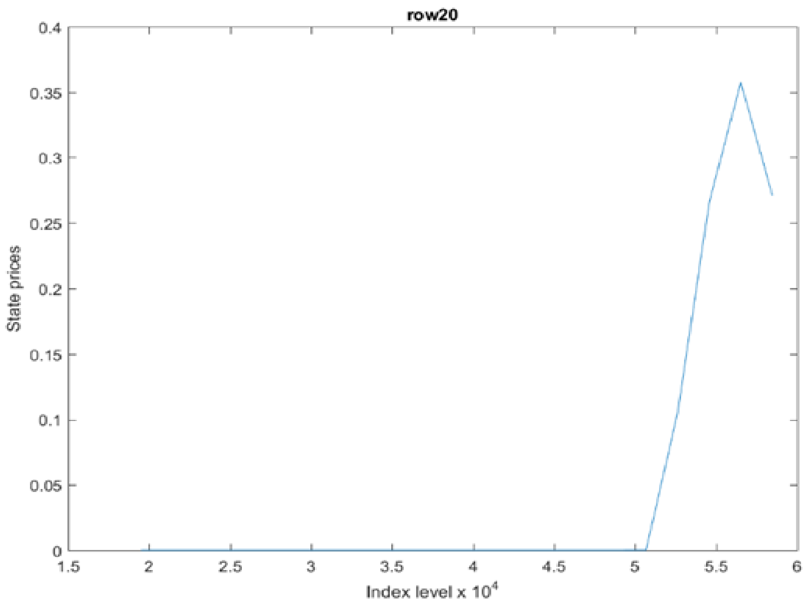
Plot D.18: State price distribution for the 18th row of the estimated state price matrix P



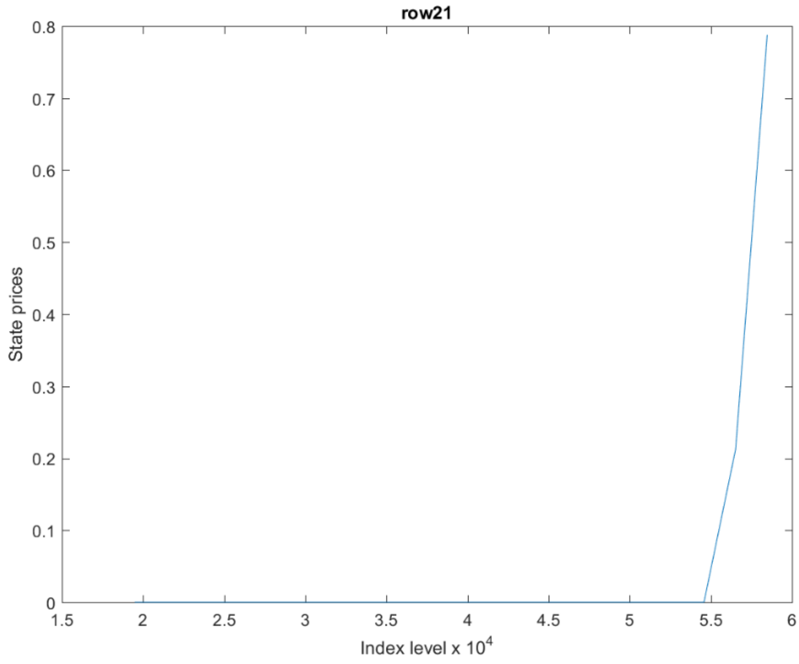
Plot D.19: State price distribution for the 19th row of the estimated state price matrix P



Plot D.20: State price distribution for the 20rst row of the estimated state price matrix P



Plot D.21: State price distribution for the 21rst row of the estimated state price matrix P



E: The table of the real world probabilities matrix F and its plots

Table E: The real world probabilities matrix F (21x21)

