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The Multi-Agent Equilibrium Problem In Consumption-Investment Utility Optimization And Numerical Applications

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Abstract

We study an economy with finite heterogeneous agents over a finite time horizon, who differ in their endowments and utilities. Each agent has individual commodity earnings streams, and is endowed with a set of productive assets which produce commodity dividend streams. The agents can trade commodity and productive assets in order to hedge the risks. The profits can be invested in financial assets. Each agent chooses a commodity consumption process and manages his portfolio in order to maximize his expected total utility from consumption of commodity, over a finite horizon subject to the constraint that his wealth at the terminal time must be nonnegative. When equilibrium prices are accepted by the individual agents during the determination of their optimal consumption and portfolio policies, the commodity is entirely consumed as it received, all productive assets are exactly owned and all financial assets are held in zero net supply. In order to replace many agents with distinct utility functions and incomes, we introduce a "representative agent", who represents their individual interests and has their aggregate income. The goal of this thesis is to establish the existence and uniqueness of “equilibrium” commodity spot price process and productive asset prices in a multi-agent economy.

Key words: Capital asset pricing, consumption/investment decisions, equilibrium, feasibility, financial/productive assets, Monte Carlo simulation, optimality, representative agent, stochastic differential equations.

Contents

Chapter 1: Introduction

1.1 Historical background.

Significant progress has been made on the development of a mathematical theory for capital asset pricing and especially on the optimal actions of single agents and the way in which the aggregation of these actions leads to prices for capital assets.

- Merton [16,17] was the first who studied the single agent optimal control problem and he produced closed form solutions for the consumption and investment policies and the agent's indirect utility, or value function, when the utility function for consumption satisfied the condition $U'(0) = \infty$. For these solutions, stock prices are treated as Geometric Brownian Motion (GBM) with constant coefficients. To address equilibrium, Merton proposed that the interest rate, the mean rates of return, and the diffusion coefficients of the stock price processes should not be constant, but should themselves be Itô processes with constant drift and diffusion coefficients. This generalized model is complex, and no comparable clear solution was produced.
- Cox, Ingersoll and Ross [3] generalized Merton's approach. They postulated an underlying Markov state process describing the economy and let the stock price coefficients to be functions of this state process.
- Karatzas, Lehoczky, Sethi and Shreve [10] determined the optimal single agent consumption and investment policies and the value function for wealth for *arbitrary, smooth, concave* utility functions of consumption which were assumed only to satisfy conditions required for the finiteness of the value function. Their paper removed the constraint $U'(0) = \infty$, carefully treated the consumption constraint $c \geq 0$, and addressed the possibility of bankruptcy. Stock prices were again modeled by constant coefficient Geometric Brownian Motion processes, and clear optimal consumption and investment formulas were obtained.
- Karatzas, Lehoczky and Shreve [11] also developed a martingale-based characterization of the optimal decisions for a single agent. This approach is appropriate to a much more general class of stock price processes, including non-Markovian models. A clear characterization of single-agent optimal consumption policies was provided for general utility functions and semimartingale stock price processes.
- The same methodology was independently developed and applied to diffusion

models by Cox & Huang [2].

As a result, the optimal behavior of a single agent is now well understood.

Equilibrium in dynamic, stochastic, multi-agent problems has been studied by numerous authors.

- The usual approach followed by Duffie [4] and Duffie &Huang [5,6] is to reduce the dynamic problem to a static one by considering *agent consumption processes* to be points in a suitable abstract space. Each agent has a preference structure defining a partial order over consumption plans. Under certain conditions, a deep fixed point theorem can be invoked to prove the existence of a solution to the static equilibrium problem. A martingale representation theorem can then be employed to create a solution to the dynamic equilibrium problem.

  There are *two drawbacks* to this approach:

  I. the basic work by Mas-Colell [15] required "uniform properness", a strong restriction on the preference ordering. In particular, this property does not allow utility functions satisfying $U'(0) = \infty$, and so such utility functions are not allowed in [4], [5] and [6]. This difficulty was overcome by Duffie and Zame [7], using a lengthy functional-analytic argument.

  II. the usual approach gives little insight into the nature of equilibrium. The optimal consumption plans and spot price processes cannot be exhibited, nor can uniqueness of the equilibrium be established.

- Both Duffie & Zame [7] and Karatzas, Lakner, Lehoczky and Shreve [9] generalize the results of Cox, Ingersoll and Ross [3] in *two important directions:*
  (1) heterogeneous agents are allowed, whereas in Cox, Ingersoll and Ross [3] all agents have the same endowments and the same utility functions.
  (2) endowment processes are adapted in a general way to an N-dimensional Brownian motion whereas in Cox, Ingersoll and Ross [3] this dependence on the underlying Brownian motion must be via a state process so that Markov.
- Duffie & Zame [7] and Karatzas, Lakner, Lehoczky and Shreve [9] both derive a formula for the endogenously determined equilibrium interest rate, which agrees with that of [3], and also formulas for the coefficients of the stock processes and the optimal consumption processes of the individual agents.
- The Cox, Ingersoll and Ross [3] interest rate formula is given in terms of an indirect utility function, derived from the single direct utility function in their model.

- Karatzas, Lehoczky and Shreve [12] were the first to bring the explicit characterization of optimal single-agent behavior for general stock price processes to bear on the multi-agent equilibrium problem. The result was an increase in knowledge about the existence and also about the uniqueness and the structure of equilibrium. The use of the optimal single-agent behavior allows a simple fixed point argument (specifically, the *Knaster- Kuratowski-Mazurkiewicz Lemma*) to be applied. The questions of existence and uniqueness are completely resolved under quite weak conditions on the agents' utility functions. They also provide tools for study of how economies which are not in equilibrium might converge to equilibrium.
- Cox, Ingersoll and Ross [3] eliminate the introduction of a "representative agent", who represents the individual interests of many agents with distinct utility functions and incomes and has their aggregate income, by the assumption that all agents have the same utility function and the same income. Under such an assumption, attention is immediately focused on a single, generic agent, and questions of existence and uniqueness of equilibrium are trivialized.
- Huang [8] on other hand, selects a set of positive weights ($\lambda_1, ..., \lambda_J$) that characterize the representative agent. Huang's goal is to study the nature of equilibrium, and he is content to assume rather than prove its existence.
- In contrast, Karatzas, Lehoczky and Shreve [12] wanted to construct equilibrium and they managed to reduce that construction to the problem of finding an appropriate representative agent, i.e., a vector ($\lambda_1, ..., \lambda_J$).

1.2 Model Description.

This thesis attempts to construct and establish uniqueness of an equilibrium in a multi-agent economy. The primitives in this model are the endowment processes and the utility functions of a finite number of agents. Heterogeneous agents are allowed; therefore agents could have different endowment streams and utility functions. Endowment processes are adapted in a general way to an underlying N-dimensional Brownian motion in a given probability space $(\Omega, \mathcal{F}, P)$.

We consider an economy in which there is a finite number, J, of agents (small investors), each of whom:

- owns productive assets which provide *commodity dividend streams* and
- receives *individual commodity income streams*, over a finite time horizon.

Agents may consume their endowment as it arrives, they may sell some portion of it to other agents, or they may buy extra endowment from other agents. The endowment however cannot be stored, and agents will wish to *hedge the variability* in their endowment processes by trading with one another. The agents can buy or sell the commodity at a certain spot price and buy or sell their shares of the productive assets. The agents can trade the productive assets in order to hedge the risks associated with the commodity endowments and with the returns from the productive assets. The proceeds can be invested in *financial assets* whose prices are modeled as semimartingales.

Each agent takes the security prices as given and his goal is to choose a commodity consumption process and manage his portfolio
(i) so as to maximize his expected total utility from consumption of this commodity over a finite horizon [0, T] of the model, subject to the constraint that his wealth at the terminal time must be nonnegative almost surely, and
(ii) so that an equilibrium market is formulated; in particular, when agents consume and invest in this market so as to maximize their expected utility of consumption at all times:

1. all of the commodity is exactly consumed as it is received (this condition codifies the concept of a "perishable" commodity),
2. all of the productive assets are exactly owned and
3. all pure hedging instruments (i.e. the financial assets) are held in zero net supply (for every buyer of a security there must be a seller).

The *equilibrium problem* is to construct a model in which security prices are determined by the law of supply and demand. In our model the multi-agent equilibrium problem arises when J agents (where J is some positive integer) have individual commodity earnings streams, and each agent is also endowed with a set of productive assets which produce commodity dividend streams. Each agent j receives his endowment over time and not initially. There is a single, infinitely-divisible commodity, and each agent wishes to maximize his expected total utility from consumption of this commodity over time.

We derive the optimal agent *consumption* and *investment* decision processes when *prices of productive assets* and *commodity spot prices* are specified. This happens when equilibrium prices of the productive assets and the commodity are accepted by the individual agents during the determination of their optimal consumption and portfolio policies.

We actually consider *two related models:*

i. in our primary model, *the moneyed model*, prices are measured in some currency (dollars) rather than in units of the commodity, and the price of an asset relative to a unit of commodity at a certain time is obtained by dividing the currency (dollar) price of the former by the currency (dollar) price of the latter,

ii. in the secondary model, *the moneyless model*, prices of assets are measured directly in units of commodity.

In both models, prices for trading the productive assets must be established *endogenously.*

Specifically, in the *moneyless model,* there is no spot price to be determined, or equivalently, the spot price is set identically equal to one. In other words, now the prices (measured in units of commodity) of the financial assets are *endogenous*, and equilibrium prices of the productive and financial assets are those which cause the same result as equilibrium prices in the moneyed model.

Because a market consisting only of productive assets may not be complete, (i.e., may not allow for hedging of all risk), we introduce *financial assets*. In the moneyed model, the prices (measured in dollars) of financial assets are *exogenous*. The price structure of these financial assets will influence

(1) the equilibrium prices of the productive assets and

(2) the equilibrium spot price of the commodity,

but will not affect the equilibrium allocation of the commodity among agents.

The *goal* of an equilibrium analysis is to establish the *existence* and *uniqueness* of equilibrium prices, and to characterize these prices as well as the consumption and investment decisions made by the individual agents. Thus, we prove the existence and uniqueness of an "equilibrium" commodity spot price process and productive asset prices.

One important step in the search for equilibrium is to introduce a "representative agent", that is, to replace the many agents with distinct utility functions and incomes by a single agent who represents their individual interests and has their aggregate income. In our model, each agent has a utility function, U_j, and we construct a "representative agent" whose utility function will play the role of the Cox, Ingersoll & Ross [3] function U. This representative agent acts as a proxy for the individual agents by receiving their aggregate endowment and solving his own optimization problem with utility function

$$U(c) = \max_{\substack{c_1 \geq 0,\dots,c_J \geq 0 \\ c_1+\dots+c_J=c}} \left[\lambda_1 U_1\left(c_1\right)+\dots\dots+\lambda_J U_J\left(c_J\right)\right], \qquad (1.2.1)$$

where the weights (λ_1, ..., λ_J) characterize the representative agent.

In order to construct equilibrium, we reduce this problem to find an appropriate representative agent, i.e., an appropriate vector (λ_1,..., λ_J). This allows for the equilibrium to be constructed in $\Re^J$ and not in some infinite-dimensional function space. Then, the optimal consumption strategies of the individual agents can be found explicitly in terms of the equilibrium values of λ_1,..., λ_J.

1.3 Thesis Overview.

This thesis is organized as follows. In section 2.1 we set out the basic idea of equilibrium in a simple, two-stage model. Although the models are considerably more complex, many of the essential features of the complex models are present in this simpler model. In section 2.2 we discuss aspects of equilibrium model with its primitives. In section 2.3 we present the financial assets and in section 2.4 we present the spot price process for the commodity. In section 3.1 we see the individual agents' optimization problems. In section 3.2 we define the notion of equilibrium. In section 3.3 we show how the absence of arbitrage opportunities, determines the price processes of the productive assets. In section 3.4 we solve the problem of section 3.1 and it remains to determine the equilibrium spot price process (in the moneyed model) or financial asset price processes (in the moneyless model) which cause the markets to clear. In section 3.5 we give a characterization of equilibrium. In section 4.1 we introduce the utility function for a "representative agent" and the explanation of how the representative agent relates to equilibrium. In section 4.2 we state the existence and uniqueness of a fixed point for a certain operator from $(0,\infty)^J$ into itself. Moreover, section 4.3 proves the existence assertion and section 4.4 establishes uniqueness. Finally, section 5.1 gives examples in which the equilibrium in both models can be computed explicitly, even when agents are allowed to have different utility functions and in section 5.2 we present numerical Monte Carlo methods for these examples.

Chapter 2: The Financial Market

2.1 The idea of equilibrium.

In order to set out the basic idea of equilibrium we use a simple two-stage model. This simpler model is useful in order to understand the more complicated models. Suppose there are J agents and each agent j receives:

i. a positive income $\hat{c}_j(1)$ of units of a certain commodity in period 1 and

ii. a second positive income $\hat{c}_j(2)$ of units of the same commodity in period 2.

The agent wishes to maximize his utility from consumption of the commodity over these 2 periods. If his consumption in period t is $c_j(t),\ t=1,2,$ then the utility is defined to be:

$$\log c_j(1) + \log c_j(2), \ \text{ where } c_j(t) \geq 0,\ t=1,2 \text{ and } \log 0 \overset{\Delta}{=} -\infty.$$

If (a) the only commodity available to the agent is his income $\hat{c}_j(1), \hat{c}_j(2)$ and

(b) the commodity is perishable (commodity not consumed in period 1 is not available in period 2), then the agent must choose:

$$c_j(1) \in \left[0, \hat{c}_j(1)\right] \quad , \quad c_j(2) \in \left[0, \hat{c}_j(2)\right].$$

Obviously, his *optimal choices* are:

$$c_j(1) = \hat{c}_j(1)\ ,\ c_j(2) = \hat{c}_j(2). \qquad (2.1.1)$$

Nevertheless, if agent j is permitted to trade with the other agents, his lot in life can probably be improved. We present how this trading is financed in a *moneyed model.* To facilitate trading, we postulate a *bond* with period t price $f_0(t) > 0$ dollars, and we postulate the *commodity* with period t spot price $\psi(t) > 0$ dollars, t = 1, 2. Then agent j can turn his endowment into units of bond ξ_j, given by

$$\xi_j \overset{\Delta}{=} \frac{\psi(1)\hat{c}_j(1)}{f_0(1)} + \frac{\psi(2)\hat{c}_j(2)}{f_0(2)}, \qquad (2.1.2)$$

and he can finance any consumption plan $c_j(1),\ c_j(2)$ as long as

$$\frac{\psi(1)\ c_j(1)}{f_0(1)} + \frac{\psi(2)\ c_j(2)}{f_0(2)} \leq \xi_j. \qquad (2.1.3)$$

The bond allows the agent to finance consumption by moving capital from one period to the other by saving or borrowing; namely by acquiring respectively a long or short position in the bond in the 1st period.

Then, we have the following optimization problem for agent j:

To maximize his utility from consumption: $\log c_j(1) + \log c_j(2)$,

subject to the constraints:

$$\frac{\psi(1)c_j(1)}{f_0(1)} + \frac{\psi(2)c_j(2)}{f_0(2)} \le \xi_j \text{ and } c_j(1) \ge 0,\ c_j(2) \ge 0.$$

The unique solution to this problem is

$$c_j^*(1) = \frac{\xi_j f_0(1)}{2\psi(1)},\ c_j^*(2) = \frac{\xi_j f_0(2)}{2\psi(2)}. \qquad (2.1.4)$$

Indeed, introducing a Lagrange multiplier $\lambda_j > 0$ for the above constraint, we have that:

$$\log c_j(1) + \log c_j(2) \le \log c_j(1) + \log c_j(2) + \lambda_j\left[\xi_j - \frac{\psi(1)\,c_j(1)}{f_0(1)} - \frac{\psi(2)\,c_j(2)}{f_0(2)}\right]$$
$$= \left[\log c_j(1) - \lambda_j \frac{\psi(1)\,c_j(1)}{f_0(1)}\right] + \left[\log c_j(2) - \lambda_j \frac{\psi(2)\,c_j(2)}{f_0(2)}\right] + \lambda_j\,\xi_j.$$

Therefore, making use of (3.4.13) for $U_j(t, x) = \log x$ so that $I_j(t,y) = 1/y$, the agent j maximizes his utility from consumption if and only if he chooses:

$$c_j^*(1) = \frac{f_0(1)}{\lambda_j \psi(1)},\quad c_j^*(2) = \frac{f_0(2)}{\lambda_j \psi(2)}$$

$$\text{and } \lambda_j\left[\xi_j - \frac{\psi(1)c_j^*(1)}{f_0(1)} - \frac{\psi(2)c_j^*(2)}{f_0(2)}\right] = 0 \Leftrightarrow \lambda_j\left(\xi_j - \frac{1}{\lambda_j} - \frac{1}{\lambda_j}\right) = 0 \Leftrightarrow \lambda_j = \frac{2}{\xi_j}.$$

Thus, a direct substitution of λ_j to the optimal consumption choices leads to (2.1.4). ■

Furthermore, a bit of algebra gives:

$$\sum_{t=1}^{2} \log \hat{c}_j(t) \le \sum_{t=1}^{2} \log c_j^*(t), \text{ with equality holding if and only if } \frac{\psi(1)\hat{c}_j(1)}{f_0(1)} = \frac{\psi(2)\hat{c}_j(2)}{f_0(2)}.$$

Proof

Using (2.1.2) and (2.1.4) we get that:

$$\sum_{t=1}^{2} \log c_j^*(t) = \log c_j^*(1) + \log c_j^*(2) = \log\left[c_j^*(1) c_j^*(2)\right] = \log \frac{\xi_j^2 f_0(1) f_0(2)}{4\psi(1)\psi(2)}$$

$$= \log\left[\left(\frac{\psi(1)\hat{c}_j(1)}{f_0(1)} + \frac{\psi(2)\hat{c}_j(2)}{f_0(2)}\right)^2 \frac{f_0(1) f_0(2)}{4\psi(1)\psi(2)}\right]$$

$$= \log\left[\frac{f_0(2)\psi(1)}{4f_0(1)\psi(2)}\hat{c}_j^2(1) + \frac{\hat{c}_j(1)\hat{c}_j(2)}{2} + \frac{f_0(1)\psi(2)}{4f_0(2)\psi(1)}\hat{c}_j^2(2)\right].$$

Therefore we have that

$$\sum_{t=1}^{2}\log c_j^*(t) \geq \sum_{t=1}^{2}\log \hat{c}_j(t)$$

$$\Leftrightarrow \log\left[\frac{f_0(2)\psi(1)}{4f_0(1)\psi(2)}\hat{c}_j^2(1) + \frac{\hat{c}_j(1)\hat{c}_j(2)}{2} + \frac{f_0(1)\psi(2)}{4f_0(2)\psi(1)}\hat{c}_j^2(2)\right] \geq \log\left(\hat{c}_j(1)\hat{c}_j(2)\right)$$

$$\Leftrightarrow \frac{f_0(2)\psi(1)}{4f_0(1)\psi(2)}\hat{c}_j^2(1) + \frac{\hat{c}_j(1)\hat{c}_j(2)}{2} + \frac{f_0(1)\psi(2)}{4f_0(2)\psi(1)}\hat{c}_j^2(2) \geq \hat{c}_j(1)\hat{c}_j(2)$$

$$\Leftrightarrow \left(\frac{\psi(1)\hat{c}_j(1)}{f_0(1)}\right)^2 - 2\left(\frac{\psi(1)\hat{c}_j(1)}{f_0(1)}\right)\left(\frac{\psi(2)\hat{c}_j(2)}{f_0(2)}\right) + \left(\frac{\psi(2)\hat{c}_j(2)}{f_0(2)}\right)^2 \geq 0 \Leftrightarrow \left(\frac{\psi(1)\hat{c}_j(1)}{f_0(1)} - \frac{\psi(2)\hat{c}_j(2)}{f_0(2)}\right)^2 \geq 0,$$

with equality holding if and only if $\frac{\psi(1)\hat{c}_j(1)}{f_0(1)} = \frac{\psi(2)\hat{c}_j(2)}{f_0(2)}$. ■

In other words, trading will generally strictly improve the lot of the j[th] agent.

The optimization problem for agent j can be stated and solved irrespectively of the choice of $f_0(1) > 0$, $f_0(2) > 0$, $\psi(1) > 0$, and $\psi(2) > 0$. However, the commodity is perishable, and its only source in each period is the aggregate income of the agents in that period. So, the *supply* in period t is:

$$\hat{c}(t) = \sum_{j=1}^{J}\hat{c}_j(t), \quad t = 1, 2.$$

From (2.1.2) and (2.1.4) the *demand* in period t is:

$$\frac{f_0(t)}{2\psi(t)}\sum_{j=1}^{J}\xi_j = \frac{f_0(t)}{2\psi(t)}\left[\frac{\psi(1)\hat{c}(1)}{f_0(1)} + \frac{\psi(2)\hat{c}(2)}{f_0(2)}\right].$$

An *equilibrium spot price* pair ($\psi(1)$, $\psi(2)$) is the one which causes supply to equal demand in each period t, that is

$$\hat{c}(1) = \frac{f_0(1)}{2\psi(1)}\left[\frac{\psi(1)\hat{c}(1)}{f_0(1)} + \frac{\psi(2)\hat{c}(2)}{f_0(2)}\right] \quad , \quad \hat{c}(2) = \frac{f_0(2)}{2\psi(2)}\left[\frac{\psi(1)\hat{c}(1)}{f_0(1)} + \frac{\psi(2)\hat{c}(2)}{f_0(2)}\right].$$

Dividing by parts, these equilibrium conditions reduce to

$$\frac{\psi(1)\hat{c}(1)}{f_0(1)} = \frac{\psi(2)\hat{c}(2)}{f_0(2)}. \qquad (2.1.5)$$

Hence, the equilibrium prices are determined up to a multiplicative constant. Replacement of (2.1.5) into (2.1.2) for $\psi(1)$ or $\psi(2)$ gives

$$\xi_j = \frac{\psi(1)}{f_0(1)}\left(\hat{c}_j(1) + \frac{\hat{c}(1)\hat{c}_j(2)}{\hat{c}(2)}\right) \quad \text{or} \quad \xi_j = \frac{\psi(2)}{f_0(2)}\left(\frac{\hat{c}(2)\hat{c}_j(1)}{\hat{c}(1)} + \hat{c}_j(2)\right),$$

and then replacement of these expressions into (2.1.4), results in

$$c_j^*(1) = \lambda_j \hat{c}(1), \quad c_j^*(2) = \lambda_j \hat{c}(2), \quad (2.1.6)$$

where

$$\lambda_j = \frac{1}{2}\left[\frac{\hat{c}_j(1)}{\hat{c}(1)} + \frac{\hat{c}_j(2)}{\hat{c}(2)}\right]. \quad (2.1.7)$$

We conclude that, although the equilibrium prices are not completely determined, the equilibrium optimal consumption plan of each agent is unique and does not depend on the bond prices. In addition, the consumption of agent j in each period is a fixed fraction λ_j of supply, and λ_j is directly related to agent j's relative importance in the economy.

We have given a complete analysis of this simple, two-stage, deterministic equilibrium model. The models are considerably more complex than this one, but many of the essential features of the complex models are already present in the simpler setting. We record here 4 ingredients of a more realistic model.

1. Agents should not have perfect knowledge of their future incomes, nor of the future spot prices. These will be modeled by stochastic processes.
2. Money which is borrowed or held between periods should incur an interest charge or could be invested, respectively, in a variety of financial instruments, some of which are more volatile than a bond. These will be called financial assets.
3. Not all agents should have the same utility from consumption. Each agent will have his own utility function, in contrast to the simple model in which each agent had the logarithmic utility function.

4. Trading opportunities and consumption decisions occur more than twice. The model is in continuous time with a finite planning horizon.

The main results for the moneyed model are essentially those witnessed in the simplified moneyed model of this section. More specifically:

- firstly, an equilibrium spot price process exists, and is unique up to a multiplicative constant.
- secondly, the equilibrium optimal consumption processes of the individual agents are unique.

Moneyless model

In the moneyed model just presented, the bond prices $f_0(t)$, $t=1,2$, were *exogenous.* For the case of a *moneyless model,* an alternate approach to equilibrium would be to set arbitrarily $\psi(1) = \psi(2) = 1$, so that there is no concept of money distinct from units of commodity, and allow the bond price to be *endogenously* determined. So equation (2.1.5) becomes:

$$\frac{\hat{c}(1)}{f_0(1)} = \frac{\hat{c}(2)}{f_0(2)}. \quad (2.1.8)$$

Thus, the equilibrium bond price $f_0(t)$ is directly proportional to the supply $\hat{c}(t)$, $t=1,2$. In the moneyless model, equations (2.1.6) and (2.1.7) are valid, as well.

2.2 The model primitives.

Definition of the equilibrium models

We introduce the following:

- An exogenous N-dimensional Brownian motion $W = \{W(t) = (W_1(t),...,W_N(t))^{\dagger}, \mathcal{F}(t); 0 \le t \le T\}$ on a probability space $(\Omega, \mathcal{F}, P)$, where † denotes transposition.

- The filtration $\{\mathcal{F}(t)\}$ is the augmentation under P of the filtration generated by W. It represents the information available to the agents at time t, and all processes which follow are assumed to be $\{\mathcal{F}(t)\}$ - adapted.

- The model has also M productive assets, and associated with each one of them is a dividend process $\{\delta_m(t); 0 \le t \le T)$, which is (i) $\{\mathcal{F}(t)\}$-adapted, (ii) bounded,

(iii) measurable, (iv) nonnegative and (v) exogenous. Ownership of one share of asset m entitles one to receive the dividend process $\delta_m(\cdot)$, which is denominated in units of the single commodity in our economy, *not in dollars*. We denote by $\delta(\cdot)$ the M-dimensional column vector whose m^{th} component is $\delta_m(\cdot)$.

- There are J agents in the economy, and each agent j has an initial endowment of $\varepsilon_{j,m}$ shares of productive asset m. We assume that $\varepsilon_{j,m} \geq 0;\ \forall\ 1 \leq j \leq J,\ 1 \leq m \leq M$ and

$$\sum_{j=1}^{J} \varepsilon_{j,m} = 1, \quad m = 1,\ldots,M. \quad (2.2.1)$$

In other words, exactly *one share* of each productive asset is owned. We denote by ε_j the M-dimensional row vector ($\varepsilon_{j,1} \ldots \varepsilon_{j,M}$) of agent j's endowments.

- In addition to his endowment, each agent j is entitled to an earnings process $\{e_j(t);\ 0 \leq t \leq T\}$ measured in units of commodity, which is: (i) $\{\mathcal{F}(t)\}$- adapted, (ii) bounded, (iii) measurable, (iv) nonnegative and (v) exogenous.

- If agent j takes no action, he will receive the income process, measured in units of commodity:

$$\hat{c}_j(t) \stackrel{\Delta}{=} e_j(t) + \varepsilon_j \delta(t), \quad 0 \leq t \leq T. \quad (2.2.2)$$

We assume that the nonnegative process $\hat{c}_j(t,\omega)$ is positive on a set of positive product (i.e., Lebesgue x P)-measure; otherwise, agent j would have no role to play in the equilibrium models. The *aggregate income* process is:

$$\hat{c}(t) = \sum_{j=1}^{J} \hat{c}_j(t) = \sum_{j=1}^{J} e_j(t) + \sum_{j=1}^{J} \varepsilon_j \delta(t) = \sum_{j=1}^{J} e_j(t) + \sum_{j=1}^{J}\sum_{m=1}^{M} \varepsilon_{j,m}\delta_m(t)$$
$$= \sum_{j=1}^{J} e_j(t) + \sum_{m=1}^{M} \delta_m(t), \quad 0 \leq t \leq T, \qquad (2.2.3)$$

thanks to (2.2.1), which satisfies

$$0 < k \leq \hat{c}(t) \leq K;\ \forall\ (t,\omega) \in [0,T] \times \Omega \quad (2.2.4)$$

for some positive constants k and K.

The Agents' Utility Functions

We suppose that each agent j is endowed with a measurable utility function: $U_j(t,c):[0,T]\times(0,\infty)\rightarrow\Re$, which quantifies the *"utility"* that he derives by consuming his wealth at the rate c > 0 at time t. For every t $\in$ [0, T], the function $U_j(t,\cdot):(0,\infty)\rightarrow\Re$ is twice continuously differentiable, strictly increasing, strictly concave, and U_j satisfies the following properties:

$$\sup_{0\le t\le T}\left|U_j(t,c)\right|<\infty,\ \inf_{0\le t\le T}U_j^{'}(t,c)>0,\ \sup_{0\le t\le T}\left|U_j^{'}(t,c)\right|<\infty,\ \forall\ c>0 \qquad (2.2.5)$$

$$\text{and}\ \lim_{c\to\infty}\sup_{0\le t\le T}U_j^{'}(t,c)=0. \qquad (2.2.6)$$

In order to prove the uniqueness of equilibrium we shall impose the additional condition

$$\frac{d}{dc}\left(c\,U_j^{'}(t,c)\right)\ge 0,\quad \forall\ c>0,\quad \forall\ t\in\left[0,\ T\right], \qquad (2.2.7)$$

that each U_j satisfies. Through direct computations, this condition is equivalent to assuming that $-c\,U_j^{''}(t,c)/U_j^{'}(t,c)$, i.e. the Arrow-Pratt measure of relative risk aversion is less than or equal to one. An immediate consequence of this condition is: $U_j^{'}(t,c)\ge U_j^{'}(t,1)/c;\ \ \forall\ c\ge 1.$ Integrating this inequality we see that

$$\lim_{c\to\infty}U_j(t,c)=\infty. \qquad (2.2.8)$$

We also impose the conditions

$$U_j(t,0)=\lim_{c\downarrow 0}U_j(t,c)=-\infty\ \text{ and }\ U_j^{'}(t,0)=\lim_{c\downarrow 0}U_j^{'}(t,c)=\infty. \qquad (2.2.9)$$

To see this, set the increasing function

$$h_j(c)=\ c\,U_j^{'}(t,c)\ \text{for}\ c>1$$

and then from the Mean Value theorem there exists

$$\xi\in(1,c):\ \frac{h_j(c)-h_j(1)}{c-1}=h_j^{'}(\xi)\ \ge\ 0\Rightarrow c\,U_j^{'}(t,c)-U_j^{'}(t,c)\ge c-1\ge 0$$

$$\Rightarrow\ U_j^{'}(t,c)\ge U_j^{'}(t,1)/c.$$

2.3 The financial assets.

The moneyed and moneyless models differ in their methods of financial trading**.** We present first the *moneyed model*.

