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Department Of Banking & Financial Management

MASTER' S THESIS TITLE:

"Utility Consumption–Investment Optimization In Complete Markets And Applications In Dynamic Programming"

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Abstract

A general consumption/ investment problem is considered for an agent whose actions cannot affect the market prices, and who strives to maximize total expected discounted utility of both consumption and/or terminal wealth. Under very general conditions on the nature of the market model and on the utility functions of the agent, it is shown how to approach the above problem by considering both simultaneously and separately the two more elementary ones of maximizing utility of consumption only and of maximizing utility of terminal wealth only, or even appropriately composing them. The optimal consumption and wealth processes are obtained quite explicitly. In the case of a market model with deterministic coefficients, the optimal portfolio and consumption rules are explicitly derived in feedback form on the current level of wealth. Furthermore, the Hamilton-Jacobi-Bellman equation of dynamic programming is developed for the value function of the above utility optimization problem. In contrast to this nonlinear partial differential equation which governs the value function, its dual value function turns out to satisfy a linear one. The Monte Carlo simulation method is used for numerical applications which aim to the computation of the value function for a wide range of initial endowments.

Key words: portfolio and consumption processes, utility functions, optimizations problems, deterministic coefficients, feedback formulae, Hamilton-Jacobi-Bellman equation, Monte Carlo simulation.

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1. Introduction

The first revolution in finance began with the 1952 publication of "Portfolio Selection," an early version of the doctoral dissertation of Harry Markowitz. This publication began a shift away from the concept of trying to identify the "best" stock for an investor, and towards the concept of trying to understand and quantify the trade-offs between risk and return inherent in an entire portfolio of stocks. The vehicle for this so-called mean-variance analysis of portfolios is linear regression; once this analysis is complete, one can then address the optimization problem of choosing the portfolio with the largest mean return, subject to keeping the risk (i.e., the variance) below a specified acceptable threshold. The implementation of Markowitz's ideas was aided tremendously by William Sharpe (1964), who developed the concept of determining covariances not between every possible pair of stocks, but between each stock and the "market". For purposes of the above optimization problem each stock could then be characterized by its mean rate of return (its "α") and its correlation with the market (its "β"). For their pioneering work, Markowitz and Sharpe shared with Merton Miller the 1990 Nobel Prize in economics, the first ever awarded work in finance.

The portfolio-selection work of Markowitz and Sharpe introduced mathematics to the "black art" of investment management. With time, the mathematics has become more sophisticated. Thanks to Robert Merton and Paul Samuelson (1964), one-period models were replaced by continuous-time, Brownian-motion-driven models, and the quadratic utility function implicit in mean-variance optimization was replaced by more general increasing, concave utility functions. Model-based mutual funds have taken a permanent seat at the table of investment opportunities offered to the public. Perhaps more importantly, the paradigm of thinking about financial markets has become a mathematical model. This affects the way we now understand issues of corporate finance, taxation, exchange-rate fluctuations, and all manner of financial issues.

Here is a high-level overview of the contents of this thesis. We take up the problem of a single agent faced with optimal consumption and investment decisions in the complete version of the market model. Tools from stochastic calculus and partial differential equations of parabolic type permit a very general treatment of the associated optimization problem. This theory can be related to Markowitz's mean-variance analysis and is ostensibly about how to "beat the market", although another important use for it is as a first step toward understanding how markets operate. Its latter use is predicated on the principle that a good model of individual behavior is to postulate that individuals act in their own best interest.

This thesis solves the problem of a single agent who begins with an initial endowment and who can consume the endowment at some rate $c(\cdot)$ and invest it in any of $d+1$ available assets, while also investing in a standard, complete market. The objective of this agent is to maximize the expected utility of consumption over the planning horizon, or to maximize the expected utility of wealth at the end of the planning horizon, or to maximize the combination of these two quantities. Except for the completeness assumption, the market model is quite general, allowing the coefficient processes to be stochastic processes that are not even assumed to be Markovian. Specializations of this model to the case of deterministic coefficients are provided in Section 4.1.

The $d+1$ *assets* or *securities* available to the agent in this paper are very general. One of them is a *bond*, a security whose instantaneous rate of return may fluctuate (possibly randomly), but which is otherwise riskless. The other assets are *stocks*, risky securities whose prices have randomly fluctuating mean rates of return $b_i(\cdot)$ and volatility coefficients $\sigma_{ij}(\cdot)$. Section 2.1 provides a careful exposition of these matters. The stock prices are driven by *d* independent Wiener processes; these represent the sources of uncertainty in the market model, which we assume to be complete in the sense of Harrison and Pliska (1981) and Bensoussan (1984). In our context, completeness amounts to nondegeneracy of the "diffusion" matrix $\alpha(\cdot)=\sigma(\cdot)\sigma^T(\cdot)$.

This condition guarantees, roughly speaking, that "there are exactly as many stocks as there are sources of uncertainty in the market model." It also enables us to construct a new probability measure under which the stocks prices, discounted at the rate $r(\cdot)$ of the bond, become local martingales; this fact is of great importance in the modern theory of financial economics, and we refer the reader to Harrison and Pliska (1981), (1983) for a fuller account of its ramifications.

The processes $r(\cdot), b_i(\cdot), \sigma_{ij}(\cdot); 1 \le i, j \le d$ and $\beta(\cdot)$, the instantaneous discount rate in the economy, will be collectively referred to as the *coefficients of the market model*. We assume that our agent is a "small investor", in that his decisions do not influence the asset prices which are treated as exogenous.

Single-agent consumption/investment problems have been investigated by a number of authors. A significant plateau was reached by Merton (1969), (1971). In the special case of a market model with constant coefficients, he found closed-form solutions for the associated Bellman equations in the infinite-horizon and the zero-bequest, finite-horizon cases, when the utility of consumption belongs to the HARA class, i.e., $U_1(c) = ac^p, -1 < p < 1, p \neq 0$ or $U_1(c) = a \log c$. The infinite-horizon model was generalized by Karatzas, Lehoczky, Sethi and Shreve (1986) who presented closed-form expressions for the value function and the optimal consumption and investment policies, corresponding to general utility functions and general assumptions concerning the effect of *bankruptcy*, a possibility altogether ignored by previous authors. The work cited thus far allows short-selling of both the bond and stocks; indeed, such short-selling is mandated by the optimal investment process. A model in which such short-selling is prohibited, but in which the interest rate on the bond and the mean rate of return on the only stock are constant and equal, was studied by Lechoczky, Sethi and Shreve (1983). An explicit overview of the problem can also be found in Karatzas and Shreve (1998).

In this thesis we take the time-horizon as finite and general utility functions for consumption and terminal wealth are allowed. Moreover, the coefficients of the market model are required only to be adapted and bounded processes. This means that stock prices can fluctuate in an almost arbitrary, not necessarily Markovian fashion. Such generality notwithstanding, explicit results for the solution of the problem are provided. The methodology that accounts for both the simplicity and generality of the obtained results is the usage of the deflator process $H(\cdot)$ of (2.17) which removes the differences in mean rates of return among the investments and thus endows certain processes with the martingale property, under the original probability measure. Such an idea appeared in the context of option pricing by the Girsanov change of probability measure, in Harrison and Kreps (1979) and was more fully developed by Harrison and Pliska (1981).

Section 2.1 describes the market model and the set of consumption and portfolio processes from which the investor in this market is free to choose. Section 2.2 introduces the notion of utility function. In Section 3.1, we formulate three optimization problems for an agent.

Section 3.2 solves the problem of an agent who seeks to maximize expected utility from consumption plus expected utility from terminal wealth. The method of solution uses the *convex dual* function (Legendre transform) of the utility function. Related to this concept, we introduce and study the convex dual of the value function for the problem of Section 3.1. Several examples are provided.

Section 3.3 considers the problem of maximization of expected utility from consumption only, and the antithetical problem of maximization of expected utility from terminal wealth only. These problems are related to that of Section 3.2 in the following way. Given an initial endowment x, an agent who wishes to maximize the expected utility of consumption plus the expected utility of terminal wealth can partition his initial endowment into two parts, x_1 and x_2, such that $x = x_1 + x_2$. Beginning with initial endowment x_1, the agent should solve the problem of maximizing expected utility from consumption only; with x_2, he should solve the problem of maximizing expected utility from terminal wealth only. The superposition of these two solutions is then the solution to the problem of maximizing expected utility from consumption plus expected utility from terminal wealth. The partition of wealth that accomplishes this decomposition of the problems is derived from Section 3.3.

The obtained results are specialized to models with deterministic coefficients in Chapter 4. For such models a Markov-based analysis is provided, including the development of the *Hamilton-Jacobi-Bellman equation* for the agent's value function and the optimal consumption and portfolio processes as feedback functions of the agent's wealth. The Hamilton-Jacobi-Bellman equation of Section 4.3 is a second-order highly non-linear parabolic partial differential equation. Furthermore, the dual value function turns out to satisfy a linear second-order parabolic partial differential equation. Illustrating examples are presented.

Numerical applications via Matlab, which elaborate on the utility maximization from consumption and/or terminal wealth, are presented in Chapter 5. We examine three specific cases, maximization of utility only from consumption (Case 1), only from terminal wealth (Case 2) and from consumption and terminal wealth, simultaneously (Case 3). Considering constant coefficients, we begin each time with a specific initial endowment and, by using charts, we represent in each case the relationship between initial endowment and the value function of the investor.

This thesis is organised as follows. Chapter 2 introduces the model of the financial market, provides a detailed discussion on consumption/portfolio process pairs, and lists our assumptions on the utility functions. Chapter Section 3 studies three optimization problems for an agent: i.e., utility maximization coming from (i) both consumption and terminal wealth, (ii) from consumption only, (iii) from terminal wealth only. A decomposition of the former problem to the last two is also established. In Chapter 4 we specialize the former problem to the case of *deterministic coefficients*. We provide the optimal consumption and portfolio strategies in explicit "feedback" formulae, and represent how the value function of the optimization problem satisfies the Hamilton-Jacobi-Bellman equation and how its dual value function satisfies a linear partial differential equation. Finally, in Chapter 5 we use the Monte Carlo simulation method to demonstrate numerical applications of the obtained results through Matlab.

2. The Financial Market

2.1 Portfolio And Consumption Processes

Let us consider a market in which d+1 assets (or "securities") are traded continuously on the fixed time-horizon $[0,T]$, $0<T<\infty$. One of these assets, called the *bond,* has a price which evolves according to the differential equation

$$dS_0(t)=r(t)\,S_0(t)\,dt, \quad S_0(0)=1, \quad 0<t\le T. \quad (2.1)$$

The remaining d assets, called *stocks*, are risky; their prices are modeled by the linear stochastic differential equations:

$$dS_i(t)=S_i(t)\left[b_i(t)\,dt+\sum_{j=1}^{d}\sigma_{ij}(t)\,dW^j(t)\right], \quad S_i(0)=s_i, \quad 0<t\le T \quad (2.2)$$

for i =1,2,…,d. Here $W=\left\{W(t)=\left(W^{(1)}(t),\cdots,W^{(d)}(t)\right)^T, \quad \mathscr{F}(t); \quad 0\le t\le T\right\}$ is a d-dimensional Brownian motion on a probability space $(\Omega,\mathscr{F},P)$ and the filtration $\{\mathscr{F}(t)\}$ is the augmentation under P of the filtration $\mathscr{F}^W(t):=\sigma(W(s);0\le s\le t), 0\le t\le T$, that is generated by W; it represents the information available to agents at time t. All the processes which follow are assumed to be $\{\mathscr{F}(t)\}$- adapted, i.e., they are not anticipative of the future.

The interest rate process, $\{r(t),\mathscr{F}(t);0\le t\le T\}$ as well as the vector of mean rates of return $\left\{b(t)=(b_1(t),\cdots,b_d(t))^T, \quad \mathscr{F}(t); \quad 0\le t\le T\right\}$ and the volatility matrix $\left\{\sigma(t)=(\sigma_{ij}(t))_{1\le i,j\le d}, \quad \mathscr{F}(t); \quad 0\le t\le T\right\}$ are assumed to be measurable, adapted and bounded, uniformly in $(t,\omega)\in[0,T]\times\Omega$. We introduce the covariance matrix $\alpha(t):=\sigma(t)\sigma^T(t)$ and assume that it is invertible for every $(t,\omega)\in[0,T]\times\Omega$.

We have now an investor who starts with some initial endowment $x\ge 0$ and invests it in the d+1 assets described above. Let $N_i(t)$ denote the number of shares of asset i owned by the investor at time t. Then $X(0)\equiv x=\sum_{i=0}^{d}N_i(0)\,s_i$, and the investor's wealth at time t is

$$X(t)=\sum_{i=0}^{d}N_i(t)\,S_i(t). \quad (2.3)$$

If the trading of shares (the adjustment of the portfolio) is allowed to take place only at discrete time points, say at $\cdots,t-h,t,t+h,\cdots$ and there is no infusion or withdrawal of funds, then

$$X(t+h)-X(t)=\sum_{i=0}^{d}N_i(t)\left[S_i(t+h)-S_i(t)\right]. \quad (2.4)$$

On the other hand, if the investor chooses at time $t+h$ to consume an amount $[(t+h)-t]\,c(t+h)$ and reduce the wealth accordingly, then (2.4) should be replaced by

$$X(t+h)-X(t)=\sum_{i=0}^{d}N_i(t)\left[S_i(t+h)-S_i(t)\right]-\left[(t+h)-t\right]c(t+h). \quad (2.5)$$

Taking $h\to 0^+$, the continuous-time analogue of (2.5) is

$$dX(t)=\sum_{i=0}^{d}N_i(t)\,dS_i(t)-c(t)\,dt.$$

By taking (2.1), (2.2), (2.3) into account and denoting by $\pi_i(t):=N_i(t)P_i(t)$ the amount invested in the stock i, $1\le i\le d$, we have

$$dX(t)=\big(X(t)r(t)-c(t)\big)dt+\sum_{i=1}^{d}\pi_i(t)\left[b_i(t)-r(t)\right]dt+\sum_{i=1}^{d}\sum_{j=1}^{d}\pi_i(t)\sigma_{ij}(t)\,dW^{(j)}(t). \quad (2.6)$$

Any component of the vector $\pi(t)$ may become negative, which is to be interpreted as short-selling that particular stock. The amount

$$\pi_0(t):=X(t)-\sum_{i=1}^{d}\pi_i(t)$$

invested in the bond may also become negative and this corresponds to borrowing at the interest rate $r(t)$.

<u>Definition 2.1</u>

A portfolio process $\pi=\left\{\pi(t)=\left(\pi_1(t),\cdots,\pi_d(t)\right)^T,\quad \mathscr{F}(t);\quad 0\le t\le T\right\}$ is a measurable, adapted, $\mathbf{R}^d$-valued process for which

$$\sum_{i=1}^{d}\int_0^T \pi_i^2(t)\,dt<\infty. \quad (2.7)$$

A consumption process $c=\left\{c(t),\quad \mathscr{F}(t);\quad 0\le t<\infty\right\}$ is a measurable, adapted process with values in $[0,\infty)$ and

$$\int_0^T c(t)\,dt<\infty. \quad (2.8)$$

The conditions (2.7), (2.8) guarantee that the stochastic differential equation (2.6) has a unique solution given by (2.9), which comes from Ito's Lemma as

$$X(t)=\exp\left(\int_0^t r(s)\,ds\right)\left\{x+\int_0^t\exp\left(-\int_0^s r(u)\,du\right)\left[\pi^T(s)\left(b(s)-r(s)\underset{\sim}{1}\right)-c(s)\right]ds\right.$$
$$\left.+\int_0^t\exp\left(-\int_0^s r(u)\,du\right)\pi^T(s)\sigma(s)\,dW(s)\right\},\qquad 0\le t\le T, \qquad (2.9)$$

where $\underset{\sim}{1}$ is the d-dimensional vector with every component equal to 1. All vectors are column vectors and transposition is denoted by the superscript T.

Definition 2.2

A pair (π, c) of portfolio and consumption processes is said to be admissible for the initial endowment $x \geq 0$ if the wealth process $X^{x,c,\pi}(\cdot) \equiv X(\cdot)$ of (2.9) satisfies

$$X(t) \geq 0, \quad 0 \leq t \leq T, \quad (2.10)$$

We denote by $\mathscr{A}(x)$ the class of all such pairs.

If $b(t) = r(t)\underset{\sim}{1};\ 0 \leq t \leq T$, then the discount factor $\exp\left\{-\int_0^t r(u)du\right\}$ exactly offsets the rate of growth of all assets and (2.9) shows that

$$Q(t) := X(t)\exp\left(-\int_0^t r(u)\,du\right) - x + \int_0^t c(s)\exp\left(-\int_0^s r(u)\,du\right)ds \quad (2.11)$$

is a stochastic integral. In other words, the process consisting of current wealth plus cumulative consumption, both properly discounted, is a local martingale. When $b(t) \neq r(t)\underset{\sim}{1}$, Q of (2.14) is no longer a local martingale under P, but becomes one under a new probability measure $\tilde{P}$ that removes the drift term $\pi^T(t)\left(b(t) - r(t)\underset{\sim}{1}\right)$ from (2.6).

Due to the boundedness of the model coefficients, the process

$$\theta(t) := (\sigma(t))^{-1}\left(b(t) - r(t)\underset{\sim}{1}\right), \quad 0 \leq t \leq T, \quad (2.12)$$

is also bounded, and let

$$Z(t) := \exp\left\{-\int_0^t \theta^T(s)\,dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2\,ds\right\}. \quad (2.13)$$

Then, from Novikov's condition, the local martingale $\{Z(t), \mathscr{F}(t);\ 0 \leq t \leq T\}$ (cf. (2.16) below)is actually a martingale. In fact the new probability measure $\tilde{P}(A) = E[Z(T)1_A]$ is such that P and $\tilde{P}$ are equivalent on $\mathscr{F}(t)$, i.e. a set in $\mathscr{F}(T)$ has $\tilde{P}$ - probability zero if and only if it has P - probability zero, and

$$\tilde{W}(t) := W(t) + \int_0^t \theta(s)\,ds \quad (2.14)$$

is a standard, d-dimensional Brownian motion under $\tilde{P}$ (Girsanov (1960) or Karatzas and Shreve (1987, paragraph 3.5)). This new probability measure $\tilde{P}$ is called an *equivalent martingale measure.* Therefore, in terms of $\tilde{W}$, we may rewrite (2.9) as:

$$\frac{X(t)}{S_0(t)} = x - \int_0^t \frac{c(u)}{S_0(u)} du + \int_0^t \frac{1}{S_0(u)} \pi^T(u)\sigma(u) d\tilde{W}(u) \quad (2.15)$$

with

$$S_0(t) = \exp\left(\int_0^t r(u)\,du\right)$$

be the solution of (2.1).