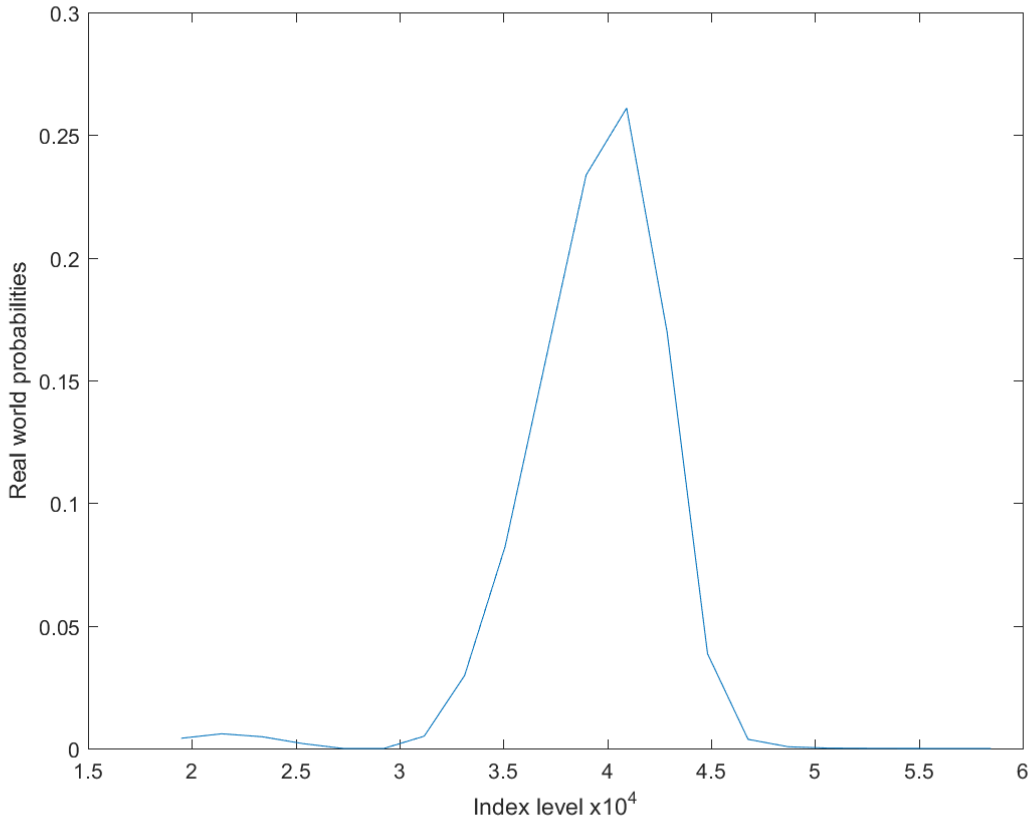
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.004 498	0.008 288	0.010 881	0.011 993	0.011 56	0.009 771	0.007 091	0.004 254	0.002 213	0.002 012
2	0.004 942	0.009 211	0.012 333	0.014 022	0.014 201	0.013 032	0.010 945	0.008 634	0.007 021	0.007 122
3	0.005 578	0.010 47	0.014 181	0.016 39	0.016 988	0.016 112	0.014 179	0.011 897	0.010 239	0.010 3
4	0.006	0.011	0.016	0.018	0.019	0.018	0.017	0.014	0.012	0.012

	268	819	117	799	715	977	005	545	66	575
5	0.006956	0.013185	0.018124	0.021379	0.022762	0.022358	0.020569	0.018149	0.016208	0.016045
6	0.007608	0.014541	0.020262	0.024377	0.026677	0.027203	0.026287	0.024599	0.02315	0.023147
7	0.008195	0.015867	0.022586	0.028018	0.031957	0.034372	0.035443	0.035597	0.035521	0.036063
8	0.008697	0.017151	0.025138	0.032452	0.038919	0.044405	0.048833	0.052198	0.054584	0.056145
9	0.009052	0.018271	0.027747	0.037479	0.047348	0.057089	0.066261	0.074236	0.080211	0.083306
10	0.002892	0.007102	0.013635	0.023142	0.035821	0.051303	0.068561	0.085848	0.100726	0.110384
11	0.00825	0.010779	0.007888	0.003038	2.23E-15	2.22E-15	0.006196	0.034735	0.09065	0.165011
12	1.65E-05	1.77E-05	1.81E-05	1.04E-05	1.60E-05	0.005452	0.023055	0.053268	0.090906	0.126844
13	6.74E-06	7.25E-06	7.66E-06	8.05E-06	6.83E-06	5.81E-06	0.003788	0.023391	0.058803	0.101014
14	1.26E-05	1.36E-05	1.45E-05	1.55E-05	1.07E-05	1.62E-05	0.004395	0.018613	0.042466	0.071583
15	0.000336	0.001074	0.002542	0.004806	0.007763	0.011281	0.015346	0.020247	0.026754	0.035982
16	0.005258	0.009702	0.012683	0.013733	0.012668	0.009723	0.005661	0.001826	0.000113	0.002659
17	0.000889	0.001843	0.002795	0.003507	0.003691	0.003187	0.002097	0.000838	2.56E-05	3.75E-05
18	4.64E-06	4.97E-06	5.35E-06	5.80E-06	6.32E-06	6.94E-06	7.66E-06	8.49E-06	9.44E-06	1.02E-05
19	7.30E-07	7.77E-07	8.30E-07	8.92E-07	9.63E-07	1.05E-06	1.14E-06	1.26E-06	1.41E-06	1.60E-06
20	8.11E-08	8.59E-08	9.13E-08	9.75E-08	1.05E-07	1.13E-07	1.22E-07	1.33E-07	1.47E-07	1.64E-07
21	1.33E-09	1.40E-09	1.48E-09	1.57E-09	1.68E-09	1.79E-09	1.93E-09	2.08E-09	2.26E-09	2.48E-09

