ESTIMATION OF GREEK SECTORS' DEFAULT CORRELATIONS. CLASSICAL CORRELATION AND USE OF COPULAS. SENSITIVITY ANALYSIS OF PORTFOLIO LOSSES.

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Abstract

This study presents a reliable and easily applicable method for the estimation of pair-wise default correlations. In order to achieve our goal we apply the notion of copula which simplifies the procedure and allows us to estimate more precisely the dependence structure. The appropriate dependence structure is applied on sectors' default probabilities to derive the joint default probabilities. The sectors' default probabilities are derived through the first-passage approach of asset value models. We examine the impact of individual default probabilities and dependence structure on joint default probability and consequently on default correlation. Finally the implications for a loan portfolio are exhibited.

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I. INTRODUCTION

Banks play an important role in the economy as intermediaries in the financial system. One of the core services that banks provide is the granting of credit. Thus their regulation guarantees the healthy operation of the financial system. Among the types of regulations exist capital requirements. But regulations also impose important restrictions on the workings of banks. Excessive capital requirements restrain credit provision needlessly and may have the opposite, than the desired, impacts on the function of the financial system. For these reasons an equilibrium point between banking supervision and bank's practices had to be found.

Basel II, which has been released from the Basel committee on Banking Supervision, brings us more closely towards this target. The instructions that are given by Basel II ensure that a bank is able to withstand both expected and unexpected losses while simultaneously show respect to good provisioning practices and internal control that banks employ. Usually banks determine the required amount of capital through a portfolio credit risk model which they use to estimate the loss distribution for a specific time horizon.

Proposed credit risk modeling techniques range from microeconomic models (e.g. CreditMetrics) to top down actuarial models (e.g. CreditRisk+). There has been extensive research in this area which showed that modeling differences are immaterial and so one model can be expressed in terms of the other (see Frey and McNeil 2000). However as Koyluoglu [1999] and Gordy [2000] show, even frequently used models of portfolio credit risk can vary widely in their estimates of economic capital due to parameter inconsistencies. According to Mingo [2000] such variation would naturally incite regulatory capital arbitrage by banking institutions.

Among the parameters that have to be estimated, so that loss distribution is rightly estimated, are those which concern the dependence structure of a loan-portfolio. This is due to the fact that we cannot assume independence of default events. Cyclical correlation and contagion effects are responsible for the absence of independence. Thus, it is important to derive default correlation.

This study focus on the estimation of sectors' default correlation. It is an attempt which is undertaken for the first time in Greece. An attempt which presents special difficulties. First of all there is not any organized primary or secondary market for the issuing and trading of debt. The losses that each bank faces from non-served loans most times are not disclosed. But even state organizations do not hold data on the loans or other securities that have not been settled.

But even if these data existed, we would have to make a number of assumptions and ignore some of the properties of default correlation in order to extract default correlation.

The most common practice both in literature and industry that is used for the derivation of default correlations, is to apply an asset value model. In asset value models default occurs if the value of a firm's assets falls below a default threshold which is given by firm's liabilities. These asset value models make assumptions about the distribution that assets follow in order to extract default probabilities. Joint default probability of two firms, is the probability that both firms' assets fall below a default threshold, the same or different for each firm. This means that joint default probability is given by a bivariate distribution. It is common practice in financial literature and practice to assume that this distribution is the normal. So the joint default probability is determined by linear correlation of assets. Zhou (2001) with the application of Bessel functions provides an analytical solution for this case and so he derives joint default probability and default correlation. Thus if we adopt linear correlation as dependence measure we also adopt the assumption about normally distributed assets. However it has been observed that assets exhibit extreme co-movements [21]. These extreme co-movements and especially those concerned with the lower tail of assets distribution cannot be captured by a normal dependence structure. It must be stated that we are particularly interested in the lower tail dependence of assets as it is the one that will determine the probability of joint defaults and normal dependence does not exhibit tail dependence. Nevertheless, it is known that the dependence structure of the joint distribution is captured from copulas [12]. Copulas are functions which associate the marginal distributions to the joint and can be viewed as a tool that can help us to model more efficiently the real data. In particular they help us to separate the margins and their dependence structure. So we can assume that one of the margins follows a particular distribution, the other margin the same distribution or another else and the dependence structure corresponds to a different distribution. This means that copulas are a general tool which can model every kind of dependence. For example, bivariate normal distribution corresponds to two normal margins with Gaussian copula. Moreover we can extract from copulas dependence measures that can depict even better the underlying dependence structure. These dependence measures include tail dependence, rank correlation etc. A further advantage of copulas is that once we estimate the appropriate one, it can be applied to arbitrary marginal distributions. A very interesting property in our case that we want to derive the joint default probability. Moreover, copulas remain the same under strictly increasing transformations of the variable of interest. Thus many authors restrict themselves to copulas in order to estimate the distribution of losses of a loan portfolio. Some of them even propose to use some other dependence measures which are derived from copulas, in contrast to those who propose asset correlation as a proxy for the default correlation. However we have to state that most models which are used for the estimation of loss distribution of a loan portfolio require strictly default correlation. This is the appropriate dependence measure for the defaults which affects the distribution of loan losses. We also have to state that default correlation is necessary for the pricing of credit derivatives.

