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Abstract

In a duopoly model between a partially privatized and a private firm we examine The Nash Equilibrium locations under spatial price discrimination with simultaneous delegation. We, further, calculate the optimal contracts. Both firms produce a perfectly substitutable good at the same marginal cost while transportation is proportional to the distance shipped. Contrary to the findings in literature when there is no delegation we show that the Nash equilibrium under delegation differs soundly from the socially optimal ocations. Furthermore, we solve the managers' location subgame for conjectural consistency and evaluate the ensuing partial equilibrium.

Περίληψη

Σε ένα δυοπωλιακό μοντέλο μεταξύ μίας μερικώς ιδιωτικής και μιας αμιγώς ιδιωτικής εταιρίας εξετάζουμε τα κατά Nash σημεία ισορροπίας του χωρικού ανταγωνισμού με ταυτόχρονη ανάθεση εξουσίας σε διευθυντές. Περαιτέρω, υπολογίζουμε τα βέλτιστα συμβόλαια. Και οι δύο εταιρίες παράγουν ένα υποκατάστατο αγαθό με το ίδιο οριακό κόστος καθώς το κόστος μετεφοράς είναι ανάλογο της απόστασης αποστολής. Σε αντίθεση με τη βιβλιογραφία όποτε δεν υπάρχει ανάθεση εκτελεστικής εξουσίας, αποδεικνύουμε ότι τα σημεία ισορροπίας κατά Nash με ανάθεση εκτελεστικής εξουσίας διαφέρουν από τις κοινωνικά βέλτιστες περιπτώσεις. Επί πλέον, λύνουμε το υποπαίγνιο των διεθυντών για συνεπείς εικασίες και υπολογίζουμε τα σημεία της μερικής ισορροπίας.

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Chapter 1

Introduction

Recently, the coexistence of public and private firms has been deemed obligatory in many aspects of the economic activity. Privatisation has become the cen-tral idea behind different political rhetorics striving for economic reforms. As a consequence in recent years scientists and practitioners have equally been extremely attentive concerning features of competition between public and private firms. Stepping aside from theoretical implications such spectrum of knowledge can be potentially used to impose different policies.

Au contraire, the leap in modern industrial organization has benefited immensely by the division between decision taking and ownership. This is done through the offer of an incentive contract presented from the owner to the manager aiming at benefiting both in terms of profit, sales and so on. The majority of representa-tive papers on the matter such as [Vikers (1985)], [Fershtman and Judd (1987)], [Sklivas (1987)] refer only to private firms. It is substantial though to be able to conclude whether the partition between control and ownership is effective in equal terms when a private and a public firm compete against each other. Although the papers discussed in the previous context pertain to private firms, it is essential to assess whether the separation of ownership and control is as effective in cases where a public company competes with a private one. A notable exception is [Heywood and Wang (2011)] Hoover [Hoover (1937)], and Lernerand Singer [Lerner and Singer (1937)], have introduced an alternative form of price competition, namely spatial price discrimination, in which transportation costs in setting delivery price schedules is born by the companies. Online purchase of products is an apt example where discrimination pricing is applied by reason of geographical location. It should be born in mind though that in generalized spatial models a plethora of options such as daily schedules of itineraries of transportation means between fixed destinations, differentiations of the ideological spectrum in newspaper editorial content, etc. can replace location as a parameter of price discrimination. The optimization of social output that is achieved through Nash's equilibrium of the standard model has been demonstrated initially for a duopoly, by Hurter and Lederer [Lederer and Hurter (1986)] and further broadened for a mixed oligopoly in a multi-varieties setting by Eleftheriou and Mihelacakis,.[Eleftheriou Michelacakis (2018)], [Eleftheriou and Michelacakis (2020)]. Despite its long – standing existence, the conjectural consistency of the canonical model of spatial price discrimination (defined by Bresnahan in [Bresnahan (1981)] has not been researched until recently. In conrast to Nash's theory in which rivals are absolute in their denial to take into consideration each other's behavior, conjectural variations are often re-garded as displaying each competitor's belief formation. It is hoped that this solution may set light into the detail of the com-petition by revealing its intrinsic rivalry. Intrinsic rivalry non visible in Nash. Full dynamic modeling regularly perceives consistency of beliefs as static, often re-ceiving criticism as a solution concept, when viewed from a strictly game theory perspective, for endeavoring to deliberate the dynamics of what is virtually a static game. Such beliefs are theoretically founded in works as for example by Cabral's,[Cabral (1995)]. Heywood and Wang,[Heywood and Wang (2016)] have been in the outset of researching the equilibrium generated by consistent loca-tion in conjectures for a private duopoly. Their research confirms that if (constant) marginal production costs and shipping travel cost are the same, the two firms collocate at the middle, farther from their socially optimal locations existing at the quartiles. Convex travel cost does not impede the verification of the inclination to close the separation gap between locations.

Michelacakis' paper [Michelacakis (2023)] is the first to analyze delegation when a public and a private firm compete against each other in a spatial context. To that end, the solution concept of conjectural consistency which has been re-searched to a lesser extent is simultaneously compared with conjectural variations to classic Nash for the location sub-game. In view of this, conjectural consistency although one shot, is regarded revealing players' belief formation in reaction to other players' choices. Accordingly, in Nash's theory each player's reaction is in-dependent of the other player's reaction.

A duopoly of a partialy privetazed and one private firm is considered with marginal production costs being the same. The parameters taken into considera-tion are the demand which is inelastic and the cost which is increased due to location and proportional to the distance shipped while both firms produce a perfectly substitutable good. The assumption of linear travel cost is maintained throughout the model. Simultaneous delegation between the owners dictates how many stages the game has. During a simultaneous delegation managers are presented with the final form of their contacts. Following this, the location is determined while at the last stage they decide on delivery price schedules, price discriminating in ac-cordance with costumer location and vie for market share or quantity. Conjectural consistency and Nash conjectures are used as solution concepts for the location sub-game with the results being compared afterwards. Diametrically opposed to private duopoly findings [Heywood and Wang (2016)], it is shown that the Nash equilibrium of the mixed duopoly model which is characterized by the absence of delegation, is consistent. This is caused by the fact that the semi-private firm con-jectures that the reaction of the public firm is independent of its own reaction. It is demonstrated that when both firms do not participate in delegation, the delegation order is immaterial to the final results since the location sub- game is decided per consistent conjectures. On the contrary, the use of Nash conjectures as a solution concept for the location sub-game presents a different situation. Barring the case where delegations of the private firm happen initially, managers are the ones who delegate the location decision in both other cases. As a result, consumer surplus exceeds the profit of both firms. Within this framework, simultaneous del-egation also causes lower delivery price schedules for all clients.

Model is presented in chapter 2. In chapter 3 we develop the model in order to find the consistent locations and in the final chapter 4 we present our conclusions.

Chapter 2

The model and the results

Two firms, R_1 semi-private owned and R_2 public, are competing in a one-dimensional linear, uniformly populated, market. Variables x and y denote the location of R_1 and R_2 respectively, with $x < y$ in $[0, 1]$ and we assuming that demand is inelastic. These two firms produce a common good at constant marginal cost w_1 and w_2 respectively.We also assume that $w_1 = w_2 = 0$. Costs of transportation are equal to td, where t is a positive scalar and d is the distance shipped. k denotes the maximum reservation price that a consumer is willing to pay for the reduced commodity and it is considered sufficiently large.
The game is played in three stages when simultaneous delegation is taking place. In the first stage, owners offer their incentive contracts to the managers. The terms of the incentive contracts are simultaneously chosen in the second stage. Both contracts a_1,a_2 are convex combinations of of profit and output. In the third and final stage managers choose locations of the market.
The determination of the location s of the indifferent consumer leads to profit function of the downstream firms. Setting respective delivered schedules equal we get $t(y - s) = t(s - x) \Rightarrow s = \frac{x+y}{2}$
The profit functions of $R_i, i = 1, 2$ are

$$\Pi_{R_1} = \int_0^x t(y - x)\, dz + \int_x^s t(x + y - 2z)\, dz \tag{2.1}$$

$$\Pi_{R_2} = \int_s^y t(2z - x - y)\, dz + \int_y^1 t(y - x)\, dz + g(x, y) \tag{2.2}$$

with

$$g(x, y) = \Pi_{R_1}(x, y) + (\int_0^s [k - t(y - z)]\, dz + \int_s^1 [k - t(z - x)]\, dz) \tag{2.3}$$

The term inside (2.3) parenthesis denotes the total consumer surplus CS and $a_1, a_2 \in (0, 1]$. Correspondingly, contractual incentives offered to managers are given by:

$$I_1(x, y) = a_1\Pi_{R_1}(x, y) + (1 - a_1)s \tag{2.4}$$

$$I_2(x, y) = a_2\Pi_{R_2}(x, y) + (1 - a_2)(1 - s) \tag{2.5}$$

Equivalently, (2.4) and (2.5) become:

$$I_1(x,y) = a_1 t[\frac{1}{4}(x+y)^2 - x^2] + \frac{1}{2}(1-a_1)(x+y) \tag{2.6}$$

$$\begin{aligned} I_2(x,y) = a_2[k + ty + \frac{1}{4}t(x+y)^2 - \frac{1}{2}t + t(A-1)x^2 - Atx + \frac{1}{2}At - ty^2] \\ + (1-a_2)(1 - \frac{x+y}{2}) \end{aligned} \tag{2.7}$$

2.1 Nash equilibrium in locations

2.1.1 Nash locations with simultaneous delegation

In this sub-section, we search for location equilibria under strategic delegation in a mixed duopoly setting. A denotes the degree of privatization, where $0 \leq A \leq 1$ and $A = 0$ stands for fully public and $A = 1$ stands for fully private. Differentiating (2.6) with respect to x, and setting derivative equal to 0 we get

$$\frac{dI_1}{dx} = \frac{3}{2}a_1 tx + \frac{1}{2}a_1 ty + \frac{1}{2}(1-a_1) = 0 \tag{2.8}$$

Now we differentiate (2.7) with respect to y and setting derivative equal to 0 we get

$$\frac{dI_2}{dy} = a_2[t + \frac{1}{2}t(x+y) - 2ty] + \frac{1}{2}(a_2 - 1) = 0 \tag{2.9}$$

The solution of system (2.8) and (2.9) returns for the two locations the formulae:

$$x^N(t : a_1, a_2) = \frac{2(t-1)a_1a_2 - a_1 + 3a_2}{8ta_1a_2} \quad = \frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1} \tag{2.10}$$

$$y^N(t : a_1, a_2) = \frac{2(3t+1)a_1a_2 - 3a_1 + a_2}{8ta_1a_2} = \quad = \frac{3t+1}{4t} - \frac{3}{8ta_2} + \frac{1}{8ta_1} \tag{2.11}$$

Equations (2.10) and (2.11) show the Nash equilibrium locations with delegation.

2.1.2 The location sub-game

To evaluate the optimum contracts we have to find the best response functions. The best of the private firm we return (2.10) and (2.11) to (2.1) and differentiate with respect to a_1 to get

$$\frac{\partial \Pi_{R_1}(a_1, a_2)}{\partial a_1} = \frac{a_1 + 5a_2 - 6a_1a_2 - 2ta_1a_2}{32ta_1^3 a_2} \tag{2.12}$$

Setting (2.16) equal to 0 gives for a_1

$$a_1(a_2) = \frac{5a_2}{-1 + 6a_2 + 2a_2 t} \tag{2.13}$$

In order to find the best response of the owner of the public firm to the choice, a_1, of the private owner we return (2.10) and (2.11) into (2.2) and differentiate with respect to a_2. This gives

$$\frac{\partial \Pi_{R_2}(a_1, a_2)}{\partial a_2} = -\frac{(A-6)a_1 + (2-3A)a_2 + 2(At + A + 2)a_1 a_2}{32 t a_1 a_2^3} \tag{2.14}$$

Setting (2.18) equal to 0 gives for a_2

$$a_2(a_1) = -\frac{(A-6)a_1}{2(At + A + 2)a_1 + (2 - 3A)} \tag{2.15}$$

Simultaneously solving for a_1 and a_2 gives the common solution.

$$a_1 = \frac{2(A-4)}{At - 3t + 2A - 8}$$

and

$$a_2 = -\frac{2(A-4)}{At + t - 2A + 8}$$

If we evaluate a_1 and a_2 in (2.10) and (2.11) we get socially optimal locations.
Location of R_1

$$\frac{A-3}{2(A-4)}$$

Location of R_2

$$\frac{A-3}{A-4}$$

Both values are acceptable because $a_1, a_1 \in (0, 1]$[1].
We can deduce that even if A=1, both firms are private, both firms depart from their socially optimal locations when the decision is taken by aggressive managers. In fact, only when A=0, i.e. one - the second - of the two firms is fully public, the public firm manages to hold on to its socially optimal location.