	11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**
1	0.004563	0.010338	0.019031	0.029398	0.039468	0.047252	0.051181	0.050245	0.044052	0.032864	0.017627
2	0.009802	0.015464	0.023736	0.033351	0.042434	0.049163	0.05213	0.050449	0.0438	0.032464	0.017346
3	0.013021	0.018812	0.027217	0.036813	0.045653	0.051959	0.054402	0.052171	0.045	0.033203	0.017692
4	0.015348	0.021426	0.030242	0.040157	0.049079	0.055229	0.057325	0.054609	0.04687	0.034461	0.018321
5	0.018791	0.024911	0.033757	0.043544	0.052123	0.057773	0.059301	0.056027	0.047796	0.034993	0.018555
6	0.025648	0.031075	0.038791	0.047099	0.054046	0.058156	0.058445	0.054382	0.045889	0.033344	0.0176
7	0.037969	0.04153	0.04626	0.050923	0.054177	0.055085	0.053067	0.047852	0.039461	0.028218	0.014749
8	0.057028	0.057274	0.05673	0.055081	0.0521	0.047756	0.042115	0.035296	0.027461	0.018806	0.009562
9	0.082779	0.078306	0.070235	0.059634	0.04790	0.036275	0.025711	0.016855	0.010031	0.005234	0.002118
10	0.11233	0.105329	0.090243	0.070142	0.048966	0.029926	0.015141	0.005579	0.001024	4.46E-05	5.16E-05
11	0.232064	0.246383	0.152934	0.033276	0.003075	0.000542	0.000114	4.48E-05	1.48E-05	4.58E-06	1.71E-06
12	0.150942	0.155781	0.14011	0.109967	0.075033	0.043277	0.019516	0.005498	0.000174	1.97E-05	6.60E-05
13	0.136737	0.154262	0.148927	0.125556	0.0945	0.064427	0.040225	0.023476	0.01336	0.007665	0.003824
14	0.099353	0.119164	0.126628	0.121322	0.107125	0.08898	0.070467	0.053485	0.038538	0.025196	0.012593
15	0.048797	0.064931	0.082276	0.097208	0.106819	0.109865	0.105961	0.095208	0.078115	0.05559	0.028954
16	0.01	0.02	0.05	0.08	0.10	0.12	0.13	0.13	0.11	0.08	0.04

	2167	9849	4495	2274	7982	6908	5517	1626	4506	4963	5423
17	0.004882	0.019325	0.044288	0.076379	0.109121	0.135781	0.150921	0.150885	0.134036	0.100926	0.054454
18	5.78E-06	0.000444	0.013231	0.041489	0.081324	0.124532	0.161003	0.180715	0.175631	0.141556	0.079996
19	1.84E-06	2.17E-06	2.06E-06	2.96E-06	0.013549	0.05957	0.129899	0.199889	0.239006	0.221429	0.136638
20	1.85E-07	2.12E-07	2.48E-07	2.99E-07	3.77E-07	3.59E-07	1.73E-05	0.105266	0.265809	0.357654	0.271251
21	2.75E-09	3.08E-09	3.49E-09	4.04E-09	4.79E-09	5.89E-09	7.63E-09	1.06E-08	1.89E-08	0.212244	0.787756

Plot E: Real world distribution for the 11th row of the matrix F



Endnotes

[i] In probability theory a **martingale** is a model of fair game where knowledge of past events never helps to predict the mean of the future winnings

[ii] **Risk neutral probability** = subjective probability * risk aversion adjustment

[iii] Marie-Esprit-Léon **Walras** was a French mathematical economist. Source: www.wikipedia.org

[iv] **Non-parametric statistics** have three characteristics.

1. Include both descriptive and inferential statistics.
2. Grows the number of parameters.
3. No assumptions about the probability distribution.

[v] **Pareto** efficiency, or Pareto optimality, is a state of **allocation** of sources in which it is impossible to make any agent to be in a better position without worsen another individual position.

[vi] In statistics, **homogeneity** and its opposite, heterogeneity, arise in describing the properties of a dataset, or several datasets. They relate to the validity of the often convenient assumption that the statistical properties of any one part of an overall dataset are the same as any other part of any one part of an overall dataset are the same as any other part

[vii] **In statistics, the method of moments** is a way of estimation of population parameters. It starts with deriving equations with the regard to the population moments to the parameters of interest. So, a sample starts to exist and the population moments are estimated by this sample. Then, the equations have a solution in the parameters of interest using this sample moments including the unknown population moments. **In probability theory, the method of moments** is a method which proves the convergence in distribution.

[viii] **The law of one price** is an economic idea assuming that a good in the economy has to be sold at the same price in all places. The elimination of any kind of arbitrage concludes in this law of one price. This law holds the basis of the theory of purchasing power parity. Source: www.wikipedia.org

[ix] **Extrapolation** is the procedure of estimating beyond the original observation range the value of a variable on the basis of its relationship with another one

[x] A Markov chain is a random process that undergoes from one state to another on a state space. The probability distribution of the next state depends only on the current state on the sequence of events that preceded it. This specific kind of "memorylessness" is called **Markov Property**

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