In this market, the exponential local martingale $Z(\cdot)$ is indeed a martingale, which permits the definition of the standard martingale measure $\tilde{P}$. We shall present the analysis of this chapter in a way that uses only the local martingale property of $Z(\cdot)$ and avoids the use of $\tilde{P}$ altogether. For this model, the following condition will be imposed:

$$\rho \le r(\cdot) \le R, \quad \rho, R > 0.$$

<u>Remark 2.3</u>

Let us define

$$\Theta(t) := \left(-\int_0^t \theta^T(s)\,dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2\,ds\right).$$

Setting, $f(x) = e^x$ and by the Ito's Lemma, we get from (2.13) that:

$$\begin{aligned} dZ(t) &= df(\Theta(t)) = f'(\Theta(t))\,d\Theta(t) + \frac{1}{2} f''(\Theta(t))\,d\Theta(t)\,d\Theta(t) \\ &= e^{\Theta(t)}\left[-\theta^T(t)\,dW(t) - \frac{1}{2}\|\theta(t)\|^2\,dt\right] + \frac{1}{2} e^{\Theta(t)} \|\theta(t)\|^2\,dt \\ &= -Z(t)\theta^T(t)\,dW(t) \qquad (2.16) \end{aligned}$$

and we define the process

$$H(t) := \frac{Z(t)}{S_0(t)}. \quad (2.17)$$

If we take (2.14)-(2.17) into account, we have from product rule:

$$\begin{aligned} d\big(H(t)X(t)\big) &= d\left(Z(t)\frac{X(t)}{S_0(t)}\right) = dZ(t)\frac{X(t)}{S_0(t)} + Z(t)\,d\left(\frac{X(t)}{S_0(t)}\right) + dZ(t)\,d\left(\frac{X(t)}{S_0(t)}\right) \\ &= -Z(t)\theta^T(t)X(t)\frac{1}{S_0(t)}dW(t) + Z(t)\left(-\frac{c(t)}{S_0(t)}dt + \frac{1}{S_0(t)}\pi^T(t)\sigma(t)\big[\theta(t)\,dt + dW(t)\big]\right) \\ &\quad - Z(t)\pi^T(t)\theta(t)\sigma(t)\frac{1}{S_0(t)}dt \\ &= -H(t)\theta^T(t)X(t)\,dW(t) - H(t)c(t)\,dt + H(t)\pi^T(t)\sigma(t)\,dW(t), \quad 0 \le t \le T, \end{aligned}$$

which through integration gives

$$H(t)X(t)+\int_0^t H(s)c(s)\,ds = x+\int_0^t H(s)\left[\sigma^T(s)\pi(s)-\theta(s)X(s)\right]^T dW(s). \quad (2.18)$$

When $(c,\pi)\in\mathscr{A}(x)$ the left-hand side of (2.18) is nonnegative, and so the Ito integral on the right side is not only a local martingale under *P*, but also a supermartingale. Thus, from the supermartingale property we have:

$$E\left[\int_0^T H(t)c(t)\,dt+H(T)X(T)\right]\le E\left[\int_0^0 H(t)c(t)\,dt+H(0)X(0)\right],$$

which implies that the so-called *budget constraint*

$$E\left[\int_0^T H(s)c(s)\,ds+H(T)X(T)\right]\le x \quad (2.19)$$

is satisfied for every $(c,\pi)\in\mathscr{A}(x)$ .The budget constraint has the satisfying interpretation that the expected "discounted" terminal wealth plus the expected "discounted" total consumption cannot exceed the initial endowment. Here the "discounting" is accomplished by the state price density process H .

Remark 2.4

Bankruptcy is an absorbing state for the wealth process $X(\cdot)$ when $(c,\pi)\in\mathscr{A}(x)$; if wealth becomes zero before time *T*, it stays there, and no further consumption or investment takes place (cf. Chung (1982, Thm. 1.4) or Karatzas and Shreve (1987, Problem 1.3.29)).

The budget constraint (2.19) is not only a necessary condition for admissibility, but is also a sufficient condition, in a sense that we now explain.

Theorem 2.5

Let $x\ge 0$ be given, let $c(\cdot)$ be a consumption process, and let ξ be a nonnegative, $\mathscr{F}(T)$ - adapted random variable such that

$$E\left[\int_0^T H(s)c(s)\,ds+H(T)\xi\right]=x. \quad (2.20)$$

Then there exists a portfolio process $\pi(\cdot)$ such that the pair (c,π) is admissible at x, i.e. $(c,\pi)\in\mathscr{A}(x)$,and $\xi = X(T)$.

Proof

Let us define $J(t) := \int_0^t H(s)c(s)ds$ and consider the nonnegative martingale

$$M(t) := E\left[J(T) + H(T)\xi / \mathscr{F}(t)\right], \quad 0 \le t \le T.$$

From (2.20) $M(0) = x$ and according to the martingale representation theorem (e.g., Karatzas and Shreve (1991), Theorem 3.4.15 and Problem 3.4.16), there is an adapted, $\mathbf{R}^d$- valued process $\psi(\cdot)$ satisfying

$$\int_0^T \|\psi(s)\|^2 ds < +\infty$$

almost surely and

$$M(t) = x + \int_0^t \psi^T(s)dW(s), \quad 0 \le t \le T.$$

Define a nonnegative process $X(\cdot)$ by

$$\begin{aligned}\frac{X(t)}{S_0(t)} &:= \frac{1}{Z(t)} E\left[\int_t^T H(s)c(s)ds + H(T)\xi / \mathscr{F}(t)\right] \\ &= \frac{1}{Z(t)}\left\{E\left[\int_0^T H(s)c(s)ds + H(T)\xi / \mathscr{F}(t)\right] - \int_0^t H(s)c(s)ds\right\} \\ &= \frac{1}{Z(t)}\left(M(t) - J(t)\right) \qquad (2.21)\end{aligned}$$

so that, $X(0) = M(0) = x$.

Furthermore, from (2.17) we have that

$$d\left(H(t)X(t)\right) = dM(t) - dJ(t) = \psi^T(t)dW(t) - H(t)c(t)dt$$

and a comparison with (2.18) shows that we should select a portfolio process $\pi(\cdot)$:

$$H(t)\left[\sigma^T(t)\pi(t) - \theta(t)X(t)\right] = \psi(t).$$

Therefore, solving for $\pi(\cdot)$ we get that $X(\cdot) \equiv X^{x,c,\pi}(\cdot)$, where

$$\pi(t) := \frac{1}{H(t)}\left(\sigma^T(t)\right)^{-1}\left[\psi(t) + \left(M(t) - J(t)\right)\theta(t)\right]. \quad (2.22)$$

Since, from (2.21), $X(t) \ge 0$ for $0 \le t \le T$ the pair (c, π) is admissible.

Finally, for t=T, we have from (2.21):

$$X(T) = \frac{S_0(T)}{Z(T)} H(T)\xi = \xi,$$

almost surely. □

2.2 Utility Functions

The agent in this chapter desires to maximize his utility. In this section we develop the properties of the utility functions we consider. We also introduce the convex dual of a utility function.

Definition 2.6

Consider a strictly increasing, strictly concave and continuously differentiable C^1 function $U:(0,\infty)\to\mathbf{R}$ with $U'(\infty)=\lim_{c\to\infty}U'(c)=0$ and $U'(0)=\lim_{c\downarrow 0}U'(c)=\infty$. A function with these properties will be called a *utility function* in the sequel. Because $U':(0,\infty)\to(0,\infty)$ is strictly decreasing, it has an inverse, $I:=(U')^{-1}:(0,\infty)\to(0,\infty)$ which is strictly decreasing, as well.

Definition 2.7

Let U be a utility function. The convex dual of U is defined as the function

$$\tilde{U}(y):=\sup_{x>0}\{U(x)-xy\},\quad y>0. \quad (2.23)$$

Lemma 2.8

Let U be a utility function and $\tilde{U}$ be its convex dual. Then:

(i) $\tilde{U}(y)\geq U(x)-xy,\quad \forall x,y>0$, where the equality holds, if and only if $x=I(y)$,

(ii) $\tilde{U}(y)=U(I(y))-yI(y)$,

(iii) $\tilde{U}'(y)=-I(y),\quad \forall y>0.$

Proof

(i) The inequality comes directly from the definition of supremum in (3.1). To compute this supremum we have:

$$\frac{\partial\{U(x)-xy\}}{\partial x}=0\Leftrightarrow U'(x)-y=0\Leftrightarrow U'(x)=y\Leftrightarrow x=I(y),$$

which makes the equality holds.

(ii) It is an immediate consequence of (i).

(iii) From (ii) we have that:

$$\tilde{U}'(y)=U'(I(y))I'(y)-I(y)-yI'(y)=yI'(y)-I(y)-yI'(y)=-I(y).$$

□

3. Utility Maximization From Consumption And/Or Terminal Wealth

3.1 The Optimization Problems

We formulate three optimization problems for an agent. This agent is sometimes called a *small investor* because his actions do not affect the prices of financial assets.

Definition 3.1

A *preference structure* is a pair of functions $U_1 : [0,T]\times(0,\infty)\to\mathbf{R}$ and $U_2 : (0,\infty)\to\mathbf{R}$ as described below:

(i) For each $t\in[0,T]$, $U_1(t,\cdot)$ is a utility function (Definition 2.6).

(ii) U_1 and U_1' (where the prime denotes differentiation with respect to the second argument) are continuous on their domain.

(iii) U_2 is a utility function.

Let an agent have an initial endowment $x > 0$ and a preference structure (U_1, U_2). The agent can consider three problems whose elements of control are the admissible consumption and portfolio processes.

Problem 3.2 (Utility from consumption)

Find an optimal pair $(c_1,\pi_1)\in\mathscr{A}_1(x)$ for the problem

$$V_1(x) := \sup_{(c,\pi)\in\mathscr{A}_1(x)} E\left[\int_0^T U_1(t,c(t))\,dt\right] \quad (3.1)$$

of maximizing expected total utility from consumption over $[0, T]$, where

$$\mathscr{A}_1(x) := \left\{(c,\pi)\in\mathscr{A}(x); E\int_0^T \min\left[0,U_1(t,c(t))\right]dt > -\infty\right\}. \quad (3.2)$$

Problem 3.3 (Utility from terminal wealth)

Find an optimal pair $(c_2,\pi_2)\in\mathscr{A}_2(x)$ for the problem

$$V_2(x) := \sup_{(c,\pi)\in\mathscr{A}_2(x)} E\left[U_2\left(X^{x,c,\pi}(T)\right)\right] \quad (3.3)$$

of maximizing expected utility from terminal wealth, where

$$\mathscr{A}_2(x) := \left\{(c,\pi) \in \mathscr{A}(x); E\min\left[0, U_2\left(X^{x,c,\pi}(T)\right)\right] > -\infty\right\}. \quad (3.4)$$

Problem 3.4 (Utility from consumption and terminal wealth)

Find an optimal pair $(c_3, \pi_3) \in \mathscr{A}_3(x)$ for the problem

$$V_3(x) := \sup_{(c,\pi)\in\mathscr{A}_3(x)} E\left[\int_0^T U_1(t, c(t))\,dt + U_2\left(X^{x,c,\pi}(T)\right)\right] \quad (3.5)$$

of maximizing expected total utility from both consumption and terminal wealth, where

$$\mathscr{A}_3(x) := \mathscr{A}_1(x) \cap \mathscr{A}_2(x). \quad (3.6)$$

Of course, since $\mathscr{A}(x) = \varnothing$ for $x < 0$, we have $\mathscr{A}_i(x) = \varnothing$ for $x < 0$ and i=1,2,3. We adopt the convention that the supremum over the empty set is - ∞ . In the next sections we shall strive to compute the *value junctions* V_1, V_2 and V_3 of these problems and to characterize (or even compute) optimal pairs $(c_i, \pi_i), i = 1, 2, 3$, that attain the suprema in (3.1), (3.3) and (3.5), respectively.

3.2 Utility From Consumption And Terminal Wealth

In order to solve Problem 3.4, for $x > 0$ we know from the budget constraint (2.19) and Theorem 2.5 that (3.7) amounts to maximizing the expression $E\left[\int_0^T U_1(t, c(t))\,dt + U_2(\xi)\right]$ over pairs (c, ξ), consisting of a consumption process $c(\cdot)$ and a nonnegative $\mathscr{F}(T)$ - measurable random variable ξ, that satisfy the budget constraint:

$$E\left[\int_0^T H(t)c(t)\,dt + H(T)\xi\right] \le x.$$

Now, if $y > 0$ is a "Lagrange multiplier" that enforces this constraint, the problem reduces to the unconstrained maximization of:

$$E\left[\int_0^T U_1(t, c(t))\,dt + U_2(\xi)\right] + y\left(x - E\left[\int_0^T H(t)c(t)\,dt + H(T)\xi\right]\right).$$

Therefore, recalling Lemma 2.8 (i), we have that:

$$\begin{aligned} E\left[\int_0^T U_1(t, c(t))\,dt + U_2(\xi)\right] &\le E\left[\int_0^T U_1(t, c(t))\,dt + U_2(\xi)\right] + y\left(x - E\left[\int_0^T H(t)c(t)\,dt + H(T)\xi\right]\right) \\ &= xy + E\int_0^T \left[U_1(t, c(t)) - yH(t)c(t)\right]dt + E\left[U_2(\xi) - yH(T)\xi\right] \\ &\le xy + E\left[\int_0^T \tilde{U}_1(t, yH(t))\,dt + \tilde{U}_2(yH(T))\right] \end{aligned}$$

with equality if and only if: $c(t) = I_1(t, yH(t)),\ 0 \le t \le T$, and $\xi = I_2(yH(T))$.

Making the appropriate substitutions in (2.20), we define now the function

$$\mathscr{X}_3(y) := E\left[\int_0^T H(t) I_1(t, yH(t))\,dt + H(T) I_2(yH(T))\right], \quad 0 < y < \infty. \quad (3.7)$$

<u>Assumption 3.5</u>

$$\mathscr{X}_3(y) < \infty, \quad \forall y \in (0,\infty).$$

<u>Remark 3.6</u>

Under this assumption, the function $\mathscr{X}_3$ inherits all the properties of the functions I_1 and I_2; namely, it is continuous on $(0,\infty)$ and strictly decreasing on $(0,\infty)$. In particular, the function $\mathscr{X}_3$ has a strictly decreasing inverse function:

$$\mathscr{Y}_3 : (0,\infty) \xrightarrow{onto} (0,\infty), \text{ so that}$$

$$\mathscr{X}_3(\mathscr{Y}_3(x)) = x, \quad \forall x \in (0,\infty). \quad (3.8)$$

Quite clearly, $y = \mathscr{Y}_3(x)$ is the only value of $y>0$ for which the above pair (c,ξ) satisfies the budget constraint with equality; (cf.(2.20)). Thus, for every $x \in (0,\infty)$, we are led to the candidate optimal wealth:

$$\xi_3 := I_2(\mathscr{Y}_3(x) H(T)) \quad (3.9)$$

and the candidate optimal consumption process:

$$c_3(t) := I_1(t, \mathscr{Y}_3(x) H(t)),\ 0 \le t \le T. \quad (3.10)$$

From (4.1), (4.2), we have:

$$E\left[\int_0^T H(u) c_3(u)\,dt + H(T)\xi_3\right] = \mathscr{X}_3(\mathscr{Y}_3(x)) = x \quad (3.11)$$

and due to Theorem 2.5 there exists a candidate optimal portfolio process $\pi_3(\cdot)$ such that $(c_3, \pi_3) \in \mathscr{A}(x)$ and $\xi_3 = X^{x,c_3,\pi_3}(T)$.

<u>Theorem 3.7</u>

Suppose that Assumption 3.5 holds, let $x \in (0,\infty)$ be given, let ξ_3 and $c_3(\cdot)$ be given by (3.9), (3.10) and let $\pi_3(\cdot)$ be such that $(c_3,\pi_3) \in \mathscr{A}(x)$ and $\xi_3 = X^{x,c_3,\pi_3}(T)$. Then $(c_3,\pi_3) \in \mathscr{A}_3(x)$ and (c_3,π_3) is optimal for Problem 3.4:

$$V_3(x) = E\left[\int_0^T U_1(t,c_3(t))\,dt + U_2\left(X^{x,c_3,\pi_3}(T)\right)\right]. \quad (3.12)$$

<u>Proof</u>

We only need to show that $(c_3,\pi_3) \in \mathscr{A}_3(x)$. Indeed, for $x = 1$ and $y = \mathscr{Y}_3(x)H(t)$, we have from that

$$U_1(t,c_3(t)) \ge U_1(t,c_3(t)) - \mathscr{Y}_3(x)H(t)c_3(t) = \tilde{U}_1(t,\mathscr{Y}_3(x)H(t)) \ge U_1(t,1) - \mathscr{Y}_3(x)H(t)$$

and

$$U_2(\xi_3) \ge U_2(\xi_3) - \mathscr{Y}_3(x)H(T)\xi_3 = \tilde{U}_2(\mathscr{Y}_3(x)H(T)) \ge U_2(1) - \mathscr{Y}_3(x)H(T).$$

Consequently,

$$\begin{aligned} E\left\{\int_0^T \min\left[0,U_1(t,c_3(t))\right]dt + \min\left[0,U_2(\xi_3)\right]\right\} &\ge \int_0^T \min\left[0,U_1(t,1)\right]dt + \min\left[0,U_2(1)\right] \\ &\quad - \mathscr{Y}_3(x)E\left[\int_0^T H(t)\,dt + H(T)\right] \\ &> -\infty \end{aligned}$$

□

<u>Remark 3.8</u>

Assume that $V_3(x) < \infty$. It follows that $c_3(\cdot)$ is the unique optimal consumption process and ξ_3 is the unique optimal terminal wealth. This implies also that $\pi_3(\cdot)$ is the unique optimal portfolio process, again up to almost everywhere equivalence.