The J agents will be buying and selling among themselves the commodity and ownerships of the productive assets, but these instruments alone may not be sufficient to allow agents to *hedge all the risk* inherent in the information pattern represented by $\{\mathcal{F}(t)\}$. This hedging occurs when agents finance their consumption strategies, and it finds its mathematical expression in the representation of $\{\mathcal{F}(t)\}$- martingales as stochastic integrals with respect to the underlying Brownian motion. The financial market has N + 1 financial assets:

- one of them is a pure discount *bond* with price $f_0(t)$ at time t,
- the remaining assets are *risky stocks* with prices per share $f_n(t)$ at time t

measured in dollars. Those assets are governed by the differential equations:

$$df_0(t) = r(t)\, f_0(t)\, dt, \quad 0 \le t \le T, \qquad (2.3.1)$$

$$df_n(t) = f_n(t)\left[b_n(t)dt + \alpha_n(t)dW(t)\right], \; 0 \le t \le T, \; n = 1,\dots,N. \qquad (2.3.2)$$

We also have the *initial condition*

$$f_n(0) = 1, \quad n = 0,\dots,N. \quad (2.3.3)$$

These differential equations have the *unique solutions:*

$$f_0(t) = e^{\int_0^t r(s)ds}, \qquad (2.3.4)$$

$$f_n(t) = e^{\int_0^t [b_n(s) - \frac{1}{2}||\alpha_n(s)||^2 ds + \int_0^t \alpha_n(s)dW(s)}, \quad n = 1,\dots,N. \quad (2.3.5)$$

These solutions are always *strictly positive*.

Proof of (2.3.4)

From (2.3.1) we have

$$\frac{df_0(t)}{f_0(t)} = r(t)\, dt \Rightarrow \int_0^t \frac{df_0(s)}{f_0(s)} = \int_0^t r(s)ds \Rightarrow \log \frac{f_0(t)}{f_0(0)} = \int_0^t r(s)ds \;\Rightarrow\; f_0(t) = e^{\int_0^t r(s)ds}.$$

Proof of (2.3.5)

Let $C(t) = \ln f_n(t) = h(f_n(t))$ for $h(f)=\ln f$, and

$$\frac{\partial h}{\partial f} = \frac{1}{f}, \quad \frac{\partial^2 h}{\partial f^2} = -\frac{1}{f^2}, \quad \frac{\partial h}{\partial t} = 0.$$

From Ito's lemma and (4.2), (4.3) we have:

$$dC(t) = \left[\frac{\partial h}{\partial f} b_n(t)\, f_n(t) + \frac{1}{2}\frac{\partial^2 h}{\partial f^2}\, \alpha_n^2(t)\; f_n^2(t) + \frac{\partial h}{\partial t}\right] dt + \left[\frac{\partial h}{\partial f}\, \alpha_n(t)\, f_n(t)\right] dW(t)$$

$$= \left[\frac{1}{f_n(t)} b_n(t) f_n(t) - \frac{1}{2} \frac{1}{f_n^2(t)} \alpha_n^2(t) f_n^2(t) \right] dt + \left[\frac{1}{f_n(t)} \alpha_n(t) f_n(t) \right] dW(t)$$

$$= \left[b_n(t) - \frac{1}{2} \alpha_n^2(t) \right] dt + \alpha_n(t)\, dW(t),$$

$$\ln f_n(t) - \ln f_n(0) = \int_0^t \left[b_n(s) - \frac{1}{2} \alpha_n^2(s) \right] ds + \int_0^t \alpha_n(s)\, dW(s) \Rightarrow f_n(t) = e^{\int_0^t [b_n(s) - \frac{1}{2} ||\alpha_n(s)||^2] ds + \int_0^t \alpha_n(s) dW(s)}.$$

Here, we denote

- by F(t) the (N + 1)-dimensional column vector of financial asset prices $(f_0(t),..., f_N(t))^\dagger$ and by f(t) the N-dimensional vector $(f_1(t),..., f_n(t))^\dagger$,
- the interest rate process $\{r(t);\ 0 \le t \le T\}$, as well as
- the vector of mean rates of return $\{b(t) = (b_1(t),..., b_n(t))^\dagger;\ 0 \le t \le T\}$ and
- the N x N volatility matrix $\alpha(t)$, whose nth row is $\alpha_n(t) = (\alpha_{n,1}(t),..., \alpha_{n,N}(t))$,

are assumed to be *measurable,* $\{\mathcal{F}(t)\}$ *-adapted, and bounded uniformly* in $(t, \omega) \in [0, T] \times \Omega$. These processes are *exogenous* in the moneyed model.

The financial assets represent contracts between agents and in equilibrium will be in zero net supply. Although they are rather arbitrarily chosen, the particular choices of $r(\cdot)$, $b(\cdot)$ and $\alpha(\cdot)$ have minimal effect on the equilibrium.

A market in which all risk can be hedged is referred to as *complete.* It is possible to obtain a complete market by introducing fewer than N + 1 financial assets, but the feasibility of this depends on the nature of the equilibrium itself. So, we have taken the convenient approach of making available enough financial assets to complete the market, regardless of the nature of the equilibrium we finally obtain.

In addition, we shall impose the *nondegeneracy assumption* that for some $\varepsilon > 0$,

$$\xi^\dagger \alpha(t) \alpha^\dagger(t) \xi \ge \varepsilon ||\xi||^2 \quad \forall \xi \in \Re^N,\ (t,\omega) \in [0,T] \times \Omega \text{ almost surely.} \quad (2.3.6)$$

The matrices $\alpha(t)$ and $\alpha^\dagger(t)$ are invertible and satisfy assumption (2.3.6), so we have,

$$||\alpha^\dagger(t,\omega)^{-1}\xi|| \le \frac{1}{\sqrt{\varepsilon}} ||\xi||,\ ||\alpha(t,\omega)^{-1}\xi|| < \frac{1}{\sqrt{\varepsilon}} ||\xi||;\ \forall\ \xi \in \Re^N, (t,\omega) \in [0,T] \times \Omega \quad (2.3.7)$$

The financial asset prices f_0 and $f_1,..., f_N$ have mean rates of return r and $b_1,..., b_N$, respectively, and we have to change the probability measure so as to make them all have the same mean rate of return. In particular, the *"relative risk" process*

$$\theta(t) = (\alpha(t))^{-1} [b(t) - r(t)\mathbf{1}_N], \quad (2.3.8)$$

where $\mathbf{1}_N$ denotes the N-dimensional vector with every component equal to one, is progressively measurable with respect to $\{\mathcal{F}(t)\}$ and thanks to (2.3.7) $||\theta||$ is bounded by some constant.

We also introduce the *"likelihood ratio" process*

$$Z(t) = e^{-\int_0^t \theta^\dagger(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds}, \ 0 \le t \le T. \quad (2.3.9)$$

It follows then from the Girsanov theorem that the exponentional supermartingale $\{Z(t), \mathcal{F}(t);\ 0 \le t \le T\}$ is actually a martingale, and the new probability measure given by

$$\tilde{P}(A) = E\left[Z(T)\cdot 1_A\right], \ \forall A \in \mathcal{F}(T), \quad (2.3.10)$$

is such that $P \text{ and } \tilde{P}$ are mutually equivalent on $\mathcal{F}(T)$. When making statements which hold almost surely, we are thus not obliged to distinguish between these two probability measures.

Moreover, from (2.3.8) we have that

$$b(t) = r(t)\mathbf{1}_N + \alpha(t)\theta(t) \Rightarrow b_n(t) = r(t) + \alpha_n(t)\theta(t),$$

and therefore from (2.3.2) we get

$$\begin{aligned} df_n(t) &= f_n(t)\left[b_n(t)dt + \alpha_n(t)dW(t)\right] \\ &= f_n(t)\left[r(t)dt + \alpha_n(t)\left(dW(t) + \theta(t)dt\right)\right] \\ &= f_n(t)\left[r(t)dt + \alpha_n(t)\, d\tilde{W}(t)\right], \end{aligned}$$

where the process

$$\tilde{W}(t) = W(t) + \int_0^t \theta(s)ds, \ 0 \le t \le T. \quad (2.3.11)$$

is a standard N-dimensional Brownian motion under the new probability measure $\tilde{P}$ (Karatzas and Shreve [13]).

The financial assets in the model will dynamically affect the value of money. The process

$$\zeta(t) = Z(t)\, e^{-\int_0^t r(u)du}, \ 0 \le t \le T, \quad (2.3.12)$$

acts as a *"deflator"*, in the sense that multiplication by $\zeta(t)$ converts wealth held at time t to the equivalent amount of wealth at time zero.

Moneyless model

The financial assets in the *moneyless model* have all the properties just listed for the moneyed model, except that they are measured in units of commodity and will be endogenously determined.

2.4 The spot price and the productive assets.

In the *moneyed model*, agents buy and sell the commodity at each time t at a spot price of $\psi(t)$ dollars per unit. This process will be determined *endogenously*, by the equilibrium considerations, in order to satisfy

$$0 < k(\psi) \le \zeta(t)\psi(t) \le K(\psi) < \infty, \quad \forall\ (t,\omega)\in[0,\ T]\times\Omega, \qquad (2.4.1)$$

where $k(\psi)$ and $K(\psi)$ are constants which may depend on ψ but not on (t,ω). Agents may also buy or sell part or all of the productive assets at their nonnegative dollar prices per share $P_1(t),\dots, P_M(t)$, which will likewise be *endogenous*. We denote by $P(t)$ the M-dimensional column vector of productive asset prices $P(t) = (P_1(t),\dots, P_M(t))^{\dagger}$. We require that for each m,

$$\zeta P_m \text{ is bounded (uniformly in t and } \omega), \ 0\le t\le T,\ m=1,\dots,M \qquad (2.4.2)$$

and P_m is a nonnegative $\{\mathcal{F}(t)\}$-semimartingale of the form:

$$dP_m(t) = \beta_m(t)dt + \alpha_m(t)\, dW(t), \quad 0 \le t \le T, m = 1,\dots,M, \qquad (2.4.3)$$

where $\{\beta(t)=(\beta_1(t),\dots,\beta_m(t))^{\dagger};\ 0\le t\le T\}$ and $\{\alpha(t)=(\alpha_{m,n}(t));\ m=1,\dots,M;\ n=1,\dots,N;\ 0\le t\le T\}$ are processes to be determined *endogenously* so that

$$\int_0^T \left[||\beta(t)|| + \sum_{m=1}^{M} ||\alpha_m(t)||^2 \right] dt < \infty, \text{ almost surely.} \qquad (2.4.4)$$

At the terminal time T, the productive asset m has paid out all its dividends and has no further value. It is required that

$$P_m(T) = 0, \text{ almost surely}, \ m = 1,\dots,M. \qquad (2.4.5)$$

The spot price process ψ contributes to the construction of equilibrium in the moneyed model.

Moneyless model

In the *moneyless model*, the spot price is equal to one, the financial assets are chosen judiciously in order to provide equilibrium, and the productive asset price processes are exactly as described for the moneyed model. We require that the financial assets satisfy (2.4.1) with $\psi = 1$; that is, ζ should be bounded from above and below by some positive constants k and K, uniformly in t and ω:

$$0 < k \le \zeta(t) \le K, \ 0 \le t \le T, \text{ a.s.} \qquad (2.4.6)$$

Chapter3: Equilibrium In A Utility Maximization problem

3.1 The optimization problem for an individual agent.

When agent j is attempting to solve his optimization problem, he acts as a price-taker. He has at his disposal the choice of

- a consumption process $\{c_j(t);\ 0 \le t \le T\}$,
- a productive asset portfolio process $\{\pi_j(t) = (\pi_{j,1}(t),\dots, \pi_{j,M}(t)\};\ 0 \le t \le T)$ and
- a financial asset portfolio process $\{\Phi_j(t) = (\varphi_{j,0}(t),\dots\dots, \varphi_{j,N}(t));\ 0 \le t \le T\}$, such that

$$\inf_{0\le t\le T} c_j(t) \ge 0, \sup_{0\le t\le T} c_j(t) < \infty \text{ almost surely,} \quad (3.1.1)$$

$$\sup_{0\le t\le T} \|\pi_j(t)\| < \infty, \int_0^t \|\Phi_j(t)\|^2 dt < \infty \text{ almost surely.} \quad (3.1.2)$$

We denote by $\varphi_j(t)$ the N-dimensional process $(\varphi_{j,1}(t),\dots, \varphi_{j,N}(t))$. The *nonnegative* consumption process represents the rate at which the agent consumes the commodity, and is thus denominated in units of the commodity. The components of the *portfolio processes* may be either positive or negative (borrowing and short-selling is permitted), and represent the j^{th} agent's positions, measured in numbers of shares, in the respective assets.

Initially, we have $\pi_{j,m}(0) = \varepsilon_{j,m}$, $m = 1,\dots,M$, and $\varphi_{j,n}(0) = 0$, $n = 0,\dots,N$, and we require that

$$\pi_j(t)P(t) + \Phi_j(t)F(t) = \varepsilon_j P(0) - \int_0^t \psi(s)c_j(s)ds + \int_0^t \psi(s)e_j(s)\,ds$$

$$+ \int_0^t \psi(s)\pi_j(s)\delta(s)ds + \int_0^t \pi_j(s)dP(s) + \int_0^t \Phi_j(s)dF(s), \quad 0 \le t \le T. \quad (3.1.3)$$

This equation is written in its *moneyed* model version**.** The *moneyless* model version is obtained by setting ψ equal to one. The integrals on the right-hand side are for:

- the decrease in wealth due to consumption,
- the increase in wealth due to earnings,
- the increase in wealth due to dividends paid by productive assets held,
- capital gains or losses from productive assets held and
- capital gains or losses from financial assets held.

We call relation (3.1.3) the *budget equation* and we denote by

$$X_j(t) = \pi_j(t)\,P(t) + \Phi_j(t)\,F(t), \quad (3.1.4)$$

the wealth of agent j at time t. Triples (c_j, π_j, Φ_j) which satisfy the budget equation are *self-financing*, in the sense that all changes in wealth are accounted for within the model. The definition of X_j from (3.1.4) leads to

$$X_j(t) = \pi_j(t)P(t) + \varphi_{j,0}(t)f_0(t) + \varphi_j(t)f(t) \Rightarrow$$

$$\varphi_{j,0}(t) = \frac{1}{f_0(t)}\,[X_j(t) - \pi_j(t)\,P(t) - \varphi_j(t)\,f(t)]. \quad (3.1.5)$$

Then, under replacement of (3.1.4) and (3.1.5) into (3.1.3) the budget equation in revised form may be given as

$$X_j(t) = \varepsilon_j P(0) + \int_0^t \psi(s)\left[e_j(s) - c_j(s)\right]ds + \int_0^t \psi(s)\pi_j(s)\,\delta(s)\,ds + \int_0^t \pi_j(s)\left[\beta(s)\,ds + \alpha(s)\,dW(s)\right] + \int_0^t \left[\varphi_{j,0}(s)\,df_0(s) + \varphi_j(s)\,df(s)\right]$$

$$= \varepsilon_j P(0) + \int_0^t \psi(s)\left[e_j(s) - c_j(s)\right]ds + \int_0^t \psi(s)\,\pi_j(s)\,\delta(s)\,ds + \int_0^t \pi_j(s)\,\beta(s)\,ds + \int_0^t \pi_j(s)\,\alpha(s)\,dW(s)$$

$$+ \int_0^t r(s)\left[X_j(s) - \pi_j(s)\,P(s) - \varphi_j(s)\,f(s)\right]ds + \int_0^t \varphi_j(s)\mathrm{diag}\big(f(s)\big)b(s)\,ds + \int_0^t \varphi_j(s)\mathrm{diag}\big(f(s)\big)\alpha(s)dW(s)$$

or equivalently

$$X_j(t) = \varepsilon_j P(0) + \int_0^t \psi(s)\left[e_j(s) - c_j(s)\right]ds + \int_0^t r(s)\,X_j(s)\,ds + \int_0^t \varphi_j(s)\mathrm{diag}\big(f(s)\big)\left[b(s) - r(s)1_N\right]ds$$

$$+ \int_0^t \pi_j(s)\left[\psi(s)\,\delta(s) + \beta(s) - r(s)\,P(s)\right]ds + \int_0^t \left[\pi_j(s)\,\alpha(s) + \varphi_j(s)\,\mathrm{diag}\,\big(f(s)\big)\,\alpha(s)\right]dW(s), \quad (3.1.6)$$

where diag (f(s)) is the N x N diagonal matrix whose diagonal entries are the components of f(s). In terms of the process $\tilde{W}$ of (2.3.11) and the coefficient processes in (2.3.1), (2.3.2) and (2.4.3) the *budget equation* becomes

$$X_j(t) = \varepsilon_j P(0) + \int_0^t \psi(s)\left[e_j(s) - c_j(s)\right]ds + \int_0^t r(s)X_j(s)ds + \int_0^t \pi_j(s)\left[\psi(s)\delta(s) + \beta(s) - r(s)P(s) - \alpha(s)\theta(s)\right]ds$$

$$+ \int_0^t \left[\pi_j(s) + \alpha(s) + \varphi_j(s)\mathrm{diag}\big(f(s)\big)\alpha(s)\right]d\tilde{W}(s). \quad (3.1.7)$$

While agents may have short-term deficits, we impose that they choose consumption and portfolio processes so that:

- for some positive constant K(c, π, φ), depending on the indicated processes but not on (t,ω), $\zeta(t)X_j(t)$ is bounded from below, that is

$$\zeta(t)X_j(t) \geq -K(c,\pi,\varphi),\ \forall t \in [0,T],\ \textit{almost surely}, \quad (3.1.8)$$

- they are subject to having non-negative wealth at the terminal time, that is

$$X_j(T) \geq 0,\ \textit{almost surely}. \quad (3.1.9)$$

Definition 3.1.1 (Feasibility)

Let a spot price process ψ satisfying (2.4.1) and a vector of productive asset prices $(P_1,..., P_M)$ of the form (2.4.3), (2.4.5) be given, where (2.4.2), (2.4.4) are also satisfied. A triple (c_j, π_j, Φ_j) of consumption, productive asset portfolio, and financial asset portfolio processes is *feasible* for agent j if (3.1.1) - (3.1.3) are satisfied, and $X_j(\cdot)$ defined by (3.1.4) or equivalently by (3.1.7), satisfies equations (3.1.8) and (3.1.9).

Definition 3.1.2 (Optimality)

A triple $(c_j^*, \pi_j^*, \Phi_j^*)$ is *optimal* for agent j if:

1. it is feasible and
2. maximizes the expected total utility from consumption

$$E\int_0^T U_j\left(t, c_j(t)\right)dt, \quad (3.1.10)$$

over all feasible triples (c_j, π_j, Φ_j) that satisfy

$$E\int_0^T \max\left\{0, -U_j(t,\ c_j(t))\right\}dt < \infty. \quad (3.1.11)$$

Thus, each agent's goal is to maximize the expected total utility from consumption over all feasible triples which satisfy (3.1.11).

3.2 The definition of equilibrium.

We can now define the notion of equilibrium. When agent j is attempting to solve his optimization problem, he acts as a price-taker. He has no influence over ψ and $(P_1,...,P_M)$. Nevertheless, in aggregate, the actions of the agents should determine the prices ψ and $(P_1,..., P_M)$ through *the law of supply and demand.* This law dictates that:

1. all the commodity be consumed as it enters the economy,
2. the aggregate demand for each productive asset be one share (which is the initial supply (cf.2.2.1)), and
3. the aggregate demand for each financial asset be zero.

Definition 3.2.1

An equilibrium in the moneyed model consists of:

i. a spot price process ψ satisfying (2.4.1),

ii. a vector of productive asset prices $(P_1,..., P_M)$ of the form (2.4.3),(2.4.5) for which (2.4.2),(2.4.4) are satisfied, and

iii. a collection of consumption, productive asset portfolio, and financial asset portfolio triples $(c_j^*, \pi_j^*, \Phi_j^*)$, $j = 1,..., J$.

Each $(c_j^*, \pi_j^*, \Phi_j^*)$ must be optimal for agent j relative to ψ and $(P_1,..., P_M)$, and for Lebesgue-almost every $t \in [0, T]$ the <u>market clearing conditions</u> almost surely

$$\sum_{j=1}^{J} c_j^*(t) = \hat{c}(t), \qquad (3.2.1)$$

$$\sum_{j=1}^{J} \pi_j^*(t) = \mathbf{1}_M^{\dagger}, \qquad (3.2.2)$$

$$\sum_{j=1}^{J} \Phi_j^*(t) = \mathbf{0}_{N+1}^{\dagger}. \qquad (3.2.3)$$

must hold. In this context:

- c_j^*, π_j^* and Φ_j^* denote the optimal processes for the j^{th} agent,
- $\mathbf{1}_M$ is the M-dimensional column vector with all components equal to 1, and
- $\mathbf{0}_{N+1}$ is the (N + 1)-dimensional column vector with all components equal to 0.

Remark 3.2.2

If $[\psi, (P_1,..., P_M), \{(c_j^*, \pi_j^*, \Phi_j^*) ; j = 1,..., J\}]$ is an equilibrium for the moneyed model, then for every j, we have from (3.2.1) that $c_j^*(t) \le \hat{c}(t)$. From conditions (2.2.4), (2.2.5) it follows that

$$E\int_0^t U_j\left(t, c_j^*(t)\right)dt < \infty, \quad j = 1, \ldots, J. \qquad (3.2.4)$$

It would not be correct to think of the collective actions of agents as determining the prices, *unless* equilibrium is essentially *unique*. Prices cannot be entirely unique, since the currency can always be revalued, which would have the effect of scaling ψ and $(P_1,..., P_M)$.

Definition 3.2.3

Suppose that for any *two equilibria*:

$[\psi, (P_1,..., P_M), \{(c_j^*, \pi_j^*, \Phi_j^*); j = 1,...J\}]$ and $[\tilde{\psi}, (\tilde{P}_1,..., \tilde{P}_M), \{(\tilde{c}_j^*, \tilde{\pi}_j^*, \tilde{\Phi}_j^*); j = 1,...J\}]$

in the *moneyed model*, there exists a positive constant γ for which

$$\psi(t) = \gamma \tilde{\psi}(t), \quad P_1(t) = \gamma \tilde{P}_1(t), \ldots\ldots, \quad P_M(t) = \gamma \tilde{P}_M(t) \qquad (3.2.5)$$

$$\text{for Lebesgue} - \text{almost every } t \in [0,\ T] \text{ and } P - \text{a.e. } \omega \in \Omega .$$

Then we say that equilibrium in the moneyed model is *unique.*

If (3.2.5) holds, then agent j faces the same optimization problem relative to $\tilde{\psi}$ and $\left(\tilde{P}_1, \ldots, \tilde{P}_M\right)$, as he does relative to ψ and $(P_1, \ldots, P_M)$. The optimal productive asset and financial asset portfolios may not be unique, but we shall show that the optimal consumption process is (Theorem 3.3.4). Thus, for the two equilibria in Definition 3.2.3, the relations of (3.2.5) will imply that for Lebesgue-almost every $t \in [0, T]$ and $P -$ a.e. $\omega \in \Omega$

$$c_j^*(t) = \tilde{c}_j^*(t). \qquad (3.2.6)$$

Furthermore, we shall show that, if $[\psi, (P_1, \ldots, P_M), \{(c_j^*, \pi_j^*, \Phi_j^*); j = 1, \ldots, J\}]$ is an equilibrium for the moneyed model and $[\tilde{\psi}, (\tilde{P}_1, \ldots, \tilde{P}_M), \{(\tilde{c}_j^*, \tilde{\pi}_j^*, \tilde{\Phi}_j^*); j = 1, \ldots, J\}]$ is an equilibrium for another moneyed model whose only difference from the first is the choice of $r(\cdot)$, $b(\cdot)$ and $a(\cdot)$, then (3.2.6) holds, although (3.2.5) may not; (cf. Corollary 4.2.3). The *conclusion* is that the exogenously selected financial assets can affect the value of money by more than a multiplicative factor, but they cannot affect the way in which real wealth, (measured in units of commodity), is ultimately distributed among the agents.

Moneyless model

The *moneyless* equilibrium model is the one obtained by requiring

$$\psi(t) = 1,\ \forall\ t \in [0, T], \text{ almost surely,} \qquad (3.2.7)$$

and regarding the financial assets as endogenous.

Definition 3.2.4

An *equilibrium in the moneyless model* consists of:

i. a vector of financial asset prices $(f_0, \ldots, f_N)$, where the processes r, b and α are: measurable, $\{\mathcal{F}(t)\}$-adapted, and bounded, and (2.3.6) holds,

ii. a vector of productive asset prices (P_1,..., P_M) of the form (2.4.3),(2.4.5) for which (2.4.2) and (2.4.4) are also satisfied, and

iii. a collection of consumption, productive asset portfolio, and financial asset portfolio triples $(c_j^*, \pi_j^*, \Phi_j^*)$, j = 1,..., J, satisfying the conditions of Definition 3.2.1 relative to $\psi \equiv 1$.

3.3 The equilibrium prices of productive assets.

In this section, we present how the absence of arbitrage opportunities, a necessary ingredient in equilibrium, determines the prices of the productive assets. In this way we are able to effectively eliminate these assets and their price processes from the model. We presuppose the existence of equilibrium and draw conclusions about the prices of the productive assets. Their associated gains processes must be martingales under the equivalent probability measure $\tilde{P}$.

Lemma 3.3.1

Let [ψ, (P_1,...,P_M), { $(c_j^*, \pi_j^*, \Phi_j^*)$; j = 1,..., J}] be an equilibrium in the moneyed model. Then

$$\psi(t)\delta(t)+\beta(t)-r(t)P(t)-\alpha(t)\theta(t)=0 \quad (3.3.1)$$

can fail only on a subset of [0,T] x Ω with zero Lebesgue x P-measure.

Proof

Let j be a given integer between 1 and J, and from (3.1.7) let

$$X_j^*(t)=\varepsilon_j P(0)+\int_0^t \psi(s)\left[e_j(s)-c_j^*(s)\right]ds+\int_0^t r(s)\, X_j^*(s)\, ds$$

$$+\int_0^t \pi_j^*(s)\left[\psi(s)\delta(s)+\beta(s)-r(s)P(s)-\alpha(s)\theta(s)\right] ds$$

$$+\int_0^t \left[\pi_j^*(s)\, \alpha(s)+\varphi_j^*(s)\, \mathrm{diag}\big(f(s)\big)\, \alpha(s)\right] d\tilde{W}(s) \qquad (3.3.2)$$

be the wealth process corresponding to optimal triple $(c_j^*, \pi_j^*, \Phi_j^*)$. The *feasibility* of $(c_j^*, \pi_j^*, \Phi_j^*)$ implies that ζX_j^* is bounded from below and $X_j^*(T) \geq 0$, almost surely.

Define

$$\pi_j(t) \overset{\Delta}{=} \operatorname{sgn}\,[\psi(t)\;\delta(t) + \beta(t)\; - r(t)\,P(t) - \alpha(t)\;\theta(t)]^{\dagger},$$

where the signum function is applied separately to each component of the above vector. Because $\alpha(\cdot)$ is invertible, there exists a unique, N-dimensional process $\varphi_j(t) = (\varphi_{j,1}(t), .., \varphi_{j,N}(t))$ such that

$$\pi_j(t)\alpha(t) + \varphi_j(t)\;\operatorname{diag}\big(f(t)\big)\alpha(t) \;=\; 0,\; 0 \le t \le T,\; \text{a.s.}$$

Set $\Phi_j(t) = (0, \varphi_{j,1}(t), \ldots, \varphi_{j,N}(t))$, $0 \le t \le T$, and define

$$c_j(t) = \frac{1}{\psi(t)}\;\pi_j\;\big[\psi(t)\;\delta(t) + \beta(t) - r(t)\,P(t) - \alpha(t)\;\theta(t)\big] \;\ge\; 0 \;\;\text{and}$$

$$\tilde{c}_j(t) = c_j^*(t) + c_j(t), \quad \tilde{\pi}_j(t) = \pi_j^*(t) + \pi_j(t), \quad \tilde{\Phi}_j(t) = \Phi_j^*(t) + \Phi_j(t),\; 0 \le t \le T.$$

X_j^* given by (3.3.2) is also the wealth process corresponding to $(\tilde{c}_j, \tilde{\pi}_j, \tilde{\Phi}_j)$. From the feasibility of $(c_j^*, \pi_j^*, \Phi_j^*)$ it is obvious that $(\tilde{c}_j, \tilde{\pi}_j, \tilde{\Phi}_j)$ is feasible, as well. By construction, $\tilde{c}_j(t) \ge c_j^*(t)$ and if (3.3.1) failed on a subset of [0, T] xΩ with positive Lebesgue x P-measure, then this inequality would be strict on this set. Due to Remark 3.2.2 and the strict monotonicity of $U_j(t,\cdot)$, we have

$$E\int_0^T U_j\big(t, c_j^*(t)\big)dt \;<\; E\int_0^T U_j\Big(t, \tilde{c}_j(t)\Big)\;dt\,,$$

which would be a violation of the optimality of $(c_j^*, \pi_j^*, \Phi_j^*)$ and (3.3.1) follows. ■

We may solve (3.3.1) for the drift in the productive assets $\beta(\cdot)$ and substitute this into (2.4.3) to obtain

$$dP(t) = [r(t)P(t) - \psi(t)\delta(t)]dt + a(t)\;d\tilde{W}(t),$$

where (2.3.11) has also been used. This linear stochastic differential equation has a unique solution, which leads to the expression for the *gains process:*

$$G(t) = P(0) + \int_0^t e^{-\int_0^s r(u)du}\;\alpha(s)\;d\tilde{W}(s) \qquad (3.3.3)$$

$$G(t) = e^{-\int_0^t r(s)ds}\;P(t) + \int_0^t e^{-\int_0^s r(u)du}\;\psi(s)\;\delta(s)\;ds\;; \qquad (3.3.4)$$

to see this compute the differential

$$d\left(e^{-\int_0^t r(s)ds} P(t)\right) = P(t)\, d\left(e^{-\int_0^t r(s)ds}\right) + e^{-\int_0^t r(s)ds} dP(t) = P(t)\, e^{-\int_0^t r(s)ds}\left(-r(t)\right)dt + e^{-\int_0^t r(s)ds}\left[\left(r(t)\,P(t) - \psi(t)\,\delta(t)\right)dt + \alpha(t)\, d\tilde{W}(t)\right]$$

$$= -e^{-\int_0^t r(s)ds}\, \psi(t)\, \delta(t)\, dt + e^{-\int_0^t r(s)ds}\, \alpha(t)\, d\tilde{W}(t)$$

and integrate on [0,t] to obtain (3.3.3) from (3.3.4).