[1]Proof at the Appendix

Chapter 3

Consistent Location

3.1 Consistent Location Conjectures

In order to find the best response functions of the managers that want to maximise their own profit with respect to their choice of location, we follow the next steps; First we differentiate (2.6) with respect to x, and setting equal to 0 we get

$$a_1 t\left[(x+y)(1+r_{21})-4x\right]+(1-a_1)(1+r_{21})=0 \tag{3.1}$$

Accordingly to the above, we differentiate (2.7) with respect to y, and setting equal to 0 we get

$$a_2 t\left[2+(x+y)(r_{12}+1)+4(A-1)xr_{12}-2Ar_{12}-4y\right]-(1-a_2)(1+r_{12})=0 \tag{3.2}$$

with $r_{12}=\frac{dx}{dy}$ and $r_{21}=\frac{dy}{dx}$ the location conjectures of firm 2 about firm 1 and firm 1 about firm 2 respectively. According to Bresnahan [Bresnahan (1981)], the advantage of the consistent conjectures equilibrium is that each firm precisely expect the competitor's location choice in response to own choice. Differentiating (3.1) with respect to y and (3.2) with respect to x we have

$$a_1 t[(r_{12}+1)(1+r_{21})-4r_{12}]=0 \tag{3.3}$$

$$a_2 t[(1+r_{21})(r_{12}+1)+4(A-1)r_{12}-4r_{21}]=0 \tag{3.4}$$

We solve the system in the following three cases:

3.1.1 A=0

If $A=0$ then $r_{12}=\frac{1}{3}$ and $r_{21}=0$

That leads to the conclusion that the semi-private firm conjectures that the public firm will choose nevertheless the same mark on the linear market to locate. Returning these values to (A.36) and (A.37) we find the equilibrium locations

$$x^C(t,a_1,a_2)=\frac{3a_1a_2t-2a_1a_2-2a_1+4a_2}{12a_1a_2t} \tag{3.5}$$

$$y^C(t,a_1,a_2) = \frac{2a_2 + 3a_2t - 2}{4a_2t} \tag{3.6}$$

Taking a closer look at the equations above, we see that remains focused to its own priorities, without considering to the semi-private rival.

3.1.2 0<A<1

- If $r_{12} = \frac{-A+3+\sqrt{A^2-10A+9}}{2A}$ and $r_{21} = \frac{-A+3+\sqrt{A^2-10A+9}}{2}$ the (3.3) and (3.4) equations system gives

$$\begin{aligned} x^C(t,a_1,a_2) =& \frac{\frac{(a_1-1)}{a_1t}\left[8A^2 - 48A + 24 - 8(A-1)\sqrt{A^2-10A+9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) + 8(A^2-6A+3)\sqrt{A^2-10A+9}} \\ &+ \frac{\frac{(1-a_2)}{a_2t}\left(8A - 24 - 8\sqrt{A^2-10A+9}\right)}{-8\left(A^3 - 11A^2 + 19A - 9\right) + 8(A^2-6A+3)\sqrt{A^2-10A+9}} \\ &+ \frac{2A\left[-4A^2 + 28A - 24 + (4A-8)\sqrt{A^2-10A+9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) + 8(A^2-6A+3)\sqrt{A^2-10A+9}} \end{aligned} \tag{3.7}$$

$$\begin{aligned} y^C(t,a_1,a_2) =& \frac{\frac{(a_2-1)}{a_2t}\cdot 16A + 2A\left[4A^2 - 4A - 4\sqrt{A^2-10A+9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) + 8(A^2-6A+3)\sqrt{A^2-10A+9}} \\ &- \frac{\frac{a_1-1}{a_1t}\left[8A^3 - 80A^2 + 160A - 72 - \left(8A^2 - 40A + 24\right)\sqrt{A^2-10A+9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) + 8(A^2-6A+3)\sqrt{A^2-10A+9}} \end{aligned} \tag{3.8}$$

To find the best response of the owners of the public firm to the choice, a_1, of the semi-private firm and the owner of the semi-private firm to the choice, a_2, of the public firm respectively the we return (3.7) and (3.8) into (2.1) and (2.2) into and differentiate with respect to a_i, where $i = 1, 2$. This gives

$$\frac{\partial \Pi_{R_1}}{\partial a_1} = -\frac{(a_1-1)}{2a_1^3t}\frac{num_{d1}}{dom_{di}} \tag{3.9}$$

$$\begin{aligned} num_{d1} =& A^6 - 22A^5 + 169A^4 - 548A^3 + 772A^2 - 480A + 108 \\ &- (A^5 - 17A^4 + 92A^3 - 184A^2 + 140A - 36)\sqrt{A^2-10A+9} \end{aligned}$$

$$dom_{d1} = (A^2-10A+9)[A^4-12A^3+36A^2-32A+9-(A^3-7A^2+9A-3)\sqrt{A^2-10A+9}]$$

$$\frac{\partial \Pi_{R_2}}{\partial a_2} = \frac{(a_2-1)}{2a_1^3t}\frac{num_{d2}}{dom_{d2}} \tag{3.10}$$

$$num_{d2} = 4A^3 + 43A^2 - 66A + 27 + (4A^2 - 17A + 9)\sqrt{A^2 - 10A + 9}$$

$$dom_{d2} = (A^2 - 10A + 9)[A^4 - 12A^3 + 36A^2 - 32A + 9 - (A^3 - 7A^2 + 9A - 3)\sqrt{A^2 - 10A + 9}]$$

Setting the above equal to zero we get

$$\begin{aligned} a_1(a_2) &= 1 \\ a_2(a_1) &= 1 \end{aligned} \tag{3.11}$$

So, the equilibrium in locations is

$$x^C = -\frac{A}{2}\frac{A^2 + 8A - 7 + (-A + 3)\sqrt{A^2 - 10A + 9}}{A^3 - 11A^2 + 19A - 9 - (A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}}$$

$$y^C = -\frac{A}{2}\frac{A^2 - 4A + 3 - (A + 1)\sqrt{A^2 - 10A + 9}}{A^3 - 11A^2 + 19A - 9 - (A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}}$$

- If $r_{12} = \frac{-A+3+\sqrt{A^2-10A+9}}{2A}$ and $r_{21} = \frac{-A+3+\sqrt{A^2-10A+9}}{2}$ the (3.3) and (3.4) equations system gives

$$\begin{aligned} x^C(t, a_1, a_2) = & \frac{\frac{(a_1-1)}{a_1 t}\left[8A^2 - 48A + 24 + 8(A - 1)\sqrt{A^2 - 10A + 9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}} \\ & - \frac{\frac{(a_2-1)}{a_2 t}\left(8A - 24 + 8\sqrt{A^2 - 10A + 9}\right)}{-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}} \\ & - \frac{2A\left[4A^2 - 28A + 24 + (4A - 8)\sqrt{A^2 - 10A + 9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}} \end{aligned} \tag{3.12}$$

$$\begin{aligned} y^C(t, a_1, a_2) = & \frac{16A \cdot \frac{(a_2-1)}{a_2 t} + 2A\left[4A^2 - 4A + 4\sqrt{A^2 - 10A + 9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}} \\ & - \frac{\frac{(a_1-1)}{a_1 t}\left[8A^3 - 80A^2 + 160A - 72 + (A^2 - 40A + 24)\sqrt{A^2 - 10A + 9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}} \end{aligned} \tag{3.13}$$

To find the best response of the owners of the public firm to the choice, a_1, of the semi-private firm and the owner of the semi-private firm to the choice,

a_2, of the public firm respectively the we return (3.9) and (3.10) into (2.1) and (2.2) into and differentiate with respect to a_i, where $i = 1, 2$. This gives

$$\frac{\partial \Pi_{R_1}}{\partial a_1} = \frac{(a_1 - 1)}{2a_1^3 t} \frac{num_{d1}}{dom_{d1}} \tag{3.14}$$

$$\begin{aligned} num_{d1} =& A^6 - 22A^5 + 169A^4 - 548A^3 + 772A^2 - 480A + 108 \\ &+ (A^5 - 17A^4 + 92A^3 - 184A^2 + 140A - 36)\sqrt{A^2 - 10A + 9} \end{aligned}$$

$$dom_{d1} = (A^2-10A+9)[A^4-12A^3+36A^2-32A+9-(A^3-7A^2+9A-3)\sqrt{A^2 - 10A + 9}]$$

$$\frac{\partial \Pi_{R_2}}{\partial a_2} = \frac{(a_2 - 1)}{2a_1^3 t} \frac{num_{d2}}{dom_{d2}} \tag{3.15}$$

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$$dom_{d2} = (A^2-10A+9)[A^4-12A^3+36A^2-32A+9+(A^3-7A^2+9A-3)\sqrt{A^2 - 10A + 9}]$$

Setting the above equal to zero we get

$$\begin{aligned} a_1(a_2) =& 1 \\ a_2(a_1) =& 1 \end{aligned} \tag{3.16}$$

So, the equilibrium in locations is

$$x^C = \frac{A}{2} \frac{A^2 - 8A + 7 + (A - 3)\sqrt{A^2 - 10A + 9}}{A^3 - 11A^2 + 19A - 9 + (A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}}$$

$$y^C = \frac{A}{2} \frac{A^2 - 4A + 3 + (A + 1)\sqrt{A^2 - 10A + 9}}{A^3 - 11A^2 + 19A - 9 + (A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}}$$

3.1.3 A=1

If $A = 1$ then $r_{12} = r_{21} = 1$ and because $A = 1$ both firms are fully private and $a_1 = a_2$ means that the rivals do not delegate. Managers, namely, act the same so the contractual incentives functions would be the same. That leads to the conclusion that the linear market is separated in two both equal parts and for that

$$x^C = \frac{1}{2} \qquad\qquad y^C = \frac{1}{2} \tag{3.17}$$

Contrary to consistent conjecture, if $a_1 = a_2 = 1$ the choices of both owners, in terms of delegations incentives, are affecting both managers' location decisions. In fact the firms locate at

$$x^N = \frac{1}{4} \qquad\qquad y^N = \frac{1}{2} \tag{3.18}$$

Chapter 4

Conclusion

We examined how the two firms compete to locate in a linear market under spatial price discrimination with delegation under Nash and consistent locations conjectures. We assumed that the degree of privatization of the semi-private firm is variable.

4.1 Synopsis and conclusions

The first to search is Nash equilibria in the game. At the location sub-game delegation is carried out simultaneously and we found that if firms delegate they are placed closer to the center and to each other. Social welfare is affected and as is also proven by Michelacakis in [Michelacakis 2023], the capital structure of the firms has an impact on the results.

It is known from the literature that without delegation we have socially efficient equilibrium, as in every spatial price discrimination model where the companies in order to set delivery price schedules tolerate transportation cost like a reformed price competition. Social optimality is also consistent in this case because the private firm conjectures that the public firm will not react according to her reaction.

On the second part where we examined all possible values of the degree of privatization via consistent conjectures and the results are promising. Beside the complexity of the calculations, we have solution to the game. More precisely, completing the final stage of the game for all the cases we can find the location of both firms. And the most expected result is the case where $A = 1$, witch means that both firms are fully private and their managers act the same. That separates market in two equal parts and both firms colocate in the center.

Even though these two results are different in general, when no delegation is taking place they become identical. Simultaneous delegation lowers delivery schedules for all costumers.

4.2 Future expansion

It would be interesting to examine a market with more firms with or without different degrees of privatization and see how each one's decision affects the decision of the others.

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Appendix A

Appendix

A.1 Profit Equations

$$\begin{aligned} \Pi_{R_1} &= \int_0^x t(y-x)\,dz + \int_x^s t(x+y-2z)\,dz \\ &= -tx^2 - ts^2 + txs + tys \end{aligned} \tag{A.1}$$

From $s = \frac{x+y}{2}$ we set in (I.1) $y = 2s - x$

$$\begin{aligned} \Pi_{R_1} &= \int_0^x t(y-x)\,dz + \int_x^s t(x+y-2z)\,dz \\ &= -tx^2 - ts^2 + txs + tys \\ &= -tx^2 - ts^2 + txs + t(2s-x)s \\ &= -tx^2 - ts^2 + txs + 2ts^2 - txs \\ &= ts^2 - tx^2 \end{aligned} \tag{A.2}$$

The profit of semi privately owned firm is

$$\Pi_{R_1} = ts^2 - tx^2 \tag{A.3}$$

$$\Pi_{R_2} = \int_s^y t(2z-x-y)\,dz + \int_y^1 t(y-x)\,dz + g(x,y) \tag{A.4}$$

with $g(x,y) = \Pi_{R_1}(x,y) + (1{-}A)[(\int_0^s [k - t(y-z)]\ dz + \int_s^1 [k - t(z-x)]\ dz)]$

$$\begin{aligned} &\int_s^y t(2z-x-y)\,dz \\ &= -t(x+y)\,[z]_s^y + 2t\left[\frac{z^2}{2}\right]_s^y \\ &= -t(x+y)(y-s) + t(y^2-s^2) \\ &= t(y^2-s^2) + t(-x-y)(y-s) \end{aligned} \tag{A.5}$$

$$\begin{aligned}&\int_y^1 t(y-x)\,dz\\&=t(y-x)\,[z]_y^1\\&=t(y-x)(1-y)\end{aligned} \tag{A.6}$$

$$\begin{aligned}&\int_0^s [k-t(y-z)]\ dz\\&=(k-ty)\,[z]_0^s+t\left[\frac{z^2}{2}\right]_0^s\\&=(k-ty)s+t\frac{s^2}{2}\\&=\frac{1}{2}ts^2+ks-tys\end{aligned} \tag{A.7}$$

$$\begin{aligned}&\int_s^1 [k-t(z-x)]\ dz\\&=(k+tx)\,[z]_s^1-t\left[\frac{z^2}{2}\right]_s^1\\&=(k+tx)(1-s)-t(\frac{1}{2}-\frac{s^2}{2})\\&=-\frac{1}{2}t(1-s^2)+k(1-s)+tx(1-s)\end{aligned} \tag{A.8}$$

From (A.3), (A.4), (A.5) and (A.6) and (A.7)

$$
\begin{aligned}
\Pi_{R_2} &= \int_s^y t(2z-x-y)\,dz + \int_y^1 t(y-x)\,dz + g(x,y) \\
&= t(y^2-s^2) + t(-x-y)(y-s) + t(y-x)(1-y) \\
&+ \frac{1}{2}ts^2 + ks - tys - \frac{1}{2}t(1-s^2) + k(1-s) + tx(1-s) \\
&= ty^2 - ts^2 - t(xy - xs + y^2 - ys) + t(y - y^2 - x + xy) \\
&+ (1-A)(ts^2 - tx^2 + \frac{1}{2}ts^2 + ks - tys - \frac{1}{2}t + \frac{1}{2}ts^2 + k - ks + tx - txs) \\
&= ty^2 - ts^2 - txy + txs - ty^2 + tys + ty - ty^2 - tx + txy \\
&+ (1-A)(2ts^2 - tx^2 - tys - \frac{1}{2}t + k + tx - txs) \\
&= -ts^2 + txs - ty^2 + tys + ty - tx + 2ts^2 - tx^2 - tys - \frac{1}{2}t + k + tx - txs \\
&- 2Ats^2 + Atx^2 + Atys + \frac{1}{2}At - Ak - Atx + Atxs \\
&= k + ty + Atys + Atxs - \frac{1}{2}t - tx^2 + ts^2 - Ak + Atx^2 - 2Ats^2 - Atx + \frac{1}{2}At - ty^2
\end{aligned}
$$