Making use of (2.17), (2.20)-(2.22), (3.9), (3.10) and (3.12), we conclude to the following result.

<u>Corollary 3.9</u>

Under the Assumption 3.5 the optimal wealth process $X_3(\cdot) = X^{x,c,\pi}(\cdot)$ is:

$$X_3(t) := \frac{1}{H(t)}E\left[\int_t^T H(s)c_3(s)\,ds + H(T)\xi_3 / \mathscr{F}(t)\right], \quad 0 \le t \le T. \quad (3.13)$$

Furthermore, the optimal portfolio $\pi_3(\cdot)$ is given by:

$$\sigma^T(\cdot)\pi_3(\cdot) = \frac{\psi_3(\cdot)}{H(\cdot)} + X_3(\cdot)\theta(\cdot), \quad (3.14)$$

in terms of the integrand $\psi_3(\cdot)$ in the stochastic integral representation $M_3(t) = x + \int_0^t \psi_3^T(s)\,dW(s)$ of the martingale

$$M_3(t) := E\left[\int_0^T H(s)c_3(s)\,ds + H(T)\xi_3 / \mathscr{F}(t)\right], \quad 0 \le t \le T. \quad (3.15)$$

The value function V_3 is then given as

$$V_3(x) = G_3(\mathscr{Y}_3(x)), \quad 0 < x < \infty, \quad (3.16)$$

where

$$G_3(y) := E\left[\int_0^T U_1(t, I_1(t, yH(t)))\,dt + U_2(I_2(yH(T)))\right], \quad 0 < y < \infty. \quad (3.17)$$

<u>Example 3.10:</u>

For every $0 \le t \le T$ let $U_1(t,x) = U_2(t,x) = \log x$ for $x > 0$. Then, $I_1(t,y) = I_2(y) = \frac{1}{y}$ for $y > 0$, and (3.7) gives

$$\mathscr{X}_3(y) = E\left[\int_0^T H(t)\,\frac{1}{yH(t)} + H(T)\frac{1}{yH(T)}\right] = \frac{T+1}{y}, \quad 0 < y < \infty$$

so that its inverse is

$$\mathscr{Y}_3(x) = \frac{T+1}{x}, \quad 0 < x < \infty.$$

The optimal terminal wealth, consumption and wealth processes are given respectively by (3.9), (3.10) and (3.13), as:

$$\xi_3 = \frac{1}{\mathscr{Y}_3(x)H(T)} = \frac{x}{(T+1)H(T)},$$

$$c_3(t) = \frac{1}{\mathscr{Y}_3(x)H(t)} = \frac{x}{(T+1)H(t)}$$

and

$$X_3(t) = \frac{1}{H(t)}E\left[\int_t^T H(s)\,\frac{x}{(T+1)H(s)}\,ds + H(T)\frac{x}{(T+1)H(T)} \Big/ \mathscr{F}(t)\right]$$

$$= \frac{x(T-t+1)}{(T+1)H(t)}, \quad 0 \le t \le T.$$

In particular, from (3.15) we have that $M_3(\cdot) = x$, so $\psi_3(t) = 0$, and the optimal portfolio, given by (3.14), is

$$\pi_3(t) = (\sigma\sigma^T)^{-1}(t)(b(t) - r(t)\underset{\sim}{1})X_3(t), \quad 0 \le t \le T, \quad (3.18)$$

where we have also used (2.12).

Furthermore, (3.16) and (3.17) show that

$$G_3(y) = E\left[\int_0^T \log\left(\frac{1}{yH(t)}\right)dt + \log\left(\frac{1}{yH(T)}\right)\right]$$
$$= -(T+1)\log y - E\left[\int_0^T \log H(t)\,dt + \log H(T)\right], \quad 0 < y < \infty,$$

and

$$V_3(x) = -(T+1)\log \mathcal{Y}_3(x) - E\left[\int_0^T \log H(t)\,dt + \log H(T)\right]$$
$$= (T+1)\log\left(\frac{x}{T+1}\right) - E\left[\int_0^T \log H(t)\,dt + \log H(T)\right], \quad 0 < x < \infty.$$

<u>Example 3.11</u>

For every $0 \le t \le T$ let $U_1(t,x) = U_2(t,x) = \frac{1}{p}x^p$ for $x > 0,\ p < 1,\ p \neq 0$.

Then, $I_1(t,y) = I_2(y) = y^{\frac{1}{p-1}}$ for $0 < y < \infty$, and (3.7) gives

$$\mathcal{X}_3(y) = E\left[\int_0^T H(t)\left(yH(t)\right)^{\frac{1}{p-1}}dt + H(T)\left(yH(T)\right)^{\frac{1}{p-1}}\right]$$
$$= y^{\frac{1}{p-1}}E\left[\int_0^T H(t)^{\frac{p}{p-1}}\,dt + H(T)^{\frac{p}{p-1}}\right] = y^{\frac{1}{p-1}}\mathcal{X}_3(1), \quad 0 < y < \infty,$$

so that its inverse is

$$\mathcal{Y}_3(x) = \left(\frac{x}{\mathcal{X}_3(1)}\right)^{p-1}, \quad 0 < x < \infty.$$

The optimal terminal wealth, consumption and wealth processes are given respectively by (3.9), (3.10) and (3.13), as:

$$\xi_3 = \frac{x}{\mathcal{X}_3(1)}\left(H(T)\right)^{\frac{1}{p-1}}, \quad c_3(t) = \frac{x}{\mathcal{X}_3(1)}\left(H(t)\right)^{\frac{1}{p-1}}$$

and

$$X_3(t) = \frac{x}{\mathcal{X}_3(1)H(t)}E\left[\int_t^T \left(H(s)\right)^{\frac{p}{p-1}}\,ds + \left(H(T)\right)^{\frac{p}{p-1}} \Big/ \mathcal{F}(t)\right].$$

Finally, from (3.16) and (3.17) we get that

$$G_3(y) = E\left\{\int_0^T \frac{1}{p}\left[\left(yH(t)\right)^{\frac{1}{p-1}}\right]^p dt + \frac{1}{p}\left[\left(yH(T)\right)^{\frac{1}{p-1}}\right]^p\right\}$$
$$= \frac{1}{p}y^{\frac{p}{p-1}}\mathcal{X}_3(1), \quad 0 < y < \infty,$$

$$V_3(x) = \frac{1}{p}\mathcal{Y}_3(x)^{\frac{p}{p-1}}\mathcal{X}_3(1) = \frac{1}{p}x^p\left(\mathcal{X}_3(1)\right)^{1-p}, \quad 0 < x < \infty.$$

We may write

$$\left(H(t)\right)^{\frac{p}{p-1}} = m(t)\Lambda(t) \quad (3.19),$$

where

$$m(t) := \exp\left\{\frac{p}{1-p}\int_0^t r(s)\,ds + \frac{p}{2(1-p)^2}\int_0^t \|\theta(s)\|^2\,ds\right\},$$

$$\Lambda(t) := \exp\left\{\frac{p}{1-p}\int_0^t \theta^T(s)\,dW(s) - \frac{p^2}{2(1-p)^2}\int_0^t \|\theta(s)\|^2\,ds\right\}.$$

If $r(\cdot), \theta(\cdot)$ are *deterministic*, then $m(\cdot)$ is deterministic and $\Lambda(\cdot)$ is a martingale,

$$d\Lambda(t) = \frac{p}{p-1}\Lambda(t)\theta^T(t)\,dW(t) \quad (3.20)$$

so,

$$E\left[\left(H(s)\right)^{\frac{p}{p-1}} \big/ \mathscr{F}(t)\right] = E\left[m(s)\Lambda(s)/\mathscr{F}(t)\right] = m(s)E\left[\Lambda(s)/\mathscr{F}(t)\right] = m(s)\Lambda(t), \quad (3.21)$$

for every $0 \le t \le s \le T$.

Furthermore, using (3.15), (3.19) and (3.20), we obtain that

$$\begin{aligned}
M_3(t) &= E\left[\int_0^T H(s)\,\frac{x}{\mathscr{X}_3(1)}\left(H(s)\right)^{\frac{1}{p-1}}ds + H(T)\frac{x}{\mathscr{X}_3(1)}\left(H(T)\right)^{\frac{1}{p-1}} \Big/ \mathscr{F}(t)\right] \\
&= \frac{x}{\mathscr{X}_3(1)}E\left[\int_0^T \left(H(s)\right)^{\frac{p}{p-1}}ds + \left(H(T)\right)^{\frac{p}{p-1}} \Big/ \mathscr{F}(t)\right] \\
&= \frac{x}{\mathscr{X}_3(1)}\left\{\int_0^t \left(H(s)\right)^{\frac{p}{p-1}}ds + \int_t^T E\left[\left(H(s)\right)^{\frac{p}{p-1}} \Big/ \mathscr{F}(t)\right]ds + E\left[\left(H(T)\right)^{\frac{p}{p-1}} \Big/ \mathscr{F}(t)\right]\right\} \\
&= \frac{x}{\mathscr{X}_3(1)}\left\{\int_0^t m(s)\Lambda(s)\,ds + \Lambda(t)\int_t^T m(s)\,ds + m(T)\Lambda(t)\right\}. \qquad (3.22)
\end{aligned}$$

Moreover, if we define $N(t) := \int_0^t m(s)ds + m(t)$, then

$$\begin{aligned}
\mathscr{X}_3(1) &= E\left[\int_0^T \left(H(t)\right)^{\frac{p}{p-1}}dt + \left(H(T)\right)^{\frac{p}{p-1}}\right] \\
&= E\left[\int_0^T m(t)\Lambda(t)\,dt + m(T)\Lambda(T)\right] \\
&= \int_0^T m(t)E\left[\Lambda(t)\right]dt + m(T)E\left[\Lambda(T)\right] \\
&= \int_0^T m(t)\,dt + m(T) = N(T);
\end{aligned}$$

the martingale property gives that $E[\Lambda(t)] = E[\Lambda(0)] = 1, \quad \forall\, 0 \le t \le T$.

Therefore, (3.22) may be equivalently written as

$$M_3(t) = \frac{x}{N(T)}\left\{\int_0^t m(s)\Lambda(s)\,ds + \Lambda(t)\left[\int_t^T m(s)\,ds + m(T)\right]\right\}$$

and by taking differentials, in conjunction with (3.20), we have

$$dM_3(t) = \frac{x}{N(T)}\left\{m(t)\Lambda(t)dt + d\left[\Lambda(t)\left(\int_t^T m(s)ds + m(T)\right)\right]\right\}$$

$$= \frac{x}{N(T)}\left\{m(t)\Lambda(t)dt + \left(\int_t^T m(s)ds + m(T)\right)\left(\frac{p}{1-p}\Lambda(t)\theta^T(t)dW(t)\right)\right.$$

$$\left. - m(t)\Lambda(t)dt\right\}$$

$$= \frac{x}{N(T)}\Lambda(t)\left(\int_t^T m(s)ds + m(T)\right)X_3(t)\frac{p}{1-p}\theta^T(t)dW(t).$$

On the other hand, performing the same computations as we did in (3.22) we get that

$$X_3(t) = \frac{x}{N(T)H(t)}\Lambda(t)\left(\int_t^T m(s)ds + m(T)\right),$$

and we conclude that

$$dM_3(t) = \frac{p}{1-p}H(t)X_3(t)\theta^T(t)dW(t).$$

But we also know that $M_3(t) = x + \int_0^t \psi_3^T(s)dW(s)$, thus a comparison with the last conclusion reveals that $\psi_3(\cdot) = \frac{p}{1-p}H(\cdot)X_3(\cdot)\theta(\cdot)$, and by solving for $\pi_3(\cdot)$ in (3.14) and recalling (2.12) we arrive at

$$\pi_3(t) = \frac{1}{1-p}\left(\sigma(t)\sigma^T(t)\right)^{-1}\left(b(t) - r(t)\underline{1}\right)X_3(t) \quad (3.23)$$

for the case of deterministic $r(\cdot)$ and $\theta(\cdot)$.

Remark 3.12

We have that $I_2(y) > 0$ for all $y > 0$, so that the random variable ξ_3 of (4.3) is strictly positive almost surely, as is the optimal wealth process $X_3(t)$ of (4.7) for $0 \le t \le T$. We can define the *portfolio proportion*

$$p_3(t) := \frac{\pi_3(t)}{X_3(t)}, \quad 0 \le t \le T,$$

a process which is obviously $\{\mathcal{F}(t)\}$-progressively measurable and satisfies $\int_0^T \|p_3(t)\|^2 dt < \infty$ almost surely. The components of the vector $p_3(t)$ represent the proportions of wealth $X_3(t)$ invested in the respective assets at time $t \in [0,T]$, and equation (2.18) for $X_3(\cdot)$ becomes

$$\frac{X_3(t)}{S_0(t)} = x - \int_0^t \frac{c_3(u)}{S_0(u)}du + \int_0^t \frac{X_3(u)}{S_0(u)}p_3^T(u)\sigma(u)\,d\widetilde{W}(u), \quad 0 \le t \le T.$$

In (3.18) and (3.23) of Examples 3.10 and 3.11, $p_3(t)$ depends on the market processes and the utility functions, but not on the wealth of the agent.

We close this section with the observation that the value function V_3 is a utility function in the sense of Definition 2.6, and we find its derivative and convex dual.

Theorem 3.13

If Assumption 3.5 holds, then the value function V_3 satisfies all the conditions of Definition 2.6 as well as

$$V_3'(x) = \mathscr{Y}_3(x), \quad \forall x \in (0,\infty), \quad (3.24)$$

$$\tilde{V}_3(y) = G_3(y) - y\mathscr{X}_3(y) = E\left[\int_0^T \tilde{U}_1(t, yH(t))\,dt - \tilde{U}_2(yH(T))\right], \quad \forall y \in (0,\infty), \quad (3.25)$$

$$\tilde{V}_3'(y) = -\mathscr{X}_3(y), \quad \forall y \in (0,\infty), \quad (3.26)$$

where $\tilde{V}_3(y) := \sup_{x>0}\{V_3(x) - xy\}, \quad y > 0. \quad (3.27)$

Proof

We first prove the concavity of V_3. Let $x_1, x_2 \in (0,\infty)$ be given and let $(c_1, \pi_1) \in \mathscr{A}_3(x_1)$, $(c_2, \pi_2) \in \mathscr{A}_3(x_2)$ also be given. It is easily verified that for $\lambda_1, \lambda_2 \in (0,1)$ with $\lambda_1 + \lambda_2 = 1$ the consumption/portfolio pair $(c,\pi) := (\lambda_1 c_1 + \lambda_2 c_2, \lambda_1\pi_1 + \lambda_2\pi_2)$ is in $\mathscr{A}_3(x)$ with $x := \lambda_1 x_1 + \lambda_2 x_2$ and $X^{x,c,\pi}(\cdot) = \lambda_1 X^{x_1,c_1,\pi_1}(\cdot) + \lambda_2 X^{x_2,c_2,\pi_2}(\cdot)$.

We know that U_1 and U_2 are concave and using (3.5) we have:

$$\begin{aligned}
\lambda_1 V_3(x_1) + \lambda_2 V_3(x_2) &= \lambda_1 E\left[\int_0^T U_1(t, c_1(t))\,dt + U_2\left(X^{x_1,c_1,\pi_1}(T)\right)\right] \\
&\quad + \lambda_2 E\left[\int_0^T U_1(t, c_2(t))\,dt + U_2\left(X^{x_2,c_2,\pi_2}(T)\right)\right] \\
&\le E\left[\int_0^T U_1(t, \lambda_1 c_1(t) + \lambda_2 c_2(t))\,dt + U_2\left(\lambda_1 X^{x_1,c_1,\pi_1}(T) + \lambda_2 X^{x_2,c_2,\pi_2}(T)\right)\right] \\
&= E\left[\int_0^T U_1(t, c(t))\,dt + U_2\left(X^{x,c,\pi}(T)\right)\right] \\
&\le V_3(x) = V_3(\lambda_1 x_1 + \lambda_2 x_2).
\end{aligned}$$

Consequently, maximizing over $(c_1, \pi_1) \in \mathscr{A}_3(x_1)$ and $(c_2, \pi_2) \in \mathscr{A}_3(x_2)$, we obtain

$$\lambda_1 V_3(x_1) + \lambda_2 V_3(x_2) \le V_3(\lambda_1 x_1 + \lambda_2 x_2).$$

□

It is easily seen from (3.16) that V_3 is strictly increasing as a composition of $\mathscr{Y}_3$ with $\mathscr{X}_3$ which are strictly decreasing. As a concave function V_3 is also continuous on $(0,\infty)$. Furthermore, if (3.24) holds, which is proved below, we have that V_3' is continuous.