Under the $\tilde{P}$-measure, with respect to which $\tilde{W}$ is a Brownian motion, the gains process G (·) is a vector of local martingales, thanks to (3.3.3). For each positive integer n, we may define

$$\sigma_n \stackrel{\Delta}{=} T \wedge inf \left\{ t \in [0,T] : \int_0^t \|\alpha(s)\|^2 ds = n \right\}$$

and we have for $0 \le t \le T$ that $G(t \wedge \sigma_n)$ is a $\tilde{P}$-martingale so

$$G(t\wedge\sigma_n) = \tilde{E}\,[G(\sigma_n) \mid \mathcal{F}(t\wedge\sigma_n)] = 1_{\{\sigma_n < t\}} G(\sigma_n) + 1_{\{\sigma_n \ge t\}} E\,[Z(\sigma_n) G(\sigma_n) \mid \mathcal{F}(t)].$$

Here we have used Lemma 3.5.3 of Karatzas, Shreve [13] to change from the conditional expectation under $\tilde{P}$ to the conditional expectation under P. Letting $n \to \infty$, and recalling (2.4.4), we have that the first term of the last expression vanishes and we obtain

$$G(t) = \lim_{n\to\infty} 1_{\{\sigma_n \ge t\}} \frac{1}{Z(t)} E\,[Z(\sigma_n)\, G(\sigma_n) / \mathcal{F}(t)], \text{ a.s.} \quad (3.3.5)$$

But on the event, $\{\sigma_n \ge t\}$,

$$\frac{1}{Z(t)} E\left[Z(\sigma_n)\, G(\sigma_n) / \mathcal{F}(t)\right] = \frac{1}{Z(t)} E\left[Z(\sigma_n) \left\{ e^{-\int_0^{\sigma_n} r(s)ds} P(\sigma_n) + \int_0^{\sigma_n} e^{-\int_0^s r(u)du} \psi(s)\, \delta(s)\, ds \right\} \Big| \mathcal{F}(t)\right]$$

$$= \frac{1}{Z(t)} E\left[\zeta(\sigma_n) P(\sigma_n) \middle| \mathcal{F}(t)\right] + \frac{1}{Z(t)} E\left[Z(\sigma_n) \middle| \mathcal{F}(t)\right] \int_0^t e^{-\int_0^s r(u)du} \psi(s)\delta(s)ds + \frac{1}{Z(t)} E\left[Z(\sigma_n) \int_t^{\sigma_n} e^{-\int_0^s r(u)du} \psi(s)\delta(s)ds \middle| \mathcal{F}(t)\right]$$

$$= \frac{1}{Z(t)} E\left[\zeta(\sigma_n) P(\sigma_n) \middle| \mathcal{F}(t)\right] + \int_0^t e^{-\int_0^s r(u)du} \psi(s)\delta(s)ds + \frac{1}{Z(t)} \int_t^{\sigma_n} E\left[Z(\sigma_n) e^{-\int_0^s r(u)du} \psi(s)\delta(s) \middle| \mathcal{F}(t)\right] ds$$

$$=\frac{1}{Z(t)}E\left[\zeta(\sigma_n)P(\sigma_n)\middle|\mathcal{F}(t)\right]+\int_0^t e^{-\int_0^s r(u)du}\psi(s)\delta(s)ds+\frac{1}{Z(t)}\int_t^{\sigma_n}E\left[E\left(Z(\sigma_n)e^{-\int_0^s r(u)du}\psi(s)\delta(s)\middle|\mathcal{F}(s)\right)\middle|\mathcal{F}(t)\right]ds$$

$$=\frac{1}{Z(t)}E\left[\zeta(\sigma_n)P(\sigma_n)\middle|\mathcal{F}(t)\right]+\int_0^t e^{-\int_0^s r(u)du}\psi(s)\delta(s)\,ds+\frac{1}{Z(t)}\int_t^{\sigma_n}E[Z(s)\,e^{-\int_0^s r(u)du}\psi(s)\,\delta(s)|\mathcal{F}(t)]$$

$$=\frac{1}{Z(t)}E\left[\zeta(\sigma_n)P(\sigma_n)\middle|\mathcal{F}(t)\right]+\int_0^t e^{-\int_0^s r(u)du}\psi(s)\,\delta(s)\,ds+\frac{1}{Z(t)}E\left[\int_t^{\sigma_n}\zeta(s)\,\psi(s)\,\delta(s)\,ds\middle|\mathcal{F}(t)\right],$$

where we have used (3.3.4) and (2.3.12) and the martingale property of $Z(\cdot)$.

From (2.4.1) and (2.4.2) we have assumed that ζ P and $\zeta\ \psi$ are bounded, and so the bounded convergence theorem asserts that the limit in (3.3.5) is

$$\frac{1}{Z(t)}E\left[\zeta(T)P(T)|\mathcal{F}(t)\right]+\int_0^t e^{-\int_0^s r(u)du}\psi(s)\,\delta(s)\,ds+\frac{1}{Z(t)}E\left[\int_t^T\zeta(s)\,\psi(s)\,\delta(s)\,ds/\mathcal{F}(t)\right]=\tilde{E}\left[G(T)|\mathcal{F}(t)\right].$$

- Therefore, G is a *martingale* under $\tilde{P}$.

The process $G(\cdot)$ at time t records the current values of the productive assets *plus* the values of the dividends paid out during [0,t], converted to dollars. All these values are discounted back to the initial time through the interest rate process $r(\cdot)$. If G is not a martingale under $\tilde{P}$, then the last Lemma implies that arbitrage opportunities exist in the trading of productive assets against financial assets.

Theorem 3.3.2

Let $[\psi,(P_1,\ldots,P_M),\{(c_j^*,\pi_j^*,\Phi_j^*);\ j=1,\ldots,J\}]$ be an equilibrium in the moneyed model. Then

$$P(t)=\tilde{E}\left[\int_t^T e^{-\int_t^s r(u)du}\psi(s)\delta(s)ds/\mathcal{F}(t)\right]=\frac{1}{\zeta(t)}E\left[\int_t^T\zeta(s)\psi(s)\delta(s)ds/\mathcal{F}(t)\right],\ 0\le t\le T,\ \text{a.s.}\quad(3.3.6)$$

The second equality of (3.3.6) is the result of changing from the $\tilde{P}$-measure to the P-measure.

Proof

From (2.4.5) and (3.3.4), we have $G(T)=\int_0^T e^{-\int_0^s r(u)du}\psi(s)\,\delta(s)\,ds$, and (3.3.4) gives for $0\le t\le T$ that

$$P(t) = e^{\int_0^t r(s)ds} G(t) - \int_0^t e^{-\int_t^s r(u)du} \psi(s)\delta(s)ds = e^{\int_0^t r(s)ds} \tilde{E}[G(T)/\mathcal{F}(t)] - \int_0^t e^{-\int_t^s r(u)du} \psi(s)\delta(s)ds$$

$$= \tilde{E}\left[\int_t^T e^{-\int_t^s r(u)du} \psi(s)\delta(s)ds / \mathcal{F}(t)\right], \text{ almost surely.}$$

■

Moneyless model

Corollary 3.3.3

Let $[(f_0,..., f_N),(P_1,...,P_M),\{(c_j^*,\pi_j^*,\Phi_j^*); j=1,..., J\}]$ be equilibrium in the *moneyless model.* Then (3.3.6) holds with ψ identically equal to one.

Theorem 3.3.2 assumes the existence of an equilibrium, which includes the assumption of the existence of a vector of productive asset prices. However, that theorem provides the formula (3.3.6) for this vector (in terms of the dividend and spot price processes), which suggests that the *existence* of productive asset prices could be a *conclusion rather than a hypothesis.* We will in fact obtain these prices via formula (3.3.6), so we must show that the prices so obtained satisfy the conditions imposed on them in Definition 3.2.1. It will be essential to represent martingales (under $\tilde{P}$) as stochastic integrals with respect to $\tilde{W}$ of (2.3.11).

Lemma 3.3.4

Let $\{Y(t), \mathcal{F}(t); 0\le t\le T\}$ be a martingale under $\tilde{P}$. Then there exists an N-dimensional process $\{H(t) = (H_1(t),..., H_n(t)), \mathcal{F}(t); 0 \le t \le T\}$ such that

$$\int_0^T ||H(t)||^2 dt < \infty, \text{ a.s. and } Y(t) = Y(0) + \int_0^t H(s)d\tilde{W}(s), \; 0 \le t \le T, \text{ a.s.} \quad (3.3.7)$$

Proof

It is mentioned that for $0 \le s \le t \le T$, we have from Bayes rule that

$$E\left[Z(t)Y(t)/\mathcal{F}(s)\right] = Z(s)\tilde{E}\left[Y(t)/\mathcal{F}(s)\right] = Z(s)Y(s), \text{ a.s.},$$

so ZY is a martingale under P. Because $\{\mathcal{F}(t)\}$ is the augmentation of the filtration generated by the Brownian motion W (under P), there exists a process $L= (L_1,..., L_N)$ such that

$$E\int_0^T ||\mathrm{L(t)}||^2 dt < \infty,\ \ \mathrm{Z}(t)\mathrm{Y}(t) = \mathrm{Y(0)} + \int_0^t \mathrm{L(s)\ dW(s)},\ \ 0 \le \mathrm{t} \le \mathrm{T},\ \text{almost surely.}$$

We have from (2.3.9) that

$$\mathrm{Z(t)} = f\left(\mathrm{M(t)}\right),\ \text{where } \mathrm{M(t)} = -\int_0^t \theta^\dagger(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds$$

and $f(x) = f'(x) = f''(x) = e^x$.

Then Itô's formula implies that

$$\begin{aligned}\mathrm{dZ(t)} &= df(M(\mathrm{t})) = f'(Mt)d(Mt) + \frac{1}{2}f''(Mt)dM(\mathrm{t})dM(\mathrm{t})\\ &= \mathrm{Z(t)}\left[-\theta^\dagger(\mathrm{t})\mathrm{dW(t)} - \frac{1}{2}||\theta(\mathrm{t})||^2 dt\right] + \frac{1}{2}\mathrm{Z(t)}||\theta(\mathrm{t})||^2 dt\\ &= -\mathrm{Z(t)}\ \theta^\dagger(\mathrm{t})\ dW(\mathrm{t}),\end{aligned}$$

and we also have that $\mathrm{d}\left[\mathrm{Z(t)Y(t)}\right] = L(t)\ dW(t)$.

$$\text{Moreover } \mathrm{Y(t)=Z(t)Y(t)}\cdot\frac{1}{\mathrm{Z(t)}} = u\left(\mathrm{Z(t)Y(t), Z(t)}\right), where\ u(v,z) = \frac{v}{z}.$$

$$So,\ \frac{\partial u}{\partial v} = \frac{1}{z},\quad \frac{\partial u}{\partial z} = -\frac{v}{z^2},\quad \frac{\partial^2 u}{\partial z \partial v} = -\frac{1}{z^2},\quad \frac{\partial^2 u}{\partial v^2} = 0,\quad \frac{\partial^2 u}{\partial z^2} = \frac{2v}{z^3},$$

and applying Ito's rule we have

$$\begin{aligned}dY(t) &= du\,(\mathrm{Z(t)\,Y(t), Z(t)})\\ &= \left\{\frac{\partial u}{\partial v}d(\mathrm{Z(t)Y(t)}) + \frac{\partial u}{\partial z}d(\mathrm{Z(t)}) + \frac{1}{2}\frac{\partial^2 u}{\partial z^2}d(\mathrm{Z(t)})d(\mathrm{Z(t)}) + \frac{\partial^2 u}{\partial z \partial v}d(\mathrm{Z(t)Y(t)})d(\mathrm{Z(t)})\right\}dt + \left\{\frac{1}{\mathrm{Z(t)}}\mathrm{L(t)} + \frac{\mathrm{Z(t)Y(t)}}{\mathrm{Z(t)}}\theta^\dagger(\mathrm{t})\right\}dW(\mathrm{t})\\ &= \left\{\frac{1}{2}\frac{\mathrm{Z(t)Y(t)}}{\mathrm{Z^2(t)}}\mathrm{Z^2(t)}||\theta(\mathrm{t})||^2 + \frac{1}{\mathrm{Z^2(t)}}\mathrm{L(t)\,Z(t)}\,\theta^\dagger(\mathrm{t})\right\}\mathrm{dt} + \left\{\frac{1}{\mathrm{Z(t)}}\mathrm{L(t)} + \frac{\mathrm{Z(t)\,Y(t)}}{\mathrm{Z(t)}}\theta^\dagger(\mathrm{t})\right\}\mathrm{dW(t)}\\ &= \left\{\mathrm{Z(t)}\,||\theta(\mathrm{t})||^2 + \frac{1}{\mathrm{Z(t)}}\mathrm{L(t)}\,\theta^\dagger(\mathrm{t})\right\}\mathrm{dt} + \left\{\frac{1}{\mathrm{Z(t)}}\mathrm{L(t)} + \frac{\mathrm{Z(t)\,Y(t)}}{\mathrm{Z(t)}}\theta^\dagger(\mathrm{t})\right\}\mathrm{dW(t)}.\end{aligned}$$

Finally, integrating on [0,t] and revoking (2.3.11) we obtain

$$\begin{aligned}\mathrm{Y(t)} &= \mathrm{Y(0)} + \int_0^t \left[\frac{1}{\mathrm{Z(t)}}\mathrm{L(s)} + \mathrm{Y(s)}\theta^\dagger(\mathrm{s})\right]\theta(\mathrm{s})\ \mathrm{ds} + \int_0^t \left[\frac{1}{\mathrm{Z(t)}}\mathrm{L(s)} + \mathrm{Y(s)}\theta^\dagger(\mathrm{s})\right] dW(s)\\ &= \mathrm{Y(0)} + \int_0^t \mathrm{H(s)d}\tilde{\mathrm{W}}(\mathrm{s}),\end{aligned}$$

where $\mathrm{H\ (t)} = \frac{1}{\mathrm{Z(t)}}\ \mathrm{L(t)} + \mathrm{Y(t)}\ \theta^\dagger(\mathrm{t}),\ \ 0 \le \mathrm{t} \le \mathrm{T}$, almost surely.

■

Theorem 3.3.5

Let a spot price ψ, satisfying (2.4.1) be given, and define P (t) = (P_1(t),.....,P_M (t))† by (3.3.6). Then, for each m = 1,..., M, P_m is a nonnegative Ito' process satisfying the conditions (2.4.2), (2.4.3) and (2.4.5), where the coefficient processes β_m and α_m satisfy (2.4.4) and (3.3.1).

Proof

From (3.3.4) and (3.3.6) we obtain

$$G_m(t)=\tilde{E}\left[\int_0^T e^{-\int_0^s r(u)du}\psi(s)\delta(s)ds/\mathcal{F}(t)\right],\quad m=1,...,M,$$

which is a martingale under $\tilde{P}$. According to Lemma 3.3.4, there exists an N-dimensional process H = (H_1,..., H_N) satisfying 3.3.7 and for which

$$G_m(t)=G_m(0)+\int_0^t H(s)d\tilde{W}(s)$$
$$=G_m(0)+\int_0^t H(s)\theta(s)ds+\int_0^t H(s)dW(s).$$

From (3.3.4) follows that

$$P_m(t)=e^{\int_0^t r(s)ds}\left[G_m(0)+\int_0^t H(s)\theta(s)ds+\int_0^t H(s)dW(s)-\int_0^t e^{-\int_0^s r(u)du}\psi(s)\delta(s)ds\right].$$

So Itô’s rule leads to

$$dP_m(t)=d\left(e^{\int_0^t r(s)ds}\right)\left[G_m(0)+\int_0^t H(s)\theta(s)ds+\int_0^t H(s)dW(s)-\int_0^t e^{-\int_0^s r(u)du}\psi(s)\delta(s)\,ds\right]$$
$$+\;e^{\int_0^t r(s)ds}\,d\left[G_m(0)+\int_0^t H(s)\,\theta(s)\,ds+\int_0^t H(s)\,dW(s)-\int_0^t e^{-\int_0^s r(u)du}\psi(s)\,\delta(s)\,ds\right]$$
$$=e^{\int_0^t r(s)ds}\,r(t)dt\left[G_m(0)+\int_0^t H(s)\theta(s)ds+\int_0^t H(s)dW(s)-\int_0^t e^{-\int_0^s r(u)du}\psi(s)\delta(s)ds\right]$$
$$+\;e^{\int_0^t r(s)ds}\left[dG_m(0)+H(t)\,\theta(t)\,dt+H(t)\,dW(t)-e^{-\int_0^t r(s)ds}\psi(t)\,\delta(t)\,dt\right]$$

$$dP_m(t) = \left(r(t)\, P_m(t) - \psi(t)\, \delta(t) + e^{\int_0^t r(s)ds} H(t)\, \theta(t) \right) dt + e^{\int_0^t r(s)ds} H(t)\, dW(t).$$

Thus, P_m has the form indicated by (2.4.3) with

$$\beta_m(t) = r(t)P_m(t) - \psi(t)\delta(t) + e^{\int_0^t r(s)ds} H(t)\theta(t), \quad \alpha_m(t) = e^{\int_0^t r(s)ds} H(t), \quad m=1,...M.$$

■

3.4 The solution of the optimization problem for an individual agent.

If a spot price process for the commodity is given (the *moneyless model* can be included by setting the spot price identically equal to one), each individual agent then faces the problem of the maximization of his expected utility from consumption. In this section, we solve this stochastic control problem. To characterize an equilibrium financial market, we let a financial market be given and study individual agent behavior in its presence. In particular, we have a fixed spot price process $\psi(\cdot)$ satisfying (2.4.1) in terms of which the productive asset price process vector P (t) = (P_1(t),.....,P_M (t)) is given by (3.3.6). We also fix an element $j \in \{1,..., J\}$. Agent j is unaware of any equilibrium considerations used to obtain ψ and P; he takes these as given and is not bound by any market clearing conditions. He also takes as given the model primitives and the financial asset price processes. We will show how agent j maximizes his expected utility of consumption.

The results of this section are stated for the *moneyed model*, but by setting ψ equal to one, similar results are obtained for the *moneyless model.*

Lemma 3.4.1

Let (c_j, π_j, Φ_j) be a feasible triple. Then the expected total value of consumption, deflated back to the original time does not exceed the expected total deflated value of income, that is

$$E\int_0^T \zeta(s)\psi(s)c_j(s)\,ds \le E\int_0^T \zeta(s)\psi(s)\hat{c}_j(s)\,ds. \quad (3.4.1)$$

This inequality can be regarded as a *budget constraint* and justifies the terminology “deflator” for the process $\zeta(\cdot)$. It mandates that the expected total value of consumption, deflated back to the original time, does not exceed the expected total deflated value of income.

Proof

Theorem 3.3.5 implies the validity of (3.3.1), and so the budget equation for the wealth of the jth agent becomes

$$X_j(t) = \varepsilon_j P(0) + \int_0^t \psi(s)\left[e_j(s) - c_j(s)\right]ds + \int_0^t r(s)\, X_j(s)ds + \int_0^t \left[\pi_j(s)\alpha(s) + \varphi_j(s)\mathrm{diag}\left(f(s)\right)\alpha(s)\right]d\tilde{W}(s)$$

for which the unique solution is

$$X_j(t) = e^{\int_0^t r(u)du}\left\{\varepsilon_j P(0) + \int_0^t e^{-\int_0^s r(u)du}\psi(s)\left[e_j(s) - c_j(s)\right]ds + \int_0^t e^{-\int_0^s r(u)du}\left[\pi_j(s)\,\alpha(s) + \varphi_j(s)\,\mathrm{diag}\left(f(s)\right)\alpha(s)\right]d\tilde{W}(s)\right\}. \quad (3.4.2)$$

For each positive integer n, let *the stopping time*

$$\tau_n = T \wedge \inf\left\{t \geq 0 : \int_0^t \left\| \pi_j(s)\ \alpha(s) + \varphi_j(s)\mathrm{diag}\left(f(s)\right)\alpha(s)\right\|^2 ds = n\right\}.$$

Because of (2.4.4),(3.1.2), and the boundedness of the coefficient processes $r(\cdot)$, $b(\cdot)$, and $\alpha(\cdot)$ appearing in (2.3.4), (2.3.5), we get $\lim_{n\to\infty}\tau_n = T$, almost surely.

From change of measure and the fact that the stochastic integral of (3.4.2) is a martingale under $\tilde{P}$ on $[0,\tau_n]$, we obtain

$$E\left[\zeta(\tau_n)X_j(\tau_n)\right] + E\int_0^{\tau_n} \zeta(s)\psi(s)c_j(s)\,ds = \tilde{E}\left[e^{-\int_0^{\tau_n} r(u)du} X_j(\tau_n)\right] + \tilde{E}\int_0^{\tau_n} e^{-\int_0^s r(u)du}\psi(s)\,c_j(s)ds$$

$$= \varepsilon_j P(0) + \tilde{E}\int_0^{\tau_n} e^{-\int_0^s r(u)du}\psi(s)e_j(s)ds = \varepsilon_j P(0) + E\int_0^{\tau_n}\zeta(s)\psi(s)e_j(s)ds.$$

If $n \to \infty$ with the use of (3.1.8), (3.1.9), Fatou's lemma, and the Monotone Convergence Theorem, it is obtained that

$$E\int_0^T \zeta(s)\psi(s)c_j(s)ds \leq \varepsilon_j P(0) + E\int_0^T \zeta(s)\psi(s)e_j(s)ds.$$

Relation (3.4.1) now follows from (2.2.2), the above relation and

$$P(0) = E\int_0^T \zeta(s)\psi(s)\delta(s)ds, \quad (3.4.3)$$

a consequence of (3.3.6). ■

Theorem 3.4.2

Conversely to Lemma 3.4.1, consider a consumption process c_j (i.e., a nonnegative, measurable, $\{\mathcal{F}(t)\}$-adapted process satisfying (3.1.1), for which (3.4.1) is valid. Then there exist a productive asset portfolio π_j and a financial asset portfolio Φ_j such that (c_j, π_j, Φ_j) is feasible. Furthermore, we can take $\pi_j \equiv \varepsilon_j$, i.e., agent j does not have to change his initial position in the productive assets.

Proof

From (2.4.1), (3.4.1), (3.4.3) and the boundedness of e_j, we introduce the random variable

$$Q_j \overset{\Delta}{=} \int_0^T e^{-\int_0^s r(u)du} \psi(s)\left[e_j(s)-c_j(s)\right]ds,$$

which is $\tilde{P}-$integrable with $\varepsilon_j P(0) + \tilde{E} Q_j \geq 0.$ According to Lemma 3.3.4, the $\tilde{P}$-martingale $\tilde{E}[Q_j / \mathcal{F}(t)]$ admits the stochastic representation

$$\tilde{E}[Q_j / \mathcal{F}(t)] = \tilde{E} Q_j + \int_0^t H(s) d\tilde{W}(s), \quad 0 \leq t \leq T, \text{a.s.},$$

where the N-dimensional, measurable process H is $\{\mathcal{F}(t)\}$-adapted and satisfies (3.3.7). We define $\pi_j \equiv \varepsilon_j$ and

$$\varphi_j(s) \overset{\Delta}{=} -\left[e^{\int_0^s r(u)du} H(s) + \varepsilon_j \alpha(s) \right] (\alpha(s))^{-1} (\text{diag}(f(s)))^{-1}, \; 0 \leq s \leq T.$$

The corresponding wealth process (cf.3.4.2)) after using Lemma 3.3.4 becomes

$$X_j(t) = e^{\int_0^t r(u)du} \left\{ \varepsilon_j \; P(0) + \int_0^t e^{-\int_0^s r(u)du} \psi(s) \left[e_j(s)-c_j(s)\right] ds - \int_0^t H(s) \, d\tilde{W}(s) \right\}$$

$$= e^{\int_0^t r(u)du} \left\{ \varepsilon_j P(0) + \tilde{E} Q_j + \tilde{E}\left[\int_t^T e^{-\int_0^s r(u)du} \psi(s)\left[c_j(s) - e_j(s)\right] ds / \mathcal{F}(t) \right] \right\}$$

$$= e^{\int_0^t r(u)du} \left\{ \varepsilon_j P(0) + \tilde{E} Q_j + \frac{1}{Z(t)} E\left[\int_t^T \zeta(s)\psi(s)\left[c_j(s)-e_j(s)\right] ds / \mathcal{F}(t) \right] \right\} \quad (3.4.4)$$

and satisfies (3.1.8) and (3.1.9). Furthermore, with $\varphi_{j,0}$ defined by (3.1.5), π_j and the

(N+1)-dimensional process $\Phi = (\varphi_{j,0}, \varphi_j)$ satisfy (3.1.2) (see (2.3.7), the strict positivity of f_n in (2.3.4) and (3.3.7)). ■

Making use of Lemma 3.4.1 and Theorem 3.4.2 we conclude that the *optimization problem* for agent j can be cast as to maximize the expected utility from consumption

$$E\int_0^T U_j\left(t, c_j(t)\right)dt,$$

subject to the *constraints:*

- ❑ $\inf_{0\le t\le T} c_j(t) \ge 0$, $\sup_{0\le t\le T} c_j(t) < \infty$, *a.s.*,
- ❑ $E\int_0^T \max\left\{0, -U_j\left(t,\ c_j(t)\right)\right\}\ dt\ < \infty,$
- ❑ $E\int_0^T \zeta(s)\psi(s)c_j(s)\ ds \le\ E\int_0^T \zeta(s)\psi(s)\hat{c}_j(s)\ ds\ \left(\text{budget constraint}\right).$

This is a problem involving the *consumption process*, but not the portfolio process. The productive asset prices do not enter this formulation of the problem; the financial asset prices enter only through ζ, which is determined by (2.3.12). Budget constraint suggests the explanation of ζ(s) ψ(s) as the dollar value at time 0 of a unit of commodity with price ψ(s) at time s. This is consistent with the explanation of ζ(s) following (2.3.12).

Now, we demonstrate the solution of the *above problem*. For each t ∈ [0, T], the function $U_j^{'}(t,\cdot)$ is strictly decreasing and satisfies (2.2.6). In order to solve this problem we introduce $I_j(t,\cdot)$ the inverse of $U_j^{'}(t,\cdot)$, i.e., a strictly decreasing, continuous mapping from $(0,\infty)$ onto $(0,\infty)$. In other words,

$$I_j(t,y) > 0,\ \ U_j^{'}\left(t,\ I_j(t,y)\right) = y,\ \ \forall\ y \in (0,\infty). \quad (3.4.5)$$

For y ∈ $(0,\infty)$, define

$$\mathcal{X}_j(y) = E\int_0^t \zeta(s)\psi(s)\ I_j\left(s, y\zeta(s)\psi(s)\right)ds. \quad (3.4.6)$$

Remark 3.4.3

The function $\mathcal{X}_j$ maps (0, ∞) into (0, ∞), is continuous, strictly decreasing, and satisfies

$$\lim_{y\downarrow 0} \mathcal{X}_j(y) = \infty, \quad (3.4.7)$$

$$\lim_{y\uparrow\infty} \mathcal{X}_j(y) = 0. \quad (3.4.8)$$

It has also a continuous, strictly decreasing inverse $\mathcal{Y}_j : (0,\infty) \to (0,\infty)$. Let

$$\xi_j = E\int_0^t \zeta(s)\,\psi(s)\,\hat{c}_j(s)\,ds, \quad (3.4.9)$$

$$\eta_j = \mathcal{Y}_j(\xi_j). \quad (3.4.10)$$

(Note that $\xi_j > 0$ because of the assumption that $\hat{c}_j(\cdot)$ is not Lebesgue x P-almost everywhere zero). We shall show that the optimal consumption process for the j^th^ agent is

$$c_j^*(t) = I_j\left(t,\ \eta_j\zeta(t)\psi(t)\right), \quad 0 \le t \le T. \quad (3.4.11)$$

Theorem 3.4.4

The unique (up to Lebesgue x P-almost everywhere equivalence) optimal consumption policy for the j^{th} agent is given by (3.4.11).