In this study we rely on the first-passage approach of asset value models to derive the sectoral individual default probabilities. We implement the notion of copula in order to model the underlying dependence structure of asset processes. We use the copula which fits better the empirical data in order to derive the joint default probability of two sectors. Thus, the whole process is reversed. Naturally we would like to have the default correlation in order to exact the joint default probability. But under this framework joint default probability is determined by assets' dependence.

Once we have derived the individual default probabilities and the joint default probability we are in position to estimate default correlations. We shed light on the impact that individual default probabilities and different copulas have on the joint default probability and default correlation .We proceed further and reveal the implications that default correlation has on a loan portfolio.

So our study comes to cover a gap in the financial literature and industry of our country despite the difficulties that are present. Our results are of particular interest for financial institutions that need default correlation in order to compute the loss distribution of their loan portfolios, regulators which want to have a view of financial institutions' risk, state organizations which might wish the development of a secondary market or to individual firms which may wish to know how their financial condition is affected from suppliers or clients. Our findings contribute to the disclosure of data that ought to be publicly available and present a method which is relatively easily applicable from every interested part. Moreover the estimation of default correlations for the Greek sectors may urge on the establishment of financial instruments which at the moment are unknown for the Greek reality such as CDOs.

Thus this study can help in the attempts for a more competitive financial system, more competitive procedures of raising capital and consequently for a more competitive Greek economy.

The remainder of the study has a cyclical structure: Section 2 introduces the basic definitions that are concerned with a loan portfolio, default correlation, its importance, the problems that posses and the main attempts to model it. Section 3 introduces the asset value models and compares the different approaches that have been developed into their framework. Section 4 is devoted to copulas and their families, parameter estimation and the selection of the appropriate one. Section 5 after presenting the procedure that was followed and the intermediate findings, returns to the notion of default correlation.

II.LOAN PORTFOLIO AND DEFAULT CORRELATION

Consider a portfolio of N loans subject to default. There are essentially two possible states a firm can be in after this time period, default or non-default. Thus we can model the state of each company i at our time horizon T as a Bernoulli random variable X_i, a so called "default indicator" defined by

$$X_i = \begin{cases} 1 & \text{if firm i is at default at time T} \\ 0 & \text{else} \end{cases}$$

and p_i shall denote the corresponding default probability (DP), i.e.

$p_i := P[X_i = 1]$ for i = 1,2,..N.

In the event of default, the lender receives only a percentage of the total debt. This percentage, called the recovery rate r, depends on the seniority of the loan. The total debt will be called exposure at default (EAD).

The loss of any obligor is then defined by a loss variable :

$L_i = EAD_i*(1-r_i)* X_i$ (1).

Now in this setting is very natural to define the expected loss (EL) of any costumer as the expectation of its corresponding loss variable L, namely

$EL_i = E[L_i] = EAD_i * (1-r_i) * E[X_i]$ (2).