We turn to (3.25). The second equation in (3.25) follows directly from Lemma 2.8(ii), (3.7) and (3.17). For the first, let $Q(y) := G_3(y) - y\mathscr{X}_3(y)$ for $0 < y < \infty$, and observe from (2.23) $\quad U_1(t,c(t)) \le \tilde{U}_1(t, yH(t)) + yH(t)c(t),\ 0 \le t \le T,$

that: $\quad U_2(X^{x,c,\pi}(T)) \le \tilde{U}_2(yH(T)) + yH(T)X^{x,c,\pi}(T)$

hold almost surely for any $y > 0, x > 0$ and $(c,\pi) \in \mathscr{A}_3(x)$. Consequently, from the second equation of (3.25), the definition of $Q(\cdot)$ and the budget constraint (2.19), we have

$$\begin{aligned} &E\left[\int_0^T U_1(t,c(t))\,dt + U_2(X^{x,c,\pi}(T))\right] \\ &\le Q(y) + yE\left[\int_0^T H(t)c(t)\,dt + H(T)X^{x,c,\pi}(T)\right] \\ &\le Q(y) + xy \qquad\qquad (3.28) \end{aligned}$$

with equality if and only if,

$$c(t) = I_1(t, yH(t)), \quad X^{x,c,\pi}(T) = I_2(yH(T)) \quad (3.29)$$

and

$$E\left[\int_0^T H(t)c(t)\,dt + H(T)X^{x,c,\pi}(T)\right] = x.$$

Taking the supremum in (3.28) over $(c,\pi) \in \mathscr{A}(x)$, we obtain $V_3(x) \le Q(y) + xy$ for all $x \in \mathbf{R}$.

$$V_3(x) \le Q(y) + xy \Rightarrow \sup_{x>0}\{V_3(x) - xy\} \le Q(y)$$

and thus $\tilde{V}_3(y) \le Q(y)$, for all $y > 0$. For the reverse inequality, observe that equality holds in (3.28) if (3.29) is satisfied and $x = \mathscr{X}_3(y) \Rightarrow y = \mathscr{Y}_3(x)$. Consequently,

$$\begin{aligned} Q(y) &= E\left[\int_0^T U_1(t, I_1(t, yH(t)))\,dt + U_2(I_2(yH(T)))\right] - xy \\ &= G_3(y) - xy = G_3(\mathscr{Y}_3(x)) - \mathscr{X}_3(y)y = V_3(x) - \mathscr{X}_3(y)y \\ &= V_3(\mathscr{X}_3(y)) - \mathscr{X}_3(y)y \\ &\le \sup_{x\in\mathbf{R}}\{V_3(x) - xy\} = \tilde{V}_3(y). \end{aligned}$$

Therefore, this gives $Q(y) = V_3(\mathscr{X}_3(y)) - \mathscr{X}_3(y)y \le \tilde{V}_3(y)$.

This completes the proof of (3.25) and shows that for $y > 0$, the maximum in (3.27) is attained by $x = \mathscr{X}_3(y)$.

To prove (3.26), we use Lemma 2.10(ii) and $\tilde{U}'(y) = -I(y), \quad \forall y > 0$ to write for any utility function U and for $0 < z < y < \infty$,

$$
\begin{aligned}
yI(y) - zI(y) - \int_z^y I(\xi)\,d\xi &= yI(y) - zI(z) + \tilde{U}(y) - \tilde{U}(z) \\
&= U(I(y)) - U(I(z)). \qquad (3.30)
\end{aligned}
$$

Therefore, applying the change of variables formula and by the change of variable, $\xi = \lambda H_0(t)$, we have:

$$
\begin{aligned}
& y\mathscr{X}_3(y) - z\mathscr{X}_3(z) - \int_z^y \mathscr{X}_3(\lambda)\,d\lambda \\
&= E\int_0^T \left[yH(t) I_1(t, yH(t)) - zH(t) I_1(t, zH(t)) - \int_{zH_0(t)}^{yH_0(t)} I_1(t,\xi)\,d\xi \right] dt \\
&\quad + E\left[yH(t) I_2(yH(T)) - zH(T) I_2(zH(T)) - \int_{zH_0(t)}^{yH_0(t)} I_2(\xi)\,d\xi \right] \\
&= E\int_0^T \left[yH(t) I_1(t, yH(t)) - zH(t) I_1(t, zH(t)) - \int_z^y H(t) I_1(t, \lambda H(t))\,d\lambda \right] dt \\
&\quad + E\left[yH(t) I_2(yH(T)) - zH(T) I_2(zH(T)) - \int_z^y I_2(\lambda H(t))\,d\lambda \right] \\
&= E\int_0^T \left[U_1(t, I_1(t, yH(t))) - U_1(t, I_1(t, zH(t))) \right] dt \\
&\quad + E\left[U_2(I_2(yH(T))) - U_2(I_2(yH(T))) \right] \\
&= G_3(y) - G_3(z) \qquad (3.31)
\end{aligned}
$$

or equivalently by (3.25) we have that

$$
\tilde{V}_3(y) - \tilde{V}_3(z) = -\int_z^y \mathscr{X}_3(\lambda)\,d\lambda, \quad 0 < z < y < \infty \quad (3.32)
$$

and (3.26) follows.

The function V can be recovered from $\tilde{V}$ by the Legendre transform inversion formula (cf.(3.27))

$$
V(t,x) = \inf_{y>0} \left\{ \tilde{V}(t,y) + xy \right\}, \quad x > 0.
$$

To compute this infimum we have:

$$
\frac{\partial \left\{ \tilde{V}_3(y) + xy \right\}}{\partial y} = 0 \Leftrightarrow \tilde{V}_3'(y) + x = 0 \Leftrightarrow \mathscr{X}_3(y) = x \Leftrightarrow \mathscr{Y}_3(x) = y,
$$

which makes the equality holds. Therefore,

$$
\begin{aligned}
V_3(x) &= \tilde{V}_3(\mathscr{Y}_3(x)) + x\mathscr{Y}_3(x), \\
V_3'(x) &= \tilde{V}_3'(\mathscr{Y}_3(x))\mathscr{Y}_3'(x) + \mathscr{Y}_3(x) + x\mathscr{Y}_3'(x) \\
&= -\mathscr{X}_3(\mathscr{Y}_3(x))\mathscr{Y}_3'(x) + \mathscr{Y}_3(x) + x\mathscr{Y}_3'(x) \\
&= \mathscr{Y}_3(x).
\end{aligned}
$$

Remark 3.14

From (3.16) we have $G_3(y) = V_3(\mathscr{X}_3(y))$ for all $y \in (0,\infty)$. If $\mathscr{X}_3'(y)$ exists, then $G_3'(y)$ also exists and is given by the formula

$$G_3'(y) = V_3'(\mathscr{X}_3(y))\mathscr{X}_3'(y) = \mathscr{Y}(\mathscr{X}_3(y))\mathscr{X}_3'(y) = y\mathscr{X}_3'(y), \quad 0 < y < \mathscr{X}_3(\infty). \quad (3.33)$$

3.3 Utility From Consumption Or Terminal Wealth

Theorem 3.17 below provides a complete solution to Problem 3.2 of maximization of expected utility from consumption alone, and Theorem 3.20 does the same for Problem 3.3 of maximization of expected utility from terminal wealth alone. This section also contains examples of these solutions and examines the dual value functions for Problems 3.2 and 3.3. Theorem 3.24 shows how to combine the solutions of these two problems to obtain the solution of Problem 3.4 of maximization of expected utility from consumption plus expected utility from terminal wealth. In particular, the dual value function for Problem 3.4 is the sum of the dual value functions for Problems 3.2 and 3.3.

Let a preference structure (U_1, U_2) be given. We define the functions

$$\mathscr{X}_1(y) := E\left[\int_0^T H(t)\, I_1(t, yH(t))\, dt\right], \quad 0 < y < \infty, \quad (3.34)$$

$$\mathscr{X}_2(y) := E\left[H(T) I_2(yH(T))\right], \quad 0 < y < \infty. \quad (3.35)$$

Assumption 3.15: $\mathscr{X}_1(y) < \infty, \quad \forall y \in (0,\infty),$

Assumption 3.16: $\mathscr{X}_2(y) < \infty, \quad \forall y \in (0,\infty).$

Just as we proved Remark 3.6, we can show that for each $i = 1,2$, under Assumptions 3.13-3.16, the function $\mathscr{X}_i$ is strictly increasing and continuous on $(0,\infty)$ with $\mathscr{X}_i(0) = \infty$ and with $\mathscr{X}_i(\infty) = 0$ and has a strictly decreasing inverse function $\mathscr{Y}_i : (0,\infty) \to (0,\infty)$, $i = 1, 2, \cdots$

The proof of the following theorem parallels the proof of Theorem 3.7 and Corollary 3.9.

Theorem 3.17 (Maximization of the expected utility from consumption)

Let Assumption 3.14 hold, let $x \in (0,\infty)$ be given, and define

$$c_1(t) := I_1(t, \mathscr{Y}_1(x) H(t)), \quad 0 \le t \le T. \quad (3.36)$$

(i) There exists a portfolio $\pi_1(\cdot)$ such that $(c_1, \pi_1) \in \mathscr{A}_1(x)$, $X^{x,c_1,\pi_1}(T) = 0$, and the pair (c_1, π_1) is optimal for Problem 3.2, i.e.,

$$V_1(x) = E\left[\int_0^T U_1(t, c_1(t))\, dt\right].$$

(ii) The optimal wealth process $X_1(t) = X^{x,c_1,\pi_1}(t)$ is

$$X_1(t) = \frac{1}{H(t)} E\left[\int_t^T H(u) c_1(u)\, du / \mathscr{F}(t)\right], \quad 0 \le t \le T. \quad (3.37)$$

(iii) The optimal portfolio $\pi_1(\cdot)$ is given by

$$\sigma^T(t) \pi_1(t) = \frac{\psi_1(t)}{H(t)} + X_1(t) \theta(t), \quad (3.38)$$

where $\psi_1(\cdot)$ is the integrand in the stochastic integral representation $M_1(t) = x + \int_0^t \psi_1^T(u)\, dW(u)$ of the martingale

$$M_1(t) := E\left[\int_0^T H(u) c_1(u)\, du / \mathscr{F}(t)\right].$$

(iv) The value function V_1 is given by

$$V_1(x) = G_1(\mathscr{Y}_1(x)), \quad x > 0, \quad (3.39)$$

where

$$G_1(y) := E\left[\int_0^T U_1(t, I_1(t, yH(t)))\, dt\right], \quad 0 < y < \infty. \quad (3.40)$$

Imitating the proof of Theorem 3.13, we can show that V_1 has the following properties.

Theorem 3.18

Let Assumption 3.15 hold, and assume $V_1(x) < \infty$ for all $x > 0$. Then V_1 satisfies all the conditions of Definition 2.6, and

$$V_1'(x) = \mathscr{Y}_1(x), \quad \forall x \in (0,\infty),$$
$$\tilde{V}_1(y) = G_1(y) - y\mathscr{X}_1(y) = E\left[\int_0^T \tilde{U}_1(t, yH(t))\, dt\right], \quad \forall y \in (0,\infty),$$
$$\tilde{V}_1'(y) = -\mathscr{X}_1(y), \quad \forall y \in (0,\infty), \quad (3.41)$$

where

$$\tilde{V}_1(y) := \sup_{x>0} \{V_1(x) - xy\}, \quad y > 0.$$

<u>Example 3.19</u> (Subsistence consumption)

For every $0 \le t \le T$ suppose $U_1(t,c) = \log c, \quad 0 < c < \infty$. Then $I_1(t,y) = 1/y$ and (3.34) gives $\mathscr{X}_1(y) = T/y$ for $0 < y < \infty$. We have $\mathscr{Y}_1(x) = T/x$ for $x > 0$ and the optimal consumption and wealth processes from (3.36), (3.37) are

$$c_1(t) = \frac{x}{TH(t)}, \quad X_1(t) = \frac{(T-t)x}{TH(t)},$$

respectively.

Furthermore, (3.39) and (3.40) show that

$$G_1(y) = -T\log y - E\int_0^T \log H(t)dt, \ y > 0,$$

$$V_1(x) = T\log x - T\log T - E\left[\int_0^T \log H(t)dt\right], \quad x > 0.$$

Finally, from Theorem 3.17(iii) we have that $M_1(\cdot) = x$, so $\psi_1(\cdot) = 0$, and the optimal portfolio given by (3.38) is

$$\pi_1(t) = \left(\sigma^T(t)\right)^{-1} X_3(t)\theta(t) = \frac{(T-t)x}{TH(t)}\left(\sigma(t)\sigma^T(t)\right)^{-1}\left(b(t) - r(t)\underset{\sim}{1}\right).$$

The analogues of Theorems 3.17, 3.18 for Problem 3.3 are the following.

<u>Theorem 3.20</u> (Maximization of utility from terminal wealth)

Let Assumption 3.16 hold, let $x \in (0,\infty)$ be given, and let

$$\xi_2 = I_2\left(\mathscr{Y}_2(x)H(T)\right).$$

(i) With $c_2 \equiv 0$, there exists a portfolio $\pi_2(\cdot)$ such that $(c_2, \pi_2) \in \mathscr{A}_2(x)$, $X^{x,c_2,\pi_2}(T) = \xi_2$, and the pair (c_2, π_2) is optimal for Problem 3.3, i.e.,

$$V_2(x) = EU_2\left(X^{x,c_2,\pi_2}(T)\right).$$

(ii) The optimal wealth process $X_2(t) = X^{x,c_2,\pi_2}(t)$ is

$$X_2(t) = \frac{1}{H(t)}E\left[H(T)\xi_2/\mathscr{F}(t)\right], \quad 0 \le t \le T.$$

(iii)The optimal portfolio $\pi_2(\cdot)$ is given by

$$\sigma^T(t)\pi_2(t) = \frac{\psi_2(t)}{H(t)} + X_2(t)\theta(t),$$

where $\psi_2(\cdot)$ is the integrand in the stochastic integral representation $M_2(t) = x + \int_0^t \psi_2{}^T(u)dW(u)$ of the martingale.

$$M_2(t) := H(t)X_2(t).$$

(iv)The value function V_2 is given by

$$V_2(x) = G_2(\mathscr{Y}_2(x)), \quad x > 0, \quad (3.42)$$

where

$$G_2(y) := EU_2(I_2(yH(T))), \quad 0 < y < \infty.$$

Theorem 3.21

Let Assumption 3.16 hold, and assume $V_2(x) < \infty$ for all $x > 0$. Then V_2 satisfies all the conditions of Definition 2.6, and

$$\begin{aligned} &V_2'(x) = \mathscr{Y}_2(x), \quad \forall x \in (0,\infty), \\ &\tilde{V}_2(y) = G_2(y) - y\mathscr{X}_2(y) = E\tilde{U}_2(yH(T)), \quad \forall y \in (0,\infty), \\ &\tilde{V}_2'(y) = -\mathscr{X}_2(y), \quad \forall y \in (0,\infty), \quad (3.43) \end{aligned}$$

where

$$\tilde{V}_2(y) := \sup_{x>0}\{V_2(x) - xy\}, \quad y > 0.$$

Remark 3.22

If $\mathscr{X}_1'(y)$ and $\mathscr{X}_2'(y)$ exist, then just as in Remark 3.14, we have for $i = 1,2,3,\cdots$,

$$G_i'(y) = y\mathscr{X}_i'(y), \quad 0 < y < \infty. \quad (3.44)$$

Example 3.23 (Portfolio insurance)

Suppose $U_2(x) = \log x, \quad 0 < x < \infty$. We have $\mathscr{X}_2(y) = 1/y$ for $0 < y < \infty$. Also, we have $\mathscr{Y}_2(x) = 1/x$ for $x > 0$, and the optimal consumption and wealth processes are $c_2(t) \equiv 0$ and

$$X_2(t) = \frac{1}{H(t)}x.$$

Finally, $G_2(y) = -\log y - E\log H(T)$, so

$$V_2(x) = \log x - E\log H(T), \quad x > 0.$$

As in Example 3.19, we can derive the optimal portfolio for $M_2(\cdot) = x$ and $\psi_2(\cdot) = 0$ to explicitly be

$$\pi_2(t) = (\sigma^T(t))^{-1} X_2(t)\theta(t) = \frac{x}{H(t)}(\sigma(t)\sigma^T(t))^{-1}(b(t) - r(t)\underset{\sim}{1}).$$

In the remainder of this section we examine the relationship among the value functions and the optimal policies for Problems 3.2-3.4. Consider an agent with initial endowment $x > 0$ who divides this wealth into two pieces, $x_1 > 0$ and $x_2 > 0$, so that $x_1 + x_2 = x$.

For the piece x_1, he constructs the optimal policy $(c_1, \pi_1) \in \mathscr{A}_1(x_1)$ of Theorem 3.17 for the problem of maximization of utility from consumption only. With the piece x_2, he constructs the optimal policy $(c_2, \pi_2) \in \mathscr{A}_2(x_2)$ of Theorem 3.20 for the problem of maximization of utility from terminal wealth only. Note that $X^{x_1, c_1, \pi_1}(T) = 0$ and $c_2(\cdot) \equiv 0$, so the superposition $(c_1 + c_2, \pi_1 + \pi_2)$ of the policies $(c_1, \pi_1), (c_2, \pi_2)$ is in $\mathscr{A}_3(x)$, results in the wealth process $X^{x,c,\pi}(t) = X^{x_1, c_1, \pi_1}(t) + X^{x_2, c_2, \pi_2}(t)$ which is governed by (2.18), and satisfies

$$V_1(x_1) + V_2(x_2) = E\left[\int_0^T U_1(t, c(t))dt + U_2\left(X^{x,c,\pi}(T)\right)\right] \leq V_3(x).$$

Therefore,

$$\sup\{V_1(x_1) + V_2(x_2); x_1 > 0, x_2 > 0, x_1 + x_2 = x\} \leq V_3(x), \quad \forall x > 0. \quad (3.45)$$

Moreover, the reverse of inequality (3.45) holds. Again, for $x > 0$, let $(c_3, \pi_3) \in \mathscr{A}_3(x)$ be the optimal policy of Theorem 3.7 for the problem of maximization of utility from consumption and terminal wealth; moreover, in conjunction with (3.11), define

$$x_1 := E\int_0^T H_0(t)c_3(t)dt, \quad x_2 := E\left[H_0(T)X^{x,c_3,\pi_3}(T)\right],$$

so that $x_1 + x_2 = x$. Theorem 2.5 and the former definition guarantee for $c(\cdot) = c_3(\cdot)$ and $\xi = 0$, the existence of a portfolio process $\hat{\pi}_1(\cdot)$ such that $X^{x_1, c_3, \hat{\pi}_1}(T) = 0$ and $(c_3, \hat{\pi}_1) \in \mathscr{A}_1(x_1)$; therefore, $E\int_0^T U_1(t, c_3(t))dt \leq V_1(x_1)$. This same theorem for $c(\cdot) = 0$ and $\xi = X^{x,c_3,\pi_3}(T)$, and the latter definition guarantee the existence of a portfolio process $\hat{\pi}_2(\cdot)$ such that with $\hat{c}_2 \equiv 0$, we have $X^{x_2, \hat{c}_2, \hat{\pi}_2}(T) = X^{x,c_3,\pi_3}(T)$ and $(\hat{c}_2, \hat{\pi}_2) \in \mathscr{A}_2(x_2)$; therefore, $EU_2\left(X^{x,c_3,\pi_3}(T)\right) = EU_2\left(X^{x_2,\hat{c}_2,\hat{\pi}_2}(T)\right) \leq V_2(x_2)$. We have then

$$\begin{aligned} V_3(x) &= E\int_0^T U_1(t, c_3(t))dt + EU_2\left(X^{x,c_3,\pi_3}(T)\right) \\ &\leq V_1(x_1) + V_2(x_2). \end{aligned}$$

Therefore, $V_3(x) \leq \sup\{V_1(x_1) + V_2(x_2) : x_1 > 0, x_2 > 0, x_1 + x_2 = x\}, \quad \forall x > 0,$ and we conclude with the following result.