Proof

According to our definitions, from equations (3.4.6), and (3.4.9) - (3.4.11) we have,

$$E\int_0^t \zeta(s)\,\psi(s)\,c_j^*(s)\,ds = \mathcal{X}_j(\eta_j) = \xi_j = E\int_0^t \zeta(s)\,\psi(s)\,\hat{c}_j(s)\,ds, \quad (3.4.12)$$

so $c_j^*(\cdot)$ satisfies (3.4.1) with equality. Let $c_j(\cdot)$ be any process satisfying (3.1.1), (3.1.11) and (3.4.1), so

$$E\int_0^T \zeta(s)\,\psi(s)\left[c_j^*(s) - c_j(s)\right] \ \ge\ 0.$$

From calculus, it is shown that

$$U_j\left(t,\ I_j(t,y)\right) - y\,I_j(t,y) = \max_{c>0}\left\{U_j(t,c) - y\,c\right\},\ \forall\ y \in (0,\infty),\ t \in [0,T], \quad (3.4.13)$$

and thus

$$E\int_0^T U_j\left(s,\ c_j(s)\right)\,ds \le E\int_0^T U_j\left(s,\ c_j(s)\right)\,ds + y_j E\int_0^T \zeta(s)\,\psi(s)\left[c_j^*(s) - c_j(s)\right]ds$$

$$\le E\int_0^T U_j\left(s,\ c_j^*(s)\right)\,ds. \quad (3.4.14)$$

So, if it is feasible, then $c_j^*(\cdot)$ is *optimal.*

There is at least one feasible consumption process; namely

$$c_j \equiv \left[E\int_0^T \zeta(s)\psi(s)\,ds \right]^{-1} \xi_j.$$

This constant process satisfies (3.4.1) with equality, and (3.1.1), (3.1.11) are also explicitly satisfied. By choosing $c_j(\cdot)$ in (3.4.14), it is concluded that $c_j^*(\cdot)$ satisfies (3.1.11). The reason why $c_j^*(\cdot)$ is the unique optimal consumption policy for agent j is that the maximum in (3.4.13) is uniquely attained at $I_j(t, y)$, ■

3.5 Characterization of Equilibrium

The issue now is how to choose $\psi(\cdot)$ so that when, for each j, $c_j^*(\cdot)$ is given by (3.4.11) relations (3.2.1)-(3.2.3) are satisfied. It turns out that the only relevant aspect of $\psi(\cdot)$ is the process $\zeta(\cdot)$ they lead to, as shown by the following Lemma.

Lemma 3.5.1

i. Let $\psi(\cdot)$, be given, such that the equilibrium conditions (3.2.1)-(3.2.3) hold. Then

$$\hat{c}(t) = \sum_{j=1}^{J} I_j(t,\eta_j\zeta(t)\psi(t)),\ 0 \le t \le T, \quad (3.5.1)$$

where η_j is defined by (3.4.10) and $\zeta(t)$ is given by (2.3.12).

ii. Conversely, suppose there exists $\psi(\cdot)$ such that the above relationship holds. Then the equilibrium conditions (3.2.1)-(3.2.3) are also satisfied.

Proof

i. For the first assertion, recall that for j = 1,...,J the optimal consumption processes $c_j^*(\cdot)$ are given by (3.4.11) and together with the spot market clearing condition (3.2.1) leads to (3.5.1).

ii. For the converse assertion, the optimal consumption processes $c_j^*(\cdot)$ are again given by (3.4.11) and together with relationship (3.5.1) relation (3.2.1) holds. From (2.2.1) and $\pi_j^* \equiv \varepsilon_j,\ \forall\ j = 1,...,J,$ relation (3.2.2) also holds

It remains to verify (3.2.3). Because for each j, $\pi_j^* \equiv \varepsilon_j$ and Φ_j^* is also given as in Theorem 3.4.2, the corresponding wealth process is given by (3.4.4):

$$X_j^*(t) = e^{\int_0^t r(u)du} \left\{ \varepsilon_j P(0) + \tilde{E} Q_j^* + \frac{1}{Z(t)} E\left[\int_t^T \zeta(s)\psi(s)\left[c_j^*(s) - e_j(s) \right] ds / \mathcal{F}(t) \right] \right\}, \quad (3.5.2)$$

$$\text{where } Q_j^* = \int_0^T e^{-\int_0^S r(u)du} \psi(s)\left[e_j(s) - c_j^*(s) \right] ds.$$

Using (2.2.3), (3.2.1) and (3.4.3), we see that

$$\sum_{j=1}^{J} \tilde{E} Q_j^* = E\int_0^T \zeta(s)\psi(s)\left[\sum_{j=1}^{J} e_j(s) - \hat{c}(s) \right] ds = -E\int_0^T \zeta(s)\ \psi(s) \sum_{m=1}^{M} \delta_m(s) ds = -\sum_{j=1}^{J} \varepsilon_j P(0). \quad (3.5.3)$$

We prove now the last equation

$$\sum_{j=1}^{J} \varepsilon_j P(0) = \sum_{j=1}^{J} \varepsilon_j\ E\int_0^T \zeta(s)\psi(s)\delta(s) ds = E\int_0^T \sum_{j=1}^{J} \zeta(s)\psi(s) \sum_{m=1}^{M} \varepsilon_{j,m}\delta_m(s)\, ds$$
$$= E\int_0^T \zeta(s)\psi(s) \sum_{m=1}^{M} \left(\sum_{j=1}^{J} \varepsilon_{j,m} \right) \delta_m(s) ds = E\int_0^T \zeta(s)\psi(s) \sum_{m=1}^{M} \delta_m(s) ds,$$

where (2.2.1) has also been invoked.

Thus, a summation over j in the above relationship for the wealth process yields:

$$\zeta(t) \sum_{j=1}^{J} X_j^*(t) = E\left[\int_0^T \zeta(s)\psi(s) \sum_{m=1}^{M} \delta_m(s) ds / \mathcal{F}(t) \right]. \quad (3.5.4)$$

From (3.1.4) and (3.3.6) we have also:

$$\zeta(t)\sum_{j=1}^{J} X_j^*(t) = \zeta(t)\sum_{m=1}^{M} P_m(t) + \zeta(t)\sum_{m=1}^{M} \Phi_j^*(t) F(t) = E\left[\int_0^T \zeta(s)\psi(s)\sum_{m=1}^{M}\delta_m(s) ds / \mathcal{F}(t) \right] + \zeta(t)\sum_{j=1}^{J}\Phi_j^*(t)\, F(t). \quad (3.5.5)$$

A comparison of equations (3.5.4) and (3.5.5) shows that

$$\sum_{j=1}^{J} \Phi_j^*(t)\ F(t) = 0, \quad 0 \le t \le T, \text{ a.s.} \quad (3.5.6)$$

Because $\pi_j^* \equiv \varepsilon_j$, (3.1.3) reduces to

$$\Phi_j^*(t)\ F(t) = \int_0^t \psi(s)\left[\hat{c}_j(s) - c_j^*(s) \right] ds + \int_0^t \Phi_j^*(s)\ dF(s),$$

which yields, in conjunction with (3.2.1)

$$0=\int_0^t \sum_{j=1}^J \Phi_j^*(s)dF(s)=\int_0^t \sum_{j=1}^J \left[\varphi_{j,0}^*(s)\, f_0(s)\, r(s)+\varphi_j^*(s)\, (\mathrm{diag}(f(s))\, b(s)\right] ds+\int_0^t \sum_{j=1}^J \varphi_j^*(s)\, \mathrm{diag}\,(f(s))\, a(s)\, dW(s).$$

The local martingale part of the right-hand side and hence also its quadratic variation

$$\int_0^t ||(\sum_{j=1}^J \varphi_j^*(s))\ \mathrm{diag}(f(s))\,\alpha(s)||^2 ds, \quad 0\le t\le T,$$

must be identically equal to zero. It follows from the nonsingularity of diag (f(s)) α(s) that:

$$\sum_{j=1}^J \varphi_j^*(t)=\mathbf{0}_N^\dagger, \text{ for a.e. } t\in[0,T], \text{ almost surely.}$$

From (3.5.5) and $f_0(t) \neq 0$, we have also that:

$$\sum_{j=1}^J \varphi_{j,0}^*(t)\ F(t)=0, \text{ for a.e. } t\in[0,T], \text{ almost surely,}$$

and this concludes the proof of (3.2.3). ■

Chapter 4: Characteristics of Equilibrium

4.1 The representative agent.

We introduce here the meaning of a “representative agent” and the utility function for a representative agent, and explain how the representative agent relates to the proof of the existence of equilibrium. For every vector $\Lambda = (\lambda_1,\ldots,\lambda_J) \in (0,\infty)^J$ let us introduce the utility function:

$$U(t,c;\Lambda) = \max_{\substack{c_1\geq 0,\ldots,c_J\geq 0 \\ c_1+\ldots\ldots\ldots+c_J=c}} \sum_{j=1}^{J} \lambda_j U_j(t,c_j);\ \forall\ (t,c)\in[0,T]\times(0,\infty).\quad (4.1.1)$$

We interpret this function as the utility function of a "representative" agent, who assigns the weights $\lambda_1,\ldots,\lambda_J$ to the utilities of the individual agents in the economy. As we will show in Lemma 4.1.1, the function U has many of the properties of $U_1, \ldots, U_J$. The problem of equilibrium can then be cast as that of determining the “right” way to assign these weights.

Lemma 4.1.1

For fixed $\Lambda \in (0,\infty)$ and $t \in [0,T]$, the function $U(t,\cdot\,;\Lambda): (0,\infty) \to \Re$ is strictly increasing and continuously differentiable, $U'(t,\cdot\,;\Lambda)$ is strictly decreasing and

$$\sup_{0\leq t\leq T}\left|U(t,\ c;\Lambda)\right| < \infty,\ \inf_{0\leq t\leq T} U'(t,c;\Lambda) > 0\ ,\ \sup_{0\leq t\leq T} U'(t,\ c;\Lambda) < \infty\ \forall\ c>0 \quad (4.1.2)$$

$$\lim_{c\to\infty}\sup_{0\leq t\leq T} U'(t,c) = 0,\qquad \lim_{c\to 0^+} U'(t,c) = \infty. \quad (4.1.3)$$

Proof

Define

$$I(t,y;\Lambda) \stackrel{\Delta}{=} \sum_{j=1}^{J} I_j(t,\frac{y}{\lambda_j}),\ 0<y<\infty. \quad (4.1.4)$$

The function $I(t,\cdot\,;\Lambda)$ is continuous and strictly decreasing on $(0,\infty)$, and maps this interval onto $(0,\infty)$. Thus, for every $c \in (0,\infty)$ there is a unique positive number $H(t,c) = H(t,c;\Lambda)$ with $I(t, H(t,c;\Lambda)) = c$, and the mapping $H(t,\cdot): (0,\infty) \to (0,\infty)$ is continuous and strictly decreasing. Let $c \in (0,\infty)$ be given, and define

$$\bar{c}_j = I_j(t,\frac{H(t,c)}{\lambda_j}),\ \ j=1,\ldots,J. \quad (4.1.5)$$

Then $\sum_{j=1}^{J} \bar{c}_j = I(t, H(t,c); \Lambda) = c$, and for each j,

$$U_j'(t, \bar{c}_j) = \frac{H(t,c)}{\lambda_j} \Leftrightarrow \lambda_j U_j'(t, \bar{c}_j) = H(t,c), \quad j = 1,\dots,J.$$

Let $c_1, \dots, c_J$ be any other nonnegative numbers with $\sum_{j=1}^{J} c_j = c$. Due to the concavity of each $U_j(t,\cdot)$ we are enabled to have

$$\begin{aligned}
\sum_{j=1}^{J} \lambda_j U_j(t, c_j) &\le \sum_{j=1}^{J} \lambda_j [U_j(t, \bar{c}_j) + (c_j - \bar{c}_j) U_j'(t, \bar{c}_j)] \\
&= \sum_{j=1}^{J} \lambda_j U_j(t, \bar{c}_j) + H(t,c) \sum_{j=1}^{J} (c_j - \bar{c}_j) \\
&= \sum_{j=1}^{J} \lambda_j U_j(t, \bar{c}_j) + H(t,c) \left(\sum_{j=1}^{J} c_j - \sum_{j=1}^{J} \bar{c}_j \right) = \sum_{j=1}^{J} \lambda_j U_j(t, \bar{c}_j).
\end{aligned}$$

We reach the conclusion that the vector $(\bar{c}_1, \dots, \bar{c}_j)$ attains the maximum in (4.1.4), i.e.

$$U(t, c; \Lambda) = \sum_{j=1}^{J} \lambda_j U_j \left(t, I_j(t, \frac{H(t,c)}{\lambda_j}) \right), \ \forall \ c \in (0, \infty). \quad (4.1.6)$$

Now each $I_j(t,\cdot)$ is differentiable, so

- $I(t,\cdot\,;\Lambda)$ is differentiable and
- $H(t,\cdot)$ is differentiable.

Furthermore, for $1 \le j \le J$ and $c \in (0,\infty)$, we have $0 < \frac{H(t,c)}{\lambda_j} < \infty$, so equation (3.4.5) gives

$$\begin{aligned}
\frac{d}{dc} \lambda_j U_j(t, I_j(t, \frac{H(t,c)}{\lambda_j})) &= U_j'(t, I_j(t, \frac{H(t,c)}{\lambda_j})) I_j'(t, \frac{H(t,c)}{\lambda_j}) H'(t,c) \\
&= \frac{1}{\lambda_j} H(t,c) I_j'(t, \frac{H(t,c)}{\lambda_j}) H'(t,c). \qquad (4.1.7)
\end{aligned}$$

The derivative of U (t, c; Λ) in (4.1.6) is thus seen to be from (4.1.7)

$$U'(t,c;\Lambda) = H(t,c)\, H'(t,c) \sum_{j=1}^{J} \frac{1}{\lambda_j} I_j'(t, \frac{H(t,c)}{\lambda_j}) = H(t,c)\, H'(t,c)\, I'\big(t, H(t,c)\big)$$

$$= H(t,c) \ \frac{d}{dc} \ I\big(t, H(t,c)\big) = H(t,c) \qquad \text{for all } c \in (0,\infty). \qquad (4.1.8)$$

From expression (4.1.8) and as $H(t,\cdot)$ is continuous, $U(t,\cdot\,;\Lambda)$ must be continuously differentiable and (4.1.8) must be valid on all of (0, ∞). In particular,

$U'(t,\cdot\,;\Lambda)$ is strictly decreasing. We can easily derive relations (4.1.2) and (4.1.3) from: (2.2.5), (2.2.6), the definition of U (t, c; Λ), and the fact that U_j (t, c) > y is equivalent to $c < I_j(t, y)$ and similarly for $U(t,\cdot\,;\Lambda)$ and $I(t,\cdot\,;\Lambda)$. ■

The properties established for $U(t,\cdot\,;\Lambda)$ in Lemma 4.1.1 are those properties, shared by each $U_j(t,\cdot)$, which were used in the derivation of the optimal consumption process for agent j.

Because of (4.1.8), the function $I(t,\cdot;\Lambda)$ of (4.1.4) satisfies

$$I(t,\ y;\Lambda) > 0,\quad U'(t,\ I(t,\ y;\Lambda);\Lambda) = y,\ \ \forall\ y \in (0,\infty). \quad (4.1.9)$$

If a spot price ψ satisfying (2.4.1) is given, then by analogy with (3.4.6) we can define

$$\mathcal{X}(y;\Lambda) \triangleq E\int_0^T \zeta(s)\psi(s)\ I(s,\ y\zeta(s)\psi(s);\Lambda)\ ds.$$

The assertions of Remark 3.4.3 are valid for $\mathcal{X}(\cdot\,;\Lambda)$, and the inverse $\mathcal{Y}(\cdot\,;\Lambda):(0,\infty)$ onto $(0,\infty)$ is continuous and strictly decreasing.

We have now that the representative agent receives the aggregate income process $\hat{c}(\cdot)$ defined in (2.2.3) and attempts to maximize his total expected utility $E\int_0^T U(t,\ c(t))\,dt$ from consumption, subject to the constraint: $E\int_0^T \zeta(s)\psi(s)c(s)\,ds \le \xi$, where

$$\xi = E\int_0^T \zeta(s)\psi(s)\hat{c}(s)\ ds. \quad (4.1.10)$$

Now with

$$\eta(\Lambda) = \mathcal{Y}(\xi;\Lambda), \quad (4.1.11)$$

the *optimal consumption process* for the representative agent is given by:

$$c^*(t;\Lambda) = I(t, \eta(\Lambda)\zeta(t)\psi(t);\ \Lambda),\ \ 0 \le t \le T. \quad (4.1.12)$$

which is the analogue of (3.4.11).

The representative agent after he computes c*(t; Λ), rather than consuming the commodity himself, he *parcels out* this consumption to the J individual agents according to the formula (4.1.5):

$$\bar{c}_j(t;\Lambda) = I_j(t, \frac{1}{\lambda_j} U'(t,\ c^*(t;\Lambda);\Lambda)),\ \ 0 \le t \le T. \quad (4.1.13)$$

Each agent j will be satisfied with this arrangement if $\bar{c}_j(t;\Lambda)$ agrees with his optimal consumption process $c_j^*(t)$ given by (3.4.11). This agreement will in fact occur from (3.4.11) and (4.1.13), given that

$$\eta_j\zeta(t)\psi(t) = \frac{1}{\lambda_j}\, U'\left(t,\, c^*(t;\Lambda);\Lambda\right),\ \ 0 \le t \le T, \quad (4.1.14)$$

and under this condition we are about to have

$$\sum_{j=1}^{J} c_j^*(t) = \sum_{j=1}^{J} \bar{c}_j(t;\Lambda) = I\left(t, U'\left(t, c^*(t;\Lambda);\Lambda\right)\right) = c^*(t;\Lambda),\ \ 0 \le t \le T.$$

It derives from (3.2.1) that a necessary condition for the *existence of equilibrium* in either the moneyed or moneyless model is

$$\hat{c}(t) = c^*(t;\Lambda),\ 0 \le t \le T,\ \text{almost surely}, \quad (4.1.15)$$

in terms of which (4.1.14) becomes

$$\zeta(t)\psi(t) = \frac{1}{\lambda_j\eta_j} U'(t, \hat{c}(t);\Lambda),\ \ 0 \le t \le T. \quad (4.1.16)$$

The equation (4.1.16) does not provide a direct formula for the deflated spot price process $\zeta\,\psi$, because the number η_j on the right-hand side depends on $\zeta\,\psi$ (see (3.4.9) and (3.4.10)) and because the vector Λ has not yet been determined.

Theorem 4.1.2 (Sufficient Conditions for Equilibrium)

Let $\Lambda = (\lambda_1,\ldots,\lambda_J) \in (0,\infty)^J$ be given, and define a spot price process $\psi(\cdot\,;\Lambda)$ by

$$\psi(t;\Lambda) = \frac{1}{\zeta(t)} U'(t, \hat{c}(t);\Lambda),\quad 0 \le t \le T\,. \quad (4.1.17)$$

Using this spot price process, for each j define $\eta_j(\Lambda)$ by (3.4.10) and $c_j^*(\cdot\,,\Lambda)$ by (3.4.11). If the vector Λ satisfies

$$\lambda_j\ \eta_j(\Lambda) = 1,\ \ \forall\ j = 1,\ldots,J, \ \ (4.1.18)$$

then

- ❖ the spot price process $\psi(\cdot\,;\Lambda)$,
- ❖ the corresponding vector of productive assets given by (3.3.6),
- ❖ the consumption processes given by

$$c_j^*(t;\Lambda) = I_j(t;\ \eta_j(\Lambda)\ U'(t, \hat{c}(t);\Lambda)),\ \ 0 \le t \le T,\quad j = 1,\ldots,J, \quad (4.1.19)$$

- the productive asset portfolio processes $\pi_j^* \equiv \varepsilon_j,\ j = 1,\dots,J$, and
- the corresponding financial asset portfolio processes $\Phi_j^*,\ j = 1,\dots,J$, given as in Theorem 3.4.2,

constitute *an equilibrium for the moneyed model.*

Proof

By assumption, $\eta_j(\Lambda)$ is the unique positive number η for which

$$E\int_0^T U'(t,\hat{c}(t);\Lambda)\, I_j(t;\eta U'(t,\hat{c}(t);\Lambda)dt = E\int_0^T U'(t,\hat{c}(t);\Lambda)\hat{c}_j(t) \text{ holds.} \quad (4.1.20)$$

In order to prove (4.1.20) we use consecutively the equations (4.1.17), (3.4.9), (3.4.10), (3.4.6), (4.1.17) and we have

$$E\int_0^T U'(t,\hat{c}(t);\Lambda)\hat{c}_j(t)\,dt = E\int_0^T \zeta(t)\,\psi(t;\Lambda)\,\hat{c}_j(t)\,dt = \xi_j(\Lambda) = \mathcal{X}_j(\eta_j(\Lambda);\Lambda) =$$

$$E\int_0^T \zeta(t)\psi(t;\Lambda)\, I_j(t;\eta_j(\Lambda)\zeta(t)\psi(t;\Lambda))dt = E\int_0^T U'(t,\hat{c}(t);\Lambda)\, I_j(t;\eta_j(\Lambda)U'(t,\hat{c}(t);\Lambda))\,dt.$$

Because of (2.2.4) and Lemma 4.1.1, $\zeta(\cdot)\psi(\cdot\,;\Lambda)$ satisfies (2.4.1). For each j, the optimality of the (c_j^*,π_j^*,Φ_j^*) follows from Theorems 3.4.2 and 3.4.4.

It remains to verify the market clearing conditions (3.2.1)-(3.2.3). From (2.2.1) we have (3.2.2). As for (3.2.1), we note from (4.1.4), (4.1.17) - (4.1.19) that

$$\sum_{j=1}^{J} c_j^*(t;\Lambda) = \sum_{j=1}^{J} I_j(t,\ \eta_j(\Lambda))\zeta(t)\psi(t;\Lambda)) = \sum_{j=1}^{J} I_j(t, \frac{1}{\lambda_j} U'(t,\hat{c}(t;\Lambda))$$

$$= I(t, U'(t,\hat{c}(t);\Lambda);\Lambda) = \hat{c}(t),\quad 0 \le t \le T,\ \text{a.s.} \qquad (4.1.21)$$

Since (3.5.1) is satisfied, Theorem 3.5.1 concludes the proof of condition (3.2.3). ■

Theorem 4.1.3 (Necessary Conditions for Equilibrium)

Let $[\psi, (P_1,\dots, P_M), \{(c_j^*,\pi_j^*,\Phi_j^*) : j = 1,.., J\}]$ be an *equilibrium for the moneyed model.* For each j, let η_j be defined by (3.4.10) and set $\Lambda=(1/\eta_1,.., 1/\eta_J)$. Then

$$\psi(t) = \frac{1}{\zeta(t)} U'(t,\hat{c}(t);\Lambda),\quad 0 \le t \le T.$$

Proof

From the equilibrium condition (3.2.1), (3.4.11) and (4.1.4) we have

$$\hat{c}(t) = \sum_{j=1}^{J} c_j^*(t) = \sum_{j=1}^{J} I_j(t, \eta_j \zeta(t)\psi(t)) = I(t, \zeta(t)\psi(t); \Lambda)$$

and thus from (4.1.9) (recalling from (2.2.4) that $\hat{c}(t) > 0$), we conclude that

$$U'(t, \hat{c}(t); \Lambda)) = \zeta(t)\psi(t), \quad 0 \le t \le T.$$

Moneyless model

Corollary 4.1.4

Let $[(f_0,..., f_N),(P_1,..., P_M), \{(c_j^*, \pi_j^*, \Phi_j^*) : j = 1,..., J\}]$ be an equilibrium for the *moneyless model*. For each j, let η_j be defined by (3.4.10) with $\psi \equiv 1$, and set $\Lambda = (1/\eta_1, \ldots\ldots, 1/\eta_J)$. Then

$$\zeta(t) = U'(t, \hat{c}(t); \Lambda), \quad 0 \le t \le T. \quad (4.1.22)$$

4.2. Existence and uniqueness of equilibrium

In Theorem 4.2.1 we state the existence and uniqueness of a fixed point for a certain operator from $(0,\infty)^J$ into itself, and we show how all the properties we desire for equilibrium in the moneyed model flow from this theorem. In this section, we also provide conditions which guarantee its existence.

Theorem 4.2.1

There exists $\Lambda \in (0,\infty)^J$ such that $\eta_j(\Lambda)$, j = 1,..., J, defined as in Theorem 4.1.2, satisfy (4.1.18). If Λ and $\tilde{\Lambda}$ are both elements of $(0,\infty)^J$ with this property, if (2.2.7) holds, then $\Lambda = \gamma \tilde{\Lambda}$ *f*or some $\gamma > 0$.

Corollary 4.2.2

There exists an *equilibrium in the moneyed model*. If (2.2.7) holds, equilibrium in the moneyed model *is unique*.

Proof

Existence follows from Theorems 4.1.2 and 4.2.1. For uniqueness, suppose that $[\psi,(P_1,\ldots,P_M),\{(c_j^*,\pi_j^*,\Phi_j^*): j=1,\ldots,J\}]$ *and* $[\tilde{\psi},(\tilde{P}_1,\ldots,\tilde{P}_M),\{(\tilde{c}_j^*,\tilde{\pi}_j^*,\tilde{\Phi}_j^*): j=1,\ldots,J\}]$ are both equilibria. According to Theorem 4.1.3, there exist $\Lambda,\tilde{\Lambda}\in(0,\infty)^J$ such that $\eta_j(\Lambda)$ and $\eta_j(\tilde{\Lambda})$, j= 1,…..,J satisfy their respective versions of (4.1.17) and

$$\psi(t)=\frac{1}{\zeta(t)}U'(t,\hat{c}(t);\Lambda),\quad \tilde{\psi}(t)=\frac{1}{\zeta(t)}U'(t,\hat{c}(t);\tilde{\Lambda}),\ 0\le t\le T,\ \text{a.s.}$$

Theorem 4.2.1 implies $\Lambda=\gamma\tilde{\Lambda}$ for some $\gamma>0$, so $\psi=\gamma\tilde{\psi}$. Theorem 3.3.2 gives then $P_m=\gamma\tilde{P}_m,\ m=1,\ldots,M.$ ■

Corollary 4.2.3

Assume (2.2.7) .Suppose also that

- $[\psi,(P_1,\ldots,P_M),\{(c_j^*,\pi_j^*,\Phi_j^*): j=1,\ldots,J\}]$ is an equilibrium for the moneyed model and
- $[\tilde{\psi},(\tilde{P}_1,\ldots,\tilde{P}_M),\{(\tilde{c}_j^*,\tilde{\pi}_j^*,\tilde{\Phi}_j^*): j=1,\ldots,J\}]$ is an equilibrium for another moneyed model *which differs from the first* only in the choice of the coefficients of the financial assets $r(\cdot)$, $b(\cdot)$ and $\alpha(\cdot)$.

Let ζ be the deflator defined by (2.3.12) for the first model and let $\tilde{\zeta}$ be the analogously defined deflator for the second model. Then for Lebesgue x P almost every (t, ω), we have,

$$\zeta(t)\psi(t)=\gamma\tilde{\zeta}(t)\tilde{\psi}(t),\quad c_j^*(t)=\tilde{c}_j^*(t),\quad j=1,\ldots,J,\ \text{for some }\gamma>0.\quad (4.2.1)$$

Proof

For $\Lambda\in(0,\infty)^J$ and $j\in\{1,\ldots,J\}$, let $\eta_j(\Lambda)$ be the unique positive number η satisfying (4.1.20). The mapping $\eta_j:(0,\infty)^J\to(0,\infty)$ depends on the model primitives of section 2.2, but not on the financial assets. According to Theorem 4.1.3, there exist $\Lambda,\tilde{\Lambda}\in(0,\infty)^J$ such that

$$\zeta(t)\psi(t) = U'(t,\hat{c}(t);\Lambda),\quad \tilde{\zeta}(t)\tilde{\psi}(t) = U'(t,\hat{c}(t);\tilde{\Lambda}),\quad 0 \le t \le T. \quad (4.2.2)$$

Indeed, by comparing (4.2.2) and (4.1.16), we conclude that these vectors

$\Lambda = (\lambda_1 ...,\lambda_J)$ and $\tilde{\Lambda} = (\tilde{\lambda}_1,.., \tilde{\lambda}_J)$ satisfy

$$\lambda_j\eta_j(\Lambda)=\tilde{\lambda}_j\, \eta_j(\tilde{\Lambda})= 1,\ j=1,\dots,J\ ,$$

and so Theorem 4.2.1 asserts the existence of $\gamma > 0$ such that $\Lambda = \gamma\tilde{\Lambda}$. It follows from (4.2.2) that $\zeta\psi=\gamma\tilde{\zeta}\tilde{\psi}$. Furthermore, the unique (by Theorem 3.4.4) optimal consumption processes are given by (4.1.19), and satisfy

$$c_j^*(t)=I_j(t,n_j(\Lambda)\zeta(t)\psi(t))= I_j(t,n_j(\tilde{\Lambda})\tilde{\zeta}(t)\tilde{\psi}(t))=\tilde{c}_j^*(t),\ \ 0 \le t \le T.$$ ■

Moneyless model

Now, we discuss about the existence of equilibrium in the *moneyless model.* The necessary condition (4.1.22) involves the Itô process ζ, and this suggests that the aggregate income process appearing on the right-hand side of (4.1.22) should also be an Itô process. The proof of the existence of equilibrium in the *moneyless model* imposes assumptions not required by the existence part of Corollary 4.2.2.