The EL can be viewed as an insurance or loss reserve in order for a financial institution to cover its losses. But holding capital as a cushion against expected losses is not enough. In fact the financial institution should also hold money for covering unexpected losses from the EL. This quantity which is called unexpected loss [UL] is defined as

$$UL = \sqrt{Var(L_i)} = \sqrt{Var[EAD_i * LGD_i * X_i]} \quad (3)$$

Without loss of generality let us assume that $EAD_i = EAD$ and $r_i = r$ for i=1,2,…N. The portfolio loss is then defined as the random variable

$$L_{PF} = \sum_{i=1}^{N} L_i = \sum_{i=1}^{N} EAD * (1-r) * X_i = EAD * (1-r) * \sum_{i=1}^{N} X_i. \quad (4)$$

Analogously we have $EL_{PF} = \sum_{i=1}^{N} EL_i =. EAD * (1-r) * \sum_{i=1}^{N} E[X_i]$ (5).

In the case of the UL, we have

$$UL_p = \sqrt{\sum_{i=1}^{N} Var(X_i)} = \sqrt{\sum Var[EAD * (1-r) * X_i]} = EAD * (1-r) * \sqrt{\sum_{i=1}^{N} Var(X_i)} \quad (6)$$

However, defining the UL of a portfolio as the risk capital saved for cases of financial distress is not the best choice, because there might be a significant likelihood that losses will exceed the portfolio's EL by more than one standard deviation of the portfolio loss. Therefore one seeks other ways to quantify risk capital, hereby taking a target level of statistical confidence into account. The most common way to quantify risk capital is the concept of economic capital (EC). For a prescribed level of confidence α it is defined as the α-quantile of the portfolio loss L_{PF} minus the EL_{PF} .

Hence the joint default probability determines the entire risk of our portfolio. For example suppose we know that each of the loans in the portfolio has a 10% probability of default over the next five years. It could be that all loans in the portfolio always default together. So there is 10% probability that all loans in the portfolio will default and 90% probability that none of them will default. This is an example of "perfect" positive default correlation. At the other extreme it could be the case that loans in the portfolio always default separately, which means that if one of them defaults, no other defaults. This would be an example of perfect negative correlation. So there is 100% probability that one and only one (if any) loan in the portfolio will default. It is obvious that the former portfolio is much more risky than the latter, even though the default probabilities of bonds in the portfolios are the same. The difference in risk profiles, which is only due to default correlation, has profound implications to investors, lenders, rating agencies and regulators. Debt backed by the former portfolio should bear a higher premium for credit risk and be rated lower. If this is a regulated entity, it should be required to have more capital.

In general while examining the joint default probability of two firms it is reasonable that when one entity defaults, the other entity may have a higher likelihood of defaulting. Perhaps both firms are experiencing pressures from the general economy, their region or their industry. It is obvious from the previous example that default correlation is very important in understanding and predicting the behavior of credit portfolios. It directly affects the profile of investors in credit risky assets and is therefore important to the creditors and regulators of these investors. Default correlation also has implications for industrial companies that expose themselves in the credit risk of their suppliers and customers through the normal course of business.

Thus we always need to take into account default correlation. Default correlation is the phenomenon that one obligor defaulting on its debt is affected by whether or not another obligor has defaulted on its debts.

We define default correlation $Corr(X_1,X_2)$ as

$$Corr(X_1,X_2)=\frac{E[X_1 * X_2]-E[X_1]*E[X_2]}{\sqrt{Var[X_1]Var[X_2]}}=$$

$$\frac{P(X_1=1 and X_2=1)-P(X_1=1)*P(X_2=1)}{\{P(X_1=1)*[1-P(X_1=1)]\}^{1/2}*\{P(X_2=1)*[1-P(X_2=1)]\}^{1/2}} \quad (7)$$

The equation holds because X_1 and X_2 are Bernoulli random variables and so we have $E(X_i)= P(X_i=1)$, $Var(X_i)= P(X_i=1)[1- P(X_i=1)]$ and $E(X_1 * X_2) = P(X_1 =1$ and $X_2 =1)$.