Theorem 3.24

Let Assumption 3.5 hold. Then

$$V_3(x) = \sup\{V_1(x) + V_2(x_2); x_1 + x_2 = x\} \quad \forall x > 0. \quad (3.46)$$

If, in addition, $V_i(x) < \infty$ for all $x > 0$ and $i = 1,2,$ then for each $x \in (0,\infty)$ the supremum in (3.46) is attained by $x_1 = \mathscr{X}_1(\mathscr{Y}_3(x)),\ x_2 = \mathscr{X}_2(\mathscr{Y}_3(x))$. In particular,

$$V_3(x) = V_1(\mathscr{X}_1(\mathscr{Y}_3(x))) + V_2(\mathscr{X}_2(\mathscr{Y}_3(x))), \quad \forall x \in (0,\infty) \quad (3.47)$$

and

$$\tilde{V}_3(y) = \tilde{V}_1(y) + \tilde{V}_2(y), \quad \forall y \in (0,\infty). \quad (3.48)$$

<u>Proof</u>

For fixed $x \in (0,\infty)$, let us consider the concave function $f:(0,x) \to \mathbf{R}$ defined by $f(x_1) := V_1(x_1) + V_2(x - x_1)$. Under the assumption that $V_i < \infty$ for $i = 1,2,$ this function is finite on its domain. Now, $f'(x_1) = \mathscr{Y}_1(x_1) - \mathscr{Y}_2(x - x_1)$ for $0 < x_1 < x$, and f' is continuous and strictly decreasing on this interval.

The maximum in (3.46) is attained by the unique value $x_1 \in (0,x)$ where $f'(x_1) = 0$, i.e., $\mathscr{Y}_1(x_1) = \mathscr{Y}_2(x - x_1)$. We check that $x_1 = \mathscr{X}_1(\mathscr{Y}_3(x))$ solves this equation, to wit,

$$\begin{aligned}\mathscr{Y}_1(\mathscr{X}_1(\mathscr{Y}_3(x))) &= \mathscr{Y}_3(x)\\ &= \mathscr{Y}_2(\mathscr{X}_2(\mathscr{Y}_3(x)))\\ &= \mathscr{Y}_2(\mathscr{X}_3(\mathscr{Y}_3(x)) - \mathscr{X}_1(\mathscr{Y}_3(x)))\\ &= \mathscr{Y}_2(x - \mathscr{X}_1(\mathscr{Y}_3(x))),\end{aligned}$$

since from (3.7), (3.34) and (3.35) we have that

$$\mathscr{X}_1(\cdot) + \mathscr{X}_2(\cdot) = \mathscr{X}_3(\cdot).$$

We have shown that the supremum in (3.46) is attained by $x_1 = \mathscr{X}_1(\mathscr{Y}_3(x)),\ x_2 = x - \mathscr{X}_1(\mathscr{Y}_3(x)) = \mathscr{X}_3(\mathscr{Y}_3(x)) - \mathscr{X}_1(\mathscr{Y}_3(x)) = \mathscr{X}_2(\mathscr{Y}_3(x))$; i.e., (3.47) holds. Equation (3.48) follows immediately from (3.25), (3.41), (3.43), and the definitions of G_i and $\mathscr{X}_i$ for $i = 1,2,3$.

□

4. A Dynamic Programming Framework

4.1 Deterministic Coefficients

In this section we specialize the results of Section 3.2 to the case of continuous, deterministic functions $r(\cdot):[0,T]\to\mathbf{R}$, $\theta(\cdot):[0,T]\to\mathbf{R}^d$ and $\sigma(\cdot):[0,T]\to L(\mathbf{R}^d:\mathbf{R}^d)$, the set of $d\times d$ matrices. In this case, stock prices and the money-market price become Markov processes. We will focus on obtaining an explicit formula for the optimal portfolio $\pi_3(\cdot)$ of Corollary 3.9, whose existence was established there but for which no useful representation apart from (3.14) was provided. We shall show that the value function for Problem 3.4 is a solution to the nonlinear, second-order parabolic Hamilton-Jacobi-Bellman partial differential equation one would expect (Theorem 4.9), and the dual value function is the unique solution of a linear second-order parabolic partial differential equation (Theorem 4.10). Several examples are provided.

We shall represent both the optimal portfolio $\pi_3(\cdot)$ and also the optimal consumption rate process $c_3(\cdot)$ in "feedback form" on the level of wealth $X_3(\cdot)$ of (3.13), i.e.,

$$c_3(t)=C(t,X_3(t)),\quad \pi_3(t)=\Pi(t,X_3(t)),\quad 0\le t\le T,\quad (4.1)$$

for suitable functions $C:[0,T]\times(0,\infty)\to[0,\infty)$ and $\Pi:[0,T]\times(0,\infty)\to\mathbf{R}^d$ (cf. Theorem 4.7), which do not depend on the initial wealth. Such a representation shows that in the case of deterministic coefficients the current level of wealth is a sufficient statistic for the utility maximization Problem 3.4: an investor who computes his optimal strategy at time t on the basis of his current wealth only, can do just as well as an investor who keeps track of the whole past and present information $\mathscr{F}(t)$ about the market! Similar results hold for Problems 3.2 and 3.3; their derivation is straightforward. Throughout this section, the following two assumptions will be in force. These are not the weakest assumptions that support the subsequent analysis, but they will permit us to proceed with a minimum of technical fuss.

<u>Assumption 4.1</u>

We have that the processes $r(\cdot),\theta(\cdot)$ and $\sigma(\cdot)$ are nonrandom, continuous (and hence bounded) functions on $[0,T]$, and $r(\cdot)$ and $\|\theta(\cdot)\|$ are in fact Holder continuous, i.e., for some $K>0$ and $\rho\in(0,1)$ we have

$$|r(t_1)-r(t_2)|\le K|t_1-t_2|^{\rho},\quad \|\theta(t_1)-\theta(t_2)\|\le K|t_1-t_2|^{\rho}$$

for all $t_1, t_2 \in [0,T]$. Furthermore, $\|\theta(\cdot)\|$ is bounded away from zero and infinity. In particular, there are positive constants κ_1, κ_2 such that

$$0 < \kappa_1 \le \|\theta(t)\| \le \kappa_2 < \infty, \quad \forall t \in [0,T],$$

almost surely.

Because of Novikov's condition (e.g., Karatzas and Shreve (1991)), Assumption 4.1 guarantees that the local martingale $Z(\cdot)$ is in fact a martingale. This permits the construction of the martingale measure $\tilde{P}$ under which the process $\tilde{W}$ of (2.14) is a Brownian motion. We have not needed this probability measure in previous sections, but we shall make use of it in this section.

Assumption 4.2

The agent's preference structure (U_1, U_2) satisfies the terminal conditions:

(i) (polynomial growth of I_1 and I_2) there is a constant $\gamma > 0$ such that

$$I_1(t,y) \le \gamma + y^{-\gamma} \quad \forall (t,y) \in [0,T] \times (0,\infty),$$
$$I_2(y) \le \gamma + y^{-\gamma} \quad \forall y \in (0,\infty);$$

(ii) (polynomial growth of $U_1 \circ I_1$ and $U_2 \circ I_2$) there is a constant $\gamma > 0$ such that

$$U_1(t, I_1(t,y)) \ge -\gamma - y^{\gamma} \quad \forall (t,y) \in [0,T] \times (0,\infty),$$
$$U_2(I_2(y)) \ge -\gamma - y^{\gamma} \quad \forall y \in (0,\infty);$$

(iii) (Holder continuity of I_1) for each $y_0 \in (0,\infty)$, there exist constants $\varepsilon(y_0) > 0$, $K(y_0) > 0$, and $\rho(y_0) \in (0,1)$ such that

$$|I_1(t,y) - I_1(t,y_0)| \le K(y_0)|y - y_0|^{\rho(y_0)} \quad \forall t \in [0,T],$$
$$\forall y \in (0,\infty) \cap (y_0 - \varepsilon(y_0), y_0 + \varepsilon(y_0));$$

(iv) either for each $t \in [0,T], I_1'(t,y) := \frac{\partial}{\partial y} I_1(t,y)$ is defined and strictly negative for all y in a set of positive Lebesgue measure, or else $I_2'(y)$ is defined and strictly negative for all y in a set of positive Lebesgue measure.

Remark 4.3

Because $I_1(t,\cdot)$ and I_2 are strictly decreasing, Assumption 4.2(i), (ii) implies the existence of a constant $\gamma > 0$ such that

$$|U_1(t, I_1(t,y))| \le \gamma + y^{\gamma} + y^{-\gamma} \quad \forall (t,y) \in [0,T] \times (0,\infty),$$
$$|U_2(I_2(y))| \le \gamma + y^{\gamma} + y^{-\gamma} \quad \forall y \in (0,\infty).$$

Furthermore, for each $y_0 \in (0,\infty)$ and $\varepsilon(y_0)$, $K(y_0)$,and $\rho(y_0)$ as in Assumption 4.2(iii), the mean value theorem implies for all $y \in (0,\infty) \cap (y_0 - \varepsilon(y_0), y_0 + \varepsilon(y_0))$ that

$$\begin{aligned}\left|U_1(t, I_1(t,y)) - U_1(t, I_1(t,y_0))\right| &\le U_1'(t, \imath(t))\left|I_1(t,y) - I_1(t,y_0)\right| \\ &\le M\, K(y_0)\left|y - y_0\right|^{\rho(y_0)},\end{aligned}$$

where $\imath(t)$ takes values between $I_1(t,y)$ and $I_1(t,y_0)$ and M is a bound on the continuous function $U_1'(t, I_1(t,\eta))$ as (t,η) ranges over the $[0,T]\times[(0,\infty)\cap(y_0 - \varepsilon(y_0), y_0 + \varepsilon(y_0))]$. In other words, $U_1 \circ I_1$ enjoys the same kind of Holder continuity posited in Assumption 4.2(iii) for I_1.

4.2 Feedback Formulae

We introduce the process

$$\begin{aligned}Y^{(t,y)}(s) &:= y\exp\left\{-\int_t^s r(u)\,du - \int_t^s \theta^T(u)\,dW(u) - \frac{1}{2}\int_t^s \|\theta(u)\|^2\,du\right\} \\ &= y\exp\left\{-\int_t^s r(u)\,du - \int_t^s \theta^T(u)\,d\tilde{W}(u) + \frac{1}{2}\int_t^s \|\theta(u)\|^2\,du\right\}, \quad t \le s \le T, \qquad (4.2)\end{aligned}$$

for any given $(t,y) \in [0,T]\times(0,\infty)$. The process $Y^{(t,y)}(\cdot)$ is a diffusion and from Ito's formula it has the linear dynamics:

$$\begin{aligned}dY^{(t,y)}(s) &= Y^{(t,y)}(s)\left[-r(s)\,ds - \theta^T(s)\,dW(s)\right] \\ &= Y^{(t,y)}(s)\left[-r(s)\,ds + \|\theta(s)\|^2\,ds - \theta^T(s)\,d\tilde{W}(s)\right]\end{aligned}$$

with $Y^{(t,y)}(t) = y$ and $Y^{(t,y)}(s) = yY^{(t,1)}(s) = yH(s)/H(t)$. With these properties in mind, and using the Markov property for $Y^{(t,y)}(\cdot)$ under the martingale measure $\tilde{P}$, as well as (3.9), (3.10) and "Bayes' rule": $\tilde{E}\left[Y(t)/\mathscr{F}(s)\right] = \dfrac{E\left[Y(t)Z(t)/\mathscr{F}(s)\right]}{Z(s)}$, we may rewrite the expression (3.13) for the optimal wealth as follows:

$$\begin{aligned}X_3(t) &= \frac{1}{Z(t)}E\left[\int_t^T Z(s)e^{-\int_t^s r(u)du} I_1\left(s, zY^{(0,1)}(s)\right)ds \right. \\ &\qquad \left. + Z(T)e^{-\int_t^T r(u)du} I_2\left(zY^{(0,1)}(T)\right)\Big/\mathscr{F}(t)\right] \\ &= \tilde{E}\left[\int_t^T e^{-\int_t^s r(u)du} I_1\left(s, Y^{(0,z)}(s)\right)ds \right. \\ &\qquad \left. + e^{-\int_t^T r(u)du} I_2\left(Y^{(0,z)}(T)\right)\Big/\mathscr{F}(t)\right] \\ &= \mathscr{X}\left(t, Y^{(0,z)}(t)\right), \quad 0 \le t \le T, \qquad (4.3)\end{aligned}$$

where $z = \mathscr{Y}_3(x)$, $\tilde{E}$ denotes expectation with respect to the martingale measure $\tilde{P}$,and $\mathscr{X} : [0,T]\times(0,\infty)\to(0,\infty)$ is given by

$$\mathscr{X}(t,y) := \tilde{E}\Big[\int_t^T e^{-\int_t^s r(u)du} I_1\left(s, yY^{(t,1)}(s)\right)ds + e^{-\int_t^T r(u)du} I_2\left(yY^{(t,1)}(T)\right)\Big]. \quad (4.4)$$

The Markov property for $Y^{(t,y)}(\cdot)$ under *P* implies that

$$E\Big[\int_t^T Y^{(t,y)}(s) I_1\left(s, Y^{(t,y)}(s)\right)ds + Y^{(t,y)}(T) I_2\left(Y^{(t,y)}(T)\right) \big/ \mathscr{F}(t)\Big]$$

is a function of $Y^{(t,y)}(t) = y$, i.e., is deterministic. Therefore,

$$\begin{aligned}
\mathscr{X}(t,y) &= \tilde{E}\Big\{\tilde{E}\Big[\int_t^T e^{-\int_t^s r(u)du} I_1\left(s, Y^{(t,y)}(s)\right)ds \\
&\qquad + e^{-\int_t^T r(u)du} I_2\left(Y^{(t,y)}(T)\right)\big/\mathscr{F}(t)\Big]\Big\} \\
&= E\Big\{\frac{Z(T)}{Z(t)} E\Big[\int_t^T e^{-\int_t^s r(u)du} Z(s) I_1\left(s, Y^{(t,y)}(s)\right)ds \\
&\qquad + e^{-\int_t^T r(u)du} Z(T) I_2\left(Y^{(t,y)}(T)\right)\big/\mathscr{F}(t)\Big]\Big\} \\
&= \frac{1}{y} E\Big\{Z(T) E\Big[\int_t^T Y^{(t,y)}(s) I_1\left(s, Y^{(t,y)}(s)\right)ds \\
&\qquad + Y^{(t,y)}(T) I_2\left(Y^{(t,y)}(T)\right)\big/\mathscr{F}(t)\Big]\Big\} \\
&= \frac{1}{y} E\Big[\int_t^T Y^{(t,y)}(s) I_1\left(s, Y^{(t,y)}(s)\right)ds \\
&\qquad + Y^{(t,y)}(T) I_2\left(Y^{(t,y)}(T)\right)\Big], \qquad (4.5)
\end{aligned}$$

and $\mathscr{X}(\cdot\,,\cdot)$ is an extension of the function $\mathscr{X}_3(\cdot) = \mathscr{X}(0,\cdot)$ defined by (3.7). We should properly write $\mathscr{X}_3(t,y)$ rather than $\mathscr{X}(t,y)$, to indicate that this function is associated with Problem 3.4. However, we do not carry out an analysis for Problems 3.2 and 3.3 under the assumption of deterministic coefficients, and hence permit ourselves the convenience of suppressing the subscript.

Lemma 4.4

Under Assumptions 4.1, 4.2, the function $\mathscr{X}$ defined by (4.4) is of class $C([0,T]\times(0,\infty))\cap C^{1,2}([0,T)\times(0,\infty))$ and solves the Cauchy problem

$$\begin{aligned}
\mathscr{X}_t(t,y) + \frac{1}{2}\|\theta(t)\|^2 y^2 \mathscr{X}_{yy}(t,y) + \left(\|\theta(t)\|^2 - r(t)\right) y \mathscr{X}_y(t,y) - r(t)\mathscr{X}(t,y) \\
= -I_1(t,y) \quad \text{on } [0,T)\times(0,\infty), \qquad (4.6)
\end{aligned}$$

$$\mathscr{X}(T,y) = I_2(y) \quad \text{on } (0,\infty). \quad (4.7)$$

Furthermore, for each $t \in [0,T)$, $\mathscr{X}(t,\cdot)$ is strictly decreasing with

$$\mathscr{X}(t,0+) = \infty \text{ and } \mathscr{X}(t,\infty) = 0. \quad (4.8)$$

Consequently, for $t \in [0,T)$, $\mathscr{X}(t,\cdot)$ has a strictly decreasing inverse function $\mathscr{Y}(t,\cdot):(0,\infty) \xrightarrow{onto} (0,\infty)$, i.e.,

$$\mathscr{X}(t,\mathscr{Y}(t,x)) = x, \quad \forall x \in (0,\infty), \quad (4.9)$$

and $\mathscr{Y}$ is of class $C^{1,2}$ on the set

$$D := \{(t,x) \in [0,T) \times (0,\infty); \quad x > 0\}. \quad (4.10)$$

For $t = T$, we have $\mathscr{X}(T,\cdot) = I_2(\cdot)$ which is strictly decreasing on the interval $(0,\infty)$, and we have $\mathscr{X}(T,\infty) = 0$. The inverse of $\mathscr{X}(T,\cdot)$ is $\mathscr{Y}(T,\cdot) := U_2'(\cdot)$ which also satisfies (4.9). The function $\mathscr{Y}$ is continuous on the set $[0,T] \times (0,\infty)$.