The profit of public owned firm is

$$
\Pi_{R_2} = k+ty+Atys+Atxs-\frac{1}{2}t-tx^2+ts^2-Ak+Atx^2-2Ats^2-Atx+\frac{1}{2}At-ty^2 \quad \text{(A.9)}
$$

A.2 Nash Equilibria in Locations

Setting $s = \frac{1}{2}x + \frac{1}{2}y$ in (A.3)

$$
\begin{aligned}
\Pi_{R_1} &= ts^2 - tx^2 \\
&= t\frac{1}{4}(x+y)^2 - tx^2
\end{aligned} \quad \text{(A.10)}
$$

Differentiating (A.9) with respect to x

$$
\frac{\partial \Pi_{R_1}}{\partial x} = \frac{\partial(\frac{1}{4}t(x+y)^2 - tx^2)}{\partial x} = \frac{1}{2}t(x+y) - 2tx \quad \text{(A.11)}
$$

Setting $s = \frac{1}{2}(x+y)$ in (A.8)

$$\Pi_{R_2} = k + ty + Atys + Atxs - \frac{1}{2}t - tx^2 + ts^2 - Ak + Atx^2 - 2Ats^2 - Atx + \frac{1}{2}At - ty^2$$

$$= k + ty + \frac{1}{2}Aty(x+y) + \frac{1}{2}Atx(x+y) - \frac{1}{2}t - tx^2$$
$$+ \frac{1}{4}t(x+y)^2 - Ak + Atx^2 - \frac{1}{2}At(x+y)^2 - Atx + \frac{1}{2}At - ty^2$$

$$= k + ty + \frac{1}{2}Atxy + \frac{1}{2}Aty^2 + \frac{1}{2}Atx^2 + \frac{1}{2}Atxy - \frac{1}{2}t - tx^2$$
$$+ \frac{1}{4}t(x^2+2xy+y^2) - Ak + Atx^2 - \frac{1}{2}At(x^2+2xy+y^2) - Atx + \frac{1}{2}At - ty^2$$

$$= k + ty - \frac{1}{2}t - tx^2 + \frac{1}{4}tx^2 + \frac{1}{2}txy + \frac{1}{4}ty^2 - Ak + Atx^2 - Atx + \frac{1}{2}At - ty^2 \tag{A.12}$$

Differentiating (A.11) with respect to y

$$\frac{\partial \Pi_{R_2}}{\partial y} = \frac{\partial(k + ty - \frac{1}{2}t - tx^2 + \frac{1}{4}tx^2 + \frac{1}{2}txy + \frac{1}{4}ty^2 - Ak + Atx^2 - Atx + \frac{1}{2}At - ty^2)}{\partial y}$$

$$= t + \frac{1}{2}tx + \frac{1}{2}ty + -2ty \tag{A.13}$$

Differentiating $I_1(x,y)$ with respect to x we get

$$\begin{aligned} &\frac{\partial(a_1 t[\frac{1}{4}(x+y)^2 - x^2])}{\partial x} + \frac{\partial(\frac{1}{2}(1-a_1)(x+y))}{\partial x} \\ &= \frac{1}{2}a_1 t(x+y) - 2a_1 tx + \frac{1}{2}(1-a_1) \\ &= -\frac{3}{2}a_1 tx + \frac{1}{2}a_1 ty + \frac{1}{2}(1-a_1) \end{aligned} \tag{A.14}$$

Differentiating $I_2(x,y)$ with respect to y we get

$$\begin{aligned} &\frac{\partial(a_2[k + ty - \frac{1}{2}t + (A-1)tx^2 + \frac{1}{4}t(x+y)^2 - Ak - Atx + \frac{1}{2}At - ty^2])}{\partial y} \\ &+ \frac{\partial((1-a_2)[1 - \frac{1}{2}(x+y)])}{\partial y} \\ &= a_2[t + \frac{1}{2}t(x+y) - 2ty] + \frac{1}{2}(a_2 - 1) \end{aligned} \tag{A.15}$$

Setting (A.14) and (A.15) equal to 0 and solving the system we get

$$
\begin{aligned}
x &= \frac{2(t-1)a_1a_2 - a_1 + 3a_2}{8ta_1a_2} = \frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1} \\
y &= \frac{2(3t+1)a_1a_2 - 3a_1 + a_2}{8ta_1a_2} = \frac{3t+1}{4t} - \frac{3}{8ta_2} + \frac{1}{8ta_1}
\end{aligned}
\tag{A.16}
$$

Absent delegation $a_1 = a_2 = 1$ and then we have

$$
\begin{aligned}
x &= \frac{1}{4} \\
y &= \frac{3}{4}
\end{aligned}
\tag{A.17}
$$

A.3 Public's firm marginal profit

Evaluating (A.16) in Π_{R_1} we get

$$
\begin{aligned}
\Pi_{R_1} &= \frac{1}{4}t(x+y)^2 - tx^2 \\
&= \frac{1}{4}t\left(\frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1} + \frac{3t+1}{4t} - \frac{3}{8ta_2} + \frac{1}{8ta_1}\right)^2 - t\left(\frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1}\right)^2 \\
&= \frac{1}{4}t\left(\frac{t-1+3t+1}{4t} + \frac{1+3}{8ta_1} - \frac{1+3}{8ta_2}\right)^2 - t\left(\frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1}\right)^2 \\
&= \frac{1}{4}t\left(1 + \frac{1}{2ta_1} - \frac{1}{2ta_2}\right)^2 - t\left(\frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1}\right)^2 \\
&= \frac{1}{16}\frac{(a_1 - a_2 - 2a_1a_2t)^2}{ta_1^2a_2^2} - \frac{1}{64}\frac{(a_1 - 3a_2 + 2a_1a_2 - 2a_1a_2t)^2}{ta_1^2a_2^2}
\end{aligned}
\tag{A.18}
$$

Now if we differentiate (A.17) with respect to a_1 we get

$$\begin{aligned}\frac{\partial \Pi_{R_1}(a_1,a_2)}{\partial a_1} &= \frac{1}{16}\frac{2(a_1-a_2-2a_1a_2t)(1-2a_2t)ta_1^2a_2^2-2ta_2^2a_1(a_1-a_2-2a_1a_2t)^2}{(ta_1^2a_2^2)^2}\\ &-\frac{1}{64}\frac{2(a_1-3a_2+2a_1a_2-2a_1a_2t)(1+2a_2-2a_2t)ta_1^2a_2^2}{(ta_1^2a_2^2)^2}\\ &-\frac{2(a_1-3a_2+2a_1a_2-2a_1a_2t)^2ta_2^2a_1}{(ta_1^2a_2^2)^2}\\ &=\frac{1}{32}\frac{1}{ta_1^3a_2}(a_1+5a_2-6a_1a_2-2a_1a_2t)\end{aligned} \tag{A.19}$$

which is the marginal profit of R_1 with respect to a_1

A.4 Semi-private's firm marginal profit

Evaluating (A.16) in Π_{R_2} we get

$$\begin{aligned}\Pi_{R_2} &= k+ty-\frac{1}{2}t-tx^2+\frac{1}{4}tx^2+\frac{1}{2}txy+\frac{1}{4}ty^2-Ak+Atx^2-Atx+\frac{1}{2}At-ty^2\\ &=k+t\left(\frac{3t+1}{4t}-\frac{3}{8ta_2}+\frac{1}{8ta_1}\right)-\frac{1}{2}t-t\left(\frac{t-1}{4t}-\frac{1}{8ta_2}+\frac{3}{8ta_1}\right)^2\\ &+\frac{1}{4}t\left(\frac{t-1}{4t}-\frac{1}{8ta_2}+\frac{3}{8ta_1}\right)^2+\frac{1}{2}t\left(\frac{t-1}{4t}-\frac{1}{8ta_2}+\frac{3}{8ta_1}\right)\left(\frac{3t+1}{4t}-\frac{3}{8ta_2}+\frac{1}{8ta_1}\right)\\ &+\frac{1}{4}t\left(\frac{3t+1}{4t}-\frac{3}{8ta_2}+\frac{1}{8ta_1}\right)^2-Ak+At\left(\frac{t-1}{4t}-\frac{1}{8ta_2}+\frac{3}{8ta_1}\right)^2\\ &-At\left(\frac{t-1}{4t}-\frac{1}{8ta_2}+\frac{3}{8ta_1}\right)+\frac{1}{2}At-t\left(\frac{3t+1}{4t}-\frac{3}{8ta_2}+\frac{1}{8ta_1}\right)^2\end{aligned} \tag{A.20}$$

Now if we differentiate (A.19) with respect to a_2 we get

$$\frac{\partial \Pi_{R_2}(a_1,a_2)}{\partial a_2} = t\frac{3}{8ta_2^2} - 2t\left(\frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1}\right)\frac{1}{8ta_2^2} + \frac{2}{4}t\left(\frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1}\right)\frac{1}{8ta_2^2}$$

$$+ \frac{1}{2}t\frac{1}{8ta_2^2}\left(\frac{3t+1}{4t} - \frac{3}{8ta_2} + \frac{1}{8ta_1}\right) + \frac{1}{2}t\left(\frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1}\right)\frac{3}{8ta_2^2}$$

$$+ \frac{2}{4}t\left(\frac{3t+1}{4t} - \frac{3}{8ta_2} + \frac{1}{8ta_1}\right)\frac{3}{8ta_2^2} + 2At\left(\frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1}\right)\frac{1}{8ta_2^2}$$

$$- At\frac{1}{8ta_2^2} - 2t\left(\frac{3t+1}{4t} - \frac{3}{8ta_2} + \frac{1}{8ta_1}\right)\frac{3}{8ta_2^2}$$

$$= -\frac{1}{32}\frac{1}{ta_1a_2^2}[(A-6)a_1 + (2-3A)a_2 + 2a_1a_2(At+A+2)] \tag{A.21}$$

which is the marginal profit of R_1 with respect to a_2
Solving now the system of (A.18) and (A.20) we get

$$a_1 = \frac{2(A-4)}{2A+At-3t-8}$$

$$a_2 = -\frac{2(A-4)}{At+t-2A+8} \tag{A.22}$$

Both values are acceptable because

$$a_1 = \frac{2(A-4)}{2A+At-3t-8} \Rightarrow$$
$$a_1 = \frac{2(A-4)}{2(A-4)+(A-3)t}$$

$$A-4<0 \quad and \quad A-3<0 \qquad \Rightarrow a_1>0$$
$$0<2(4-A)<2(4-A)+(3-A)t \qquad \Rightarrow a_1<1$$

$$a_2 = -\frac{2(A-4)}{At+t-2A+8} \Rightarrow$$
$$a_2 = -\frac{2(A-4)}{-2(A-4)+(A+1)t}$$

$$-(A-4)>0 \quad and \quad -2(A-4)>0 \quad and \quad A+1>0 \qquad \Rightarrow a_2>0$$
$$0<-2(A-4)<-2(A-1)+(A+1)t \qquad \Rightarrow a_2<1$$

Now we evaluate (A.21) in (A.16)

$$
\begin{aligned}
x &= \frac{t-1}{4t} - \frac{1}{8ta_2} + \frac{3}{8ta_1} \\
&= \frac{t-1}{4t} - \frac{2A+t-2A}{8t2(A-4)} + \frac{3(2A+At-3t-8)}{8t2(A-4)} \\
&= \frac{A-3}{2(A-4)}
\end{aligned}
\tag{A.23}
$$

$$
\begin{aligned}
y &= \frac{3t+1}{4t} - \frac{3}{8ta_2} + \frac{1}{8ta_1} \\
&= \frac{3t+1}{4t} - \frac{3(2A+t-2A)}{8t2(A-4)} + \frac{2A+t-2A}{8t2(A-4)} \\
&= \frac{A-3}{A-4}
\end{aligned}
\tag{A.24}
$$

A.5 Consistent Location Conjectures with Delegation

$$
\begin{aligned}
I_1(x,y) &= a_1\Pi_{R_1} + (1-a_1)s \\
&= a_1[ts^2 - tx^2] + (1-a_1)s \\
&= a_1t\left[\frac{(x+y)^2}{4} - x^2\right] + (1-a_1)\frac{(x+y)}{2}
\end{aligned}
\tag{A.25}
$$

Now we differentiate I_1 with respect to x and we have

$$
\begin{aligned}
\frac{\partial I_1}{\partial x} &= a_1t\left[\frac{2(x+y)}{4}(1+r_{21}) - 2x\right] + \frac{(1-a_1)(1+r_{21})}{2} \\
&= a_1t\left[\frac{(x+y)(1+r_{21})}{2} - 2x\right] + \frac{(1-a_1)(1+r_{21})}{2}
\end{aligned}
\tag{A.26}
$$

where $r_{21} = \frac{dy}{dx}$.