Default correlation analysis plays a critical role in determining joint default probability-the probability of multiple defaults. From equation (7) we have

$P\ (X_1=1 \text{ and } X_2=1) = E(X_1 * X_2) = E(X_1) * E(X_2) + Corr(X_1, X_2)\sqrt{Var(X_1)Var(X_2)}$. (8)

We must have in mind that the joint default probability of two firms is

$P\ (X_1=1 \text{ and } X_2=1) = P\ (X_1=1) * P(X_2=1)$

if and only if we assume independence of default events.

We discern three cases

- If $Corr(X_1, X_2) = 0$ the third term it vanishes and there is independence of defaults
- If $Corr(X_1, X_2) > 0$ the two counterparties are interrelated in the sense that the default of one party increases the likelihood that the other will default too
- If $Corr(X_1, X_2) < 0$ we can treat the two counterparties as a hedge of the ther since if one of them defaults the possibility that the other will default is decreased.

Theoretically correlation can range from -1 to 1. But for binomial events like default, the range of possible default correlations is dictated by the default probabilities of the two credits. If both credits do not have the same default probability, they cannot have perfect correlation. Only if the default probability of two credits were 50% would it be mathematically possible for default correlation to range fully from -1 to +1. In the case that the default probabilities were 20% default correlation would be equal to one if and only if the joint default probability was 20%. If joint default probability was zero, then default correlation would be equal to -0.25. Thus default correlation would have a range of possible values [-0.25, 1] since the probability of any event cannot take negative values.

The number of the correlations that we have to estimate in a loan portfolio is (N-1)N/2 . An impossible task if $N \to \infty$, as it happens with a bank's portfolio. So emerges naturally the idea to group the loans having as criterion the creditworthiness of the customer. This is possible through the internal ratings of banks or the publicly available ratings from the rating agencies. Now the default correlations that we have to estimate are reduced dramatically.

With enough data and the strong assumption that default probability is constant in each rating class not only for its members but also from period to period we can calculate historic default correlations.

To compute say the default correlation of two B-rated companies over one year, we set P(A) and P(B) equal to the historic average one year default rate for B-rated companies. We compute P(A and B) by first counting the number of companies rated B at the beginning of a year that subsequently defaulted over the particular year. We then calculate all possible pairs of such defaulting B-rated companies. If X is the number of B-rated companies defaulting in a year, the possible pairs are: $\frac{X*(X-1)}{2}$.

We next calculate all possible pairs of B-rated companies whether or not they default, using the same formula $\frac{Y*(Y-1)}{2}$ where Y is the number of B-rated companies available to default. The joint default probability of B-rated companies in a particular year is: $\frac{[X*(X-1)]/2}{[Y*(Y-1)]/2}$.

The average of this statistic is taken over available years to determine P(A and B).

However the assumption of constant default probability turns out to be unsupportable. In fact default probabilities for different rating categories change from year to year. Varying default probability, a simple and plausible alternative explanation of fluctuating default rates throws doubt to the empirical default correlations.

Analysts' of Credit Suisse First Boston (CSFB) take into account this problem by assuming zero default correlation, while they allow default probability to fluctuate. In particular in CreditRisk$^+$ the default indicator X_i of each firm i is taken conditionally independent on its Bernoulli parameter p_i ,where p_i itself is random and described by a factor model. That is (X_i | $p_1 \ldots p_n$) independent ~Ber (p_i). It is assumed that there exist K risk factors $R_1, R_2, \ldots, R_K$ which describe the variability of the default probabilities p_i. These factors are taken to be independent Gamma distributed.

The link between the (p_i) and the (R_j) is given by the following factor model:

$p_i = \overline{p}_i \sum_{j=1}^{K} a_{ij} R_j$, i=1,2,…N and $(R_j)_j \sim Gam(1,\sigma_j^2)$, $\sum_{j=1}^{K} a_{ij} = 1 \ \forall i$.

It is clear that the factor loadings a_{ij},measure the sensitivity of obligor i to the risk R_j. In order to perfom a sector analysis, the authors of CreditRisk+ propose apportioning an obligor' s systematic risk across a mixture of independent sectors. These sectors could be industry, regional sectors or rating groups. However such an approach is difficult to realize in practice. Moreover the correct modeling of correlations of default risk between sectors is very important for examining the effects of diversification on active portfolio management.