Theorem 4.2.4

Assume that for j=1,….,J there are (i) bounded, (ii) measurable, (iii) $\{\mathcal{F}(t)\}$-adapted processes $\mu_j(\cdot)$ and $\rho_j(\cdot)$ taking values in $\Re$ and $\Re^N$ respectively, such that:

$$\hat{c}_j(t) = \hat{c}_j(0) + \int_0^t \mu_j(s)ds + \int_0^t \rho_j^{\dagger}(s)\, dW(s),\ \ 0 \le t \le T,$$

where $\hat{c}_j(0)$ is a deterministic, nonnegative constant. We also define that

$$\hat{c}(t) = \sum_{j=1}^{J}\hat{c}_j(t),\ \ \mu(t) = \sum_{j=1}^{J}\mu_j(t),\quad \rho(t) = \sum_{j=1}^{J}\rho_j(t),\quad 0 \le t \le T;$$ in other words

there are (i) bounded, (ii) measurable, (iii) $\{\mathcal{F}(t)\}$-adapted processes $\{\mu(t); 0 \le t \le T\}$ and $\{\rho(t); 0 \le t \le T\}$, taking values in $\Re$ and $\Re^N$ respectively, such that:

$$\hat{c}(t) = \hat{c}(0) + \int_0^t \mu(s)ds + \int_0^t \rho^{\dagger}(s)\, dW(s),\ \ 0 \le t \le T,\ \text{a.s.} \quad (4.2.3)$$

The processes $\mu(\cdot)$ and $\rho(\cdot)$ are assumed to be such that (2.2.4) holds, and for j=1,...,J the derivatives $(\partial^2/\partial t\partial c)U_j, (\partial^2/\partial c^2)U_j$ *and* $(\partial^3/\partial c^3)U_j$ exist and are continuous on [0, T] x $(0,\infty)$.Then there exists an equilibrium for the *moneyless model.*

Proof

For each j:

- $I_j(t,\cdot)$ is a strictly decreasing function from $(0,\infty)$ onto $(0,\infty)$ and
- $U_j'(t, I_j(t, y)) = y, \ \forall \ (t, y) \in [0, T] \times (0,\infty)$.

It follows from our assumptions that $\partial I_j / \partial t$, $\partial I_j / \partial y$ and $\partial I_j / \partial y^2$ are continuous on $\left[0,T\right] \times (0,\infty)$. As a result, for each $\Lambda \in (0,\infty)^J$: $I(t,\cdot;\Lambda)$ defined by (4.1.4) is a strictly decreasing function from $(0,\infty)$ onto $(0,\infty)$, and $U'(t, I(t, y; \Lambda); \Lambda) = y \ \forall (t, y) \in \left[0,T\right] \times (0,\infty)$. Furthermore, $\partial I(t,y;\Lambda)/\partial t$, $\partial I(t,y;\Lambda)/\partial y$ and $\partial^2 I(t, y; \Lambda)/\partial y^2$ are continuous, so

$$U_t'(t,c;\Lambda) = \frac{\partial^2}{\partial t\partial c}U(t,c;\Lambda),\ U''(t,c;\Lambda) = \frac{\partial^2}{\partial c^2}U(t,c;\Lambda),\ U'''(t,c;\Lambda) = \frac{\partial^3}{\partial c^3}U(t,c;\Lambda)$$

are continuous on $\left[0,T\right] \times (0,\infty)$. In particular, for any $\Lambda \in (0,\infty)^J$, (4.2.3) and Itô's lemma imply that

$$dU'(t,\hat{c}(t);\Lambda) = \left[U_t'(t,\hat{c}(t);\Lambda) + \mu(t)U''(t,c;\Lambda) + \frac{1}{2}\|\rho(t)\|^2 U'''(t,c;\Lambda)\right]dt + U'''(t,c;\Lambda)\rho^\dagger(t)dW(t). \quad (4.2.4)$$

We now choose the proper Λ. A restatement of the existence part of Theorem 4.2.1 is that there exists $\Lambda = (\lambda_1,...,\lambda_J) \in (0,\infty)^J$ for which equation (4.1.20) holds with $\eta = 1/\lambda_j$, $j = 1,...,J$. But $U'(t, c; \Lambda)$ is positive homogeneous in Λ, so we can choose this Λ to also satisfy the normalization condition

$$U'(0,\hat{c}(0);\Lambda) = 1. \quad (4.2.5)$$

With this choice of Λ, define the *bounded processes*

$$r(t) \stackrel{\Delta}{=} -\frac{1}{U'(t,\hat{c}(t);\Lambda)}\left[U_t'(t,\hat{c}(t);\Lambda) + \mu(t)U''(t,\hat{c}(t);\Lambda) + \frac{1}{2}\|\rho(t)\|^2 U'''(t,\hat{c}(t);\Lambda)\right], \quad (4.2.6)$$

$$\theta(t) \stackrel{\Delta}{=} -\frac{1}{U'(t,\hat{c}(t);\Lambda)}U''(t,\hat{c}(t);\Lambda)\rho(t). \quad (4.2.7)$$

From (4.2.4) , (4.2.6) and (4.2.7) we have:

$$dU'(t,\hat{c}(t);\Lambda) = U'(t,\hat{c}(t);\Lambda)\left[-r(t)dt - \theta^{\dagger}(t)dW(t)\right],$$

which combined with (4.2.7), gives

$$U'(t,\hat{c}(t);\Lambda) = e^{-\int_0^t r(s)\,ds - \int_0^t \theta^{\dagger}(s)\,dW(s) - \frac{1}{2}\int_0^t ||\,\theta(s)||^2 ds}.$$

If the interest rate r is given by (4.2.6) and the mean rates of return $b(\cdot)$ and the volatility matrix $\alpha(\cdot)$ are chosen so that $\alpha^{-1}(t)\ \left(b(t) - r(t)\ \mathbf{1}_N\right)$ agrees with θ(t) defined by (4.2.7), then the deflator process ζ(t) of (2.3.12) agrees with $U'(t,\hat{c}(t);\Lambda)$.
We choose such a $b(\cdot)$ and $\alpha(\cdot)$ and we define the corresponding financial asset prices $(f_0,..., f_N)$ by (2.3.4), (2.3.5).

Having defined $(f_0,..., f_N)$, we can consider these prices as exogenous and seek equilibrium in the moneyed model. Theorem 4.1.2 implies its existence with $\psi \equiv 1$. But this leads immediately to equilibrium in the *moneyless model*. ■

Remark 4.2.5

The construction of equilibrium in the proof of Theorem 4.2.4 provides the formulas (4.2.6) and (4.2.7) for the interest rate $r(\cdot)$ and the relative risk $\theta(\cdot) = \alpha^{-1}(\cdot)\left(b(\cdot) - r(\cdot)\ \mathbf{1}_N\right)$ but not for $\alpha(\cdot)$ and $b(\cdot)$ separately. The necessary condition of Corollary 4.1.4 shows that as long as equilibrium in the moneyless model exists, these formulas must hold. These formulas have been obtained by Duffie and Zame [7] and Cox, Ingresoll and Ross [3].

4.3 Proof of Existence.

We will demonstrate in this section that there exists $\Lambda = (\lambda_1,..., \lambda_J) \in (0,\infty)^J$ such that the numbers $\eta_1(\Lambda),..., \eta_J(\Lambda)$ determined by (4.1.20) satisfy $\lambda_J\eta_J(\Lambda) = 1$. In other words, we find $\Lambda \in (0,\infty)^J$ such that

$$E\int_0^T U'(t,\hat{c}(t);\Lambda)\ [I_j(t,\frac{1}{\lambda_j}U'(t,\hat{c}(t);\Lambda) - \hat{c}_j(t)]dt = 0,\ \forall\ j = 1,\dots,J. \quad (4.3.1)$$

This is the existence part of Theorem 4.2.1.

Let $e^{(1)},..., e^{(n)}$ denote the elementary vectors of $\Re^J$, and let A = {1,..., J}. Suppose B $\subseteq$ A, then Φ_B denotes the convex hull of the elementary vectors

{$e^{(i)}$; i $\in$ B}, i.e., $\Phi_B = \left\{\sum_{i\in B}\lambda_i e^{(i)}; \lambda_i \geq 0 \ \forall \text{ i, and } \sum_{i\in B}\lambda_i = 1\right\}$. We define

$$\Phi_A^+ = \left\{\sum_{i\in A}\lambda_i e^{(i)}; \lambda_i > 0 \ \forall \text{i, and } \sum_{i\in A}\lambda_i = 1\right\}.$$

We define for $\Lambda \in \Phi_A^+$ and $j = 1,..., J$

$$R_j(\Lambda) = E\int_0^T U'(t,\hat{c}(t);\Lambda)\ [I_j(t,\frac{1}{\lambda_j}U'(t,\hat{c}(t);\Lambda) - \hat{c}_j(t)]\ dt, \text{ if } \lambda_j > 0. \quad (4.3.2)$$

We can extend R_j to Φ_A by continuity. Indeed, one merely needs to adopt the convention that for every $y \in (0,\infty)$, $I_j(t,\ y/\lambda_j) = 0$ if $\lambda_j = 0$.

Then (4.1.4) defines a continuous function $I(t, y;\cdot)$ on Φ_A , and $U(t,c;\cdot)$ defined on Φ_A by (4.1.1) still satisfies (4.1.9). It follows that $U'(t,c;\cdot)$ is continuous on Φ_A , as is R_j defined above. Note, that since the origin is not in Φ_A , for each $\Lambda \in \Phi_A$, the function $U(t,\cdot\ ;\Lambda)$ has all the properties set out in Lemma 4.1.1. In particular, $U'(t,c;\Lambda) > 0$ for every $t \in [0,T]$ and $c \geq 0$. It follows that if $\Lambda \in \Phi_A$ and one of the components of Λ, say the j^{th}, is zero, then

$$R_j(\Lambda) = -E\int_0^T U'(t,\hat{c}(t);\Lambda)\hat{c}_j(t)\ dt < 0, \text{ if } \lambda_j = 0. \quad (4.3.3)$$

We now prove the existence of a solution in $(0,\infty)^J$ to (4.3.1).

Lemma 4.3.1

There exists a vector Λ* in Φ_A^+ such that $R_j(\Lambda^*) = 0$ for each $j = 1,..., J$.

Proof

For j = 1,...,J, define the closed set $F_j \overset{\Delta}{=} \left\{\Lambda \in \Phi_A;\ R_j(\Lambda) \geq 0\right\}$. From (4.1.4) and (4.1.9), we see that

$$\sum_{j=1}^{J} R_j(\Lambda) = 0, \ \forall\ \Lambda \in \Phi_A. \quad (4.3.4)$$

Therefore, each Λ in Φ_A must be in at least one F_j, for otherwise this sum would be strictly negative. Suppose there were a Λ in Φ_A which was not in $\bigcup_{j\in A} F_j$. This would imply $R_j(\Lambda) < 0$ and thus $\sum_{j=1}^{J} R_j(\Lambda) < 0$, a contradiction to (4.3.4). Similarly, if $B \subseteq A$ and $\Lambda \in \Phi_B$ then Λ must be in $\bigcup_{j\in B} F_j$, for otherwise (4.3.3) would imply that $\sum_{j=1}^{J} R_j(\Lambda) < 0$.

By the Knaster- Kuratowski-Mazurkiewicz Theorem [1,pg.26] $\bigcap_{j\in B} F_j$ is nonempty. Choose Λ^* in this intersection. Then $R_j(\Lambda^*) \geq 0$ for every j, but because of (4.3.4) we must in fact have $R_j(\Lambda^*) = 0$ for $j = 1,...,J$. In light of (4.3.3), Λ^* must be in $\Phi_A^+ \subseteq (0,\infty)^J$. ■

4.4. Proof of uniqueness.

We next turn our attention to the question of uniqueness. In order to prove the uniqueness of equilibrium, we shall assume in this section that the additional condition of (2.2.7) holds for each U_j. This condition (2.2.7) is equivalent to the assumption

$$h_j(t,y) \overset{\Delta}{=} y\, I_j(t,y) \text{ is non-increasing in y.} \quad (4.4.1)$$

To prove the direct implication we consider $y_1, y_2 \in (0,\infty)$ such that

$$\begin{aligned} I(t,y_1) > I(t,y_2) &\Rightarrow I(t,y_1)\, U_j'(I_j(t,y_1)) \geq I(t,y_2)\, U_j'(I_j(t,y_2)) \\ &\Rightarrow y_1 I(t,y_1) \geq y_2 I(t,y_2) \end{aligned}$$

where we have used the decreasing monotonicity of I and (2.2.7). The opposite direction is proved similarly.

The equation (4.4.1) leads to the following uniqueness result.

Theorem 4.4.1

Assume that for every $t \in [0, T]$, the function $U_j(t,\cdot) : (0,\infty) \to \Re$ is twice continuously differentiable, strictly increasing, strictly concave, and U_j satisfies the conditions (2.2.5)-(2.2.9). Then the solution $\Lambda \in (0,\infty)$ of (4.1.20) is unique up to multiplication by a positive constant.

Proof

We introduce the usual partial order in $(0,\infty)^J : \Lambda \le M$ if and only if $\lambda_j \le \mu_j, \forall j \in \{1,...,J\}$. We write $\Lambda < M$ if $\Lambda \le M$ and $\Lambda \ne M$. In particular, notice in (4.1.4) the implications

$$\Lambda \le M \Rightarrow I(t,y,\Lambda) \le I(t,y,M) \ \forall \ (t,y) \in [0,T] \times (0,\infty). \quad (4.4.2)$$

Let Λ and $\tilde{\Lambda}$ be two solutions of (4.1.20) and define

$$\gamma \overset{\Delta}{=} \max_{1 \le j \le J} \frac{\lambda_j}{\tilde{\lambda}_j} \text{ and } M = (\mu_1, \ ..., \mu_J) \overset{\Delta}{=} \gamma \tilde{\Lambda},$$

so M is a solution of (4.1.20) and $\Lambda \le M$. If $\Lambda = M$, then $\tilde{\Lambda}$ is indeed a positive multiple of Λ. Therefore, it suffices to rule out the case $\Lambda < M$.

Suppose that $\Lambda < M$. From (4.4.2) we obtain

$$U'(t, \hat{c}(t);\Lambda) < U'(t, \hat{c}(t);M) \ \forall (t,\omega) \in \ [0,T] \times \Omega. \quad (4.4.3)$$

Choose an integer $j \in \{1,....,J\}$ satisfying $\lambda_j = \eta \tilde{\lambda}_j$ (and $\lambda_j = \mu_j$ hence also).

We have

$$E\int_0^T h_j(t, \frac{1}{\lambda_j} U'(t, \hat{c}(t);\Lambda) \, dt \ \ge E\int_0^T h_j(t, \frac{1}{\mu_j} U'(t, \hat{c}(t);M) \, dt,$$

$$E\int_0^T \frac{1}{\lambda_j} U'(t, \hat{c}(t);\Lambda) \ \hat{c}_j(t) \, dt \ < \ E\int_0^T \frac{1}{\lambda_j} U'(t, \hat{c}(t);M) \ \hat{c}_j(t) \, dt,$$

where h_j is given by (4.4.1). Taking the difference of these two relations, we obtain

$$\frac{1}{\lambda_j} R_j(\Lambda) > \frac{1}{\mu_j} R_j(M).$$

But Λ and M both solve (4.1.20), so $R_j(\Lambda) = R_j(M) = 0$, and a contradiction is obtained. Thus, we obtain that $\Lambda = \gamma \tilde{\Lambda}$. ■

Chapter 5: Applications

5.1 Examples.

We provide three examples in which equilibrium in both models can be computed clearly. In every example, firstly we deal with the moneyed model and afterwards with the moneyless model. We also provide one special case in which the computation of closed form solutions to the equilibrium problem can be done, even if agents have different utility functions.

Example 5.1.1 (Logarithmic utility functions)

If the number of agents J is arbitrary, and each agent has the same time-independent utility function U_j (t, c) = log c, then I_j (t, y) = 1/y, and the optimal consumption process given by (3.4.11) is

$$c_j^*(t) = \frac{1}{\eta_j \zeta(t)\psi(t)}, \ 0 \le t \le T,$$

where η_j is chosen so that (see (3.4.6) and (3.4.9))

$$\mathcal{X}_j(\eta_j) = \frac{T}{\eta_j} = \xi_j := E\int_0^T \zeta(t)\psi(t)\hat{c}_j(t)dt.$$

Equivalently,

$$c_j^*(t) = \frac{\xi_j}{T\zeta(t)\psi(t)}, \ 0 \le t \le T. \quad (5.1.1)$$

This expression could be compared with (2.1.4). According to (3.2.1), equilibrium in the moneyed model requires

$$\frac{1}{T\zeta(t)\psi(t)}\sum_{j=1}^{J}\xi_j = \sum_{j=1}^{J} c_j^*(t) = \hat{c}(t), \ 0 \le t \le T,$$

which leads to

$$E\int_0^T \zeta(s)\psi(s)\hat{c}(s)ds = T\,\zeta(t)\psi(t)\hat{c}(t),$$

and from which we draw the conclusion that there is a constant γ > 0 such that

$$\gamma = \zeta(t)\,\psi(t)\,\hat{c}(t), \ 0 \le t \le T. \quad (5.1.2)$$

This expression could be compared with (2.1.5). Replacement of (5.1.2) into (5.1.1) results in

$$c_j^*(t) = \lambda_j \hat{c}(t), \ 0 \le t \le T, \quad (5.1.3)$$

where from (5.1.2) also follows that:

$$\lambda_j = \frac{\xi_j}{T\gamma} = \frac{E\int_0^T \zeta(t)\psi(t)\hat{c}_j(t)dt}{T\gamma} = \frac{E\int_0^T \frac{\gamma}{\hat{c}(t)}\hat{c}_j(t)dt}{T\gamma} = \frac{1}{T}E\int_0^T \frac{\hat{c}_j(t)}{\hat{c}(t)}. \quad (5.1.4)$$

The expressions (5.1.3) and (5.1.4) could be compared with (2.1.6) and (2.1.7). The vector $\Lambda = (\lambda_1,..., \lambda_J)$ is a fixed point of L defined by (4.3.1), (4.3.2), (4.3.4). From (4.1.4) and (5.1.4) follows that

$$I(t, y; \Lambda) = \sum_{j=1}^{J} I_j(t, \frac{y}{\lambda_j}) = \frac{\sum_{j=1}^{J}\lambda_j}{y} = \frac{1}{y}, \quad y>0,$$

$$U'(c;\Lambda) = I^{-1}(c;\Lambda) = \frac{1}{c}, \quad c>0.$$

Thus from (4.3.2)

$$R_j(\Lambda) = E\int_0^T \frac{1}{\hat{c}(t)}\left[\lambda_j\hat{c}(t) - \hat{c}_j(t)\right]dt = \lambda_j T - E\int_0^T \frac{\hat{c}_j(t)}{\hat{c}(t)}dt = 0, \; j=1,\ldots,J.$$

Moneyless model

In the *moneyless model*, we suppose that $\hat{c}(\cdot)$ is given by (4.2.3). From (4.2.6) and (4.2.7), in conjunction with

$$U'(t,\hat{c}(t);\Lambda) = \frac{1}{\hat{c}(t)}, \quad U''(t,\hat{c}(t);\Lambda) = -\frac{1}{\hat{c}^2(t)}, \quad U'''(t,\hat{c}(t);\Lambda) = \frac{2}{\hat{c}^3(t)}.$$

we get that

$$r(t) = \frac{\mu(t)}{\hat{c}(t)} + \frac{||\rho(t)||^2}{\hat{c}^2(t)}, \quad \text{a.s, } 0 \le t \le T \text{ and}$$

$$\theta(t) = \frac{\rho(t)}{\hat{c}(t)}, \quad \text{a.s, } 0 \le t \le T,$$

respectively.

Example 5.1.2 (Power utility functions)

Let $\delta \in (0,1)$ be given, and let each agent have the utility function $U_j(t,c) = c^{\delta}$.

Then $U_j'(t,c) = \delta c^{\delta-1}$. Thus, $I_j(t,y) = (y/\delta)^{1/(\delta-1)}$ and the optimal consumption process given by (3.4.11) is

$$c_j^*(t) = \left[\frac{\eta_j}{\delta}\zeta(t)\psi(t)\right]^{1/(\delta-1)}, \quad 0 \le t \le T,$$

where η_j is chosen so that (see (3.4.6) and (3.4.9))

$$\mathcal{X}_j(\eta_j) = \left[\frac{\eta_j}{\delta}\right]^{1/(\delta-1)} E\int_0^T [\zeta(t)\psi(t)]^{\delta/(\delta-1)} dt = \xi_j := E\int_0^T \zeta(t)\psi(t)\hat{c}_j(t)dt.$$

In other words,

$$c_j^*(t) = \left[E\int_0^T [\zeta(t)\psi(t)]^{\delta/\delta-1} dt\right]^{-1} \xi_j [\zeta(t)\psi(t)]^{1/(\delta-1)}, \quad 0 \le t \le T. \quad (5.1.5)$$

Due to (3.2.1), equilibrium in the *moneyed model* requires

$$\sum_{j=1}^J c_j^*(t) = \left[E\int_0^T [\zeta(t)\psi(t)]^{\delta/(\delta-1)} dt\right]^{-1} \left(\sum_{j=1}^J \xi_j\right) [\zeta(t)\psi(t)]^{1/(\delta-1)} = \hat{c}(t), \quad 0 \le t \le T,$$

from which we draw the conclusion that there is a constant $\gamma > 0$ such that

$$[\zeta(t)\psi(t)]^{1/(1-\delta)}\hat{c}(t) = \gamma, \quad 0 \le t \le T. \quad (5.1.6)$$

Replacement of (5.1.6) into (5.1.5) gives us

$$c_j^*(t) = \lambda_j^{1/(1-\delta)}\hat{c}(t), \quad 0 \le t \le T, \qquad (5.1.7)$$

$$\text{where } \lambda_j = \left[\frac{E\int_0^T \hat{c}^{\delta-1}(t)\hat{c}_j(t)dt}{E\int_0^T \hat{c}^{\delta}(t)dt}\right]^{1-\delta}. \quad (5.1.8)$$

In case we set $\delta = 0$ formulas (5.1.2)-(5.1.4) are obtained in (5.1.6)-(5.1.8). The vector $\Lambda = (\lambda_1,\ldots, \lambda_J)$ is a fixed point of L defined by (4.3.1),(4.3.2) and (4.3.4).

From (4.1.4)

$$I(t, y; \Lambda) = \sum_{j=1}^J I_j\left(t, \frac{y}{\lambda_j}\right) = \sum_{j=1}^J \left(\frac{y}{\lambda_j\delta}\right)^{\frac{1}{\delta-1}} = \left(\frac{y}{\delta}\right)^{\frac{1}{\delta-1}} \sum_{j=1}^J \left(\frac{1}{\lambda_j}\right)^{\frac{1}{\delta-1}} = \left(\frac{y}{\delta}\right)^{\frac{1}{\delta-1}} \sum_{j=1}^J \lambda_j^{\frac{1}{1-\delta}} = \left(\frac{y}{\delta}\right)^{\frac{1}{\delta-1}}.$$

which also leads to

$$U'(c;\Lambda) = I^{-1}(c;\Lambda) = \delta c^{\delta-1}, \text{ for } c > 0$$

and thus

$$R_j(\Lambda) = E\int_0^T \delta\hat{c}^{\delta-1}(t)\left[\lambda_j^{1/(1-\delta)}\hat{c}(t) - \hat{c}_j(t)\right] dt = 0, \quad j = 1,\ldots,J.$$

Moneyless model

In the *moneyless model,* we also assume that $\hat{c}$ is given by (4.2.3). We have that

$$U'(t,\hat{c}(t);\Lambda)=\delta\hat{c}(t)^{\delta\text{-}1},\ \ U''(t,\hat{c}(t);\Lambda)=\delta\,(\delta\text{-}1)\ \hat{c}(t)^{\delta\text{-}2},\ U'''(t,\hat{c}(t);\Lambda)=\delta\,(\delta\text{-}1)\,(\delta\text{-}2)\ \hat{c}(t)^{\delta\text{-}3}.$$

Thus, formulas (4.2.6) and (4.2.7) become

$$r(\mathrm{t}) =(1-\delta)\ \frac{\mu(t)}{\hat{c}(\mathrm{t})}-\frac{(1-\delta)\ (2-\delta)\ \left\|\rho(\mathrm{t})\right\|^2}{2\hat{c}^2(\mathrm{t})},\ \ a.s,\ \ 0\le \mathrm{t}\le \mathrm{T}\text{ and}$$

$$\theta(t)=(1-\delta)\ \frac{\rho(\mathrm{t})}{\hat{\mathrm{c}}(\mathrm{t})},\quad a.s,\ \ 0\le \mathrm{t}\le \mathrm{T}.$$

Example 5.1.3 ($U_1(c)$ = log c, $U_2(c) = c^{1/2}$)

If agents have different utility functions, one cannot in general compute closed form solutions to the equilibrium problem. One special case in which this computation can be done is the model with two agents, i.e. J = 2, and

$$\mathrm{U}_1(\mathrm{c})=\log \mathrm{c},\ \ \mathrm{U}_2(\mathrm{c})=\sqrt{c}$$

Then

$$\mathrm{I}_1(\mathrm{y})=1/y\quad\text{and}\quad \mathrm{I}_2(\mathrm{y})=\frac{1}{4y^2}.$$

From (4.1.4)

$$\mathrm{I}(\mathrm{y};\Lambda)=\mathrm{I}_1\left(\frac{\mathrm{y}}{\lambda_1}\right)+\mathrm{I}_2\left(\frac{\mathrm{y}}{\lambda_2}\right)=\frac{\lambda_1}{\mathrm{y}}+\frac{1}{4}\frac{\lambda_2^2}{y^2}.$$

In order to compute the inverse of I we solve for y the equation

$$\mathrm{I}(\mathrm{y};\Lambda)=\mathrm{c} \Leftrightarrow -\mathrm{c}\ \mathrm{y}^2+\lambda_1\ \mathrm{y}+\frac{1}{4}\lambda_2^2\ =\ 0$$

whose positive solution is

$$y=\frac{\lambda_1}{2c}+\frac{\lambda_1}{2c}\left[1+\sqrt{1+c\left(\frac{\lambda_2}{\lambda_1}\right)^2}\,\right].$$

Therefore

$$U'(\hat{c}(t);\lambda_1,\lambda_2)=\frac{\lambda_1}{2\,\hat{c}(t)}\left[1+\sqrt{1+\hat{c}(t)\left(\frac{\lambda_2}{\lambda_1}\right)^2}\,\right]$$

and then the optimal consumption rates are given by (3.4.11) as

$$c_1^*(t) = \mathrm{I}_1\ (t, \eta_1 \zeta(t)\psi(t)) = \mathrm{I}_1\ (t, \frac{1}{\lambda_1} U'(\hat{c}(t); \lambda_1, \lambda_2)) = \frac{\lambda_1}{U'(\hat{c}(t); \lambda_1, \lambda_2)} = \frac{2\hat{c}(t)}{1 + \sqrt{1 + \hat{c}(t)\left(\lambda_2 / \lambda_1\right)^2}}\ ,$$

$$c_2^*(t) = \mathrm{I}_2\ (t, \eta_2 \zeta(t)\psi(t)) = \mathrm{I}_2\ (t, \frac{1}{\lambda_2} U'(\hat{c}(t); \lambda_1, \lambda_2)) = \left[\frac{1}{2} \frac{\lambda_2}{U'(\hat{c}(t); \lambda_1, \lambda_2)}\right]^2 = \left[\frac{\frac{\lambda_2}{\lambda_1}\hat{c}(t)}{1 + \sqrt{1 + \hat{c}(t)(\lambda_2 / \lambda_1)^2}}\right]^2.$$

For j=1 from (4.3.1) we have that

$$\mathrm{E}\int_0^{\mathrm{T}} U'(t, \hat{c}(t); \Lambda) \left[\frac{\lambda_1}{\mathrm{U}'(\mathrm{t}, \hat{c}(\mathrm{t}); \Lambda)} - \hat{c}_1(\mathrm{t})\right] dt = 0 \iff \mathrm{T} - \mathrm{E}\int_0^{\mathrm{T}} \frac{1}{2\hat{c}(t)} \left[1 + \sqrt{1 + \hat{c}(\mathrm{t})\left(\frac{\lambda_2}{\lambda_1}\right)^2}\right] \hat{c}_1(\mathrm{t})\ \mathrm{dt} = 0,$$

which determines λ_2/λ_1.

Then the normalization condition (4.2.5) gives:

$$\lambda_1 = \frac{2\hat{c}(0)}{1 + \sqrt{1 + \hat{c}(0)\left(\frac{\lambda_2}{\lambda_1}\right)^2}},$$

from which now both λ_1 and λ_2 can be found.

Example 5.1.4 (Constant aggregate income)

Let the number of agents J be arbitrary, and we assume that each agent j has his individual, time-independent utility function $U_j(c)$. In addition, assume that there is a positive number $\hat{c}$ such that $P\left[\sum_{j=1}^{J} \hat{c}_j(t) = \hat{c}\right] = 1,\ \ 0 \le \mathrm{t} \le \mathrm{T}$. We show that in the moneyed model the equilibrium deflated spot price $\zeta(t)\ \psi(t)$ is constant, and each agent's optimal equilibrium consumption is constant and equal to

$$\mathrm{c}_{\mathrm{j}}^*(\mathrm{t}) = \frac{1}{\mathrm{T}}\ \mathrm{E}\int_0^{\mathrm{T}} \hat{c}_{\mathrm{j}}(t)dt,\ \ \mathrm{j} = 1, ..., \mathrm{J}. \quad (5.1.9)$$

To do this, we define $\Lambda = (\lambda_1, ..., \lambda_J)$, where

$$\lambda_{\mathrm{j}} = \frac{1}{\mathrm{U}_{\mathrm{j}}'(\mathrm{c}_{\mathrm{j}}^*)}. \quad (5.1.10)$$

Due to (4.1.4),

$$I(1;\Lambda)=\sum_{j=1}^{J} I_j\left(\frac{1}{\lambda_j}\right)=\sum_{j=1}^{J} c_j^* = \hat{c}, \quad j=1,\ldots,J,$$

and thus

$$U'(\hat{c};\Lambda)=1=\lambda_j U_j'(c_j^*).$$

From (4.3.1) we have

$$R_j(\Lambda)=T\, I_j\left(\frac{1}{\lambda_j}\right)-E\int_0^T \hat{c}_j(t)\, dt=0,$$

so Λ is a fixed point of the operator L defined by (4.3.4). With $\eta_j(\Lambda)$, j = 1,..., J, as described in Theorem 4.1.2, relation (4.1.18) holds. It follows from that theorem that $\psi(t) \triangleq 1/\zeta(t)$ is the (unique up to a multiplicative constant) equilibrium spot price and $c_j^* = I_j(1/\lambda_j)$ is the (unique) optimal equilibrium consumption for agent j. Agents' income processes can be random and time-varying, so although their optimal equilibrium consumption processes are constant, they will need nonconstant portfolio processes to finance this consumption.