Setting the equation $\frac{\partial I_1}{\partial x} = 0$ we get

$$
\begin{aligned}
&a_1t\left[\frac{(x+y)(1+r_{21})}{2} - 2x\right] + \frac{(1-a_1)(1+r_{21})}{2} = 0 \\
&a_1t\left[(x+y)(1+r_{21}) - 4x\right] + (1-a_1)(1+r_{21}) = 0
\end{aligned}
\tag{A.27}
$$

$$
\begin{aligned}
I_2(x,y) &= a_2 \Pi_{R_2} + (1-a_2)(1-s) \\
&= a_2 \left[k + ty + Atys + Atxs - \frac{1}{2}t - tx^2 + ts^2 - Ak + Atx^2 - 2Ats^2 - Atx + \frac{1}{2}At - ty^2 \right] \\
&+ (1-a_2)(1-s) \\
&= a_2 \left[(1-A) + ty + Ats(x+y) - \frac{1}{2}t + t(A-1)x^2 + t(1-2A)s^2 - Atx + \frac{1}{2}At - ty^2 \right] \\
&+ (1-a_2)(1-s) \\
&= a_2 \left[(1-A)k + ty + \frac{At(x+y)^2}{2} - \frac{1}{2}t + t(A-1)x^2 + \frac{t(1-2A)(x+y)^2}{4} - Atx + \frac{1}{2}At - ty^2 \right] \\
&+ (1-a_2)(1 - \frac{x+y}{2}) \\
&= a_2 \left[(1-A)k + ty + \frac{2At + t(1-2A)}{4}(x+y)^2 - \frac{1}{2}t + t(A-1)x^2 - Atx + \frac{1}{2}At - ty^2 \right] \\
&+ (1-a_2)(1 - \frac{x+y}{2}) \\
&= a_2 \left[(1-A)k + ty + \frac{t(x+y)^2}{4} - \frac{1}{2}t + t(A-1)x^2 - Atx + \frac{1}{2}At - ty^2 \right] \\
&+ (1-a_2)(1 - \frac{x+y}{2})
\end{aligned} \tag{A.28}
$$

Now we differentiate I_2 with respect to y and we have

$$\begin{aligned}\frac{\partial I_2}{\partial y} &= a_2\left[t + \frac{2t(x+y)(r_{12}+1)}{4} + 2t(A-1)xr_{12} - Atr_{12} - 2ty\right] \\ &+ (1-a_2)\left(-\frac{r_{12}+1}{2}\right) \\ &= a_2\left[t + \frac{t(x+y)(1+r_{12})}{2} + 2t(A-1)xr_{12} - Atr_{12} - 2ty\right] \\ &- \frac{(1-a_2)(1+r_{12})}{2}\end{aligned} \tag{A.29}$$

where $r_{12} = \frac{dx}{dy}$.

Setting the equation $\frac{\partial I_2}{\partial y} = 0$ we get

$$\begin{aligned}a_2t\left[1 + \frac{(x+y)(r_{12}+1)}{2} + 2(A-1)xr_{12} - Ar_{12} - 2y\right] - \frac{(1-a_2)(1+r_{12})}{2} = 0 \\ a_2t\left[2 + (x+y)(r_{12}+1) + 4(A-1)xr_{12} - 2Ar_{12} - 4y\right] - (1-a_2)(1+r_{12}) = 0\end{aligned} \tag{A.30}$$

Differentiating A.27 with respect to y we get

$$\frac{\partial\left(a_1t[(x+y)(1+r_{21}) - 4x - (1-a_1)(1+r_{21})]\right)}{\partial y} = a_1t[(r_{12}+1)(1+r_{21}) - 4r_{12}] \tag{A.31}$$

Setting A.31 equal to 0 we have

$$a_1t[(r_{12}+1)(1+r_{21}) - 4r_{12}] = 0 \tag{A.32}$$

Differentiating A.30 with respect to x we get

$$\begin{aligned}&\frac{\partial\left(a_2t\left[2 + (x+y)(r_{12}+1) + 4(A-1)xr_{12} - 2Ar_{12} - 4y\right] - (1-a_2)(1+r_{12})\right)}{\partial x} \\ &= a_2t[(1+r_{21})(r_{12}+1) + 4(A-1)r_{12} - 4r_{21}]\end{aligned} \tag{A.33}$$

Setting A.33 equal to 0 we have

$$a_2t[(1+r_{21})(r_{12}+1) + 4(A-1)r_{12} - 4r_{21}] = 0 \tag{A.34}$$

Solving the system of equations A.32 and A.34

$$\begin{aligned}(r_{12}+1)(r_{21}+1) - 4r_{12} = 0 \\ (r_{21}+1)(r_{12}+1) + 4(A-1)r_{12} - 4r_{21} = 0\end{aligned} \tag{A.35}$$

From A.35 we have

$$4r_{12} + 4(A-1)r_{12} - 4r_{21} = 0 \tag{A.36}$$

That leads to $r_{21} = Ar_{12}$. If we substitute that to A.35 we get

$$\begin{aligned}&(r_{12}+1)(Ar_{12}+1) - 4r_{12} = 0 \Rightarrow\\&Ar_{12}^2 + r_{12} + Ar_{12} + 1 - 4r_{12} = 0 \Rightarrow\\&Ar_{12}^2 + (A-3)r_{12} + 1 = 0\end{aligned} \tag{A.37}$$

For equation A.37 we have the three following cases

1. **A=0**

   If $A = 0$ then

   $$\begin{aligned}&-3r_{12} + 1 = 0 \Rightarrow\\&r_{12} = \frac{1}{3} \quad \text{and} \quad r_{21} = 0\end{aligned}$$

   Returning these values to A.27 we get

   $$\begin{aligned}&a_1t[(x+y)(1+0) - 4x] + (1-a_1)(1+0) = 0 \Rightarrow\\&a_1t(x+y-4x)(1-a_1) = 0 \Rightarrow\\&a_1t(y-3x) + (1-a_1) = 0 \Rightarrow\\&y - 3x = \frac{a_1 - 1}{a_1t}\end{aligned} \tag{A.38}$$

   Returning these values to A.29 we get

   $$\begin{aligned}&a_2t\left[2 + (x+y)\left(\frac{1}{3}+1\right) + 4(0-1)x\frac{1}{3} - 2\cdot 0\cdot\frac{1}{3} - 4y\right] - (1-a_2)\left(1+\frac{1}{3}\right) = 0 \Rightarrow\\&a_2t\left[2 + (x+y)\frac{4}{3} - \frac{4}{3}x - 4y\right] - (1-a_2)\frac{4}{3} = 0 \Rightarrow\\&a_2t\left(2 + \frac{4}{3}x + \frac{4}{3}y - \frac{4}{3}x - 4y\right) - (1-a_2)\frac{4}{3} = 0 \Rightarrow\\&a_2t\left(2 - \frac{8}{3}y\right) - (1-a_2)\frac{4}{3} = 0 \Rightarrow\\&a_2t(6-8y) - (1-a_2)4 = 0 \Rightarrow\\&6 - 8y = \frac{(1-a_2)4}{a_2t} \Rightarrow\\&-8y = \frac{(1-a_2)4}{a_2t} - 6 \Rightarrow\\&-8y = \frac{(1-a_2)4 - 6a_2t}{a_2t} \Rightarrow\\&y = \frac{4a_2 + 6a_2t - 4}{8a_2t} \Rightarrow\\&y = \frac{2a_2 + 3a_2t - 2}{4a_2t}\end{aligned} \tag{A.39}$$

Returning A.39 in A.38 we have

$$
\begin{aligned}
x &= \frac{a_1 - 1 - a_1 t y}{-3a_1 t} \\
&= \frac{a_1 - 1 - a_1 t \frac{2a_2 + 3a_2 t - 2}{4a_2 t}}{-3a_1 t} \\
&= \frac{4a_1a_2 - 4a_2 - 2a_1a_2 - 3a_1a_2t + 2a_1}{-12a_1a_2t} \\
&= \frac{2a_1a_2 - 3a_1a_2t + 2a_1 - 4a_2}{-12a_1a_2t} \\
&= \frac{3a_1a_2t - 2a_1a_2 - 2a_1 + 4a_2}{12a_1a_2t}
\end{aligned} \tag{A.40}
$$

Evaluating (A.39) and (A.40) in (A.10) and (A.12) we get

$$
\Pi_{R_1} = \frac{1}{48}\frac{4a_1^2 - 4a_2^2 + 8a_1a_2^2 - 8a_1^2a_2 - 12a_1^2a_2t + 12a_1^2a_2^2t + 9a_1^2a_2^2t^2}{a_1^2a_2^2t} \tag{A.41}
$$

$$
\Pi_{R_2} = \frac{1}{24}\frac{-4a_1^2 - 2a_2^2 + 8a_1^2a_2 + 4a_1a_2^2 - 6a_1^2a_2^2 + 24a_1^2a_2^2kt - 3a_1^2a_2^2t^2}{a_1^2a_2^2t} \tag{A.42}
$$

Differentiating (A.41) with respect to a_1 we get

$$
\frac{\partial \Pi_{R_1}}{\partial a_1} = -\frac{a_1 - 1}{6a_1^3 t} \tag{A.43}
$$

Differentiating (A.41) with respect to a_2 we get

$$
\frac{\partial \Pi_{R_2}}{\partial a_2} = -\frac{a_2 - 1}{3a_2^3 t} \tag{A.44}
$$

Setting (A.43) and (A.44) equal to 0 and solving with respect to a_1 and a_2 we get respectively

$$
\begin{aligned}
a_1(a_2) &= 1 \\
a_2(a_1) &= 1
\end{aligned} \tag{A.45}
$$

Now we return (A.45) into (A.39) and (A.40) and we get

$$x^C = \frac{1}{4}$$
$$y^C = \frac{3}{4}$$

2. **0<A<1**

Solving the quadratic equation in respect to r_{12} we have

$$\begin{aligned}\Delta = (A-3)^2 - 4A &= A^2 - 6a + 9 - 4a \\ &= A^2 - 10A + 9 \\ &= (A-1)(A-9)\end{aligned} \tag{A.46}$$

Because $0 < A < 1$, the discriminant of the quadratic equation is greater to 0 and we can calculate the solutions.

$$r_{12} = \frac{-(A-3) \pm \sqrt{A^2 - 10A + 9}}{2A} \tag{A.47}$$

- If $r_{12} = \frac{-A+3+\sqrt{A^2-10A+9}}{2A}$ and $r_{21} = \frac{-A+3+\sqrt{A^2-10A+9}}{2}$

$$a_1t[(x+y)(1+r_{21}-4x]+(1-a_1)(1+r_{21})=0 \Rightarrow$$
$$a_1t[(x+y)(1+r_{21}-4x]=(a_1-1)(1+r_{21})=0 \Rightarrow$$
$$(x+y)(1+r_{21}-4x)=\frac{(a_1-1)(1+r_{21})}{a_1t} \Rightarrow$$
$$(1+r_{21}-4)x+(1+r_{21})y=\frac{(a_1-1)(1+r_{21})}{a_1t} \Rightarrow$$
$$(r_{21}-3)x+(r_{21}+1)y=\frac{(a_1-1)(1+r_{21})}{a_1t} \Rightarrow$$

$$\left(\frac{-A+3+\sqrt{A^2-10A+9}}{2}-3\right)x+\left(\frac{-A+3+\sqrt{A^2-10A+9}}{2}+1\right)y=$$
$$\frac{(a_1-1)\left(1+\frac{-A+3+\sqrt{A^2-10A+9}}{2}\right)}{a_1t} \Rightarrow$$

$$\left(-A-3+\sqrt{A^2-10A+9}\right)x+\left(-A+5+\sqrt{A^2-10A+9}\right)y=$$
$$\frac{(a_1-1)\left(-A+5+\sqrt{A^2-10A+9}\right)}{a_1t}$$

$$a_2t\left[2+(x+y)(r_{12}+1)+4(A-1)r_{12}x-2Ar_{12}-4y\right]-(1-a_2)(1+r_{12})=0 \Rightarrow$$

$$2+(x+y)(r_{12}+1)+4(A-1)r_{12}x-2Ar_{12}-4y=\frac{(1-a_2)(1+r_{12})}{a_2t} \Rightarrow$$

$$(x+y)(r_{12}+1)+4(A-1)r_{12}x-4y=\frac{(1-a_2)(1+r_{12})}{a_2t}+2Ar_{12}-2 \Rightarrow$$

$$\left[(4A-3)r_{12}+1\right]x+(r_{12}-3)y=\frac{(1-a_2)(1+r_{12})}{a_2t}+2Ar_{12}-2 \Rightarrow$$

$$\left[(4A-3)\frac{-A+3+\sqrt{A^2-10A+9}}{2A}+1\right]x+\left(\frac{-A+3+\sqrt{A^2-10A+9}}{2A}-3\right)y=$$
$$\frac{(1-a_2)\left(1+\frac{-A+3+\sqrt{A^2-10A+9}}{2A}\right)}{a_2t}+2A\frac{-A+3+\sqrt{A^2-10A+9}}{2A}-2 \Rightarrow$$

$$\left[\frac{(4A-3)\left(-A+3+\sqrt{A^2-10A+9}\right)+2A}{2A}\right]x+\left(\frac{-A+3+\sqrt{A^2-10A+9}-6A}{2A}\right)y$$
$$=\frac{(1-a_2)\left(2A-A+3+\sqrt{A^2-10A+9}\right)}{2Aa_2t}+2A\frac{-A+3+\sqrt{A^2-10A+9}}{2A}-2\Rightarrow$$

$$\left(-4A^2+17A-9+(4A-3)\sqrt{A^2-10A+9}\right)x+\left(-7A+3+\sqrt{A^2-10A+9}\right)y=$$
$$=\frac{(1-a_2)\left(A+3+\sqrt{A^2-10A+9}\right)}{a_2t}+2A\left(-A+3+\sqrt{A^2-10A+9}\right)-4A\Rightarrow$$

$$=\frac{(1-a_2)\left(A+3+\sqrt{A^2-10A+9}\right)}{a_2t}+2A\left(-3A+3+\sqrt{A^2-10A+9}\right) \tag{A.48}$$