The extension of CreditRisk+ model by examining correlations between industries and the derivation of a formula for the unexpected loss and risk contributions has been done with the method of Bürgisser et al.[1998]

Particularly they defined the expected and unexpected loss for a sector as follows: EL = $\sum_{A} p_A . v_A$ (9) and

$$UL = \sigma^2 \, EL^2 + \sum_{A} p_A . v_A^2 \quad (10)$$

Futrher on they derived the unexpected loss for N industry sectors.

$$UL^2 = \sum_{K} \sigma_\kappa^2 EL_k^{\;2} + \sum_{k.l} Cor_{k,\ell} \;\; \sigma_\kappa \; \sigma_\ell \;\; EL_k \;\; EL_\ell + \sum_{A} p_A . v_A^2 \quad (11)$$

Where σ^2 the relative default variance of the equivalent sector, p_A the default probability of the portfolio and v_A the analogue exposure.

The first two sums of equation (11) represent the risk due to systematic changes in the industries, reflected by the relative default variances, and the correlations between them. The third sum is the risk contribution due to statistical nature of default events, which is important for small portfolios or for cases with low systematic risk.

The relative default variance σ^2 is determined by the following equation:

$$\sigma^2 \, EL^2 = \sum_{K} \sigma_\kappa^2 EL_k^{\;2} + \sum_{k.l} Cor(\gamma_\kappa, \gamma_\ell) \;\; \sigma_\kappa \; \sigma_\ell \;\; EL_k \;\; EL_\ell \, . \quad (12)$$

The relative default variance σ^2 is used in the Gamma-distribution in CreditRisk+ in order to calculate the portfolio loss distribution and the overall economic capital.

The correlation between sectors –if positive as usually observed in the economy-increases the systematic risk contribution and becomes very important when working with many sectors. From now on, when refer to sector we mean industry sector. From a risk management perspective, the definition of industry sectors allows to diversify credit risk. The degree to which this diversification is successful depends on the strength of correlation between the sectors. Moreover the correlations between sectors PDs crucially influence the CreditVaR and hence the economic capital.

As Bürgisser et al recommend in order to estimate the risk parameters –the relative default variance and the correlation of default events between sectors-one can choose between three approaches (1) using industry specific time series of historically observed default events,(2)using asset values of firms to derive asset correlations, which are then linked to the relevant default correlations through the Merton's model and (3)using a factor model that relates the relative number of default events to macro-economic drivers

When we estimate default correlations from historical data there are not enough time-series data available to accurately estimate them. The estimation of cross-correlations is difficult due to the "curse of dimensionality". If the length T of the available time series is comparable to the number K of industry sectors, the number of estimated correlation coefficients is of the same order as the number of input parameters with the result of large estimation errors. Rosenow et al present evidence that the PD correlations for K=20 industry sectors are well captured by a one-factor model. However, even for the one-factor model the parameter estimation is subject to large statistical fluctuations and gives rise to a considerable uncertainty in the CreditVaR. Moreover, we cannot be sure that past history reflects the current reality. Not only default probability, but also default correlation changes through time. Last but not least in order to model default we need firm-specific information.

The problem when using an econometric model of firm value is that it depends on how well the econometric model captures firm's potential market value changes. We are faced always with the danger that we have omitted an important factor. Moreover we cannot be sure that the sensitivity to a factor is stable through time. Additionally a factor can become significant while it had not been in the past or another may not be important any more.

At the other hand, asset value models provide us with a stable through time method that explains default and they use firm specific information. They also provide us with an intuitive perspective about the causes of default. In order to model default correlation in this study we will be based on asset value models.

III. ASSET VALUE MODELS

In 1974, Merton wrote a seminal paper that explained how the then recently presented Black-Scholes model could be applied to the pricing of corporate debt. Many extensions of this model followed. This family of models is sometimes referred to as the family of structural models of corporate bond prices, and views prices of corporate debt and equity as portfolios of options on the fundamental value or asset value of the firm. Extensions of the original model relate to e.g. sub-ordination arrangements, indenture provisions and default before maturity (Black and Cox 1976), stochastic interest rates (Longstaf and Schwartz 1995) or an optimally chosen capital structure (e.g. Leland 1994), to name but a few.