Moneyless model

In the *moneyless model,* the equilibrium ζ is constant, so r and θ are both identically zero. ■

Example 5.1.5

Let J = 2 and define

$$U_1(t,c)=\begin{cases}\log c & 0\le t\le T/2,\\ \log(c+1) & T/2<t\le T,\end{cases}$$

$$U_2(t,c)=\begin{cases}\log(c+1) & 0\le t\le T/2,\\ \log c & T/2<t\le T,\end{cases}$$

Direct computation reveals that

$$I_1(t,y)=\begin{cases}1/y & 0\le t\le T/2,\ y>0,\\ \left(\frac{1-y}{y}\right)^+ & T/2<t\le T,\ y>0,\end{cases}$$

$$I_2(t,y)=\begin{cases}\left(\frac{1-y}{y}\right)^+ & 0\le t\le T/2,\ y>0,\\ 1/y & T/2<t\le T,\ y>0,\end{cases}$$

$$U'(t,c;\lambda_1,\lambda_2)=\begin{cases} \lambda_1/c & 0\le t\le T/2, \quad 0<c\le \lambda_1/\lambda_2, \\ \dfrac{\lambda_1+\lambda_2}{c+1} & 0\le t\le T/2, \quad c>\lambda_1/\lambda_2, \\ \lambda_2/c & T/2<t\le T, \quad 0<c\le \lambda_2/\lambda_1, \\ \dfrac{\lambda_1+\lambda_2}{c+1} & T/2<t\le T, \quad 0<c\le \lambda_2/\lambda_1, \end{cases}$$

and so if $0<c\le \min\{\lambda_1/\lambda_2, \lambda_2/\lambda_1\}$, we have

$$c_1(t,c;\lambda_1,\lambda_2)=\mathrm{I}_1(\mathrm{t},\frac{1}{\lambda_1}\mathrm{U}'(\mathrm{t,c};\lambda_1,\lambda_2))=\begin{cases} c, & 0\le t\le T/2, \\ 0, & T/2<t\le T, \end{cases}$$

$$c_2(t,c;\lambda_1,\lambda_2)=\mathrm{I}_2(\mathrm{t},\frac{1}{\lambda_2}\mathrm{U}'(\mathrm{t,c};\lambda_1,\lambda_2))=\begin{cases} 0, & 0\le t\le T/2, \\ c, & T/2<t\le T. \end{cases}$$

Now take the income processes to be

$$\hat{c}_1(t)=\begin{cases} 1/2, & 0\le t\le T/2, \\ 0, & T/2<t\le T, \end{cases}$$

$$\hat{c}_2(t)=\frac{1}{2}-\hat{c}_1(t), \qquad 0\le t\le T.$$

If $\lambda_1>0$, $\lambda_2>0$ are chosen to satisfy

$$\min\left\{\frac{\lambda_1}{\lambda_2},\frac{\lambda_2}{\lambda_1}\right\}\ge 1/2, \qquad (5.1.11)$$

then the equilibrium conditions and equation (4.3.1)

$$\int_0^T U'(t,\frac{1}{2},\lambda_1,\lambda_2)\, c_j(t,\frac{1}{2},\lambda_1,\lambda_2)\, dt=\int_0^T U'(t,\frac{1}{2},\lambda_1,\lambda_2)\, \hat{c}_j(t)\, dt, \quad j=1,2, \qquad (5.1.12)$$

are satisfied. In particular, the corresponding equilibrium spot price from (4.1.17) is

$$\psi(t)=\begin{cases} \dfrac{2\lambda_1}{\zeta(t)}, & 0\le t\le \dfrac{T}{2}, \\ \dfrac{2\lambda_2}{\zeta(t)}, & \dfrac{T}{2}<t\le T, \end{cases} \qquad (5.1.13)$$

which is not determined up to a multiplicative constant since the selection of λ_1, λ_2 is feasible up to (5.1.11). In fact, (5.1.12) can be used to show that all the equilibrium spot price processes are given by (5.1.13), where (5.1.11) is satisfied. Consequently, the unique optimal equilibrium consumption processes are

$$c_1^*(t)=\hat{c}_1(t), \quad c_2^*(t)=\hat{c}_2(t), \quad 0\le t\le T.$$

5.2 Numerical Monte Carlo Methods

Application 5.2.1

For the Example 5.1.1, we assume that there are two agents, and each agent has the same time-independent utility function U_j (t, c) = log c. Then I_j (t, y) = 1/y. Thus, we have

$$U_1(t,c) = \log c \text{ and } U_2(t,c) = \log c, \quad (5.2.1)$$

and then

$$I_1(t,y) = 1/y \text{ and } I_2(t,y) = 1/y. \quad (5.2.2)$$

From (4.1.4) and (5.2.2) we have

$$I(t,y;\Lambda) = \sum_{j=1}^{2} I_j(t,\frac{y}{\lambda_j}) = I_1(t,\frac{y}{\lambda_1}) + I_2(t,\frac{y}{\lambda_2}) = \frac{\lambda_1+\lambda_2}{y}. \quad (5.2.3)$$

Thus, we have the inverse of $I(t,y;\Lambda)$ given by

$$U'(t,y;\Lambda) = \frac{\lambda_1+\lambda_2}{y}. \quad (5.2.4)$$

We also have that

$$I_1\left(t, \frac{1}{\lambda_1} U'(t,y;\Lambda)\right) = \frac{\lambda_1}{U'(t,y;\Lambda)} = \frac{\lambda_1 y}{\lambda_1+\lambda_2}, \quad (5.2.5)$$

$$I_2\left(t, \frac{1}{\lambda_2} U'(t,y;\Lambda)\right) = \frac{\lambda_2}{U'(t,y;\Lambda)} = \frac{\lambda_2 y}{\lambda_1+\lambda_2}. \quad (5.2.6)$$

So, we will try to find $\lambda_1, \lambda_2 \in (0,\infty)$ such that equation (4.3.1) for j=1,2 holds, thus from (4.3.2)

$$R_j(\Lambda) = E\int_0^T U'(t,\hat{c}(t);\Lambda)\left[I_j(t,\frac{1}{\lambda_j}U'(t,\hat{c}(t);\Lambda) - \hat{c}_j(t)\right]dt = 0, \quad j=1,2.$$

In other words, we must find those $\lambda_1, \lambda_2 \in (0,\infty)$ in order to have:

$$R_1(\Lambda) = E\int_0^T \frac{\lambda_1+\lambda_2}{\hat{c}(t)}\left(\frac{\lambda_1 \hat{c}(t)}{\lambda_1+\lambda_2} - \hat{c}_1(t)\right)dt = \lambda_1 T - E\int_0^T \frac{\lambda_1+\lambda_2}{\hat{c}(t)}\hat{c}_1(t)dt = 0 \quad (5.2.7)$$

and $$R_2(\Lambda) = E\int_0^T \frac{\lambda_1+\lambda_2}{\hat{c}(t)}\left(\frac{\lambda_2 \hat{c}(t)}{\lambda_1+\lambda_2} - \hat{c}_2(t)\right)dt = \lambda_2 T - E\int_0^T \frac{\lambda_1+\lambda_2}{\hat{c}(t)}\hat{c}_2(t)dt = 0. \quad (5.2.8)$$

STEP 1

First of all, we assume that the income process $\hat{c}_j$ (t) follows Geometric Brownian Motion so we have

$$d\,\hat{c}_j(t) = \mu_j\,\hat{c}_j(t)\,dt + \sigma_j\,\hat{c}_j(t)\,dW(t). \quad (5.2.9)$$

Using Ito's lemma, we may transform (5.2.9) into the following form:

$$d\log \hat{c}_j(t) = (\mu_j - \tfrac{1}{2}\sigma_j)dt + \sigma_j dW(t).$$

Using properties of the lognormal distribution and recalling the properties of the standard Wiener process we obtain

$$d\,\hat{c}_{j_{t+dt}} = \hat{c}_{jt}\, e^{(\mu_j - \frac{\sigma_j^2}{2} dt + \sigma_j \varepsilon \sqrt{dt})}, \quad (5.2.10)$$

where $\varepsilon \sim \mathcal{N}(0,1)$ is a standard normal random variable and dt is the time step.

Based on equation (5.2.10) it is easy to generate sample paths for the income process. A straightforward code to generate sample paths of prices following geometric Brownian motion is given in M-file 1. The M-file IncomePaths yields a matrix of sample paths, where the replications are stored row by row and columns correspond to time instants. The first column contains the same value, the initial price, for all sample paths. For the i agent we have to provide the above function with the initial price of income (C_i), the drift (mu_i), the volatility $(sigma_i)$, the time horizon (T), the number of time steps (NSteps), and the number of replications (NRepl) for i=1,2. The time horizon (T), the number of time steps (NSteps), and the number of replications (NRepl) are same for both agents.

The last parameter NRepl is the number of replications, i.e., samples we want to take, while NSteps represents the number of values generated at each sample path. With few samples, we see quite some variability in the estimate, which starts looking reasonable when the number of samples is increased considerably. Clearly, we cannot yield just a point estimate: we should also compute some confidence interval for the estimate. Eventually, we come up with an appropriate number of samples that are needed in order to attain a given precision.

For instance, let us generate a one-year sample path for the income process of the first agent, with an initial price of income \$70, drift 0.1, and volatility 0.3 (on a yearly basis), assuming that the time step is one day. In addition, let us generate a one-year sample path for the income process of the second agent, with an initial price of income $30, drift 0.2, and volatility 0.4 (on a yearly basis), assuming that the time step is one day.

- A MATLAB function to generate sample paths of income process for each agent is displayed in M-file 1.

M-file1

```
function CPaths=IncomePaths(C,mu,sigma,T,NSteps,NRepl)
CPaths = zeros (NRepl, 1+NSteps);
CPaths(:,1)= C1;
dt = T/NSteps;
nudt = (mu1-0.5*sigma1^2)*dt;
sidt =sigma1*sqrt(dt);
for i=1:NRep1
   for j=1:NSteps
      CPaths(i,j+1)=CPaths(i,j)*exp(nudt + sidt*randn);
   end
end
```

STEP 2

Our next step is to create the equation (5.2.7) with the use of Matlab.

- We name “A1” the first term on the right hand side of equation (5.2.7) and we calculate it with a MATLAB function which is displayed in M-file 2.

M-file2

```
function Price=A1(el1,T)
Price=el1*T;
```

- We name “B1” the second term on the right hand side of equation (5.2.7) and we calculate it with a MATLAB function which is displayed in M-file 3.

M-file3

```
function [P,CI]=B1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRepl)
Price= zeros(NRepl, 1);
for i=1:NRep1
   CPaths1=IncomePaths (C1,mu1,sigma1,T,NSteps,1);
```

```
    CPaths2=IncomePaths (C2,mu2,sigma2,T,NSteps,1);
Price(i)=(el1+el2)*mean(CPaths1(2:NSteps)/(CPaths1(2:NSteps)+CPaths2(2:NSteps)
));
end
[P,aux,CI]=normfit(Price);
```

- In order to calculate R_1 (Λ) we use the MATLAB function "AR1", which deducts the function "B1" from function "A1", and is displayed in M-file 4.

M-file4

```
function Price=AR1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRepl)
Price=A1(el1,T)-B1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRepl);
```

Then, we will try to find $\lambda_1, \lambda_2 \in (0,\infty)$ such that equation $R_1(\lambda_1,\lambda_2)=0$. Thus, we construct Table 1, with the use of Matlab, with the prices of equation $R_1(\lambda_1,\lambda_2)$ when $\lambda_1,\lambda_2 \in [0,1]$.

Table 1

λ_1 / λ_2	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0	-	-0,069	-0,138	-0,207	-0,276	-0,345	-0,412	-0,482	-0,551	-0,621	-0,689
0,1	0,031	-0,038	-0,107	-0,175	-0,244	-0,313	-0,382	-0,451	-0,519	-0,589	-0,659
0,2	0,062	-0,007	-0,075	-0,144	-0,213	-0,282	-0,351	-0,420	-0,487	-0,557	-0,626
0,3	0,093	0,025	-0,044	-0,114	-0,182	-0,251	-0,320	-0,388	-0,458	-0,526	-0,596
0,4	0,125	0,056	-0,013	-0,082	-0,151	-0,219	-0,289	-0,357	-0,426	-0,494	-0,564
0,5	0,156	0,087	0,017	-0,050	-0,119	-0,189	-0,257	-0,326	-0,396	-0,465	-0,531
0,6	0,186	0,118	0,048	-0,021	-0,088	-0,158	-0,227	-0,295	-0,364	-0,433	-0,502
0,7	0,218	0,150	0,081	0,012	-0,058	-0,126	-0,195	-0,263	-0,333	-0,403	-0,471
0,8	0,250	0,180	0,111	0,043	-0,027	-0,095	-0,164	-0,233	-0,303	-0,370	-0,440
0,9	0,281	0,212	0,140	0,075	0,016	-0,064	-0,132	-0,201	-0,272	-0,338	-0,407
1	0,311	0,242	0,174	0,104	0,036	-0,034	-0,103	-0,170	-0,238	-0,308	-0,378

From Table 1 we can see that R_1 is close to zero when

- λ_1=0.2 and λ_2=0.1,
- λ_1=0.4 and λ_2=0.2,
- λ_1=0.5 and λ_2=0.2,
- λ_1=0.9 and λ_2=0.4 etc.

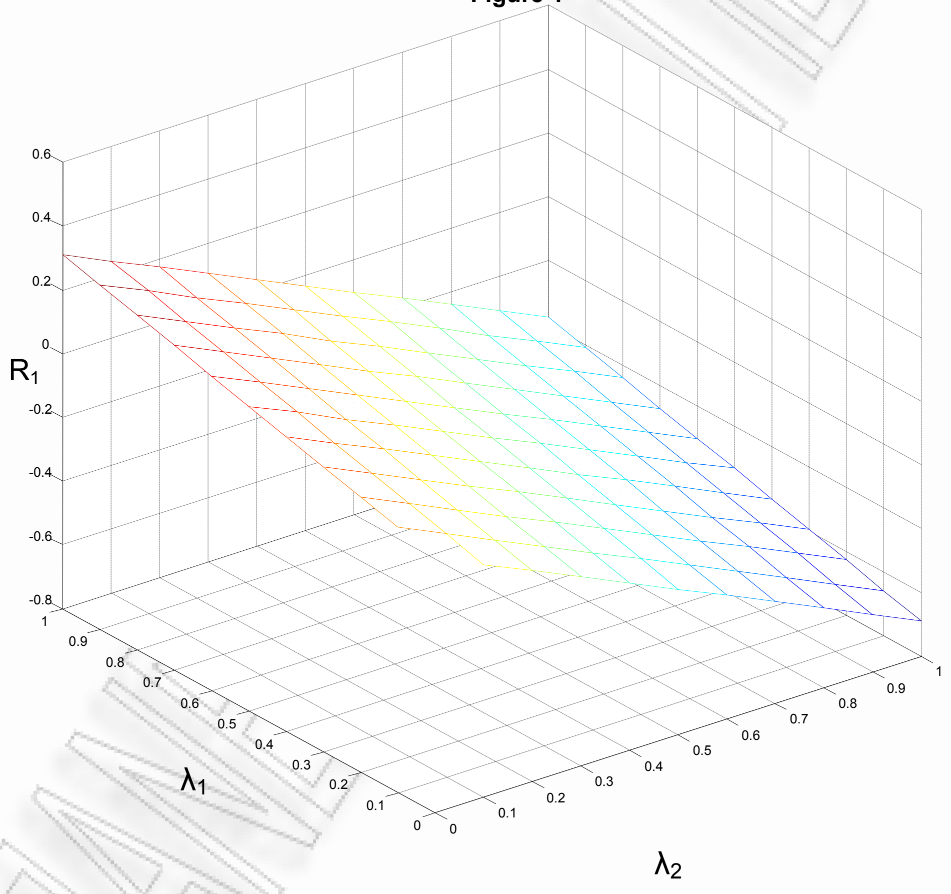
We can also see from Table1 that when λ_1=0.7 and λ_2=0.3, then $R_1 = 0.012$. From the corresponding cell of this Table, we can also understand that in order to have

$$R_1(\lambda_1, \lambda_2) = 0$$

we should take a price a little bit smaller than 0.7 for λ_1, a price a little bit bigger than 0.3 for λ_2, or a combination of them. But in order to have equilibrium, the prices of $\lambda_1$and λ_2 that makes R_1 to be zero, should make R_2 to be zero, as well.

In Figure 1, we can see the prices of R_1 when $\lambda_1, \lambda_2 \in [0,1]$. Figure 1 arises from Table 1. The higher price for R_1 is equal to 0.311 when λ_1=1 and λ_2=0 and the lower price for R_1 is equal to -0,689 when λ_1=0 and λ_2=1.

Figure 1



Running the code with the choice of λ_1=0.6886 and λ_2=0.3114 and the choice of the other parameters as we have already said yields

Price=AR1(0.6886,0.3114,70,30,0.1,0.2,0.3,0.4,1,365,100000),

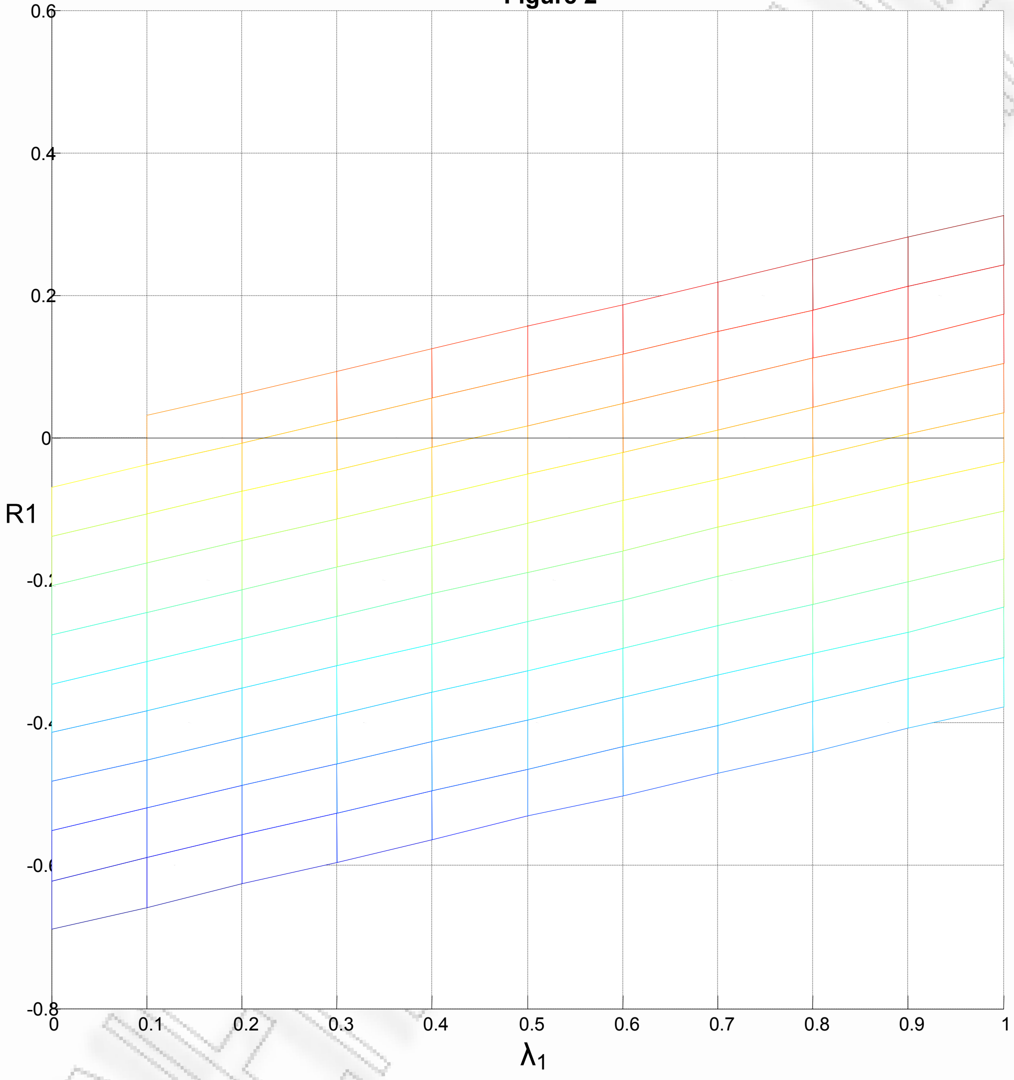
Price = 2.1085e-004 ≈ 0.

With the use of Matlab functions and specifically from M-file 4 if we assume that λ_1=0.6886 and λ_2=0.3114 for a number of 100000 replications we find out that

$$R_1(0.6886, 0.3114) = 0.$$

In Figure 2, we can see all the pairs of (λ_1, λ_2) that make R_1 to be zero. As we can see in Figure 2, $R_1(\lambda_1, \lambda_2)$ is close to zero when λ_1=0.7 and λ_2=0.3.

Figure 2



STEP 3

Our next step is to create the equation (5.2.8) with the use of Matlab.

- We name "A2" the first term on the right hand side of equation (5.2.8) and we calculate it with a MATLAB function which is displayed in M-file 5.

M-file5

```
function Price=A2(el2,T)
Price=el2*T;
```

- We name "B2" the second term on the right hand side of equation (5.2.8) and we calculate it with a MATLAB function which is displayed in M-file 6.

M-File6

```
function [P,CI]=B2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
Price= zeros(NRep1, 1);
for i=1:NRep1
    CPaths1=IncomePaths (C1,mu1,sigma1,T,NSteps,1);
    CPaths2=IncomePaths (C2,mu2,sigma2,T,NSteps,1);
Price(i)=(el1+el2)*mean(CPaths2(2:NSteps+1)/(CPaths1(2:NSteps+1)+CPaths2(2:NS
teps+1)));
end
[P,aux,CI]=normfit(Price);
```

- In order to calculate $R_2(\Lambda)$ we use the MATLAB function "AR2", which deducts the function "B2" from function "A2", and is displayed in M-file 7.

M-File7

```
function Price=AR2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
Price=A2(el2,T)-B2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1);
```

Then, we will try to find $\lambda_1, \lambda_2 \in (0, \infty)$ such that equation $R_2(\lambda_1, \lambda_2) = 0$.

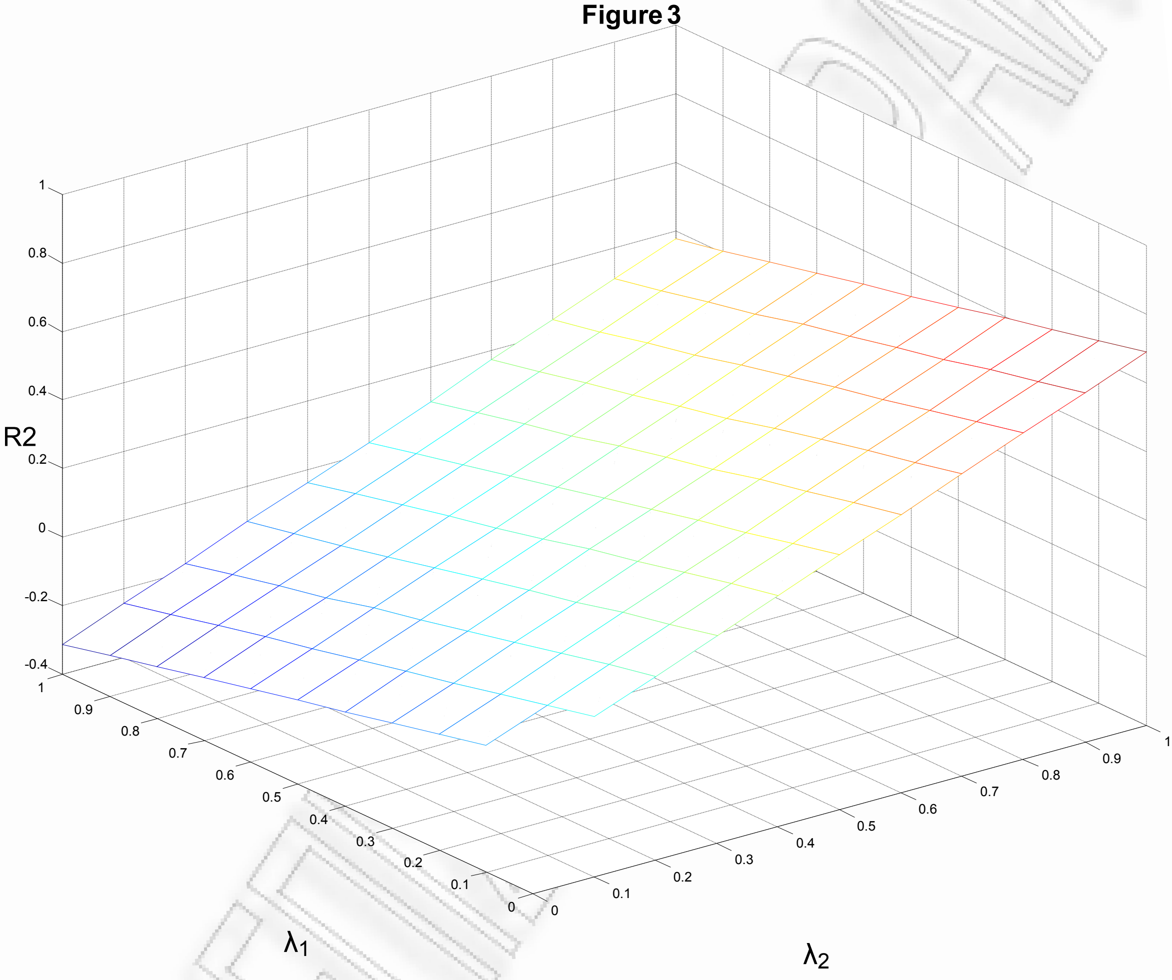
Thus, we construct Table 2, with the use of Matlab, with the prices of equation $R_2(\lambda_1, \lambda_2)$ when $\lambda_1, \lambda_2 \in [0,1]$.

Table2

λ_1 / λ_2	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0	-	0,069	0,138	0,207	0,275	0,344	0,413	0,483	0,551	0,619	0,689
0,1	-0,031	0,037	0,107	0,175	0,246	0,313	0,382	0,451	0,519	0,588	0,658
0,2	-0,062	0,007	0,075	0,145	0,213	0,282	0,350	0,420	0,489	0,558	0,626
0,3	-0,093	-0,025	0,044	0,113	0,183	0,251	0,321	0,390	0,456	0,525	0,599
0,4	-0,125	-0,056	0,012	0,082	0,153	0,220	0,289	0,357	0,427	0,494	0,563
0,5	-0,155	-0,086	-0,018	0,050	0,119	0,188	0,258	0,324	0,395	0,462	0,532
0,6	-0,186	-0,118	-0,049	0,020	0,087	0,157	0,227	0,295	0,363	0,435	0,502
0,7	-0,218	-0,148	-0,082	**-0,011**	0,058	0,125	0,192	0,263	0,330	0,401	0,470
0,8	-0,249	-0,180	-0,113	-0,043	0,027	0,094	0,162	0,233	0,302	0,370	0,439
0,9	-0,282	-0,212	-0,142	-0,074	-0,015	0,062	0,131	0,203	0,269	0,340	0,406
1	-0,311	-0,242	-0,174	-0,103	-0,037	0,031	0,099	0,174	0,242	0,310	0,379

From Table 2 we can see that, if we have λ_1=0.2 and λ_2=0.1, or λ_1=0.4 and λ_2=0.2, or λ_1=0.5 and λ_2=0.2, or λ_1=0.9 and λ_2=0.4 then R_2 is close to zero, as well. In addition, when we have λ_1=0.7 and λ_2=0.3, then $R_2 = -0.011$. In Figure 3, we can see all the prices that R_2 takes when $\lambda_1, \lambda_2 \in [0,1]$.

Figure 3



We also understand that in order to have R_2=0, we should take again a price a little bit smaller than 0.7 for λ_1, a price a little bit bigger than 0.3 for λ_2 or a combination. Running the code with the choice of λ_1=0.6886 and λ_2=0.3114 and the choice of the other parameters as we have already said yields

Price=AR2(0.6886,0.3114,70,30,0.1,0.2,0.3,0.4,1,365,100000),

Price = -1.3338e-004 ≈ 0.