Solving the system of equations A.43 and A.44 by using the determinant method we have

$$D=\begin{vmatrix} d_1 & d_2 \\ d_3 & d_4 \end{vmatrix}$$

where

$$d_1=-A-3+\sqrt{A^2-10A+9}$$
$$d_2=-A+5+\sqrt{A^2-10A+9}$$
$$d_3=-4A^2+17A-9+(4A-3)\sqrt{A^2-10A+9}$$
$$d_4=-7A+3+\sqrt{A^2-10A+9}$$

$$
\begin{aligned}
D = &\left(-A-3+\sqrt{A^2-10A+9}\right)\left(-7A+3+\sqrt{A^2-10A+9}\right) \\
&-\left(-A+5+\sqrt{A^2-10A+9}\right)\cdot\left(-4A^2+17A-9+(4A-3)\sqrt{A^2-10A+9}\right)
\end{aligned}
$$

$$
\begin{aligned}
= &\, 7A^2-3A-A\sqrt{A^2-10A+9}+21A-9\cancel{-3\sqrt{A^2-10A+9}}-7A\sqrt{A^2-10A+9} \\
&+\cancel{3\sqrt{A^2-10A+9}}+\sqrt{A^2-10A+9}^2-[4A^3-17A^2+9A \\
&-4A^2\sqrt{A^2-10A+9}+3A\sqrt{A^2-10A+9}-20A^2+85A-45 \\
&+20A\sqrt{A^2-10A+9}-15\sqrt{A^2-10A+9}-4A^2\sqrt{A^2-10A+9} \\
&+17A\sqrt{A^2-10A+9}-9\sqrt{A^2-10A+9}+(4A-3)\sqrt{A^2-10A+9}^2]
\end{aligned}
$$

$$
\begin{aligned}
= &\, 7A^2+18A-\cancel{9}-8A\sqrt{A^2-10A+9}+A^2-10A+\cancel{9}-(4A^3-37A^2+94A \\
&+40\sqrt{A^2-10A+9}-8A^2\sqrt{A^2-10A+9}-24\sqrt{A^2-10A+9}-45+4A^3 \\
&-40A^2+36A-3A^2+30A-27)
\end{aligned}
$$

$$
\begin{aligned}
= &\, 8A^2+8A-8A\sqrt{A^2-10A+9}-(8A^3-80A^2+160A-72 \\
&+40A\sqrt{A^2-10A+9}-8A^2\sqrt{A^2-10A+9}-24\sqrt{A^2-10A+9})
\end{aligned}
$$

$$
\begin{aligned}
= &\, 8A^2+8A-8A\sqrt{A^2-10A+9}-8A^3+80A^2-160A+72 \\
&-40A\sqrt{A^2-10A+9}+8A^2\sqrt{A^2-10A+9}+24\sqrt{A^2-10A+9}
\end{aligned}
$$

$$
\begin{aligned}
= &\, -8A^3+88A^2-152A+72+8A^2\sqrt{A^2-10A+9} \\
&-48A\sqrt{A^2-10A+9}+24\sqrt{A^2-10A+9}
\end{aligned}
$$

$$
\begin{aligned}
&= -8A^3+88A^2-152A+72+\left(8A^2-48A+24\right)\sqrt{A^2-10A+9} \\
&= -8\left(A^3-11A^2+19A-9\right)+8(A^2-6A+3)\sqrt{A^2-10A+9}
\end{aligned}
\tag{A.49}
$$

$$
D_x = \begin{vmatrix} d_{x1} & d_{x2} \\ d_{x3} & d_{x4} \end{vmatrix}
$$

where

$$
\begin{aligned}
d_{x1} &= \frac{(a_1-1)\left(-A+5+\sqrt{A^2-10A+9}\right)}{a_1 t} \\
d_{x2} &= -A+5+\sqrt{A^2-10A+9} \\
d_{x3} &= \frac{(1-a_2)\left(A+3+\sqrt{A^2-10A+9}\right)}{a_2 t} + \frac{4Aa_2 t\left(-A+2+\sqrt{A^2-10A+9}\right)}{a_2 t} \\
d_{x4} &= -7A+3+\sqrt{A^2-10A+9}
\end{aligned}
$$

$$
\begin{aligned}
D_x = {} & \frac{(a_1-1)}{a_1 t}\left(-A+5\sqrt{A^2-10A+9}\right)\left(-7A+3+\sqrt{A^2-10A+9}\right) \\
& - \left(-A+5+\sqrt{A^2-10A+9}\right)\left[\frac{(1-a_2)}{a_2 t}\left(A+3+\sqrt{A^2-10A+9}\right)\right. \\
& + 2A\left(-3A+3+\sqrt{A^2-10A+9}\right)]
\end{aligned}
$$

$$
\begin{aligned}
= {} & \frac{(a_1-1)}{a_1 t}\left(7A^2-3A-A\sqrt{A^2-10A+9}-35A+15+5\sqrt{A^2-10A+9}\right. \\
& - 7A\sqrt{A^2-10A+9}+3\sqrt{A^2-10A+9}+\sqrt{A^2-10A+9}^2) \\
& + \left(A-5-\sqrt{A^2-10A+9}\right)\left[\frac{(1-a_2)}{a_2 t}\left(A+3+\sqrt{A^2-10A+9}\right)\right. \\
& + 2A\left(-3A+3+\sqrt{A^2-10A+9}\right)]
\end{aligned}
$$

$$
\begin{aligned}
= {} & \frac{(a_1-1)}{a_1 t}\left[8A^2-48A+24-8(A-1)\sqrt{A^2-10A+9}\right] \\
& + \frac{(1-a_2)}{a_2 t}\left(A-5-\sqrt{A^2-10A+9}\right)\left(A+3+\sqrt{A^2-10A+9}\right) \\
& + 2A\left(A-5-\sqrt{A^2-10A+9}\right)\left(-3A+3+\sqrt{A^2-10A+9}\right)
\end{aligned}
$$

$$
\begin{aligned}
= {} & \frac{(a_1-1)}{a_1 t}\left[8A^2-48A+24-8(A-1)\sqrt{A^2-10A+9}\right] \\
& + \frac{(1-a_2)}{a_2 t}(A^2+3A+\cancel{A\sqrt{A^2-10A+9}}-5A-15-5\sqrt{A^2-10A+9} \\
& - \cancel{A\sqrt{A^2-10A+9}}-3\sqrt{A^2-10A+9}-\sqrt{A^2-10A+9}^2) \\
& + 2A(-3A^2+3A+A\sqrt{A^2-10A+9}+15A-15-5\sqrt{A^2-10A+9} \\
& + 3A\sqrt{A^2-10A+9}-3\sqrt{A^2-10A+9}-\sqrt{A^2-10A+9}^2)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(a_1-1)}{a_1 t}\left[8A^2-48A+24-8(A-1)\sqrt{A^2-10A+9}\right] \\
&\quad + \frac{(1-a_2)}{a_2 t}(\cancel{A^2}+3A-5A-15-5\sqrt{A^2-10A+9}-3\sqrt{A^2-10A+9} \\
&\quad - \cancel{A^2}+10A-9)+2A[-3A^2+18A+(4A-8)\sqrt{A^2-10A+9}-15 \\
&\quad - A^2+10A-9] \\
&= \frac{(a_1-1)}{a_1 t}\left[8A^2-48A+24-8(A-1)\sqrt{A^2-10A+9}\right] \\
&\quad + \frac{(1-a_2)}{a_2 t}\left(8A-24-8\sqrt{A^2-10A+9}\right) \\
&\quad + 2A\left[-4A^2+28A-24+(4A-8)\sqrt{A^2-10A+9}\right]
\end{aligned} \tag{A.50}
$$

$$D_y = \begin{vmatrix} d_{y1} & d_{y2} \\ d_{y3} & d_{y4} \end{vmatrix}$$

where

$$
\begin{aligned}
d_{y1} &= -A-3+\sqrt{A^2-10A+9} \\
d_{y2} &= \frac{(a_1-1)\left(-A+5+\sqrt{A^2-10A+9}\right)}{a_1 t} \\
d_{y3} &= -4A^2+17A-9+(4A-3)\sqrt{A^2-10A+9} \\
d_{y4} &= \frac{(1-a_2)\left(A+3+\sqrt{A^2-10A+9}\right)}{a_2 t} + 2A\left(-3A+3+\sqrt{A^2-10A+9}\right)
\end{aligned}
$$

$$
\begin{aligned}
D_y = {} & \left(-A-3+\sqrt{A^2-10A+9}\right)\left[\frac{(1-a_2)}{a_2t}\left(A+3+\sqrt{A^2-10A+9}\right)\right. \\
& \left. + 2A\left(-3A+3+\sqrt{A^2-10A+9}\right)\right] - \left[-4A^2+17A-9\right. \\
& \left. + (4A-3)\sqrt{A^2-10A+9}\right]\frac{(a_1-1)}{a_1t}\left(-A+5+\sqrt{A^2-10A+9}\right) \\
\\
= {} & \frac{(a_2-1)}{a_2t}\left(A+3-\sqrt{A^2-10A+9}\right)\left(A+3+\sqrt{A^2-10A+9}\right) \\
& + 2A\left(A+3-\sqrt{A^2-10A+9}\right)\left(3A-3-\sqrt{A^2-10A+9}\right) - \frac{(a_1-1)}{a_1t} \\
& \cdot\left[4A^2-17A+9-(4A-3)\sqrt{A^2-10A+9}\right]\left(A-5-\sqrt{A^2-10A+9}\right) \\
\\
= {} & \frac{(a_2-1)}{a_2t}\left(\cancel{A^2}+6A+\cancel{9}-\cancel{A^2}+10A-\cancel{9}\right) + 2A\left(A+3-\sqrt{A^2-10A+9}\right)\cdot \\
& \left(3A-3-\sqrt{A^2-10A+9}\right) - \frac{(a_1-1)}{a_1t}\left[4A^2-17A+9\right. \\
& \left. -(4A-3)\sqrt{A^2-10A+9}\right]\left(A-5-\sqrt{A^2-10A+9}\right) \\
\\
= {} & \frac{(a_2-1)}{a_2t}\cdot 16A + 2A\left(3A^2-3A-A\sqrt{A^2-10A+9}+9A-9-3\sqrt{A^2-10A+9}\right. \\
& \left. -3A\sqrt{A^2-10A+9}+3\sqrt{A^2-10A+9}+\sqrt{A^2-10A+9}^2\right) - \frac{(a_1-1)}{a_1t}\cdot \\
& \left[4A^3-20A^2-4A^2\sqrt{A^2-10A+9}-17A^2+85A+17A\sqrt{A^2-10A+9}\right. \\
& + 9A-45-9\sqrt{A^2-10A+9}-\left(4A^2-3A\right)\sqrt{A^2-10A+9} \\
& \left. + (20A-15)\sqrt{A^2-10A+9}+(4A-3)\sqrt{A^2-10A+9}^2\right] \\
\\
= {} & \frac{(a_2-1)}{a_2t}\cdot 16A + 2A\left[3A^2+6A-4A\sqrt{A^2-10A+9}-\cancel{9}+A^2-10A+\cancel{9}\right] \\
& - \frac{(a_1-1)}{a_1t}\left[4A^3-37A^2+94A-45+(-4A^2+17A-9-4A^2+3A+20A\right. \\
& \left. -15)\cdot\sqrt{A^2-10A+9}+(4A-3)(A^2-10A+9)\right] \\
\\
= {} & \frac{(a_2-1)}{a_2t}\cdot 16A + 2A\left[4A^2-4A-4A\sqrt{A^2-10A+9}\right] \\
& - \frac{(a_1-1)}{a_1t}\left[4A^3-37A^2+94A-45-(8A^2-40A+24)\sqrt{A^2-10A+9}\right. \\
& \left. + 4A^3-40A^2+36A-3A^2+30A-27\right]
\end{aligned}
$$

$$= \frac{(a_2-1)}{a_2 t} \cdot 16A + 2A\left[4A^2 - 4A - 4A\sqrt{A^2-10A+9}\right]$$
$$- \frac{a_1-1}{a_1 t}\left[8A^3 - 80A^2 + 160A - 72 - \left(8A^2-40A+24\right)\sqrt{A^2-10A+9}\right] \quad \text{(A.51)}$$

– If $D \neq 0$ then we have $x(a_1,a_2,t) = \frac{D_x}{D}$ and $y(a_1,a_2,t) = \frac{D_y}{D}$

$$x = \frac{\frac{(a_1-1)}{a_1 t}\left[8A^2-48A+24-8(A-1)\sqrt{A^2-10A+9}\right]}{-8\left(A^3-11A^2+19A-9\right)+8(A^2-6A+3)\sqrt{A^2-10A+9}}$$
$$+ \frac{\frac{(1-a_2)}{a_2 t}\left(8A-24-8\sqrt{A^2-10A+9}\right)}{-8\left(A^3-11A^2+19A-9\right)+8(A^2-6A+3)\sqrt{A^2-10A+9}}$$
$$+ \frac{2A\left[-4A^2+28A-24+(4A-8)\sqrt{A^2-10A+9}\right]}{-8\left(A^3-11A^2+19A-9\right)+8(A^2-6A+3)\sqrt{A^2-10A+9}} \quad \text{(A.52)}$$

$$y = \frac{\frac{(a_2-1)}{a_2 t}\cdot 16A + 2A\left[4A^2-4A-4\sqrt{A^2-10A+9}\right]}{-8\left(A^3-11A^2+19A-9\right)+8(A^2-6A+3)\sqrt{A^2-10A+9}}$$
$$- \frac{\frac{a_1-1}{a_1 t}\left[8A^3-80A^2+160A-72-\left(8A^2-40A+24\right)\sqrt{A^2-10A+9}\right]}{-8\left(A^3-11A^2+19A-9\right)+8(A^2-6A+3)\sqrt{A^2-10A+9}} \quad \text{(A.53)}$$