Proof

Consider the Cauchy problem

$$\begin{aligned} u_t(t,\eta) + \frac{1}{2}\|\theta(t)\|^2 u_{\eta\eta}(t,\eta) + \left(\frac{1}{2}\|\theta(t)\|^2 - r(t)\right) u_\eta(t,\eta) - r(t)u(t,\eta) \\ = -I_1(t,e^\eta), \quad 0 \le t < T, \quad \eta \in \mathbf{R}, \quad (4.11) \\ u(T,\eta) = I_2(e^\eta), \quad \eta \in \mathbf{R}. \quad (4.12) \end{aligned}$$

The classical theory of partial differential equations (e.g., Friedman (1964)) implies that there is a function u of class $C([0,T] \times \mathbf{R}) \cap C^{1,2}([0,T) \times \mathbf{R})$ satisfying (4.11), (4.12) and has exponential growth.

We fix $(t,y) \in [0,T) \times (0,\infty)$ and use Ito's rule in conjunction with (4.2) under $\widetilde{P}$ and (4.11) to compute

$$\begin{aligned} d\left[e^{-\int_t^s r(u)du} u\left(s, \log Y^{(t,y)}(s)\right)\right] &= d\left(e^{-\int_t^s r(u)du}\right) u\left(s,\log Y^{(t,y)}(s)\right) + e^{-\int_t^s r(u)du} du\left(s,\log Y^{(t,y)}(s)\right) \\ &= -e^{-\int_t^s r(u)du} I_1\left(s,\log Y^{(t,y)}(s)\right) ds \\ &\quad - e^{-\int_t^s r(u)du} u_\eta\left(s,\log Y^{(t,y)}(s)\right) \theta^T(s) d\tilde{W}(s). \quad (4.13) \end{aligned}$$

Integrating (4.13) on $[t,T]$ we have

$$\begin{aligned} e^{-\int_t^T r(u)du} u\left(T,\log Y^{(t,y)}(T)\right) - u(t,\log y) = -\int_t^T e^{-\int_t^s r(u)du} I_1\left(s,\log Y^{(t,y)}(s)\right) ds \\ - \int_t^T e^{-\int_t^s r(u)du} u_\eta\left(s,\log Y^{(t,y)}(s)\right) \theta'(s) d\tilde{W}(s). \end{aligned}$$

Furthermore, making use of arguments involving suitable stopping times, we may treat the local martingale term as to be a true martingale under $\widetilde{P}$. Therefore, by taking expectations, this term vanishes and we obtain:

$$u(t,\log y)=\tilde{E}\left[\int_t^T e^{-\int_t^s r(u)du} I_1\left(s,Y^{(t,y)}(s)\right)ds+e^{-\int_t^T r(u)du} I_2\left(Y^{(t,y)}(T)\right)\right]$$
$$=\mathscr{X}(t,y). \quad (4.14)$$

It follows immediately that $\mathscr{X}$ is of class $C([0,T]\times(0,\infty))\cap C^{1,2}([0,T)\times(0,\infty))$ and

$$u_t(t,\log y)=\mathscr{X}_t(t,y),$$
$$u_\eta(t,\log y)=y\mathscr{X}_y(t,y),$$
$$u_{\eta\eta}(t,\log y)=y^2\mathscr{X}_{yy}(t,y)+y\mathscr{X}_y(t,y).$$

Substituting the latter into (4.11) and (4.12) we have that

$$\mathscr{X}_t(t,y)+\frac{1}{2}\|\theta(t)\|^2\left[y^2\mathscr{X}_{yy}(t,y)+y\mathscr{X}_y(t,y)\right]$$
$$+\left(\frac{1}{2}\|\theta(t)\|^2-r(t)\right)y\mathscr{X}_y(t,y)-r(t)\mathscr{X}(t,y)=-I_1(t,y),$$

$$\mathscr{X}(T,y)=I_2(y),$$

which are (4.6) and (4.7).

The function $\mathscr{X}(t,y)$ inherits the properties of I_1 and I_2 and the rest assertions of the Lemma follow from the implicit function theorem. Moreover, from the implicit function theorem we have the existence of the function $\mathscr{Y}$ that satisfies (4.9) for all $t\in[0,T]$, is of class $C^{1,2}$ on D, and is continuous on $\{(t,x)\in[0,T]\times(0,\infty)\}$. □

Remark 4.5

From (4.13), for t=0, and (4.14) we have

$$d\left(e^{-\int_0^s r(u)du}\mathscr{X}\left(s,Y^{(0,y)}(s)\right)\right)=e^{-\int_0^s r(u)du}\left[-I_1\left(s,Y^{(0,y)}(s)\right)\right.$$
$$\left.-Y^{(0,y)}(s)\mathscr{X}_y\left(s,Y^{(0,y)}(s)\right)\theta^T(s)d\tilde{W}(s)\right],$$

which leads to the following useful integral formula for $0\le t\le T$, $y>0$:

$$e^{-\int_0^t r(u)du}\mathscr{X}\left(t,Y^{(0,y)}(t)\right)+\int_0^t e^{-\int_0^s r(u)du} I_1\left(s,Y^{(0,y)}(s)\right)ds$$
$$=\mathscr{X}(0,y)-\int_0^t e^{-\int_0^s r(u)du}Y^{(0,y)}(s)\mathscr{X}_y\left(s,Y^{(0,y)}(s)\right)\theta^T(s)d\tilde{W}(s). \quad (4.15)$$

<u>Remark 4.6</u>

The proof of Lemma 4.4 also shows that $\mathscr{X}$ is the unique $C([0,T]\times(0,\infty))\cap C^{1,2}([0,T)\times(0,\infty))$ solution to the Cauchy problem (4.6), (4.7) among those functions f satisfying a suitable exponential growth condition.

We now derive the feedback form for optimal consumption and investment.

<u>Theorem 4.7</u>

Under the Assumptions 4.1 and 4.2, the feedback form (4.1) for the optimal consumption/portfolio process pair (c_3, π_3) for Problem 3.4 is given by

$$C(t,x) := I_1\left(t, \mathscr{Y}(t,x)\right), \quad (4.16)$$

$$\Pi(t,x) := -\left(\sigma^T(t)\right)^{-1}\theta(t)\frac{\mathscr{Y}(t,x)}{\mathscr{Y}_x(t,x)}, \quad (4.17)$$

for $0 \le t \le T$ and $x \in (0,\infty)$.

<u>Proof</u>

We have from (4.3) that the optimal wealth $X_3(t)$ at time $t \in [0,T]$ is $\mathscr{X}\left(t, Y^{(0,\mathscr{Y}(x))}(t)\right)$. In other words,

$$Y^{(0,\mathscr{Y}(0,x))}(t) = \mathscr{Y}\left(t, X_3(t)\right),$$

and (3.10) becomes

$$c_3(t) = I_1\left(t, \mathscr{Y}(0,x)Y^{(0,1)}(t)\right) = I_1\left(t, \mathscr{Y}\left(t, X_3(t)\right)\right),$$

which establishes (4.16). With $y = \mathscr{Y}(0,x)$ and using (4.3), we may write (4.15) as

$$\frac{X_3(t)}{S_0(t)} + \int_0^t \frac{c_3(s)}{S_0(s)}ds = x - \int_0^t \frac{\mathscr{Y}\left(s, X_3(s)\right)}{S_0(s)}\mathscr{X}_y\left(s, \mathscr{Y}\left(s, X_3(s)\right)\right)\theta^T(s)d\tilde{W}(s). \quad (4.18)$$

But from (4.9), we have $\mathscr{X}_y\left(t, \mathscr{Y}(t,x)\right) = 1/\mathscr{Y}_x(t,x)$ for all $x > 0$ and a comparison of (4.18) with the wealth equation (2.15) shows that the optimal portfolio satisfies

$$\pi_3^T(t)\sigma(t) = -\theta^T(s)\frac{\mathscr{Y}\left(t, X_3(t)\right)}{\mathscr{Y}_x\left(t, X_3(t)\right)},$$

justifying (4.17). □

4.3 Hamilton-Jacobi-Bellman Equation

Finally, we develop the Hamilton-Jacobi-Bellman (HJB) equation associated with Problem 3.4. To do that, we must extend the value function V_3 of (3.5) to include the time variable. Given $(t,x)\in[0,T]\times\mathbf{R}^+$, and given a consumption/portfolio process pair $(c(\cdot),\pi(\cdot))$, the wealth process $X^{t,x,c,\pi}(\cdot)$ corresponding to (c,π) with initial condition (t,x) is given by (cf. (2.15))

$$e^{-\int_t^s r(u)du}X^{t,x,c,\pi}(s)=x-\int_t^s e^{-\int_t^u r(v)dv}c(u)\,du$$
$$+\int_t^s e^{-\int_t^u r(v)dv}\pi^T(u)\sigma(u)\,d\tilde{W}(u),\quad t\le s\le T.\quad(4.19)$$

We say that (c,π) is admissible at (t,x) and write $(c,\pi)\in\mathscr{A}(t,x)$ if $X^{t,x,c,\pi}(s)\ge 0$ almost surely for all $s\in[t,T]$. We set

$$\mathscr{A}_3(t,x):=\Big\{(c,\pi)\in\mathscr{A}(t,x);\quad E\int_t^T\min\big[0,U_1(s,c(s))\big]ds$$
$$+E\Big(\min\big[0,U_2\big(X^{t,x,c,\pi}(T)\big)\big]\Big)>-\infty\Big\}$$

and define

$$V(t,x):=\sup_{(c,\pi)\in\mathscr{A}_3(t,x)}E\Big[\int_t^T U_1(s,c(s))\,ds+U_2\big(X^{t,x,c,\pi}(T)\big)\Big].\quad(4.20)$$

Because we do not consider the time-dependent variations of Problems 3.2 and 3.3, we allow ourselves the convenience of writing $V(t,x)$ rather than $V_3(t,x)$ in (4.20).

By analogy with (3.17), we introduce the function

$$G(t,y):=E\Big[\int_t^T U_1\Big(s,I_1\big(s,yY^{(t,1)}(s)\big)\Big)ds+U_2\Big(I_2\big(yY^{(t,1)}(T)\big)\Big)\Big],$$
$$(t,y)\in[0,T]\times(0,\infty),\quad(4.21)$$

so that $G(0,\cdot)=G_3(\cdot)$. Under Assumptions 4.1 and 4.2, we have (cf. (3.16))

$$V(t,x)=G\big(t,\mathscr{Y}(t,x)\big),\quad \text{if } x>0.\quad(4.22)$$

Of course, the value function $V_3(\cdot)$ of (3.5) is $V(0,\cdot)$ and

$$V(T,x)=U_2(x),\quad\forall x>0.\quad(4.23)$$

In particular, $V(t,x)<\infty$ for all $(t,x)\in[0,T]\times\mathbf{R}$. Moreover,

$$\lim_{x\downarrow 0}V(t,x)=\lim_{y\to\infty}G(t,y)=\int_t^T U_1(s,0^+)\,ds+U_2(0^+).\quad(4.24)$$

Lemma 4.8

Under Assumptions 4.1 and 4.2, the function G defined by (4.21) is of class $C([0,T]\times(0,\infty))\cap C^{1,2}([0,T)\times(0,\infty))$ and, among such functions that also satisfy the previously mentioned suitable growth condition of Remark 4.6, is the unique solution to the Cauchy problem

$$G_t(t,y)+\frac{1}{2}\|\theta(t)\|^2 y^2 G_{yy}(t,y)-r(t)yG_y(t,y)$$
$$=-U_1(t,I_1(t,y)) \quad \text{on } [0,T)\times(0,\infty), \quad (4.25)$$
$$G(T,y)=U_2(I_2(y)) \quad \text{on } (0,\infty). \quad (4.26)$$

Furthermore,

$$G(t,y)-G(t,z)=y\mathscr{X}(t,y)-z\mathscr{X}(t,z)$$
$$-\int_z^y \mathscr{X}(t,\lambda)\,d\lambda, \quad 0<z<y<\infty, \quad (4.27)$$
$$G_y(t,y)=y\mathscr{X}_y(t,y),$$
$$G_{yy}(t,y)=\mathscr{X}_y(t,y)+y\mathscr{X}_{yy}(t,y),\ 0\le t<T, \quad y>0. \quad (4.28)$$

Proof

The proof of (4.25) and (4.26) is like the proof of (4.6) and (4.7), except that now we use Remark 4.3 and take $u:[0,T]\times\mathbf{R}\to\mathbf{R}$ to be the $C([0,T]\times(0,\infty))\cap C^{1,2}([0,T)\times(0,\infty))$ solution of the Cauchy problem

$$u_t(t,\eta)+\frac{1}{2}\|\theta(t)\|^2 u_{\eta\eta}(t,\eta)-\left(r(t)+\frac{1}{2}\|\theta(t)\|^2\right)u_\eta(t,\eta)$$
$$=-U_1\left(t,I_1\left(t,e^\eta\right)\right), \quad 0\le t<T, \quad \eta\in\mathbf{R}, \quad (4.29)$$
$$u(T,\eta)=U_2\left(t,I_2\left(t,e^\eta\right)\right), \quad \eta\in\mathbf{R}. \quad (4.30)$$

Ito's rule, (4.2) under P, and (4.29) imply that (cf. (4.13))

$$du\left(s,\log Y^{(t,y)}(s)\right)=-U_1\left(s,I_1\left(s,Y^{(t,y)}(s)\right)\right)ds$$
$$-u_\eta\left(s,\log Y^{(t,y)}(s)\right)\theta^T(s)\,dW(s),$$

and so

$$u(t,\log y)=E\int_t^T U_1\left(s,I_1\left(s,Y^{(t,y)}(s)\right)\right)ds+EU_2\left(I_2\left(Y^{(t,y)}(T)\right)\right)=G(t,y).$$

Consequently, G solves the Cauchy problem (4.25), (4.26).

Equation (4.27) is just (3.31) with initial time t rather than initial time zero. Equation (4.28) follows from differentiation of (4.27). Uniqueness follows as in Remark 4.6.

□

Theorem 4.9 (Hamilton-Jacobi-Bellman equation)

Under Assumptions 4.1 and 4.2, the value function $V(t,x)$ of (4.22) is of class $C^{1,2}$ on the set D of (4.10), continuous on the set $\{(t,x)\in[0,T]\times(0,\infty)\}$, and satisfies the boundary conditions (4.23), (4.24). Furthermore, V satisfies the Hamilton-Jacobi-Bellman equation of dynamic programming:

$$V_t(t,x)+\max_{\substack{0\le c<\infty\\ \pi\in\mathbf{R}^d}}\Big[\frac{1}{2}\|\sigma^T(t)\pi\|^2 V_{xx}(t,x)$$
$$+\big(r(t)x-c+\pi^T\sigma(t)\theta(t)\big)V_x(t,x)+U_1(t,c)\Big]=0 \quad \text{on } D. \quad (4.31)$$

In particular, the maximization in (4.31) is achieved by the pair $(C(t,x),\Pi(t,x))$ of (4.16), (4.17).