Structural models have found applications in risk management, in central banks, or in pricing.

Classical approach

On his seminal work Merton R. makes the following assumptions in order to make use of the Black-Scholes pricing model in valuing corporate securities.

1) There are no transactions costs, taxes or problems with indivisibilities of assets.
2) There are a sufficient number of investors with comparable wealth levels so that each investor believes that he can buy and sell as much of an asset as he wants at the market price.
3) There exists an exchange market for borrowing and lending at the same rate of interest.
4) Short-sales of all assets, with full use of the proceeds, are allowed.
5) Trading in assets takes place continuously in time.
6) The Modigliani-Miller theorem that the value of the firm is invariant to its capital structure obtains.
7) The Term-structure is "flat" and known with certainty i.e., the price of a riskless discount bond which promises a payment of one dollar at time τ in the future is $P(\tau) = \exp[-r\tau]$ where r is the (instantaneous) riskless rate of interest, the same for all time.

To calculate the probability of default, we make assumptions about the distribution of assets at debt maturity under the physical probability P. The standard model for the evolution of asset prices over time is geometric Brownian motion.

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t \text{ , } V_0 > 0.$$

Where $\mu \in \Re$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and W is a standard Brownian motion. Setting $m = \mu - 1/2\ \sigma^2$, Ito' s lemma implies that $V_t = V_0 e^{mt+\sigma W t}$.

Since W_T is normally distributed with mean zero and variance T , default probabilities P(T) are given by

$$P(T) = P(V_T < K) = P(\sigma W_T < \log L - mT) = \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) \quad (13)$$

Where $L = K/V_0$ is the initial leverage ratio and Φ is the standard normal distribution function.

Hence the default time is a discrete random variable given by

$$\tau = \begin{cases} T & \text{if } V_T < K \\ \infty & \text{if else} \end{cases}$$

If a firm goes default the limited liability feature of equity means that the equity holders have the right but not the obligation, to pay off the debt holders and take over the remaining assets of a firm. That is the debt holders essentially own the firm until their liabilities are paid off in full by the equity holders. Thus equity can be viewed as a call option on the firm's assets with strike price equal to the book value of the firm's debts (payable at time T).

Assuming that the firm can neither repurchase shares nor issue new senior debt, we discern the following situations:

1) if the asset value V_T exceeds or equals the face value K of the bonds, the bond holders will receive their promised payment K and the shareholders will get the remaining $V_T - K$.

2) if the asset value V_T is less than K, the ownership of the firm will be transferred to the bondholders, who lose the amount $K - V_T$. Equity is worthless

because of limited liability. Summarizing, the value of the bond issue B^T_T at time T is given by

B^T_T =min (K, V_T) =K- max (0,K- V_T).

This payoff is equivalent to that of a portfolio composed of a default free-loan with face value K maturing at T and a short European put position on the assets of the firm with strike K and maturity T.

The value of the equity E_T at time T is given by $E_T = \max(0, V_T - K)$, which is equivalent to the payoff of a European call option on the assets of the firm with strike K and maturity T.

Pricing equity and credit risky debt reducing to pricing European options. We consider the classical Black-Scholes setting. The financial market is frictionless, trading takes place continuously in time risk-free interest rates r>0 are constant and firm assets follow geometric Brownian motion. Also, the value of the firm is a traded asset. The equity value is given by the Black-Scholes call option formula C:

$E_0 = C(\sigma, T, K, r, V_0) = V_0\,\Phi(d_+) - e^{-rT} K\,\Phi(d_-)$ (14) ,

where $d_{\pm} = \dfrac{(r \pm \frac{1}{2}\sigma^2)T - \log L}{\sigma\sqrt{T}}$

We note that the equity pricing function is monotone in firm volatility σ: equity holders always benefit from an increase in firm volatility.

While riskfree zero coupon bond prices are just kept K exp (-rT) with T being the bond maturity, the value of the corresponding credit-risky bonds is

B^T_0 = K exp (-rT) – P (σ,T,K,r, V_0) (15)

where P is the the Black-Scholes vanilla put option formula.