Therefore, with the use of Matlab functions and specifically from M-file7 if we assume that λ_1=0.6886 and λ_2=0.3114 and NRepl=100000 we find that

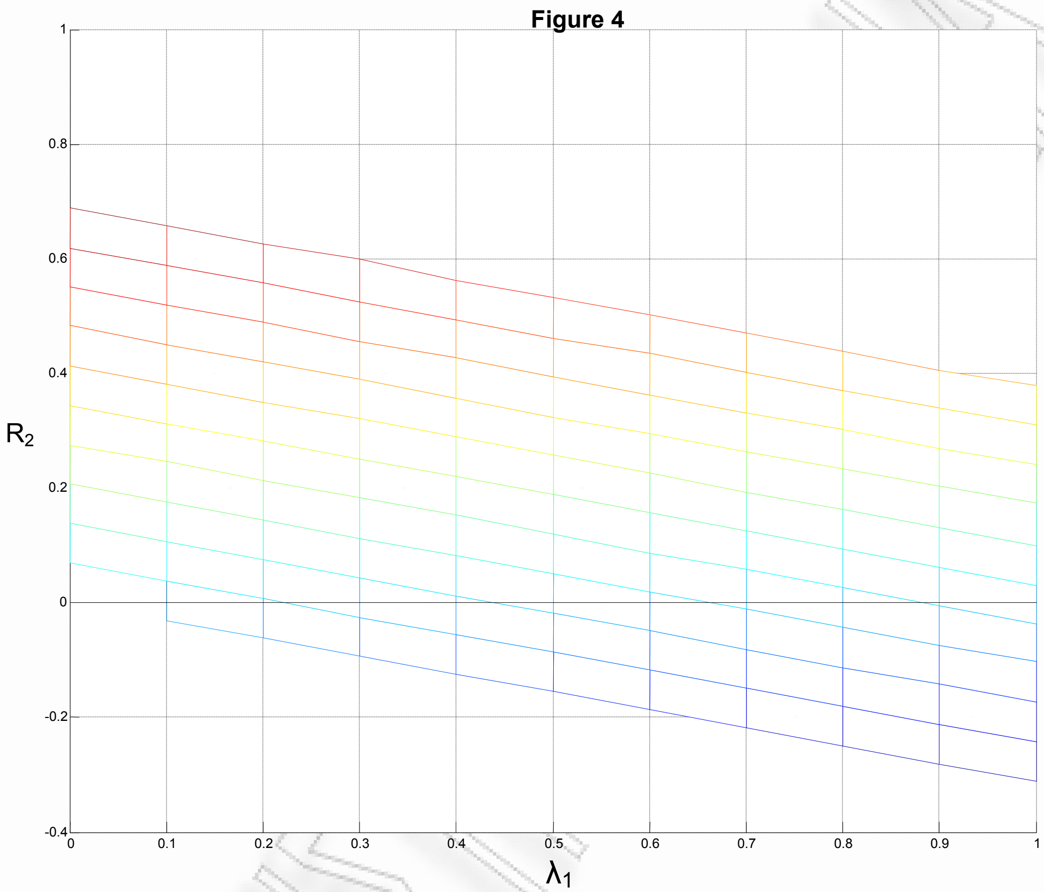
$$R_2(0.6886, 0.3114) = 0.$$

Thus,

$$R_1(0.6886, 0.3114) = R_2(0.6886, 0.3114) = 0.$$

As we can see in Figure 4, $R_2(\lambda_1, \lambda_2)$ is also close to zero when λ_1=0.7 and λ_2=0.3.

Figure 4



Consequently, we observe that the prices of λ_1=0.6886 and λ_2=0.3114 make R_1 and R_2 to be zero, simultaneously.

STEP 4

Now, we will find out if our results satisfy the theory. From (5.1.4)

$$\lambda_1 = \frac{1}{T} E \int_0^T \frac{\hat{c}_1(t)}{\hat{c}(t)} \text{ and } \lambda_2 = \frac{1}{T} E \int_0^T \frac{\hat{c}_2(t)}{\hat{c}(t)}.$$

- In order to calculate λ_1 (according to equation 5.1.4) we use a MATLAB function which is displayed in M-file 8.

M-file8

```
function [P,CI]=lamda1(C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
l1=zeros(NRep1,1);
for i=1:NRep1
   CPaths1=IncomePaths (C1,mu1,sigma1,T,NSteps,1);
```

```
   CPaths2=IncomePaths (C2,mu2,sigma2,T,NSteps,1);
l1(i)=1/T*mean(CPaths1(2:NSteps)/(CPaths1(2:NSteps)+CPaths2(2:NSteps)));
end
[P,aux,CI]=normfit(l1);
```

In particular, we assume that

i. the sample path for the income process of the first agent has an initial price $70, drift 0.1, and volatility 0.3 and the time step is one day,
ii. the sample path for the income process of the second agent has an initial price $30, drift 0.2, and volatility 0.4 and the time step is one day.

Running the code with the choice of the parameters as we have already said yields

$$[P,CI] = \text{lamda1}(70,30,0.1,0.2,0.3,0.4,1,365,100000),$$
$$P = 0.6886 \quad \text{and} \quad CI = [0.6884\ 0.6888].$$

Thus, we find out that $\lambda_1 = 0.6886$ and the estimating confidence interval is between 0.6884 and 0.688, so precision is attained. Therefore, our result for λ_1 is verified.

- In order to calculate λ_2 (according to equation 5.1.4) we use a MATLAB function which is displayed in M-file 9.

<u>M-file9</u>

```
function [P,CI]=lamda2(C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
l2=zeros(NRep1,1);
for i=1:NRep1
   CPaths1=IncomePaths (C1,mu1,sigma1,T,NSteps,1);
   CPaths2=IncomePaths (C2,mu2,sigma2,T,NSteps,1);
   l2(i)=1/T*mean(CPaths2(2:NSteps)/(CPaths1(2:NSteps)+CPaths2(2:NSteps)));
end
[P,aux,CI]=normfit(l2);
```

Running the code with the choice of the parameters as we have already said yields

$$[P,CI] = \text{lamda2}(70,30,0.1,0.2,0.3,0.4,1,365,100000),$$
$$P = 0.3114 \quad \text{and} \quad CI = [0.3111 \quad 0.3117].$$

Thus, we find out that $\lambda_2 = 0.3114$ and the confidence interval is between 0.3111 and 0.3117, so precision is attained. Therefore, our result for λ_2 is also verified.

- Consequently, we find out that our results for λ_1 and λ_2 satisfy the theory.

Application 5.2.2

For the Example 5.1.2, we assume that there are two agents, and each agent has the utility function $U_j(t,c)= c^{\delta}$, where $\delta \in (0,1)$. Then $I_j(t,y)=(y/\delta)^{1/(\delta-1)}$.

Thus, we have

$$U_1(t,c)=U_2(t,c)=c^{\delta}, \qquad (5.2.11)$$

and then

$$I_1(t,y)=(y/\delta)^{1/(\delta-1)} \text{ and } I_2(t,y)=(y/\delta)^{1/(\delta-1)}. \qquad (5.2.12)$$

From (4.1.4) and (5.2.12) we have

$$I(t,y;\Lambda)=\sum_{j=1}^{2} I_j(t,\frac{y}{\lambda_j})=I_1(t,\frac{y}{\lambda_1})+I_2(t,\frac{y}{\lambda_2})=\left(\frac{y}{\lambda_1\delta}\right)^{1/(\delta-1)}+\left(\frac{y}{\lambda_2\delta}\right)^{1/(\delta-1)}.$$

We assume that δ=1/2 so we have

$$I(t,y;\lambda_1,\lambda_2)=\left(\frac{2y}{\lambda_1}\right)^{-2}+\left(\frac{2y}{\lambda_2}\right)^{-2}=\frac{\lambda_1^2+\lambda_2^2}{4y^2}, \qquad (5.2.13)$$

and thus, we have the inverse of I (t, y;Λ) given by

$$U'(t,y;\lambda_1,\lambda_2)=\frac{\sqrt{\lambda_1^2+\lambda_2^2}}{2\sqrt{y}}. \qquad (5.2.14)$$

We also have that

$$I_1\left(t,\frac{1}{\lambda_1}U'(t,y;\Lambda)\right)=\left(\frac{\lambda_1}{2U'(t,y;\Lambda)}\right)^2=\frac{y\lambda_1^2}{(\lambda_1^2+\lambda_2^2)}, \qquad (5.2.15)$$

$$I_2\left(t,\frac{1}{\lambda_2}U'(t,y;\Lambda)\right)=\left(\frac{\lambda_2}{2U'(t,y;\Lambda)}\right)^2=\frac{y\lambda_2^2}{(\lambda_1^2+\lambda_2^2)}. \qquad (5.2.16)$$

So, we will try to find $\lambda_1,\lambda_2 \in (0,\infty)$ such that equation (4.3.1) for j=1,2 holds,

Thus from (4.3.2)

$$R_j(\Lambda)=E\int_0^T U'(t,\hat{c}(t);\Lambda)\left[I_j(t,\frac{1}{\lambda_j}U'(t,\hat{c}(t);\Lambda)-\hat{c}_j(t)\right]dt=0, \qquad j=1,2.$$

In other words, we must find those $\lambda_1,\lambda_2 \in (0,\infty)$ in order to have:

$$R_1(\Lambda)=E\int_0^T \frac{\sqrt{\lambda_1^2+\lambda_2^2}}{2\sqrt{\hat{c}(t)}}\left(\frac{\lambda_1^2}{\lambda_1^2+\lambda_2^2}\hat{c}(t)-\hat{c}_1(t)\right)dt=\left(E\int_0^T \frac{\lambda_1^2}{2\sqrt{\lambda_1^2+\lambda_2^2}}\sqrt{\hat{c}(t)}-E\int_0^T \frac{\sqrt{\lambda_1^2+\lambda_2^2}}{2\sqrt{\hat{c}(t)}}\hat{c}_1(t)\right)dt=0 \qquad (5.2.17)$$

and $R_2(\Lambda)=E\int_0^T \frac{\sqrt{\lambda_1^2+\lambda_2^2}}{2\sqrt{\hat{c}(t)}}\left(\frac{\lambda_2^2}{\lambda_1+\lambda_2}\hat{c}(t)-\hat{c}_2(t)\right)dt=\left(E\int_0^T \frac{\lambda_2^2}{2\sqrt{\lambda_1^2+\lambda_2^2}}\sqrt{\hat{c}(t)}-E\int_0^T \frac{\sqrt{\lambda_1^2+\lambda_2^2}}{2\sqrt{\hat{c}(t)}}\hat{c}_2(t)\right)dt=0.$ (5.2.18)

STEP 1

Our first step is to create the equation (5.2.17) with the use of Matlab.

- We name "MeanUdifxI1" the first term on the right hand side of (5.2.17) and we calculate it with a MATLAB function which is displayed in M-file 10.

M-file10

```
function  [P,CI] = MeanUdifxI1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,
NRepl)
MeanA1=zeros(NRepl,1);
for i=1:NRepl
   CPaths1=IncomePaths(C1,mu1,sigma1,T,NSteps,1);
   CPaths2=IncomePaths(C2,mu2,sigma2,T,NSteps,1);
MeanA1(i)=(el1^2/(2*sqrt(el1^2+el2^2)))*mean(sqrt(CPaths1(2:NSteps+1)+CPaths2
(2:NSteps+1)));
end
[P,aux,CI]=normfit(MeanA1);
```

- We name "MeanUdifxC1" the second term on the right hand side of (5.2.17) and we calculate it with a MATLAB function which is displayed in M-file 11.

M-file11

```
function [P,CI] = MeanUdifxC1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,
NRepl)
MeanB1=zeros(NRepl,1);
for i=1:NRep1
   CPaths1=IncomePaths(C1,mu1,sigma1,T,NSteps,1);
   CPaths2=IncomePaths(C2,mu2,sigma2,T,NSteps,1);
MeanB1(i)=((sqrt(el1^2+el2^2))/2)*mean(CPaths1(2:NSteps+1)/sqrt(CPaths1(2:NSte
ps+1)+CPaths2(2:NSteps+1)));
end
[P,aux,CI]=normfit(MeanB1);
```

- In order to calculate R_1 (Λ) we use the MATLAB function"R1", which deducts the function "MeanUdifxC1" from function "MeanUdifxI1", and is displayed in M-file 12.

M-file12

```
function Price=R1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
Price=MeanUdifxI1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)-
MeanUdifxC1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1);
```

Then, we will try to find $\lambda_1, \lambda_2 \in (0,\infty)$ such that equation $R_1(\lambda_1,\lambda_2)=0$. Thus, we construct Table 3, with the use of Matlab, with the prices of equation $R_1(\lambda_1,\lambda_2)$ when $\lambda_1,\lambda_2 \in [0,1]$.

Table 3

λ_1 / λ_2	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0	NaN	-0,356	-0,713	-1,068	-1,425	-1,781	-2,140	-2,499	-2,851	-3,208	-3,566
0,1	0,158	-0,139	-0,566	-0,965	-1,347	-1,719	-2,080	-2,446	-2,815	-3,173	-3,529
0,2	0,318	0,125	-0,278	-0,714	-1,136	-1,535	-1,932	-2,312	-2,694	-3,070	-3,432
0,3	0,477	0,341	**0,001**	-0,418	-0,855	-1,285	-1,705	-2,113	-2,504	-2,892	-3,272
0,4	0,639	0,531	0,252	-0,131	-0,556	-0,996	-1,424	-1,852	-2,266	-2,674	-3,082
0,5	0,796	0,713	0,473	0,132	-0,269	-0,699	-1,133	-1,571	-1,996	-2,420	-2,839
0,6	0,950	0,884	0,674	0,375	**0,006**	-0,405	-0,840	-1,271	-1,709	-2,138	-2,569
0,7	1,117	1,052	0,878	0,596	0,255	-0,136	-0,544	-0,976	-1,414	-1,843	-2,279
0,8	1,270	1,214	1,061	0,817	0,500	0,125	-0,261	-0,688	-1,120	-1,557	-1,992
0,9	1,430	1,373	1,242	1,019	0,729	0,384	**0,000**	-0,403	-0,820	-1,261	-1,698
1	1,589	1,548	1,423	1,220	0,945	0,623	0,265	-0,124	-0,544	-0,963	-1,400

From Table 3 we can see that

$$\text{when } \lambda_1 = 0.3 \text{ and } \lambda_2 = 0.2, \text{ then } R_1 = 0.001.$$
$$\text{when } \lambda_1 = 0.6 \text{ and } \lambda_2 = 0.4, \text{ then } R_1 = 0.006.$$
$$\text{when } \lambda_1 = 0.9 \text{ and } \lambda_2 = 0.4, \text{ then } R_1 = 0.000.$$

Therefore, we can understand that when $\lambda_1 \approx 1.5 \cdot \lambda_2$ then R_1 is close to zero. Then, we will try to find the price of R_1 when λ_1=0.8311 and λ_2=0.5569, which satisfies the relation $\lambda_1 \approx 1.5 \cdot \lambda_2$.

Running the code with the choice of λ_1=0.8311 and λ_2=0.5569 and NRepl=100000 and the choice of the other parameters as we have already said yields

Price=R1(0.8311,0.5569,70,30,0.1,0.2,0.3,0.4,1,365,100000),

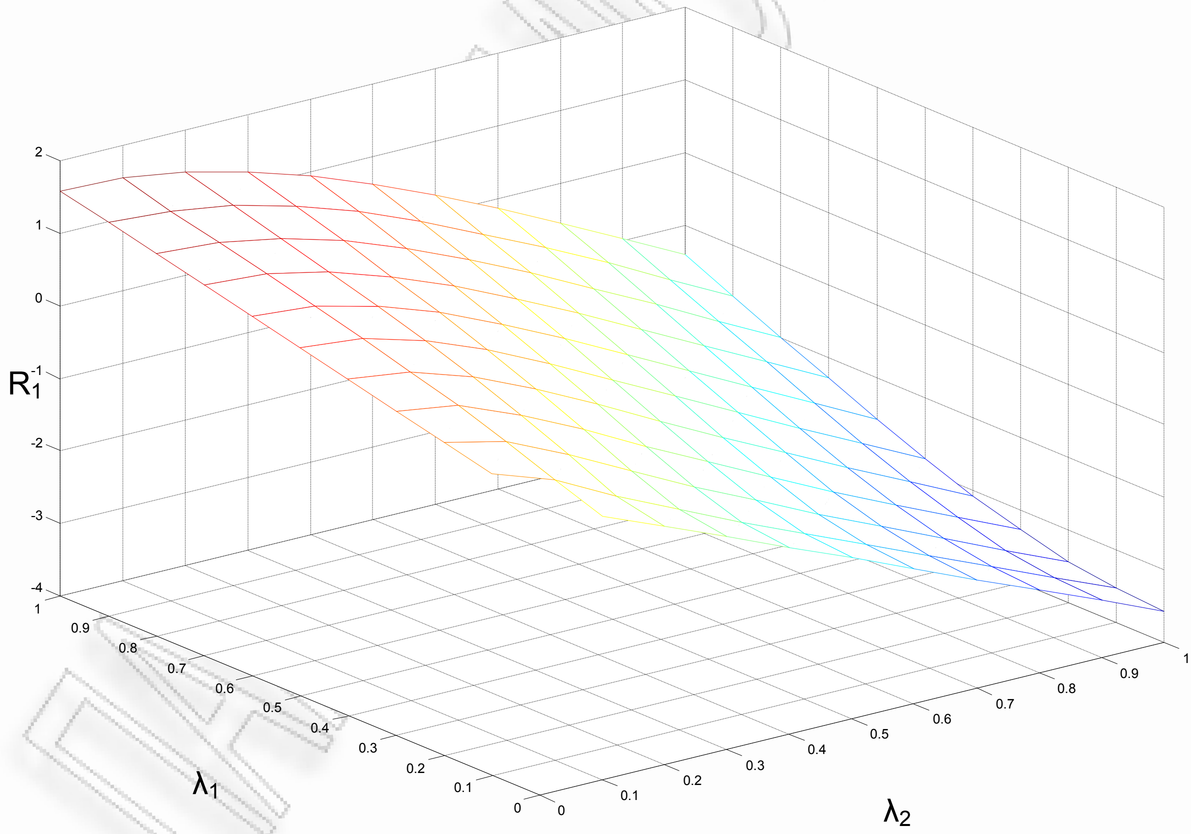
Price =-0.0093.

Thus, with the use of Matlab functions and specifically from M-file 12 we find that

$$R_1(0.8311, 0.5569) \approx 0.$$

In Figure 5, we can see all the prices that R_1 takes when $\lambda_1, \lambda_2 \in [0,1]$. Figure 5 arises from Table 3. The higher price for R_1 is equal to 1.589 when λ_1=1 and λ_2=0. The lower price for R_1 is equal to -3.566 when λ_1=0 and λ_2=1.

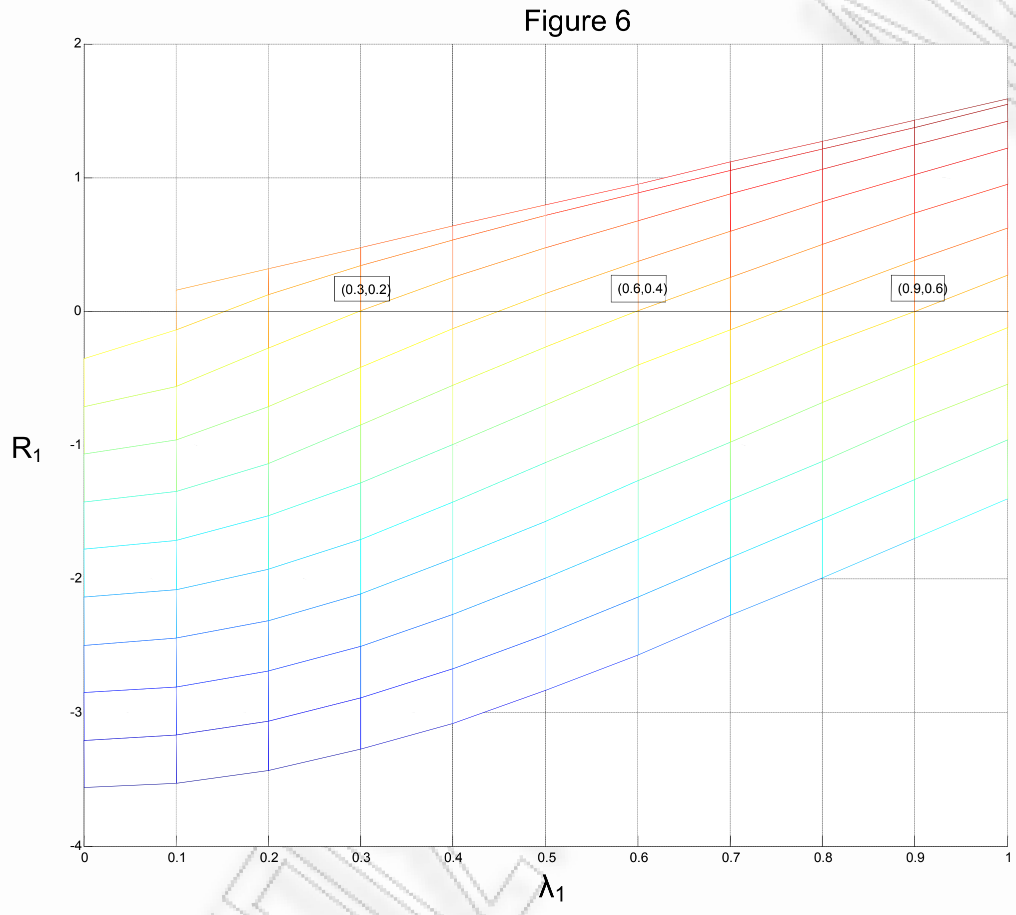
Figure 5



In Figure 6 we can also see the prices of R_1, but it is easier to understand whether R_1 is closer to zero. For this reason, we have added in Figure 6 the horizontal line R_1=0.

As we can see in Figure 6, R_1 is close to zero for the following pairs of (λ_1, λ_2):

(0.3,0.2), (0.6,0.4), (0.9,0.6) etc. Of course there are many pairs of (λ_1, λ_2) that makes R_1 equal to zero and as we have seen with the use of Matlab, another pair is (0.8311,0.5569).



STEP 2

Our next step is to create the equation (5.2.18) with the use of Matlab.

- We name "MeanUdifxI2" the first term on the right hand side of (5.2.18) and we calculate it with a MATLAB function which is displayed in M-file 13.

M-file13

```
function [P,CI] = MeanUdifxI2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,
NRep1)
MeanA2=zeros(NRep1,1);
for i=1:NRep1
CPaths1=IncomePaths(C1,mu1,sigma1,T,NSteps,1);
CPaths2=IncomePaths(C2,mu2,sigma2,T,NSteps,1);
```

```
MeanA2(i)=(el2^2/(2*sqrt(el1^2+el2^2)))*mean(sqrt(CPaths1(2:NSteps+1)+CPaths2
(2:NSteps+1)));
end
[P,aux,CI]=normfit(MeanA2);
```

- We name "MeanUdifxC2" the second term on the right hand side of (5.2.18) and we calculate it with a MATLAB function which is displayed in M-file 14.

M-file14

```
function [P,CI] = MeanUdifxC2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,
NRep1)
MeanB2=zeros(NRep1,1);
for i=1:NRep1
   CPaths1=IncomePaths(C1,mu1,sigma1,T,NSteps,1);
   CPaths2=IncomePaths(C2,mu2,sigma2,T,NSteps,1);
MeanB2(i)=((sqrt(el1^2+el2^2))/2)*mean(CPaths2(2:NSteps+1)/sqrt(CPaths1(2:NSte
ps+1)+CPaths2(2:NSteps+1)));
end
[P,aux,CI]=normfit(MeanB2);
```

- In order to calculate $R_2(\Lambda)$ we use the MATLAB function "R2", which deducts the function "MeanUdifxC2" from function "MeanUdifxI2", and is displayed in M-file 15.

M-file15

```
function Price=R2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
Price=MeanUdifxI2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)-
MeanUdifxC2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1);
```

Then, we will try to find $\lambda_1, \lambda_2 \in (0,\infty)$ such that equation $R_2(\lambda_1,\lambda_2)=0$. Thus, we construct Table 4, with the use of Matlab, with the prices of equation $R_2(\lambda_1,\lambda_2)$ when $\lambda_1,\lambda_2 \in [0,1]$.

Table 4

λ_1 / λ_2	**0**	**0,1**	**0,2**	**0,3**	**0,4**	**0,5**	**0,6**	**0,7**	**0,8**	**0,9**	**1**
0	-	0,354	0,707	1,059	1,414	1,776	2,124	2,485	2,836	3,203	3,540
0,1	-0,161	0,136	0,563	0,960	1,341	1,708	2,066	2,437	2,790	3,153	3,516
0,2	-0,321	-0,129	0,275	0,707	1,124	1,527	1,915	2,297	2,675	3,037	3,416
0,3	-0,484	-0,346	**-0,009**	0,410	0,845	1,273	1,684	2,090	2,486	2,882	3,254
0,4	-0,644	-0,538	-0,259	0,127	0,548	0,976	1,412	1,823	2,247	2,650	3,043
0,5	-0,803	-0,720	-0,488	-0,144	0,250	0,694	1,116	1,552	1,977	2,399	2,810
0,6	-0,966	-0,897	-0,689	-0,390	**-0,017**	0,385	0,819	1,256	1,696	2,120	2,541
0,7	-1,133	-1,064	-0,893	-0,612	-0,279	0,113	0,528	0,962	1,393	1,836	2,258
0,8	-1,288	-1,234	-1,083	-0,844	-0,515	-0,156	0,247	0,668	1,091	1,518	1,946
0,9	-1,450	-1,404	-1,262	-1,040	-0,747	-0,408	**-0,030**	0,383	0,790	1,240	1,667
1	-1,607	-1,571	-1,440	-1,242	-0,966	-0,644	-0,288	0,108	0,516	0,935	1,371

From Table 4 we can see that R_2 is close to zero for the pairs (0.3,0.2) , (0,6.0.4) and (0.9,0.6). In Figure 7, we can see all the prices that R_2 takes when $\lambda_1, \lambda_2 \in [0,1]$.

Figure 7

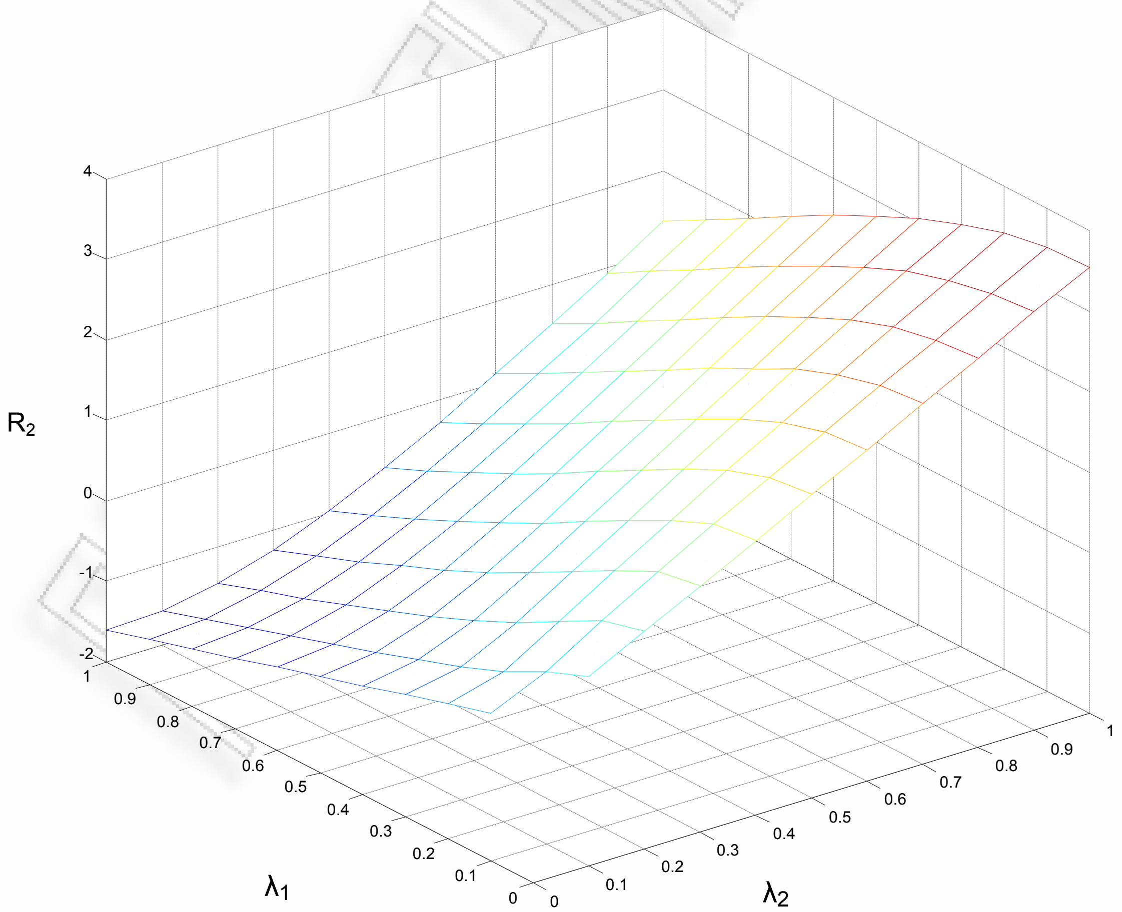


Figure 7 arises from Table 4. The higher price for R_2 is equal to 3.540 when λ_1=0 and λ_2=1. The lower price for R_2 is equal to -1.607 when λ_1=1 and λ_2=0.