Evaluating A.53 and A.54 in A.10 we get

$$\Pi_1 = frac_{\Pi_{11}} + frac_{\Pi_{12}} \quad \text{(A.54)}$$

where

$$frac_{\Pi_{11}} = -\frac{1}{4a_1^2a_2^2t}\frac{num_{11}}{dom_{11}}$$

$$\begin{aligned} num_{11} =& 3a_1^2 - 12a_2^2 - 6a_1^2a_2 + 24a_1a_2^2 - 9a_1^2a_2^2 + 40Aa_2^2 - 80Aa_1a_2^2 + 40Aa_1^2a_2^2 \\ &- 40A^2a_2^2 + 80A^2A - 1a_2^2 - 40A^2a_1^2a_2^2 + 12A^3a_2^2 - 24A^3a_1a_2^2 + 12A^3a_1^2a_2^2 \\ &- A^4a_2^2 + 2A^4a_1a_2^2 - A^4a_1^2a_2^2 + 8Aa_1^2a_2t - 4A^2a_1^2a_2t - 8Aa_1^2a_2^2 + 4A^2a_1^2a_2^2t \\ &+ 5A^2a_1^2a_2^2t^2 - 6A^3a_1^2a_2^2t^2 + A^4a_1^2a_2^2t^2 \end{aligned}$$

$$dom_{11} = A^4 - 12A^3 + 36A^2 - 32A + 9 - (A^3 - 7A^2 + 9A - 3)\sqrt{A^2-10A+9}$$

$$frac_{\Pi_{12}} = -\frac{\sqrt{A^2-10A+9}}{4a_1^2a_2^2t}\frac{num_{12}}{dom_{12}}$$

$$num_{12} = -9a_1^2 + 36a_2^2 + 5Aa_1^2 + 18a_1^2a_2 - 10Aa_1^2a_2 - 72a_1a_2^2 + 27a_1^2a_2^2 - 140Aa_2^2 + 280Aa_1a_1^2 - 135Aa_1^2a_2^2 + 184A^2a_2^2 - 368A^2a_1a_2^2 + 184A^2a_1^2a_2^2 - 92A^3a_2^2 + 184A^3a_1a_2^2 - 92A^3a_1^2a_2^2 + 17A^4a_2^2 - 34A^4a_1a_2^2 + 17A^4a_1^2a_2^2 - A^5a_2^2 + 2A^5a_1a_2^2 - A^5a_1^2a_2^2 - 24Aa_1^2a_2t + 28A^2a_1^2a_2t - 4A^3a_1^2a_2t + 24Aa_1^2a_2^2t - 28A^2a_1^2a_2^2t + 4A^3a_1^2a_2^2t - 17A^2a_1^2a_2^2t^2 + 27A^3a_1^2a_2^2t^2 - 11A^4a_1^2a_2^2t^2 + A^5a_1^2a_2^2t^2$$

$$dom_{12} = (A^2-10A+9)[A^4-12A^3+36A^2-32A+9-(A^3-7A^2+9A-3)\sqrt{A^2-10A+9}]$$

Evaluating A.53 and A.54 in A.12 we get

$$\Pi_2 = frac_{\Pi_{21}} + frac_{\Pi_{22}} \tag{A.55}$$

where

$$frac_{\Pi_{21}} = \frac{1}{4a_1^2a_2^2t}\frac{num_{21}}{dom_{21}}$$

$$num_{21} = -3a_1^2 - 24a_2^2 + 4aA_1^2 + 6a_1^2a_2 - 8Aa_1^2a_2 + 48a_1a_2^2 - 27a_1^2a_2^2 + 84Aa_2^2 - 168Aa_1a_2^2 + 88Aa_1^2a_2^2 - 92A^2a_2^2 + 184A^2a_1^2a_2^2 + 32A^3a_2^2 - 64A^3a_1a_2^2 + 32A^3a_1^2a_2^2 - 3A^4a_2^2 + 6A^4a_1a_2^2 - 3A^4a_1^2a_2^2 - 36a_1a_2^2t + 36a_1^2a_2^2t + 136Aa_1a_2^2t - 136Aa_1^2a_2^2t - 168A^2a_1a_2^2t + 168A^2a_1^2a_2^2t + 72A^3a_1a_2^2t - 72A^3a_1^2a_2^2t - 8A^4a_1a_2^2t + 8A^4a_1^2a_2^2t + 36a_1^2a_2^2kt - 164Aa_1^2a_2^2kt + 272A^2a_1^2a_2^2kt - 192A^3a_1^2a_2^2kt + 52A^4a_1^2a_2^2kt - 4A^5a_1^2a_2^2kt - 18a_1^2a_2^2t^2 + 82Aa_1^2a_2^2t^2 - 130A^2a_1^2a_2^2t^2 + 83A^3a_1^2a_2^2t^2 - 18A^4a_1^2a_2^2t^2 + A^5a_1^2a_2^2t^2$$

$$dom_{21} = A^4-12A^3+36A^2-32A+9-(A^3-7A^2+9A-3)\sqrt{A^2-10A+9}$$

$$frac_{\Pi_{22}} = \frac{\sqrt{A^2-10A+9}}{4a_1^2a_2^2t}\frac{num_{22}}{dom_{22}}$$

$$num_{22} = -9a_1^2 - 72a_2^2 + 17Aa_1^2 - 4A^2a_1^2 + 18a_1^2a_2 - 34Aa_1^2a_2 + 8A^2a_1^2a_2 + 144a_1a_2^2 - 81a_1^2a_2^2 + 292Aa_2^2 - 584Aa_1a_2^2 + 309Aa_1^2a_2^2 - 408A^2a_2^2 + 816A^2a_1a_2^2 - 412A^2a_1^2a_2^2 + 228A^3a_2^2 - 456A^3a_1a_2^2 + 228A^3a_1^2a_2^2 - 47A^4a_2^2 + 94A^4a_1a_2^2 - 47A^4a_1^2a_2^2 + 3A^5a_2^2 - 6A^5a_1a_2^2 + 3A^5a_1^2a_2^2 - 108a_1a_2^2t + 108a_1^2a_2^2t + 468Aa_1a_2^2t - 468Aa_1^2a_2^2t - 720A^2a_1a_2^2t + 720A^2a_1^2a_2^2t + 464A^3a_1a_2^2t - 464A^3a_1^2a_2^2t - 112A^4a_1a_2^2t + 112A^4a_1^2a_2^2t + 8A^5a_1a_2^2t - 8A^5a_1^2a_2^2t + 108a_1^2a_2^2kt - 552Aa_1^2a_2^2kt + 1068A^2a_1^2a_2^2kt - 976A^3a_1^2a_2^2kt + 420A^4a_1^2a_2^2kt - 72A^5a_1^2a_2^2kt + 4A^6a_1^2a_2^2kt - 54a_1^2a_2^2t^2 + 276Aa_1^2a_2^2t^2 - 520A^2a_1^2a_2^2t^2 + 441A^3a_1^2a_2^2t^2 - 165A^4a_1^2a_2^2t^2 + 23A^5a_1^2a_2^2t^2 - A^6a_1^2a_2^2t^2$$

$dom_{22} = (A^2-10A+9)[A^4-12A^3+36A^2-32A+9-(A^3-7A^2+9A-3)\sqrt{A^2-10A+9}]$

Differentiating (A.55) with respect to a_1 we get

$$\frac{\partial \Pi_{R_1}}{\partial a_1} = -\frac{(a_1-1)}{2a_1^3 t}\frac{num_{d1}}{dom_{di}} \quad \text{(A.56)}$$

$$\begin{aligned} num_{d1} =& A^6 - 22A^5 + 169A^4 - 548A^3 + 772A^2 - 480A + 108 \\ &- (A^5 - 17A^4 + 92A^3 - 184A^2 + 140A - 36)\sqrt{A^2-10A+9} \end{aligned}$$

$dom_{d1} = (A^2-10A+9)[A^4-12A^3+36A^2-32A+9-(A^3-7A^2+9A-3)\sqrt{A^2-10A+9}]$

Differentiating (A.56) with respect to a_2 we get

$$\frac{\partial \Pi_{R_2}}{\partial a_2} = \frac{(a_2-1)}{2a_1^3 t}\frac{num_{d2}}{dom_{d2}} \quad \text{(A.57)}$$

$$num_{d2} = 4A^3 + 43A^2 - 66A + 27 + (4A^2 - 17A + 9)\sqrt{A^2-10A+9}$$

$dom_{d2} = (A^2-10A+9)[A^4-12A^3+36A^2-32A+9-(A^3-7A^2+9A-3)\sqrt{A^2-10A+9}]$

Setting (A.57) and (A.58) equal to 0 and solving with respect to a_1 and a_2 we get respectively

$$\begin{aligned} a_1(a_2) =& 1 \\ a_2(a_1) =& 1 \end{aligned} \quad \text{(A.58)}$$

Now we return (A.59) into (A.53) and (A.54) and we get

$$x^C = -\frac{A}{2}\frac{A^2+8A-7+(-A+3)\sqrt{A^2-10A+9}}{A^3-11A^2+19A-9-(A^2-6A+3)\sqrt{A^2-10A+9}}$$

$$y^C = -\frac{A}{2}\frac{A^2-4A+3-(A+1)\sqrt{A^2-10A+9}}{A^3-11A^2+19A-9-(A^2-6A+3)\sqrt{A^2-10A+9}}$$

- If $D = 0$ then we have

$$\begin{aligned} &-8\left(A^3-11A^2+19A-9\right) + 8(A^2-6A+3)\sqrt{A^2-10A+9} = 0 \Rightarrow \\ &A^3-11A^2+19A-9 = (A^2-6A+3)\sqrt{A^2-10A+9} \Rightarrow \\ &\left(A^3-11A^2+19A-9\right)^2 = \left(A^2-6A+3\right)^2\left(A^2-10A+9\right) \Rightarrow \end{aligned}$$

$$A = 1 \qquad or \qquad A = 9 \qquad or \qquad A = \frac{2}{3} \quad \text{(A.59)}$$

In this case $0 < A < 1$ so the acceptable value is $A = \frac{2}{3}$.

$$D_x = \frac{(a_1 - 1)}{a_1 t}\left[8\left(\frac{2}{3}\right)^2 - 48\cdot\frac{2}{3} + 24 - 8\left(\frac{2}{3} - 1\right)\sqrt{\left(\frac{2}{2}\right)^2 - 10\cdot\frac{2}{3} + 9}\right]$$
$$+ \frac{(1 - a_2)}{a_2 t}\left[8\cdot\frac{2}{3} - 24 - 8\sqrt{\left(\frac{2}{2}\right)^2 - 10\cdot\frac{2}{3} + 9}\right]$$
$$+ 4\cdot\frac{2}{3}\left[-\left(\frac{2}{3}\right)^2 + 17\cdot\frac{2}{3} - 19 + \left(2\cdot\frac{2}{3} - 7\right)\sqrt{\left(\frac{2}{3}\right)^2 - 10\cdot\frac{2}{3} + 9}\right]$$

$$= \frac{(a_1 - 1)}{a_1 t}\cdot 0 + \frac{(1 - a_2)}{a_2 t}\cdot(-32) - \frac{1264}{27}$$

$$= 32\cdot\frac{(a_2 - 1)}{a_2 t} - \frac{576}{27} \neq 0 \tag{A.60}$$

That means that the system has no solution.

- If $r_{12} = \frac{-A+3-\sqrt{A^2-10A+9}}{2A}$ and $r_{21} = \frac{-A+3-\sqrt{A^2-10A+9}}{2}$

$$a_1 t[(x + y)(1 + r_{21} - 4x] + (1 - a_1)(1 + r_{21}) = 0 \Rightarrow$$
$$a_1 t[(x + y)(1 + r_{21} - 4x] = (a_1 - 1)(1 + r_{21}) = 0 \Rightarrow$$
$$(x + y)(1 + r_{21} - 4x = \frac{(a_1 - 1)(1 + r_{21})}{a_1 t} \Rightarrow$$
$$(1 + r_{21} - 4)x + (1 + r_{21})y = \frac{(a_1 - 1)(1 + r_{21})}{a_1 t} \Rightarrow$$
$$(r_{21} - 3)x + (r_{21} + 1)y = \frac{(a_1 - 1)(1 + r_{21})}{a_1 t} \Rightarrow$$

$$\left(\frac{-A + 3 - \sqrt{A^2 - 10A + 9}}{2} - 3\right)x + \left(\frac{-A + 3 - \sqrt{A^2 - 10A + 9}}{2} + 1\right)y = \frac{(a_1 - 1)\left(1 + \frac{-A+3-\sqrt{A^2-10A+9}}{2}\right)}{a_1 t} \Rightarrow$$

$$\left(A + 3 + \sqrt{A^2 - 10A + 9}\right)x + \left(A - 5 + \sqrt{A^2 - 10A + 9}\right)y = \frac{(a_1 - 1)\left(A - 5 + \sqrt{A^2 - 10A + 9}\right)}{a_1 t} \tag{A.61}$$

$$a_2t\left[2+(x+y)(r_{12}+1)+4(A-1)r_{12}x-2Ar_{12}-4y\right]-(1-a_2)(1+r_{12})=0 \Rightarrow$$

$$2+(x+y)(r_{12}+1)+4(A-1)r_{12}x-2Ar_{12}-4y=\frac{(1-a_2)(1+r_{12})}{a_2t} \Rightarrow$$

$$(x+y)(r_{12}+1)+4(A-1)r_{12}x-4y=\frac{(1-a_2)(1+r_{12})}{a_2t}+2Ar_{12}-2 \Rightarrow$$

$$\left[(4A-3)r_{12}+1\right]x+(r_{12}-3)y=\frac{(1-a_2)(1+r_{12})}{a_2t}+2Ar_{12}-2 \Rightarrow$$