Proof

Differentiating (4.9), we obtain for $(t,x)\in D$,

$$\mathscr{X}_t(t,\mathscr{Y}(t,x))+\mathscr{X}_y(t,\mathscr{Y}(t,x))\mathscr{Y}_t(t,x)=0,$$
$$\mathscr{X}_y(t,\mathscr{Y}(t,x))\mathscr{Y}_x(t,x)=1.$$

Differentiating (4.22) and using the formula (4.28), we obtain for $(t,x)\in D$,

$$V_t(t,x)=G_t(t,\mathscr{Y}(t,x))+G_y(t,\mathscr{Y}(t,x))\mathscr{Y}_t(t,x),$$
$$\begin{aligned}V_x(t,x)&=G_y(t,\mathscr{Y}(t,x))\mathscr{Y}_x(t,x)\\ &=\mathscr{Y}(t,x)\mathscr{X}_y(t,\mathscr{Y}(t,x))\mathscr{Y}_x(t,x)\\ &=\mathscr{Y}(t,x),\end{aligned}$$
$$V_{xx}(t,x)=\mathscr{Y}_x(t,x).$$

Using these formulas, we can rewrite the left-hand side of (4.31) as

$$G_t(t,\mathscr{Y}(t,x))+G_y(t,\mathscr{Y}(t,x))\mathscr{Y}_t(t,x)+r(t)x\mathscr{Y}(t,x)$$
$$+\max_{0\le c<\infty}\big[U_1(t,c)-c\mathscr{Y}(t,x)\big]$$
$$+\max_{\pi\in\mathbf{R}^d}\Big[\frac{1}{2}\|\sigma^T(t)\pi\|^2\mathscr{Y}_x(t,x)+\pi^T\sigma(t)\theta(t)\mathscr{Y}(t,x)\Big]. \quad (4.32)$$

Both expressions to be maximized are strictly concave. Setting their derivatives equal to zero, we verify that (4.16) and (4.17) provide the maximizing values of c and π, respectively. Substitution of these values converts the expression of (4.32) into

$$G_t(t,\mathscr{Y}(t,x))+G_y(t,\mathscr{Y}(t,x))\mathscr{Y}_t(t,x)+r(t)x\mathscr{Y}(t,x)$$
$$+U_1\big(t,I_1(t,\mathscr{Y}(t,x))\big)-\mathscr{Y}(t,x)I_1(t,\mathscr{Y}(t,x))$$
$$-\frac{1}{2}\|\theta(t)\|^2\frac{\mathscr{Y}^2(t,x)}{\mathscr{Y}_x(t,x)}.$$

We can also use (4.28) and the first two identities of this proof to write this expression in the form:

$$G_t\left(t,\mathscr{Y}(t,x)\right)-\mathscr{Y}(t,x)\mathscr{X}_t\left(t,\mathscr{Y}(t,x)\right)+r(t)x\mathscr{Y}(t,x)+U_1\left(t,I_1\left(t,\mathscr{Y}(t,x)\right)\right)$$
$$-\mathscr{Y}(t,x)I_1\left(t,\mathscr{Y}(t,x)\right)-\frac{1}{2}\|\theta(t)\|^2\mathscr{Y}^2(t,x)\mathscr{X}_y\left(t,\mathscr{Y}(t,x)\right).$$

Setting $y=\mathscr{Y}(t,x)$, so that, $x=\mathscr{X}(t,y)$ we can use (4.28) and (4.25) to write this in the simpler form:

$$-\frac{1}{2}\|\theta(t)\|^2 yG_{yy}(t,y)+r(t)yG_y(t,y)-y\mathscr{X}_t(t,y)+r(t)y\mathscr{X}(t,y)$$
$$-yI_1(t,y)-\frac{1}{2}\|\theta(t)\|^2 y^2\mathscr{X}_y(t,y),$$

which through (4.28) equals to

$$-y\Big[\mathscr{X}_t(t,y)+\frac{1}{2}\|\theta(t)\|^2 y^2\mathscr{X}_{yy}(t,y)+\left(\|\theta(t)\|^2-r(t)\right)y\mathscr{X}_y(t,y)$$
$$-r\mathscr{X}(t,y)+I_1(t,y)\Big].$$

According to Lemma 4.4, this last expression is zero. □

Theorem 4.9 provides only a necessary condition for the value function V; it is not claimed that V is the only function that is of class $C^{1,2}$ on D and satisfies (4.31) with boundary conditions (4.23), (4.24). In order to make such a uniqueness assertion, one would have also to impose additional technical condition on V. Instead of pursuing this approach, it is easier to derive a necessary and sufficient condition for the convex dual of V, defined by the formula

$$\tilde{V}(t,y):=\sup_{x\in\mathbf{R}}\{V(t,x)-xy\},\quad y\in\mathbf{R}.$$

In contrast to the nonlinear partial differential equation (4.31), which governs the value function V, the dual value function $\tilde{V}$ satisfies the linear partial differential equation (4.36) below. Then the function V can be recovered from $\tilde{V}$ by the Legendre transform inversion formula

$$V(t,x)=\inf_{y\in\mathbf{R}}\{\tilde{V}(t,y)+xy\},\quad x\in\mathbf{R}.$$

<u>Theorem 4.10</u>

(Convex dual of $V(t,\cdot)$): Let Assumptions 4.1 and 4.2 hold. Then, for each $t \in [0,T]$ the function $V(t,\cdot)$ satisfies all the conditions of Definition 3.1, and

$$V_x(t,x) = \mathscr{Y}(t,x), \quad \forall x \in (0,\infty), \quad (4.33)$$

$$\begin{aligned}\tilde{V}(t,y) &= G(t,y) - y\mathscr{X}(t,y) \\ &= E\left[\int_t^T \tilde{U}_1\left(s, yY^{(t,1)}(s)\right)ds + \tilde{U}_2\left(yY^{(t,1)}(T)\right)\right], \quad \forall y \in (0,\infty), \quad (4.34)\end{aligned}$$

$$\tilde{V}_y(t,y) = -\mathscr{X}(t,y), \quad \forall y \in (0,\infty). \quad (4.35)$$

Moreover, $\tilde{V}$ is of class $C([0,T]\times(0,\infty)) \cap C^{1,2}([0,T)\times(0,\infty))$ and is the unique solution of the Cauchy problem

$$\tilde{V}_t(t,y) + \frac{1}{2}\|\theta(t)\|^2 y^2\tilde{V}_{yy}(t,y) - r(t)y\tilde{V}_y(t,y) = -\tilde{U}_1(t,y)$$

$$\text{on } [0,T)\times(0,\infty), \quad (4.36)$$

$$\tilde{V}(t,y) = \tilde{U}_2(y), \quad y \in (0,\infty). \quad (4.37)$$

<u>Proof</u>

All the claims (4.33)-(4.35) made here for fixed $t \in [0,T)$ are contained in Theorem 3.12, for $t = 0$. When $t = T$, (4.33)-(4.35) and (4.37) follow directly from the definitions. Equation (4.34), Lemma 4.4, and Lemma 4.8 show that $\tilde{V}$ has the claimed degree of smoothness. Equations (4.34), (4.25), (4.6), and (4.35) yield (4.36). □

The following examples illustrate the use of Theorem 4.10 to compute the value function and the optimal consumption and portfolio processes in feedback form.

<u>Example 4.11</u>

Fix $p \in (-\infty,1)\setminus\{0\}$ and set

$$U_1(t,x) = U_2(x) = \frac{x^p}{p}, \quad x > 0.$$

Then,

$$I_1(t,y) = I_2(t,y) = y^{1/(p-1)},$$

$$\tilde{U}_1(t,y) = \frac{1-p}{p}y^{p/(p-1)}, \quad 0 \le t \le T, \quad y > 0,$$

$$\tilde{U}_2(y) = \frac{1-p}{p}y^{p/(p-1)}, \quad y > 0.$$

In light of the form of $\tilde{U}_1$ and $\tilde{U}_2$, we seek a solution $\tilde{V}$ of (4.36),(4.37) of the form

$$\tilde{V}(t,y) = \frac{1-p}{p}k(t)y^{p/(p-1)}, \quad (4.38)$$

subject to the function $k:[0,T]\to\mathbf{R}$ to be determined. Differentiating (4.38) we have:

$$\tilde{V}_t(t,y)=\frac{1-p}{p}k'(t)\,y^{p/(p-1)},$$
$$\tilde{V}_y(t,y)=-k(t)\,y^{1/(p-1)},$$
$$\tilde{V}_{yy}(t,y)=-\frac{1}{p-1}k(t)\,y^{(2-p)/(p-1)}.$$

Substituting these into (4.36), (4.37) we obtain:

$$\frac{1-p}{p}k'(t)\,y^{p/(p-1)}-\frac{1}{2}\|\theta(t)\|^2\frac{1}{p-1}k(t)\,y^{p/(p-1)}+r(t)k(t)\,y^{p/(p-1)}=-\frac{1-p}{p}y^{p/(p-1)}$$

or

$$\frac{1-p}{p}y^{p/(p-1)}\left[k'(t)+k(t)\left(\frac{1}{2}\|\theta(t)\|^2\frac{p}{(1-p)^2}+\frac{p}{1-p}r(t)\right)\right]=-\frac{1-p}{p}y^{p/(p-1)},$$

and

$$\frac{1-p}{p}k(T)\,y^{p/(p-1)}=\frac{1-p}{p}y^{p/(p-1)}.$$

Since these relationships should hold $\forall\ y>0$, it shows that (4.38) is a solution of (4.36), (4.37) if and only if

$$k'(t)+\alpha(t)k(t)=-1,\quad 0\le t\le T,$$

and

$$k(T)=1,$$

where

$$\alpha(t):=\frac{p}{(1-p)^2}\left[\frac{1}{2}\|\theta(t)\|^2+r(t)(1-p)\right].\quad (4.39)$$

From these conditions, we see that

$$k(t)=e^{\int_t^T\alpha(s)ds}\left[1+\int_t^T e^{-\int_s^T\alpha(u)du}ds\right].\quad (4.40)$$

The function $\tilde{V}$, defined by (4.38) and (4.40), is the unique solution of (4.36) and (4.37).

From (4.34) and (4.35) we have that,

$$\mathscr{X}(t,y)=k(t)\,y^{1/(p-1)},\quad G(t,y)=\frac{1}{p}k(t)\,y^{p/(p-1)},\ 0\le t\le T,\quad y>0,$$

and consequently, for $0\le t\le T$,

$$\mathscr{Y}(t,x)=\left(\frac{x}{k(t)}\right)^{p-1},\quad \forall x>0,$$

$$V(t,x)=\frac{1}{p}k(t)\left(\frac{x}{k(t)}\right)^{p},\quad \forall x>0,$$

thanks to (4.22). The optimal consumption and portfolio in feedback form of (4.16), (4.17) are

$$C(t,x)=\frac{x}{k(t)},\quad \forall x>0,$$

$$\Pi(t,x)=\left(\sigma'(t)\right)^{-1}\theta(t)\frac{x}{1-p},\quad \forall x>0.$$

Example 4.12

Set $U_1(t,x)=U_2(x)=\log x,\quad x>0,$

then,

$$I_1(t,y)=I_2(y)=\frac{1}{y},$$

$$\tilde{U}_1(t,y)=-\log y-1,\quad 0\le t\le T,\quad y>0,$$

$$\tilde{U}_2(y)=-\log y-1,\quad y>0.$$

In view of the form of $\tilde{U}_1$ and $\tilde{U}_2$, we seek a solution $\tilde{V}$ of (4.36), (4.37) of the form

$$\tilde{V}(t,y)=-k(t)\log y-m(t).$$

Differentiating the latter, we have:

$$\tilde{V}_t(t,y)=-k'(t)\log y-m'(t),$$

$$\tilde{V}_y(t,y)=-k(t)\frac{1}{y},$$

$$\tilde{V}_{yy}(t,y)=y^2k(t).$$

Substituting these into (4.36), (4.37) we obtain:

$$-k'(t)\log y-m'(t)+\frac{1}{2}\|\theta(t)\|^2k(t)+r(t)k(t)=\log y+1,$$

$$\forall t\in[0,T) \text{ and } \forall y\in(0,\infty),$$

which leads to

$$m'(t)=\frac{1}{2}\|\theta(t)\|^2k(t)+r(t)k(t)-1,$$

$$k'(t)=-1,$$

and

$$-k(T)\log y-m(T)=-\log y-1,$$

which yields

$$k(T)=m(T)=1.$$

This function $\tilde{V}$ solves (4.36), (4.37) if and only if

$$k(t)=T-t+1,$$

$$m(t)=1+\int_t^T\left[1-(T-s+1)\left(r(s)+\|\theta(t)\|^2/2\right)\right]ds.$$

Again, this function $\widetilde{V}$ is the unique solution of (4.36), (4.37), and from (4.34), (4.35) we have that

$$\mathscr{X}(t,y)=\frac{k(t)}{y},\quad G(t,y)=k(t)(1-\log y)-m(t).$$

Consequently, for $0\le t\le T$,

$$\mathscr{Y}(t,x)=\frac{k(t)}{x},\quad \forall x>0,$$

$$V(t,x)=k(t)\log\left(\frac{x}{k(t)}\right)+k(t)-m(t),\quad \forall x>0,$$

thanks to (4.22), and the feedback formulae of (4.16) and (4.17) are

$$C(t,x)=\frac{x}{k(t)},\quad \forall x>0,$$

$$\Pi(t,x)=\left(\sigma^{T}(t)\right)^{-1}\theta(t)x,\quad \forall x>0.$$

5. Monte Carlo Numerical Applications With Matlab

The starting point for the application of Monte Carlo method is the generation of sample paths of the underlying factors. In our cases, that will be described below, we need a whole path or, at least, a sequence of values at given time instants. We consider the geometric Brownian motion model:

$$dH(t) = -rH(t)dt - \theta H(t)dW(t),$$

that corresponds to the process of (2.17) for constant coefficients, and from this equation we get

$$dH(t) = -rH(t)dt - \theta H(t)\sqrt{\delta t}\ \varepsilon, \quad (5.1)$$

where δt is the discretization step and $\varepsilon \sim \mathcal{N}(0,1)$. The marginal distribution of each value $H(i) = H(i\delta t)$ is normal. Actually, taking a very small δt we may reduce the error, but this is time consuming. With complicated stochastic differential equations, we may have to generate the whole sample path, even if we are only interested in values at maturity.

To simulate the path of the product $H(t)I(t, yH(t))$, as we will see below, over an interval $[0,T]$, we must discretize time with a time step δt. Equation (5.1) is particularly useful as it can be integrated exactly according to (4.2), yielding:

$$H(t+\delta t) = H(t)\exp\left[\left(-\frac{\theta^2}{2} - r\right)\delta t - \theta\sqrt{\delta t}\ \varepsilon\right], \quad (5.2)$$

where $\varepsilon \sim \mathcal{N}(0,1)$ is a standard normal random variable. Based on equation (5.2), it is fairly easy to generate sample paths for the product $H(t)I(t, yH(t))$.

We will use Matlab to create functions for three specific cases. In Case 1, we will see an example of maximization of utility only from consumption. In Case 2, an example of maximization of utility only from terminal wealth, and finally, in Case 3 an example of maximization of utility only from consumption and terminal wealth. In each case, we have to provide the equation (5.2) with inputs and an initial value. Firstly, by (2.17) we have that $H(0) = 1$. In addition, we will use constant coefficients which are, the rate of the bond r, with $r = 13\%$, and the market price of risk θ - theta, with theta = 11%. Moreover, the time horizon T, will be for one year, T=1. Finally, the number of time steps NSteps, with Nsteps = 365, and the number of replications NRep1, with NRep1 = 100000.

Case 1 (Maximization of utility from consumption)

In this case we will formulate in practice the maximization of the expected discounted utility only from consumption. Because utility comes only from consumption, it is plausible that one should strive to compute the net effect of the latter, corresponding to a wide range of initial endowments.

For every $0 \le t \le T$ let $U_1(t,x) = \log x + 2\sqrt{x}$ for $x > 0$. Then, $I_1(t,y) = \left(2y + \sqrt{4y+1} + 1\right) / 2y^2$ for $y > 0$. Also, from (3.34) we know that, $\mathcal{X}_1(y) := E\left[\int_0^T H(t) I_1(t, yH(t))\,dt\right], \quad 0 < y < \infty.$

With the help of Matlab we create function $\mathcal{X}_1(y)$ which is used to give us the Monte Carlo simulation. The Matlab codes are presented in Tables 5.1 and 5.2.

Table 5.1

```
function [Final,I2]=Paths(H0,r,theta,T,NSteps,NRep1,y)

HPaths=zeros(NRep1, 1+NSteps);
HPaths(:,1)=H0;
Final=zeros(NRep1, NSteps);
I2=zeros(NRep1, NSteps);
dt=T/NSteps;
nudt=(-(0.5*theta^2)-r)*dt;
sidt=theta*sqrt(dt);

for i=1:NRep1
   for j=1:NSteps
      HPaths(i,j+1)=HPaths(i,j)*exp(nudt - sidt*randn);
      HT = HPaths(i,j+1);
      I2(i,j)=(2*y*HT+sqrt((4*y*HT)+1)+1)./(2*(y*HT).^2);
      Final(i,j) = HT * I2(i,j);
   end
end
```

We generate one hundred thousands one-year sample paths for the product $H(t)I_1(t, yH(t))$ with initial price 1 monetary unit, r = 11%, and theta = 13%, assuming that the time step is 365 days. A straightforward code to generate sample paths of the product $H(t)I_1(t, yH(t))$ following geometric Brownian motion is given in the above figure. The function Paths yields a matrix of sample paths, where the replications are stored row by row and columns correspond to time instants.

Table 5.2

```
function [P,CI]=Integral(H0,r,T,theta,NSamples,NRep1,y)
M=zeros(NRep1,1);
for i=1:NRep1
   Final=Paths(H0,r,theta,T,NSamples,NRep1,y);
   M(i)=mean(Final(i, :));
end

[P,aux,CI]=normfit(M);
```

In this code NSamples is the number N of sampled points to compute the arithmetic average, which should not be confused with the number of replications NRep1. In this case, we have to generate whole sample paths; we need samples only by specified time instants, but we may still have to generate a large amount of data.

Also, this code is based on one nested “for” loop, which makes NRep1 times the computation $\int_0^T H(t)I_1(t, yH(t))dt \approx \frac{1}{N}\left(\sum_{i=1}^{N} H(t_i)I_1(t_i, yH(t_i))\right)$ for $t_i = i\delta t$ and $\delta t = T/N$, where $T = 1$ and $N = 100000$. Finally, “normfit” returns the mean of the latter NRep1 values.

We know that, for the initial endowment x we have, $\mathscr{X}_1(y) = x$. Having each time a different initial endowment, we find the corresponding y that verifies the latter. Also, from (3.40) we have that, $G_1(y) := E\left[\int_0^T U_1(t, I_1(t, yH(t)))dt\right], \quad 0 < y < \infty$. We create this function, with Matlab and from (3.39) we take the corresponding value function $V_1(x) = G_1(\mathscr{Y}(x)), \quad x > 0$. The Matlab code is presented in Table 5.3.