Running the code with the choice of λ_1=0.8311 and λ_2=0.5569 and NRepl=100000 and the choice of the other parameters as we have already said yields

Price=R2(0.8311,0.5569,70,30,0.1,0.2,0.3,0.4,1,365,100000),

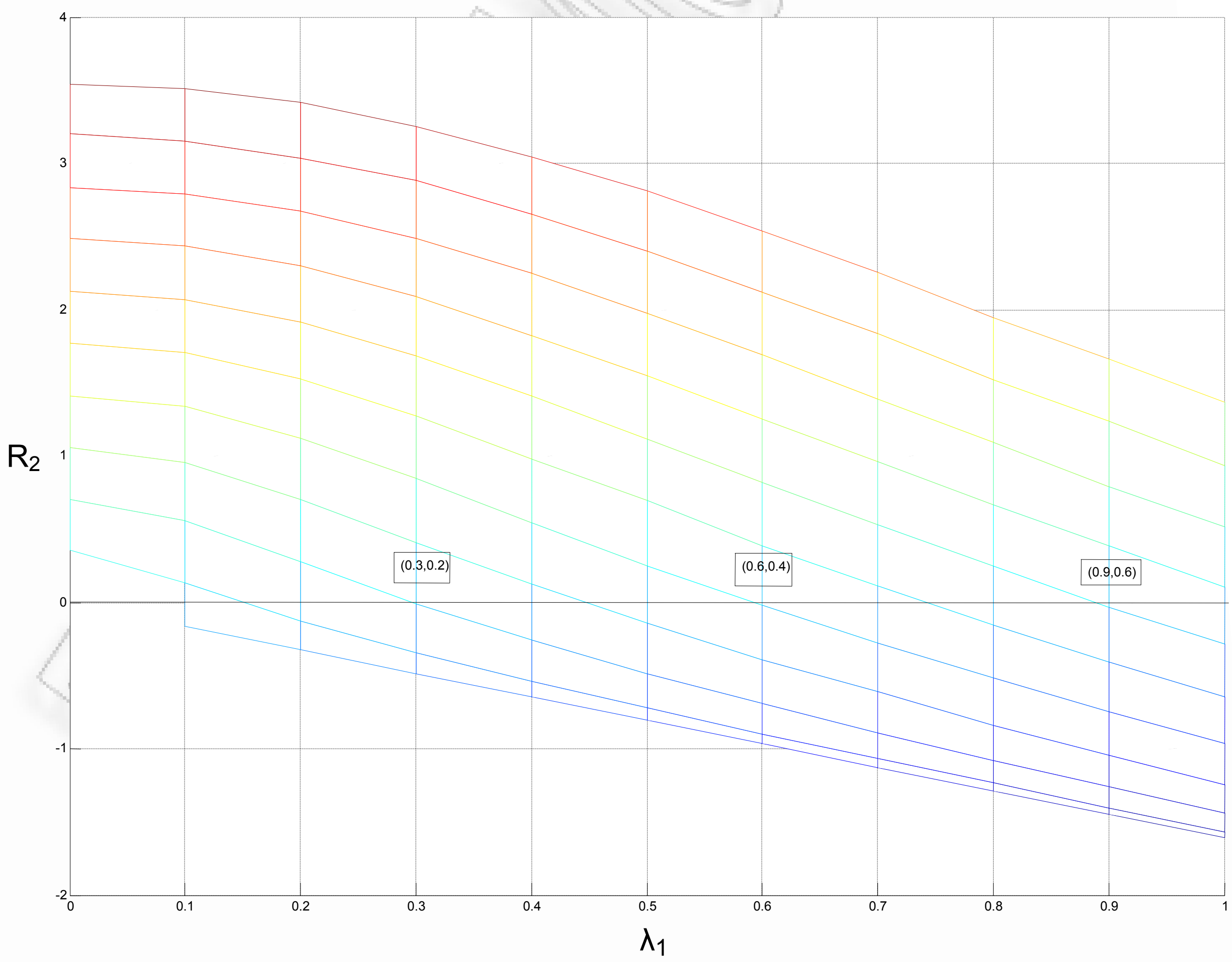
Price =-0.0098.

With the use of Matlab functions and specifically from M-file 15 we find out that

$$R_2(0.8311, 0.5569) \approx 0.$$

In Figure 8, we have added the horizontal line R_2=0. As we can see, R_2 is close to zero for the pairs (0.3,0.2), (0.6,0.4) and (0.9,0.6). Of course there are many pairs of (λ_1,λ_2) that makes R_2 equal to zero and as we have seen, (0.8311, 0.5569) is one of them.

Figure 8



STEP 3

Now, we will find out if our results satisfy the theory. From (5.1.8)

$$\lambda_1 = \left[\frac{E\int_0^T c^{\delta-1}(t)c_1(t)dt}{E\int_0^T c^{\delta}(t)dt}\right]^{1-\delta} \quad \text{and} \quad \lambda_2 = \left[\frac{E\int_0^T c^{\delta-1}(t)c_2(t)dt}{E\int_0^T c^{\delta}(t)dt}\right]^{1-\delta}.$$

- In order to calculate λ_1 (according to equation 5.1.8) we use a MATLAB function which is displayed in M-file 16.

M-file 16

```
function Price=lone(C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
lone=zeros(NRep1,1);
for i=1:NRep1
   CPaths1=IncomePaths(C1,mu1,sigma1,T,NSteps,1);
   CPaths2=IncomePaths(C2,mu2,sigma2,T,NSteps,1);
arithmhths=mean((CPaths1(2:NSteps))/sqrt(CPaths1(2:NSteps)+CPaths2(2:NSteps));
paronomasths=mean(sqrt(CPaths1(2:NSteps)+CPaths2(2:NSteps)));
   lone(i)=sqrt(arithmhths/paronomasths);
end
Price=mean(lone);
```

Running the code with 100000 replications and the choice of the other parameters as we have already said Matlab yields

Price = lone(70,30,0.1,0.2,0.3,0.4,1,365,100000),

Price = 0.8311.

Finally, we find out that the price of λ_1 is equal to 0.8311. Therefore, our result for λ_1 is verified.

- In order to calculate λ_2 we use a MATLAB function which is displayed in M-file 17

M-file 17

```
function Price=ltwo(C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
ltwo=zeros(NRep1,1);
```

```
for i=1:NRep1
CPaths1=IncomePaths(C1,mu1,sigma1,T,NSteps,1);
CPaths2=IncomePaths(C2,mu2,sigma2,T,NSteps,1);
arithmhths=mean((CPaths2(2:NSteps))/sqrt(CPaths1(2:NSteps)+CPaths2(2:NSteps)));
paronomasths=mean(sqrt(CPaths1(2:NSteps)+CPaths2(2:NSteps)));
   ltwo(i)=sqrt(arithmhths/paronomasths);
end
Price=mean(ltwo);
```

Running the code with 100000 replications and the choice of the other parameters as we have already said Matlab yields

Price= ltwo(70,30,0.1,0.2,0.3,0.4,1,365,100000),

Price =0.5569.

Finally, we find out that $\lambda_2 = 0.5569$. Therefore, our result for λ_2 is verified, as well.

- Consequently, we find out that our results for λ_1 and λ_2 satisfy the theory.

Application 5.2.3

For the Example 5.1.3, we assume that there are two agents, where agent 1 has the utility function $U_1(t,c) = \log c$ and agent 2 has the utility function $U_2(t,c) = \sqrt{c}$.

Then

$$I_1(t,y) = (1/y) \text{ and } I_2(t,y) = (1/4y^2). \quad (5.2.19)$$

From (4.1.4) and (5.2.19) we have

$$I(t,y;\Lambda) = \sum_{j=1}^{2} I_j(t,\frac{y}{\lambda_j}) = I_1(t,\frac{y}{\lambda_1}) + I_2(t,\frac{y}{\lambda_2}) = \left(\frac{\lambda_1}{y}\right) + \left(\frac{1}{4}\frac{\lambda_2^2}{y}\right). \quad (5.2.20)$$

Thus, we have the inverse of I (t, y; Λ) given by

$$U'(t,y;\Lambda) = \frac{\lambda_1}{2y}\left[1 + \sqrt{1 + y\left(\frac{\lambda_2}{\lambda_1}\right)^2}\right]. \quad (5.2.21)$$

We also have that

$$I_1\left(t, \frac{1}{\lambda_1} U'(t,y;\Lambda)\right) = \frac{\lambda_1}{U'(t,y;\Lambda)} = \frac{2y}{1 + \sqrt{1 + y(\lambda_2/\lambda_1)^2}}, \quad (5.2.22)$$

$$I_2\left(t,\frac{1}{\lambda_2}U'(t,y;\Lambda)\right)=\left[\frac{1}{2}\frac{\lambda_2}{U'(t,y;\Lambda)}\right]^2=\left[\frac{\frac{\lambda_2}{\lambda_1}y}{1+\sqrt{1+y(\lambda_2/\lambda_1)^2}}\right]^2. \quad (5.2.23)$$

So, we will try to find $\lambda_1,\lambda_2\in(0,\infty)$ such that equation (4.3.1) for j=1,2 holds, thus from (4.3.2)

$$R_j(\Lambda)=E\int_0^T U'(t,\hat{c}(t);\Lambda)\,[I_j(t,\frac{1}{\lambda_j}U'(t,\hat{c}(t);\Lambda)-\hat{c}_j(t)]\,dt=0,\quad j=1,2.$$

In other words, we must find those $\lambda_1,\lambda_2\in(0,\infty)$ in order to have

$$R_1(\Lambda)=\lambda_1 T-E\int_0^T\frac{\lambda_1}{2\hat{c}(t)}\left[1+\sqrt{1+\hat{c}(t)\left(\frac{\lambda_2}{\lambda_1}\right)^2}\right]\hat{c}_1(t)\,dt=0 \quad (5.2.24)$$

and

$$R_2(\Lambda)=E\int_0^T\frac{\lambda_1}{2\hat{c}(t)}\left[1+\sqrt{1+\hat{c}(t)\left(\frac{\lambda_2}{\lambda_1}\right)^2}\right]\left[\frac{\frac{\lambda_2}{\lambda_1}\hat{c}(t)}{1+\sqrt{1+\hat{c}(t)(\lambda_2/\lambda_1)^2}}\right]^2-E\int_0^T\frac{\lambda_1}{2\hat{c}(t)}\left[1+\sqrt{1+\hat{c}(t)\left(\frac{\lambda_2}{\lambda_1}\right)^2}\right]\hat{c}_2(t)\,dt=0. \quad (5.2.25)$$

STEP 1

Our first step is to create the equation (5.2.24) with the use of Matlab.

- We name "alpha1" the first term on the right hand side of (5.2.24) and we calculate it with a MATLAB function which is displayed in M-file 18.

M-file 18

```
function Price=alpha1(el1,T)
Price=el1*T;
```

- We name "beta1" the second term on the right hand side of (5.2.24) and we calculate it with a MATLAB function which is displayed in M-file 19.

M-file 19

```
function  [P,CI]=beta1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
beta1=zeros(NRep1,1);
for i=1:NRep1
CPaths1=IncomePaths(C1,mu1,sigma1,T,NSteps,1);
```

```
CPaths2=IncomePaths(C2,mu2,sigma2,T,NSteps,1);
beta1(i)=mean(CPaths1(2:NSteps+1))*el1*(1+sqrt(1+(CPaths1(2:NSteps+1)+CPaths2(2:NSteps+1))*(el2^2/el1^2)))/(2*(CPaths1(2:NSteps+1)+CPaths2(2:NSteps+1)));
end
 [P,aux,CI]=normfit(beta1)
```

- In order to calculate $R_1(\Lambda)$ we use the MATLAB function “ER1”, which deducts the function “beta1” from function “alpha1” and is displayed in M-file 20.

M-file20

```
function Price=ER1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
Price=alpha1(el1,T)-beta1(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1);
```

Then, we will try to find $\lambda_1, \lambda_2 \in (0, \infty)$ such that equation $R_1(\lambda_1, \lambda_2) = 0$. Thus, we construct Table 5, with the use of Matlab, with the prices of equation $R_1(\lambda_1, \lambda_2)$ when $\lambda_1, \lambda_2 \in [0,1]$.

Table 5

λ_1 / λ_2	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
0,1	0,032	-0,289	-0,641	-0,993	-1,345	-1,695	-2,051	-2,404	-2,756	-3,112	-3,459
0,2	0,064	-0,227	-0,576	-0,927	-1,278	-1,631	-1,982	-2,335	-2,687	-3,041	-3,395
0,3	0,096	-0,169	-0,515	-0,864	-1,214	-1,568	-1,920	-2,271	-2,624	-2,979	-3,329
0,4	0,128	-0,115	-0,455	-0,804	-1,155	-1,504	-1,855	-2,213	-2,565	-2,910	-3,270
0,5	0,160	-0,062	-0,397	-0,742	-1,092	-1,441	-1,792	-2,145	-2,499	-2,844	-3,202
0,6	0,191	**-0,012**	-0,339	-0,682	-1,031	-1,379	-1,730	-2,082	-2,434	-2,784	-3,140
0,7	0,224	0,036	-0,284	-0,624	-0,971	-1,322	-1,667	-2,018	-2,367	-2,719	-3,077
0,8	0,255	0,081	-0,228	-0,566	-0,909	-1,254	-1,606	-1,955	-2,306	-2,660	-3,012
0,9	0,287	0,126	-0,177	-0,509	-0,852	-1,202	-1,545	-1,901	-2,243	-2,589	-2,941
1	0,318	0,169	-0,123	-0,452	-0,791	-1,136	-1,481	-1,836	-2,181	-2,537	-2,882

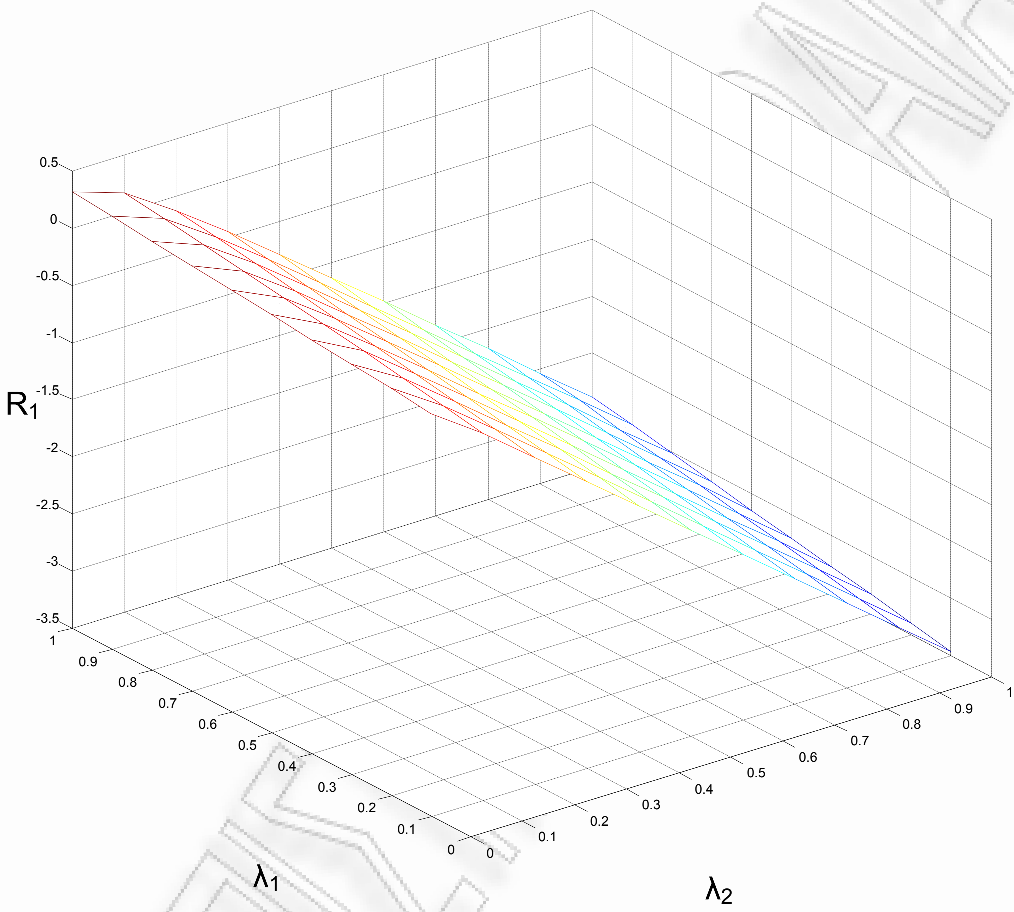
We can see from Table 5 that when $\lambda_1 = 0.6$ and $\lambda_2 = 0.1$, then R_1=-0.012. From the corresponding cell of this Table, we can also understand that in order to have

$$R_1(\lambda_1, \lambda_2) = 0$$

we should take a price a little bit bigger than 0.6 for λ_1.

In Figure 9, we can see the prices of R_1 when $\lambda_1, \lambda_2 \in [0,1]$. Figure 9 arises from Table 5. The higher price for R_1 is equal to 0.318 when $\lambda_1=1$ and $\lambda_2=0$. The lower price for R_1 is equal to -3,459 when $\lambda_1=0.1$ and $\lambda_2=1$.

Figure 9



Running the code with the choice of $\lambda_1=0.633$ and $\lambda_2=0.1$ and NRepl=100000 and the choice of the other parameters as we have already said yields

Price=ER1(0.633,0.1,70,30,0.1,0.2,0.3,0.4,1,365,100000),

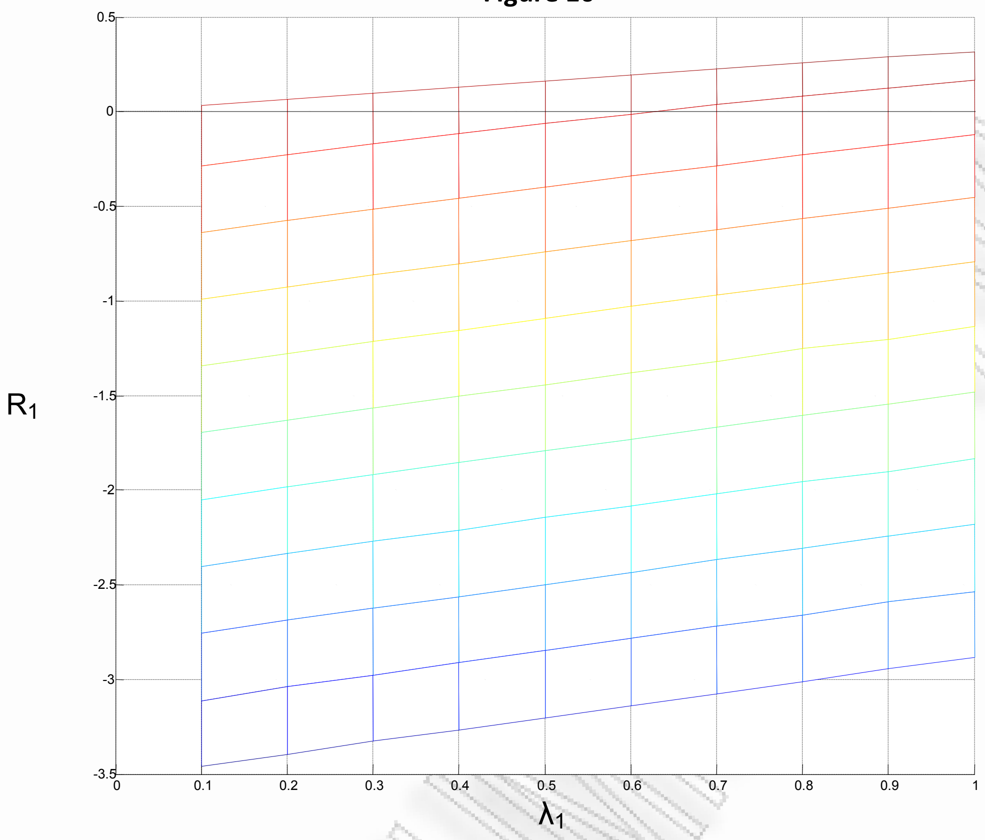
Price=0.0033≈0.

Thus, with the use of Matlab functions and specifically from M-file 20 we find that

$$R_1(0.633, 0.1) \approx 0$$

In Figure 10, we have added the horizontal line $R_1=0$. As we can see in Figure 10 $R_1(\lambda_1, \lambda_2)$ is close to zero when $\lambda_1 \approx 0.6$ and $\lambda_2=0.1$.

Figure 10



STEP 2

Our next step is to create the equation (5.2.25) with the use of Matlab.

- We name "alpha2" the first term on the right hand side of (5.2.25) and we calculate it with a MATLAB function which is displayed in M-file 21.

M-file 21

```
function Price=alpha2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
alpha2=zeros(NRep1,1);
for i=1:NRep1
CPaths1=IncomePaths(C1,mu1,sigma1,T,NSteps,1);
CPaths2=IncomePaths(C2,mu2,sigma2,T,NSteps,1);
alpha2(i)=el1*(1+sqrt(1+(CPaths1(2:NSteps+1)+CPaths2(2:NSteps+1))*(el2^2/el1^2
)))/(2*(CPaths1(2:NSteps+1)+CPaths2(2:NSteps+1)))*((el2/el1)*(CPaths1(2:NSteps+
1)+CPaths2(2:NSteps+1))/(1+sqrt(1+(CPaths1(2:NSteps+1)+CPaths2(2:NSteps+1))*(
el2^2/el1^2))))^2;
 end
Price=mean(alpha2)
```

- We name "beta2" the second term on the right hand side of (5.2.25) and we calculate it with a MATLAB function which is displayed in M-file 22.

M-file 22

```
function  [P,CI]=beta2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
beta2=zeros(NRep1,1);
for i=1:NRep1
CPaths1=IncomePaths(C1,mu1,sigma1,T,NSteps,1);
CPaths2=IncomePaths(C2,mu2,sigma2,T,NSteps,1);
beta2(i)=mean(CPaths2(2:NSteps+1))*el1*(1+sqrt(1+(CPaths1(2:NSteps+1)+CPaths
2(2:NSteps+1))*(el2^2/el1^2)))/(2*(CPaths1(2:NSteps+1)+CPaths2(2:NSteps+1)));
end
[P,aux,CI]=normfit(beta2);
```

- In order to calculate $R_2(\Lambda)$ we use the MATLAB function "ER2", which deducts the function "beta2" from function "alpha2" and is displayed in M-file 23.

M-file 23

```
function Price=ER2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)
Price=alpha2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1)-
beta2(el1,el2,C1,C2,mu1,mu2,sigma1,sigma2,T,NSteps,NRep1);
```

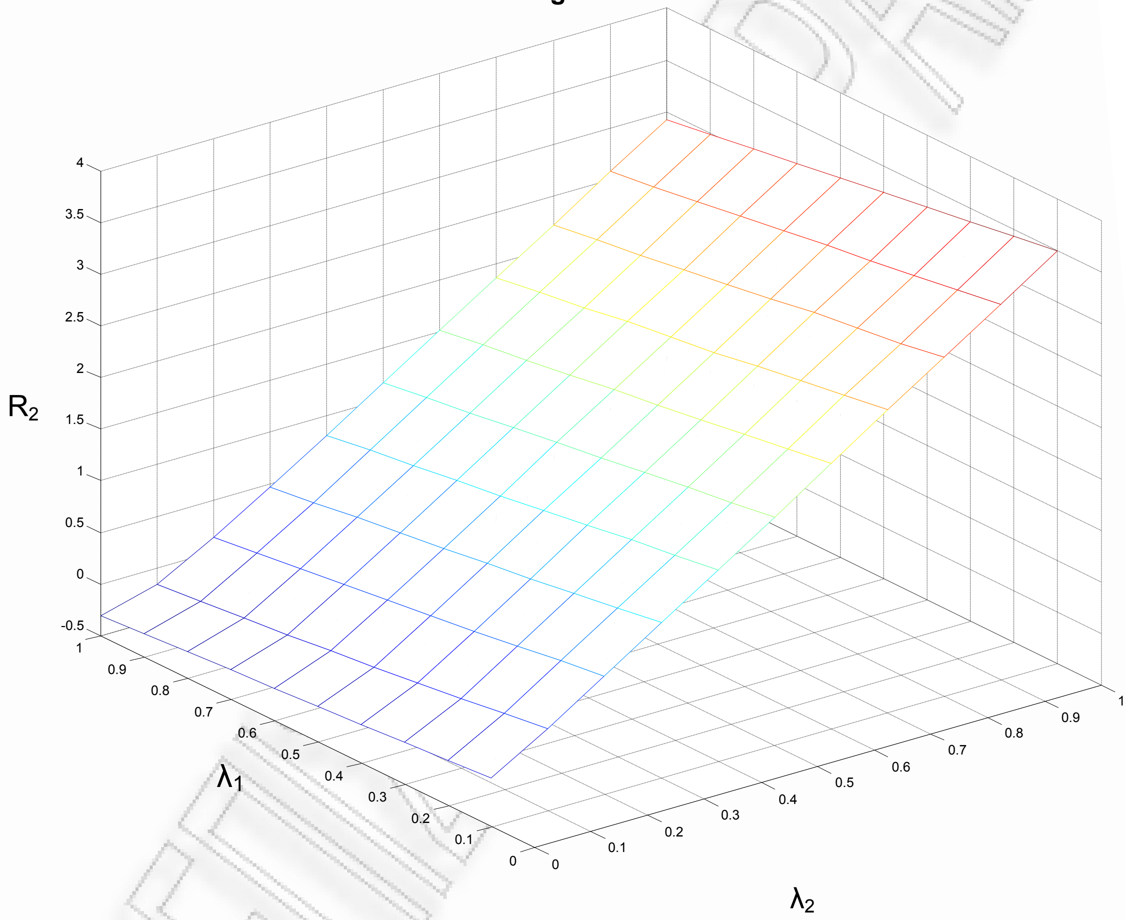
Then, we will try to find $\lambda_1, \lambda_2 \in (0,\infty)$ such that equation $R_2(\lambda_1,\lambda_2)=0$. Thus, we construct Table 6, with the use of Matlab, with the prices of equation $R_2(\lambda_1,\lambda_2)$ when $\lambda_1, \lambda_2 \in [0,1]$.

Table 6

λ_1 / λ_2	**0**	**0,1**	**0,2**	**0,3**	**0,4**	**0,5**	**0,6**	**0,7**	**0,8**	**0,9**	**1**
0	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
0,1	-0,031	0,293	0,648	1,005	1,364	1,714	2,073	2,433	2,789	3,146	3,501
0,2	-0,061	0,233	0,585	0,941	1,294	1,651	2,011	2,369	2,718	3,075	3,442
0,3	-0,093	0,175	0,523	0,876	1,236	1,591	1,944	2,304	2,658	3,013	3,373
0,4	-0,123	0,119	0,463	0,817	1,172	1,529	1,880	2,242	2,600	2,951	3,311
0,5	-0,154	0,068	0,405	0,756	1,111	1,467	1,816	2,172	2,532	2,889	3,246
0,6	-0,185	0,018	0,348	0,696	1,048	1,403	1,755	2,114	2,470	2,827	3,181
0,7	-0,216	-0,028	0,293	0,640	0,987	1,343	1,695	2,056	2,404	2,761	3,116
0,8	-0,247	-0,074	0,239	0,580	0,924	1,276	1,628	1,985	2,338	2,697	3,046
0,9	-0,277	-0,117	0,189	0,524	0,872	1,219	1,571	1,925	2,285	2,633	2,986
1	-0,308	-0,159	0,135	0,466	0,810	1,161	1,514	1,863	2,219	2,573	2,924

As we can see from Table 6 R_2 is close to zero when $\lambda_1 = 0.6$ and $\lambda_2 = 0.1$. In this case, R_2 =0.018. In Figure 11, we can see the prices of R_2 when $\lambda_1, \lambda_2 \in [0,1]$. Figure 11 arises from Table 6. The higher price for R_2 is equal to 3.501 when λ_1=0.1 and λ_2=1. The lower price for R_1 is equal to -0,308 when λ_1=1 and λ_2=0.

Figure 11



Running the code with the choice of λ_1=0.633 and λ_2=0.1 and NRepl=100000 and the choice of the other parameters as we have already said yields

Price=ER2(0.633,0.1,70,30,0.1,0.2,0.3,0.4,1,365,100000),

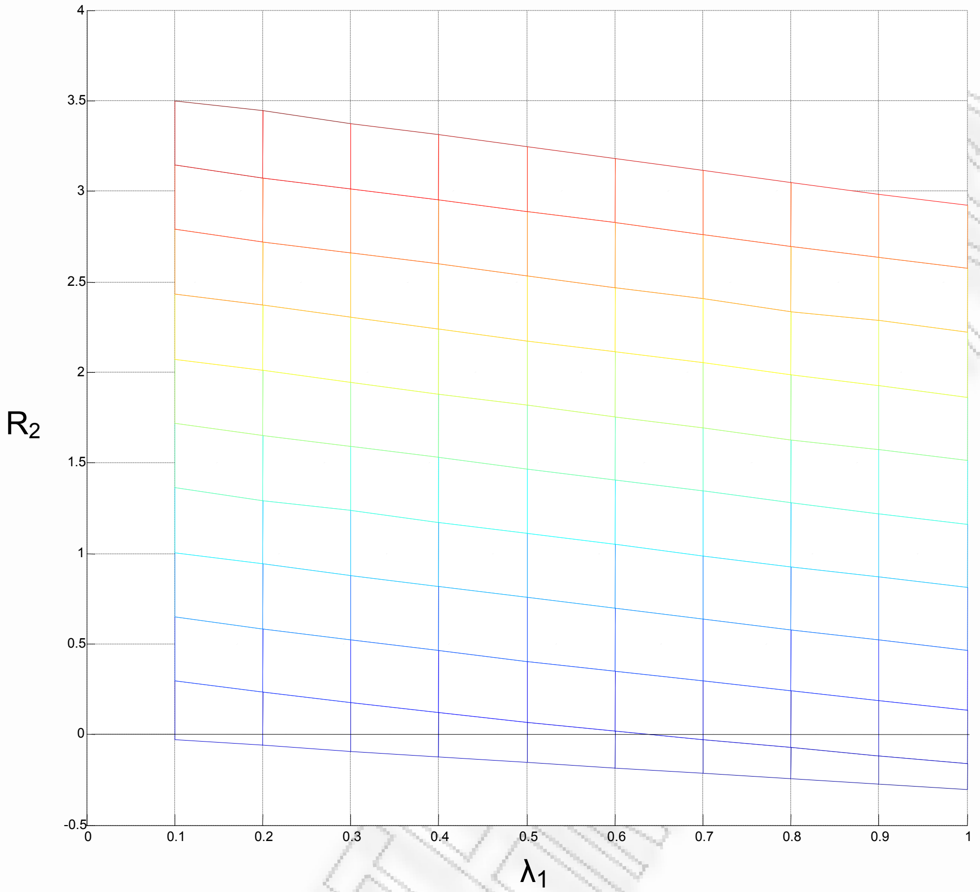
Price=0.0036≈0.

Thus, with the use of Matlab functions and specifically from M-file 23 we find that

$$R_2(0.633, 0.1) \approx 0.$$

As we can see in Figure 12, $R_2(\lambda_1, \lambda_2)$ is closer to zero when $\lambda_1 \approx 0.6$ and $\lambda_2 = 0.1$. From this Figure we can also see that the higher price for R_2 is when λ_1=0.1 and λ_2=1 and the lower price for R_1 is when λ_1=1 and λ_2=0.

Figure 12



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