$$\left[(4A-3)\frac{-A+3-\sqrt{A^2-10A+9}}{2A}+1\right]x+\left(\frac{-A+3-\sqrt{A^2-10A+9}}{2A}-3\right)y= \frac{(1-a_2)\left(1+\frac{-A+3-\sqrt{A^2-10A+9}}{2A}\right)}{a_2t}+2A\frac{-A+3-\sqrt{A^2-10A+9}}{2A}-2 \Rightarrow$$

$$\left[\frac{(4A-3)\left(-A+3-\sqrt{A^2-10A+9}\right)+2A}{2A}\right]x+\left(\frac{-A+3-\sqrt{A^2-10A+9}-6A}{2A}\right)y =\frac{(1-a_2)\left(2A-A+3-\sqrt{A^2-10A+9}\right)}{2Aa_2t}+2A\frac{-A+3-\sqrt{A^2-10A+9}}{2A}-2 \Rightarrow$$

$$\left[4A^2-17A+9+(4A-3)\sqrt{A^2-10A+9}\right]x+\left(7A-3+\sqrt{A^2-10A+9}\right)y= \frac{(a_2-1)\left(A+3-\sqrt{A^2-10A+9}\right)}{a_2t}+2A\left(3A-3+\sqrt{A^2-10A+9}\right) \tag{A.62}$$

$$D=\begin{vmatrix} d_1 & d_2 \\ d_3 & d_4 \end{vmatrix}$$

where

$$d_1=A+3+\sqrt{A^2-10A+9}$$
$$d_2=A-5+\sqrt{A^2-10A+9}$$
$$d_3=4A^2-17A+9+(4A-3)\sqrt{A^2-10A+9}$$
$$d_4=7A-3+\sqrt{A^2-10A+97}A-3+\sqrt{A^2-10A+9}$$

$$
\begin{aligned}
D = & \left(A+3+\sqrt{A^2-10A+9}\right)\left(7A-3+\sqrt{A^2-10A+9}\right) \\
& - \left(A-5+\sqrt{A^2-10A+9}\right)\cdot\left(4A^2-17A+9+(4A-3)\sqrt{A^2-10A+9}\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \, 7A^2-3A+A\sqrt{A^2-10A+9}+21A-9+3\sqrt{A^2-10A+9} \\
& + (7A-3)\sqrt{A^2-10A+9}+\sqrt{A^2-10A+9}^2 \\
& - \Big[4A^3-17A^2+9A+(4A^2-3A)\sqrt{A^2-10A+9}-20A^2 \\
& + 85A-45-(20A-15)\sqrt{A^2-10A+9}+(4A^2-17A+9)\sqrt{A^2-10A+9} \\
& + (4A-3)\sqrt{A^2-10A+9}^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
= & \, 8A^2+8A+(A+\cancel{3}+7A-\cancel{3})\sqrt{A^2-10A+9}-\cancel{9}+\cancel{9} \\
& - \big[4A^3-37A^2+94A-45+(4A^2-3A-20A+15+4A^2-17A+9) \\
& \cdot\sqrt{A^2-10A+9}+(4A-3)\sqrt{A^2-10A+9}\big]
\end{aligned}
$$

$$
\begin{aligned}
= & \, 8A^2+8A+8A\sqrt{A^2-10A+9}-\big[4A^3-37A^2+94A-45 \\
& + (8A^2-40A+24)\sqrt{A^2-10A+9}+4A^3-40A^2+36A-3A^2+36A-27\big]
\end{aligned}
$$

$$
\begin{aligned}
= & \, 8A^2+8A+8A\sqrt{A^2-10A+9}-\big[8A^3-80A^2+160A-72 \\
& + (8A^2-40A+24)\sqrt{A^2-10A+9}\big]
\end{aligned}
$$

$$
\begin{aligned}
= & \, 8A^2+8A+8A\sqrt{A^2-10A+9}-8A^3+80A^2-160A+72 \\
& - (8A^2-40A+24)\sqrt{A^2-10A+9}
\end{aligned}
$$

$$
= -8\left(A^3-11A^2+19A-9\right)-8\left(A^2-6A+3\right)\sqrt{A^2-10A+9} \tag{A.63}
$$

$$
D_x = \begin{vmatrix} d_{x1} & d_{x2} \\ d_{x3} & d_{x4} \end{vmatrix}
$$

where

$$
\begin{aligned}
d_{x1} &= \frac{(a_1-1)\left(A-5+\sqrt{A^2-10A+9}\right)}{a_1 t} \\
d_{x2} &= A-5+\sqrt{A^2-10A+9} \\
d_{x3} &= \frac{(a_2-1)\left(A+3-\sqrt{A^2-10A+9}\right)}{a_2 t}+2A\left(3A-3+\sqrt{A^2-10A+9}\right) \\
d_{x4} &= 7A-3+\sqrt{A^2-10A+9}
\end{aligned}
$$

$$
\begin{aligned}
D_x = & \frac{(a_1-1)}{a_1 t}\left(A-5+\sqrt{A^2-10A+9}\right)\left(7A-3+\sqrt{A^2-10A+9}\right) \\
& -\left(A-5+\sqrt{A^2-10A+9}\right)\left[\frac{(a_2)-1}{a_2 t}\left(A+3-\sqrt{A^2-10A+9}\right)\right. \\
& \left.+2A\left(3A-3+\sqrt{A^2-10A+9}\right)\right] \\
= & \frac{(a_1-1)}{a_1 t}\left[7A^2-3A+A\sqrt{A^2-10A+9}-35A+15-5\sqrt{A^2-10A+9}\right. \\
& \left.+(7A-3)\sqrt{A^2-10A+9}+\sqrt{A^2-10A+9}^2\right]-\left(A-5+\sqrt{A^2-10A+9}\right) \\
& \cdot\frac{(a_2-1)}{a_2 t}\left(A+3-\sqrt{A^2-10A+9}\right)-2A\left(A-5+\sqrt{A^2-10A+9}\right) \\
& \cdot\left(3A-3+\sqrt{A^2-10A+9}\right) \\
= & \frac{(a_1-1)}{a_1 t}\left[8A^2-48A+24+(8A-8)\sqrt{A^2-10A+9}\right] \\
& -\frac{(a_2-1)}{a_2 t}\left[8A-24+(-\cancel{A}+5+\cancel{A}+3)\sqrt{A^2-10A+9}\right] \\
& -2A\left[4A^2-28A+24+(A-5+3A-3)\sqrt{A^2-10A+9}\right] \\
= & \frac{(a_1-1)}{a_1 t}\left[8A^2-48A+24+(8A-8)\sqrt{A^2-10A+9}\right] \\
& -\frac{(a_2-1)}{a_2 t}\left(8A-24+8\sqrt{A^2-10A+9}\right) \\
& -2A\left[4A^2-28A+24+(4A-8)\sqrt{A^2-10A+9}\right]
\end{aligned}
\tag{A.64}
$$

$$
D_y = \begin{vmatrix} d_{y1} & d_{y2} \\ d_{y3} & d_{y4} \end{vmatrix}
$$

where

$$
\begin{aligned}
d_{y1} &= A+3+\sqrt{A^2-10A+9} \\
d_{y2} &= \frac{(a_1-1)\left(A-5+\sqrt{A^2-10A+9}\right)}{a_1 t} \\
d_{y3} &= 4A^2-17A+9+(4A-3)\sqrt{A^2-10A+9} \\
d_{y4} &= \frac{(a_2-1)\left(A+3-\sqrt{A^2-10A+9}\right)}{a_2 t}+2A\left(3A-3+\sqrt{A^2-10A+9}\right)
\end{aligned}
$$

$$D_y = \left(A+3+\sqrt{A^2-10A+9}\right)\left[\frac{(a_2-1)}{a_2t}\left(A+3-\sqrt{A^2-10A+9}\right)\right.$$
$$+2A\left(3A-3+\sqrt{A^2-10A+9}\right)] - [4A^2-17A+9$$
$$+(4A-3)\sqrt{A^2-10A+9}]\frac{(a_1-1)}{a_1t}\left(-A+5+\sqrt{A^2-10A+9}\right)$$

$$= \frac{(a_2-1)}{a_2t}\left(A+3+\sqrt{A^2-10A+9}\right)\left(A+3-\sqrt{A^2-10A+9}\right)$$
$$+2A\left(A+3+\sqrt{A^2-10A+9}\right)\cdot\left(3A-3+\sqrt{A^2-10A+9}\right)$$
$$-\frac{(a_1-1)}{a_1t}\left[4A^3-17A^2+9A+(4A^2-3A)\sqrt{A^2-10A+9}\right.$$
$$-20A^2+85A-45-(20A-15)\sqrt{A^2-10A+9}+(4A^2-17A+9)$$
$$\cdot\sqrt{A^2-10A+9}+(4A-3)\sqrt{A^2-10A+9}^2]$$

$$= \frac{(a_2-1)}{a_2t}\left(\cancel{A^2}+6A+\cancel{9}-\cancel{A^2}+10A-\cancel{9}\right)+2A\left[3A^2-3A+A\sqrt{A^2-10A+9}\right.$$
$$+9A-9+3\sqrt{A^2-10A+9}+(3A-3)\sqrt{A^2-10A+9}+\sqrt{A^2-10A+9}^2]$$
$$-\frac{(a_1-1)}{a_1t}[4A^3-37A^2+94A-45+(4A^2-3A-20A+15+4A^2-17A+9)$$
$$\cdot\sqrt{A^2-10A+9}^2+(4A-3)(A^2-10A+9)$$

$$= 16A\frac{(a_2-1)}{a_2t}+2A\left[4A^2-4A+4\sqrt{A^2-10A+9}\right]$$
$$-\frac{(a_1-1)}{a_1t}\left[4A^3-37A^2+94A-45+(8A^2-40A+24)\sqrt{A^2-10A+9}\right.$$
$$+4A^3-40A^2+36A-3A^2+30A-27]$$

$$= 16A\frac{(a_2-1)}{a_2t}+2A\left[4A^2-4A+4\sqrt{A^2-10A+9}\right]$$
$$-\frac{(a_1-1)}{a_1t}\left[8A^3-80A^2+160A-72+(8A^2-40A+24)\sqrt{A^2-10A+9}\right] \tag{A.65}$$

– If $D \neq 0$ then we have $x(a_1,a_2,t)=\frac{D_x}{D}$ and $y(a_1,a_2,t)=\frac{D_y}{D}$

$$x = \frac{\frac{(a_1-1)}{a_1 t}\left[8A^2 - 48A + 24 + 8(A-1)\sqrt{A^2-10A+9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}} - \frac{\frac{(a_2-1)}{a_2 t}\left(8A - 24 + 8\sqrt{A^2-10A+9}\right)}{-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}} - \frac{2A\left[4A^2 - 28A + 24 + (4A-8)\sqrt{A^2-10A+9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}} \tag{A.66}$$

$$y = \frac{16A \cdot \frac{(a_2-1)}{a_2 t} + 2A\left[4A^2 - 4A + 4\sqrt{A^2-10A+9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}} - \frac{\frac{(a_1-1)}{a_1 t}\left[8A^3 - 80A^2 + 160A - 72 + (A^2 - 40A + 24)\sqrt{A^2-10A+9}\right]}{-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}} \tag{A.67}$$

Evaluating A.67 and A.68 in A.10 we get

$$\Pi_1 = frac_{\Pi_{11}} + frac_{\Pi_{12}} \tag{A.68}$$

where

$$frac_{\Pi_{11}} = -\frac{1}{4a_1^2 a_2^2 t}\frac{num_{11}}{dom_{11}}$$

$$\begin{aligned} num_{11} =& 3a_1^2 - 12a_2^2 - 6a_1^2 a_2 + 24a_1 a_2^2 - 9a_1^2 a_2^2 + 40Aa_2^2 - 80Aa_1 a_2^2 + 40Aa_1^2 a_2^2 \\ & - 40A^2 a_2^2 + 80A^2 a_1 a_2^2 - 40A^2 a_1^2 a_2^2 + 12A^3 a_2^2 - 24A^3 a_1 a_2^2 \\ & + 12A^3 a_1^2 a_2^2 - A^4 a_2^2 + 2A^4 a_1 a_2^2 - A^4 a_1^2 a_2^2 + 8Aa_1^2 a_2 t - 4A^2 a_1^2 a_2 t \\ & - 8Aa_1^2 a_2^2 t + 4A^2 a_1^2 a_2^2 t + 5A^2 a_1^2 a_2^2 t^2 - 6A^3 a_1^2 a_2^2 t^2 + A^4 a_1^2 a_2^2 t^2 \end{aligned}$$

$$dom_{11} = A^4 - 12A^3 + 36A^2 - 32A + 9 + (A^3 - 7A^2 + 9A - 3)\sqrt{A^2-10A+9}$$

$$frac_{\Pi_{12}} = -\frac{\sqrt{A^2-10A+9}}{4a_1^2 a_2^2 t}\frac{num_{12}}{dom_{12}}$$

$$\begin{aligned} num_{12} =& -9a_1^2 + 36a_2^2 + 5Aa_1^2 + 18a_1^2 a_2 - 10Aa_1^2 a_2 - 72a_1 a_2^2 + 27a_1^2 a_2^2 \\ & - 140Aa_2^2 + 280Aa_1 a_2^2 - 135Aa_1^2 a_2^2 + 184A^2 a_2^2 - 368A^2 a_1 a_2^2 \\ & + 184A^2 a_1^2 a_2^2 - 92A^3 a_2^2 + 184A^3 a_1 a_2^2 - 92A^3 a_1^2 a_2^2 + 17A^4 a_2^2 \\ & - 34A^4 a_1 a_2^2 + 17A^4 a_1^2 a_2^2 - A^5 a_2^2 + 2A^5 a_1 a_2^2 - A^5 a_1^2 a_2^2 - 24Aa_1^2 a_2^2 t \\ & + 28A^2 a_1^2 a_2 t - 4A^3 a_1^2 a_2 t + 24Aa_1^2 a_2^2 t - 28A^2 a_1^2 a_2^2 t + 4A^3 a_1^2 a_2^2 t \\ & - 17A^2 a_1^2 a_2^2 t^2 + 27A^3 a_1^2 a_2^2 t^2 - 11A^4 a_1^2 a_2^2 t^2 + A^5 a_1^2 a_2^2 t^2 \end{aligned}$$

$$dom_{12} = (A^2{-}10A{+}9)[A^4{-}12A^3{+}36A^2{-}32A{+}9{+}(A^3{-}7A^2{+}9A{-}3)\sqrt{A^2-10A+9}]$$