Table 5.3

```
function [P,CI]=Integral2(H0,r,T,theta,NSteps,NRep1,y)


U2=zeros(NRep1, NSteps);
[Final,I2]=Paths(H0,r,theta,T,NSteps,NRep1,y);
U2 =log(I2)+2*sqrt(I2);


for i=1:NRep1
   M(i)=mean(U2(i, :));
end


[P,aux,CI]=normfit(M);
```

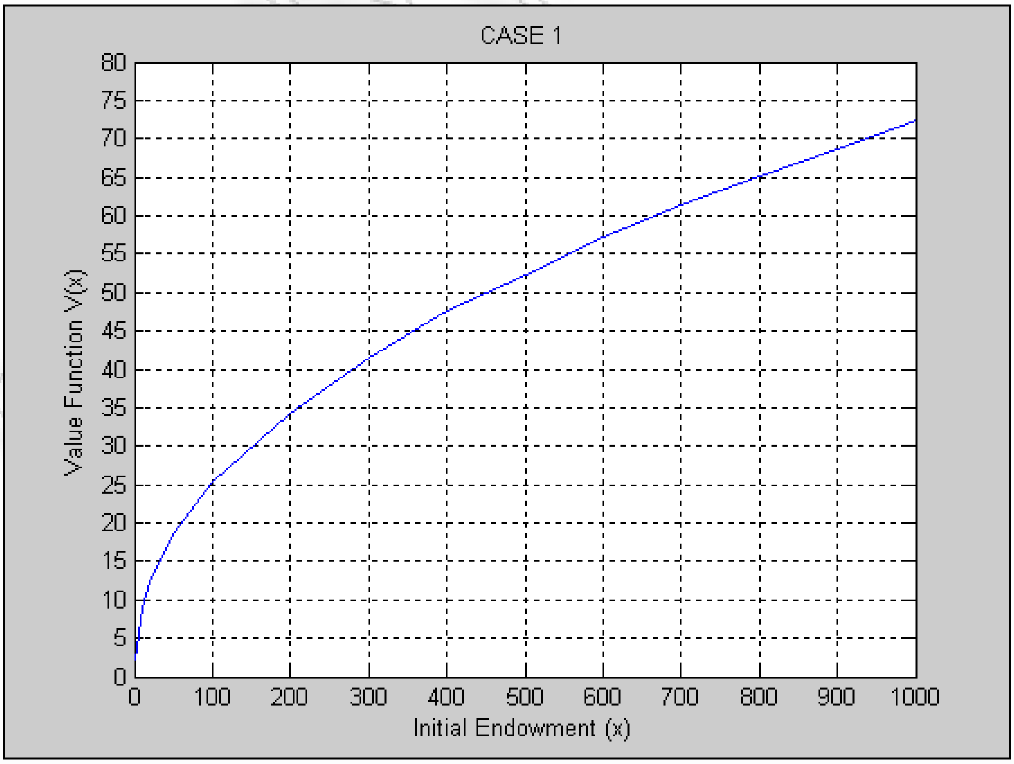
This code is based again on "for" loop, which makes NRep1 times the computation $\int_0^T U_1(t, I_1(t, yH(t)))dt \approx \frac{1}{N}\left(\sum_{i=1}^{N} U_1(t_i, I_1(t_i, yH(t_i)))\right)$ for $t_i = i\delta t$ and $\delta t = T/N$, where $T = 1$ and $N = 100000$. Finally, again "normfit" returns the mean of the latter NRep1 values.

In addition, by giving specific prices to our inputs, with the help of Matlab, we found for each initial endowment x the corresponding y, which verifies $\mathscr{X}_1(y) = x$, and the corresponding value function $V_1(x)$. Finally, we create a chart which has on each axis the results for the initial endowment and the value function, respectively. The above results are presented in Table 5.4 and Figure 5.5.

Table 5.4

Initial Endowment x	Corresponding y	Value Function $V_1(x)$
1	2,03	2,15
10	0,427	8,9
20	0,282	12,3
50	0,167	18,6
100	0,114	25,3
200	0,0781	34,3
300	0,0633	41,5
400	0,0542	47,6
500	0,0484	52,3
600	0,0439	57,1
700	0,0405	61,4
800	0,0379	65,1
900	0,0356	68,6
1000	0,0337	72,3

Figure 5.5



Case 2 (Maximization of utility from terminal wealth)

In this case, with a utility function U_2 as in Case 1, the problem now is to maximize the expected discounted utility from terminal wealth. Because, utility comes now only from terminal wealth, it is quite reasonable again to estimate the corresponding value function for various values of initial endowments.

For every $0 \le t \le T$ let $U_2(t,x) = \log x + 2\sqrt{x}$ for $x > 0$. Then, $I_2(y) = \left(2y + \sqrt{4y+1} + 1\right)/2y^2$ for $y > 0$. Also, from (3.35) we know that, $\mathscr{X}_2(y) := E[H(T)I_2(yH(T))], \quad 0 < y < \infty$. With the help of Matlab we create function $\mathscr{X}_2(y)$ which is used to give us a simpler Monte Carlo simulation. The Matlab code is presented in Table 5.6.

Table 5.6

```
function [Price,CI,I2]=GBM(H0,r,T,theta,NRep1,y)

nudT=(-(0.5*theta^2)-r)*T;
siT=theta*sqrt(T);
HT=H0*exp(nudT-siT*randn(NRep1,1));
I2=(2*y*HT+sqrt((4*y*HT)+1)+1)./(2*(y*HT).^2);

[Price,VarPrice,CI]= normfit(HT.*I2);
```

Here, we compute firstly NRep1 times the value of, $H(T) = H(0)\exp\left[\left(-\frac{\theta^2}{2} - r\right)T - \theta\sqrt{T}\,\varepsilon\right]$, where, $H(0) = 1$ and $T = 1$, for each line of the table that we have create. Finally, with "normfit", we average these values and compute the mean $E(H(T)I_2(yH(T)))$, having NRep1=100000.

We know that, for the initial endowment x we have, $\mathscr{X}_2(y) = x$. Having each time a different initial endowment, we find the corresponding y that verifies the latter. Also, from Theorem 3.20 (iv) we have that, $G_2(y) := EU_2(I_2(yH(T))), \quad 0 < y < \infty$. We create this function, with Matlab and from (3.42) we take the corresponding value function $V_2(x) = G_2(\mathscr{Y}_2(x)), \quad x > 0$. The Matlab code is presented in Table 5.7.

Table 5.7

```
function [EU,CI]=V2(H0,r,T,theta,NRep1,y)

[Price,CI,I2]=GBM(H0,r,T,theta,NRep1,y);
U2=log(I2)+2*sqrt(I2);

[EU,VarPrice,CI]= normfit(U2);
```

In this code, we compute NRep1 values of expression,

$$U_2(I_2(yH(T))) = \log I_2(yH(T)) + 2\sqrt{I_2(yH(T))},$$

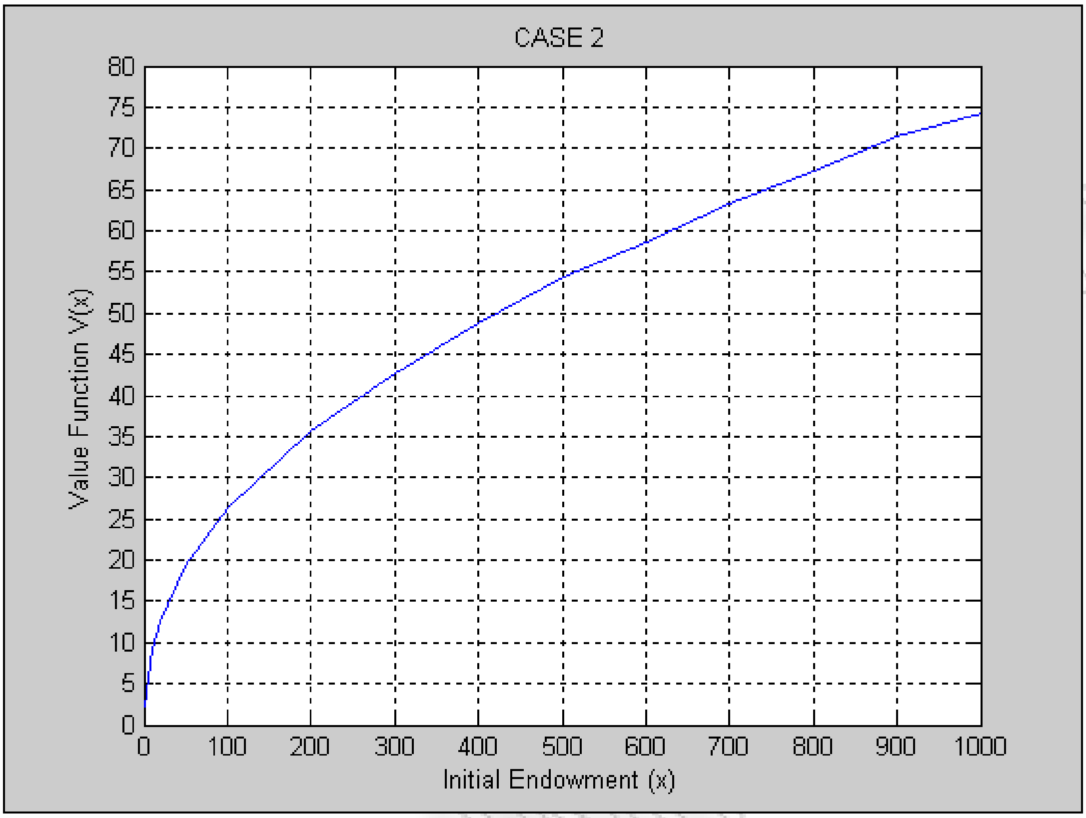
for each line of the table that we have, and with “normfit” we compute its expectation, having NRep1=100000.

Moreover, by giving specific prices to our inputs, with the help of Matlab, we found for each initial endowment x the corresponding y, which verifies $\mathscr{X}_2(y) = x$, and the corresponding value function $V_2(x)$. Finally, we create a chart which has on each axis the results for the initial endowment and the value function, respectively. The above results are presented in Table 5.8 and Figure 5.9.

Table 5.8

Initial Endowment x	Corresponding y	Value Function $V_2(x)$
1	2,07	2,3
10	0,44	9,2
20	0,29	12,6
50	0,172	19,2
100	0,117	26,2
200	0,081	35,8
300	0,065	42,7
400	0,056	49,2
500	0,0497	54,4
600	0,0453	58,7
700	0,0421	63,2
800	0,0388	67,2
900	0,0365	71,5
1000	0,0348	74,3

Table 5.9



Case 3 (Maximization of utility from both consumption and terminal wealth)

A stochastic control problem, which is arguably more interesting than those we saw in previous cases, concerns the maximization of the total expected discounted utility from both consumption and terminal wealth.

For every $0 \le t \le T$ let $U_1(t,x) = U_2(t,x) = \log x + 2\sqrt{x}$ for $x > 0$. Then, $I_1(t,y) = I_2(y) = \left(2y + \sqrt{4y+1} + 1\right)/2y^2$ for $y > 0$. From (3.7) we know that, $\mathcal{X}_3(y) := E\left[\int_0^T H(t) I_1(t, yH(t))dt + H(T) I_2(yH(T))\right]$, $0 < y < \infty$. By combining the codes in Tables 5.2 and 5.6 from Cases 1 and 2, respectively, we create function $\mathcal{X}_3(y)$ which is used to give us the Monte Carlo. Matlab code represented in Table 5.10.

Table 5.10

```
function X3=montecarlo(H0,r,T,theta,NSamples,NRep1,y)

[P,CI]=Integral(H0,r,T,theta,NSamples,NRep1,y);
[Price,CI,I2]=GBM(H0,r,T,theta,NRep1,y);

X3=P+Price;
```

We know that, for the initial endowment x we have, $\mathscr{X}_3(y) = x$. Having each time a different initial endowment, we find the corresponding y that verifies the latter. Also, from (3.17) we have that, $G_3(y) := E\left[\int_0^T U_1(t, I_1(t, yH(t)))dt + U_2(I_2(yH(T)))\right], \quad 0 < y < \infty.$ We create this function, by taking together the codes in Tables 5.3 and 5.7 from Cases 1 and 2, respectively and from (3.16) we take the corresponding value function $V_3(x) = G_3(\mathscr{Y}_3(x)), \quad 0 < x < \infty.$ The Matlab code is presented in Table 5.11.

Table 5.11

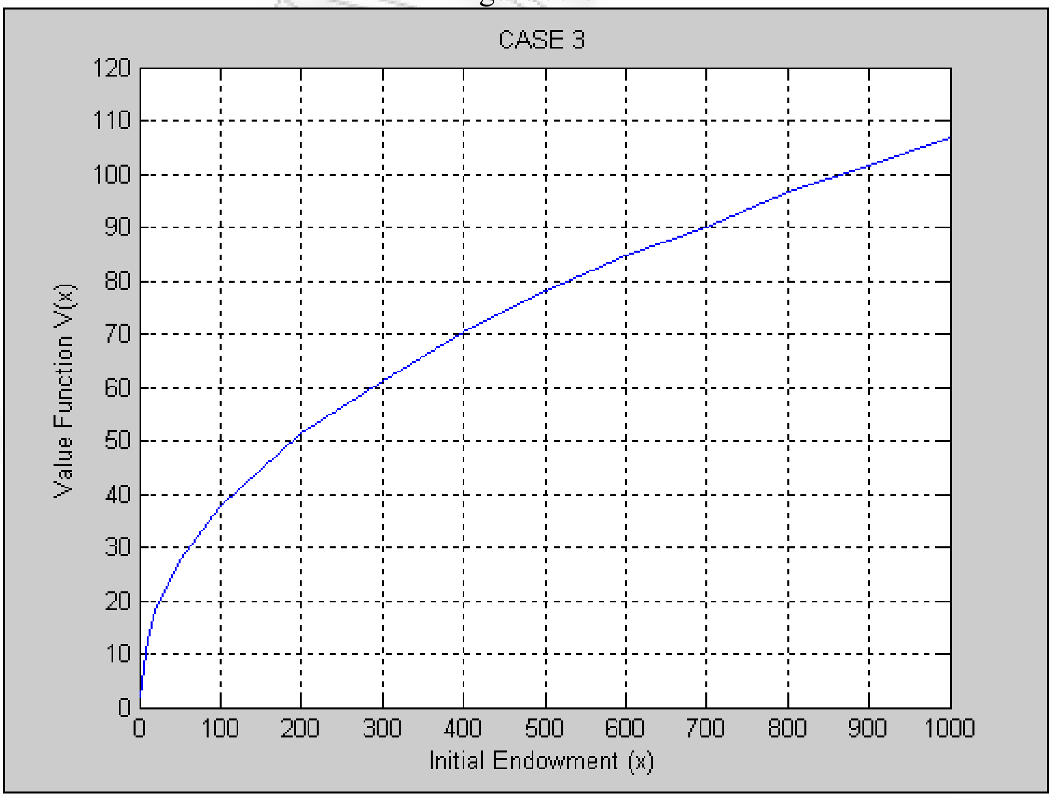
```
function V3=montecarlo2(H0,r,T,theta,NSteps,NRep1,y)

[P,CI]=Integral2(H0,r,T,theta,NSteps,NRep1,y);
[EU,CI]=V2(H0,r,T,theta,NRep1,y);

V3=P+EU;
```

Furthermore, by giving specific prices to our inputs, with the help of Matlab, we found for each initial endowment x the corresponding y, which verifies $\mathscr{X}_3(y) = x$, and the corresponding value function $V_3(x)$. Finally, we create a chart which has on each axis the results for the initial endowment and the value function, respectively. The above results are presented in Table 5.12 and Figure 5.13.

Table 5.12

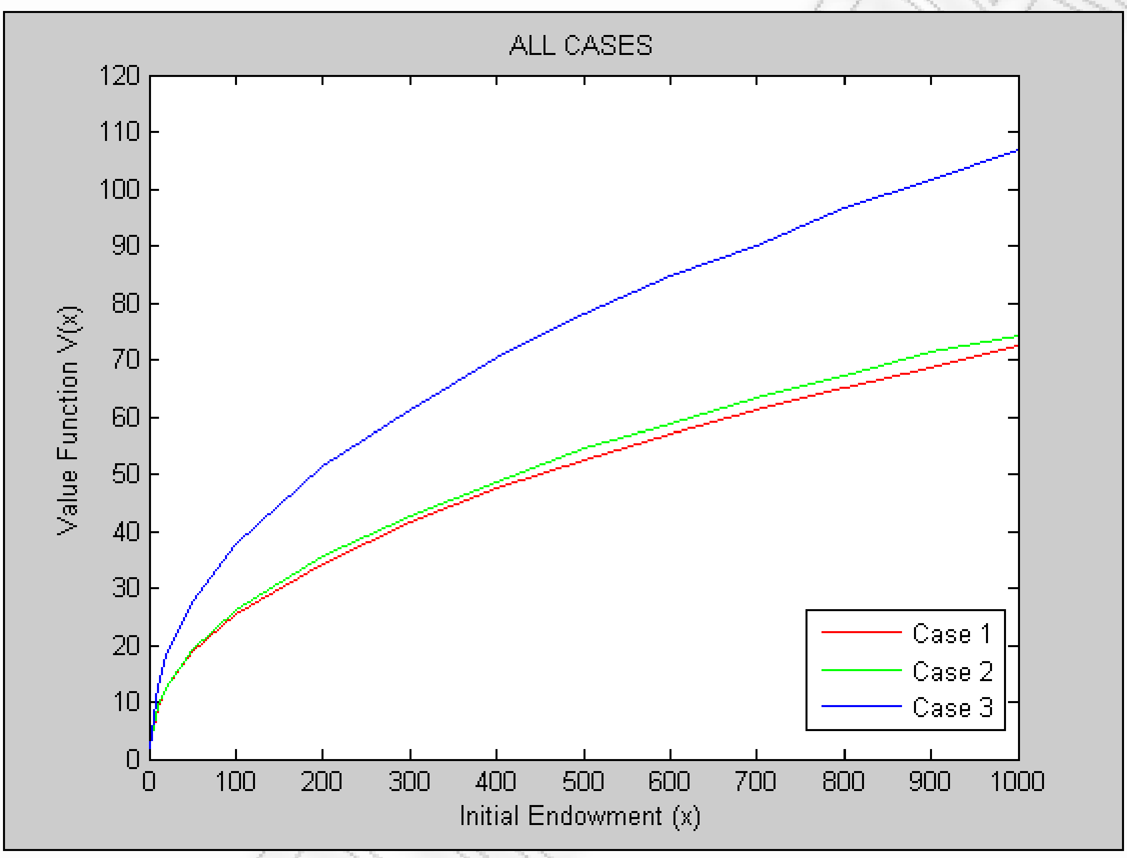
Initial Endowment x	Corresponding y	Value Function $V_3(x)$
1	3,49	1,78
10	0,67	12,8
20	0,431	18,1
50	0,25	27,5
100	0,169	37,7
200	0,1147	51,4
300	0,093	61,3
400	0,079	70,4
500	0,07	78,2
600	0,064	84,6
700	0,059	90,1
800	0,055	96,5
900	0,0519	101,7
1000	0,049	106,8

Figure 5.13



To sum up, we present the above three cases in one chart to see the differences between each case. Our results are presented in Figure 5.14. Obviously, the value functions V_1, V_2 and V_3 satisfy all the conditions of a utility function as obtained in Theorems 3.18, 3.21 and 3.13, respectively.

Figure 5.14



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