Evaluating A.67 and A.69 in A.12 we get

$$\Pi_2 = frac_{\Pi_{21}} + frac_{\Pi_{22}} \tag{A.69}$$

where

$$frac_{\Pi_{21}} = \frac{1}{4a_1^2a_2^2t}\frac{num_{21}}{dom_{21}}$$

$$\begin{aligned} num_{21} =& -3a_1^2 - 24a_2^2 + 4Aa_1^2 + 6a_1^2a_2 - 8Aa_1^2a_2 + 48a_1a_2^2 - 27a_1^2a_2^2 + 84Aa_2^2 \\ & - 168Aa_1a_2^2 + 88Aa_1^2a_2^2 - 92A^2a_2^2 + 184A^2a_1a_2^2 - 92A^2a_1^2a_2^2 + 32A^3a_2^2 \\ & - 64A^3a_1a_2^2 + 32A^3a_1^2a_2^2 - 3A^4a_2^2 + 6A^4a_1a_262 - 3A^4a_1^2a_2^2 - 36a_1a_2^2t + 36a_1^2a \\ & + 136Aa_1a_2^2t - 136Aa_1^2a_2^2t - 168A^2a_1a_2^2t + 168A^2a_1^2a_2^2t + 72A^3a_1a_2^2t \\ & - 72A^3a_1^2a_2^2t - 8A^4a_1a_2^2t + 8A^4a_1^2a_2^2t + 36a_1^2a_2^2kt - 164Aa_1^2a_2^2kt \\ & + 272A^2a_1^2a_2^2kt - 192A^3a_1^2a_2^2kt + 52A^4a_1^2a_2^2kt - 4A^5a_1^2a_2^2kt - 18a_1^2a_2^2t^2 \\ & 82Aa_1^2a_2^2t^2 - 130A^2a_1^2a_2^2t^2 + 83A^3a_1^2a_2^2t^2 - 18A^4a_1^2a_2^2t^2 + A^5a_1^2a_2^2t^2 \end{aligned}$$

$$dom_{21} = A^4{-}12A^3{+}36A^2{-}32A{+}9{+}(A^3{-}7A^2{+}9A{-}3)\sqrt{A^2-10A+9}$$

$$frac_{\Pi_{12}} = \frac{\sqrt{A^2-10A+9}}{4a_1^2a_2^2t}\frac{num_{12}}{dom_{12}}$$

$$\begin{aligned} num_{22} =& 9a_1^2 + 72a_2^2 - 17Aa_1^2 + 4A^2a_1^2 - 18a_1^2a_2 + 34Aa_1^2a_2 - 8A^2a_1^2a_2 - 144a_1a_2^2 \\ & + 81a_1^2a_2^2 - 292Aa_2^2 + 584Aa_1a_2 - 309Aa_1^2a_2^2 + 408A^2a_2^2 - 816A^2a_1a_2^2 \\ & + 412A^2a_1^2a_2^2 - 228A^3a_2^2 + 456A^3a_1a_2^2 - 228A^3a_1^2a_2^2 + 47A^4a_2^2 - 94A^4a_1a_2^2 \\ & + 47A^4a_1^2a_2^2 - 3A^5a_2^2 + 6A^5a_1a_2^2 - 3A^5a_1^2a_2^2 + 108a_1a_2^2t - 108a_1^2a_2^2t \\ & - 468Aa_1a_2^2t + 468Aa_1^2a_2^2t + 720A^2a_1a_2^2t - 720A^2a_1^2a_2^2t - 464A^3a_1a_2^2t \\ & + 464A^3a_1^2a_2^2t + 112A^4a_1a_2^2t - 112A^4a_1^2a_2^2t - 8A^5a_1a_2^2t + 8A^5a_1^2a_2^2t \\ & - 108a_1^2a_2^2kt + 552Aa_1^2a_2^2kt - 1068A^2a_1^2a_2^2kt + 976A^3a_1^2a_2^2kt - 420A^4a_1^2a_2^2kt \\ & + 72A^5a_1^2a_2^2kt - 4A^6a_1^2a_2^2kt + 54a_1^2a_2^2t^2 - 276Aa_1^2a_2^2t^2 + 520A^2a_1^2a_2^2t^2 \\ & - 441A^3a_2^2a_2^2t^2 + 165A^4a_1^2a_2^2t^2 - 23A^5a_1^2a_2^2t^2 + A^6a_1^2a_2^2t^2 \end{aligned}$$

$$dom_{22} = (A^2{-}10A{+}9)[A^4{-}12A^3{+}36A^2{-}32A{+}9{+}(A^3{-}7A^2{+}9A{-}3)\sqrt{A^2-10A+9}]$$

Differentiating (A.69) with respect to a_1 we get

$$\frac{\partial \Pi_{R_1}}{\partial a_1} = \frac{(a_1-1)}{2a_1^3t}\frac{num_{d1}}{dom_{d1}} \tag{A.70}$$

$$
\begin{aligned}
num_{d1} =& A^6 - 22A^5 + 169A^4 - 548A^3 + 772A^2 - 480A + 108 \\
&+ (A^5 - 17A^4 + 92A^3 - 184A^2 + 140A - 36)\sqrt{A^2 - 10A + 9}
\end{aligned}
$$

$$
dom_{d1} = (A^2-10A+9)[A^4-12A^3+36A^2-32A+9-(A^3-7A^2+9A-3)\sqrt{A^2 - 10A + 9}]
$$

Differentiating (A.56) with respect to a_2 we get

$$
\frac{\partial \Pi_{R_2}}{\partial a_2} = \frac{(a_2 - 1)}{2a_1^3 t} \frac{num_{d2}}{dom_{d2}} \tag{A.71}
$$

$$
num_{d2} = 4A^3 + 43A^2 - 66A + 27 + (4A^2 - 17A + 9)\sqrt{A^2 - 10A + 9}
$$

$$
dom_{d2} = (A^2-10A+9)[A^4-12A^3+36A^2-32A+9+(A^3-7A^2+9A-3)\sqrt{A^2 - 10A + 9}]
$$

Setting (A.71) and (A.72) equal to 0 and solving with respect to a_1 and a_2 we get respectively

$$
\begin{aligned}
a_1(a_2) =& 1 \\
a_2(a_1) =& 1
\end{aligned} \tag{A.72}
$$

Now we return (A.73) into (A.67) and (A.68) and we get

$$
x^C = \frac{A}{2} \frac{A^2 - 8A + 7 + (A-3)\sqrt{A^2 - 10A + 9}}{A^3 - 11A^2 + 19A - 9 + (A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}}
$$

$$
y^C = \frac{A}{2} \frac{A^2 - 4A + 3 + (A+1)\sqrt{A^2 - 10A + 9}}{A^3 - 11A^2 + 19A - 9 + (A^2 - 6A + 3)\sqrt{A^2 - 10A + 9}}
$$

– If $D = 0$ then we have

$$
\begin{aligned}
&-8\left(A^3 - 11A^2 + 19A - 9\right) - 8(A^2 - 6A + 3)\sqrt{A^2 - 10A + 9} = 0 \Rightarrow \\
&A^3 - 11A^2 + 19A - 9 = -(A^2 - 6A + 3)\sqrt{A^2 - 10A + 9} \Rightarrow \\
&(A^3 - 11A^2 + 19A - 9)^2 = (A^2 - 6A + 3)^2(A^2 - 10A + 9) \Rightarrow
\end{aligned}
$$

$$
A = 0 \qquad or \qquad A = 1 \qquad or \qquad A = 9 \tag{A.73}
$$

So there are no acceptable values.

3. **A=1** If $A = 1$ then $r_{12} = r21 = 1$ and we have

$$\begin{aligned} & a_1 t\left[(x+y)(1+1)-4x\right]+(1-a_1)(1+1)=0 \Rightarrow \\ & a_1 t\left[2(x+y)-4x\right]=2(a_1-1) \Rightarrow \\ & -2a_t x+2a_1 t y=2(a_1-1) \Rightarrow \\ & -a_1 t x+a_1 t y=a_1-1 \end{aligned} \tag{A.74}$$

$$\begin{aligned} & a_2 t[2+(x+y)(1+1)-4(1-1)x\cdot 1-4-4y]-(1-a_2)(1+1)=0 \Rightarrow \\ & a_2 t[2+2(x+y)-4-4y]=2(1-a_2) \Rightarrow \\ & 2a_2 t+2a_2 t(x+y)-4a_2 t-4a_2 t y=2(1-a_2) \Rightarrow \\ & 2a_2 t x-2a_2 t y=2(1-a_2)-2a_2 t+4a_2 t \Rightarrow \\ & a_2 t x-a_2 t y=(1-a_2)-a_2 t+2a_2 t \Rightarrow \end{aligned} \tag{A.75}$$

Solving the system of equations (A.61) and (A.62) we have

$$D = \begin{vmatrix} -a_1 t & a_1 t \\ a_2 t & -a_2 t \end{vmatrix} = 0$$

A.6 Consistent Location Conjectures without Delegation

In the without delegation case, we assume that $a_1 = a_2 = a$ and the equations from Section 5 will be transformed as the following

1. **A=0**

$$y = \frac{2a + 3at - 2}{4at} \tag{A.76}$$

$$\begin{aligned} x &= \frac{3a\cdot a\cdot t-2a\cdot a-2a+4a}{12a\cdot a\cdot t} \\ &= \frac{3a^2 t-2a^2+2a}{12a^2 t} \end{aligned} \tag{A.77}$$

2. **0<A<1**
   For the two possible values of r_{12} and r_{21} we have

   - If $r_{12} = \frac{-A+3+\sqrt{A^2-10A+9}}{2A}$ and $r_{21} = \frac{-A+3+\sqrt{A^2-10A+9}}{2}$

$$x = \frac{\frac{(a-1)}{at}\left[8A^2 - 48A + 24 - 8(A-1)\sqrt{A^2-10A+9} - 8A + 24 + 8\sqrt{A^2-10A+9}\right]}{-8(A^3 - 11A^2 + 19A - 9) + 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}} + \frac{4A\left[-A^2 + 17A + 9 + (2A-7)\sqrt{A^2-10A+9}\right]}{-8(A^3 - 11A^2 + 19A - 9) + 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}}$$

$$= \frac{\frac{(a-1)}{at}\left[8A^2 - 56A + 48 - 8(A-2)\sqrt{A^2-10A+9} - 8A + 24 + 8\sqrt{A^2-10A+9}\right]}{-8(A^3 - 11A^2 + 19A - 9) + 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}} + \frac{4A\left[-A^2 + 17A + 9 + (2A-7)\sqrt{A^2-10A+9}\right]}{-8(A^3 - 11A^2 + 19A - 9) + 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}}$$

(A.78)

$$y = \frac{\frac{(1-a)}{at}\left[8A^3 - 80A^2 + 144A - 72 + (8A^2 - 40A + 24)\sqrt{A^2-10A+9}\right]}{-8(A^3 - 11A^2 + 19A - 9) + 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}} + \frac{4A\left[2A^2 - 9A + 3 - (2A+1)\sqrt{A^2-10A+9}\right]}{-8(A^3 - 11A^2 + 19A - 9) + 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}}$$

(A.79)

- If $r_{12} = \frac{-A+3-\sqrt{A^2-10A+9}}{2A}$ and $r_{21} = \frac{-A+3-\sqrt{A^2-10A+9}}{2}$

$$= \frac{\frac{(a-1)}{at}\left[8A^2 - 56A + 48 + 8(A-2)\sqrt{A^2-10A+9} - 8A + 24 + 8\sqrt{A^2-10A+9}\right]}{-8(A^3 - 11A^2 + 19A - 9) + 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}} - \frac{4A\left[2A^2 - 17A + 9 + (2A-7)\sqrt{A^2-10A+9}\right]}{-8(A^3 - 11A^2 + 19A - 9) - 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}}$$

(A.80)

$$y = \frac{\frac{(1-a)}{at}\left[8A^3 - 80A^2 + 144A - 72 + (8A^2 - 40A + 24)\sqrt{A^2-10A+9}\right]}{-8(A^3 - 11A^2 + 19A - 9) - 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}} + \frac{4A\left[2A^2 - 9A + 3 + (2A+1)\sqrt{A^2-10A+9}\right]}{-8(A^3 - 11A^2 + 19A - 9) - 8(A^2 - 6A + 3)\sqrt{A^2-10A+9}}$$

(A.81)

3. **A=0**

$$D = \begin{vmatrix} -at & at \\ at & at \end{vmatrix} = 0$$

So we have infinite solutions

And for the Nash equilibrium case,
We differentiate Π_{R_1} with respect to x that gives

$$\frac{1}{2}t(x+y) - 2tx \tag{A.82}$$

We differentiate Π_{R_2} with respect to y that gives

$$\frac{1}{2}At(x+y) + \frac{1}{2}Aty + \frac{1}{2}Atx + \frac{1}{2}t(x+y) - 2ty \tag{A.83}$$

Solving the system of (A.68) and (A.69), the firms locate at

$$x^C(t:1,1) = \frac{1}{4} \tag{A.84}$$

$$y^C(t:1,1) = \frac{3}{4} \tag{A.85}$$