The credit spread is the difference between the yield on a defaultable bond and the yield an otherwise equivalent default-free zero bond. It gives the excess return demanded by bond investors to bear the potential default losses. Since the yield y(t,T) on a bond with a price b(t,T) satisfies b(t,T) = exp(-y (t,T)(T-t)), we have for the credit spread S(t,T) at time t, $S(t,T) = -\dfrac{1}{T-t}\log\left(\dfrac{B_t^T}{\bar{B}_t^T}\right)$ where $\bar{B}_t^T$ is the price of a default-free bond maturing at T. The term structure of credit spreads is the schedule of S(t,T) against T, holding t fixed.

In the classical approach, a firm defaults if its value is below the face value of the debt at maturity.Thus $A_i = \frac{\log(V_1^i / V_0^i) - m_i}{\sigma_i} < B_i = \frac{\log L_i - m_i}{\sigma_i}$ where A_i is the standardized asset return and B_i is the standardized face value of the debt, which is called the distance to default. The vector ($A_1 \ldots A_n$) is Gaussian with mean vector zero and covariance matrix $\Sigma = p_{ij}$, $p_{ij} = \mathrm{Cov}(W_1^i, W_1^j)$ being the asset correlation.

We obtain for the joint probability of firm 1 to default at time T_1 (the fixed debt maturity)and firm 2 to default at T_2 .

$$P(T_1, T_2) = P(V_{T_i}^1 < K_1, V_{t_2}^2 < K_2) = \Phi_2\left(p, \frac{\log L_1 - m_1 T_1}{\sigma_1\sqrt{T_1}}, \frac{\log L_2 - m_2 T_2}{\sigma_2\sqrt{T_2}}\right) \quad (16)$$

where $L_i = K_1 / V_0^i$ and $\Phi_2 = (r,.,.)$ is the bivariate standard normal distribution function with linear correlation parameter $|r| < 1$ given by

$$\Phi_2(r, a, b) = \int_{-\infty}^{a}\int_{-\infty}^{b} \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(\frac{2rxy - x^2 - y^2}{2(1-r^2)}\right) dxdy \quad (17)$$

It is important to note that the industry credit portfolio models provided by Moody's KMV ("PortfolioManager") and RiskMetrics ("CreditMetrics") which belong in the category of asset-value models particularly adopt the classical approach . Their mathematical structure can briefly described in the following way. Let the state of the economy be described by a random vector $\mathbf{X} = (X_1, \ldots, X_m)$ with $m \ll n$ and define the conditional default probability $p_i(X) = E[Y_i|X] = P[Y_i = 1|X]$. Then Y is called a Bernoulli mixture model with factor vector **X** , if conditionally on **X** , the Y_i are independent Bernoulli random variables with success probability $p_i(X)$. We introduce the multifactor linear model $W_1^i = a_i' X + b_i Z_i$ for asset returns. Here **X** is normal with zero mean vector and covariance matrix $\bar{\Sigma}$, the Z_i are independent and standard normal, $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{im})$ is a vector of constant factor weights, and b_i is a constant as well.

First-passage approach

In the classical approach, firm value can dwindle to almost nothing without triggering default. This is unfavourable to bondholders. Bond indenture provisions often include safety covenants that give bond investors the right to reorganize a firm if its value falls bellow a given barrier. Suppose the default barrier D is a constant valued in (0, V_0). Then the default time τ is a continuous random variable valued in $(0,\infty]$ given by $\tau = \inf \{ t > 0: V_T < D\}$.The default probabilities are calculated as

$P(T)= P(M_T <D)= P[\min_{s\leq t}(ms+\sigma W_s)<\log(d/V_o)]$ (18)

where M is the historical low of firm values $M_t = \min_{s\leq t} V_s$

Since the distribution of the historical low of an arithmetic Brownian motion is inverse Gaussian, we have

$$P(T) = \Phi\left(\frac{\log(D/V_o) - mT}{\sigma\sqrt{T}} \right) + \left(\frac{D}{V_o} \right)^{2m/\sigma^2} \Phi\left(\frac{\log(D/V_o) + mT}{\sigma\sqrt{T}} \right) \quad (19)$$

Assuming that the firm can neither repurchase shares nor issue new senior debt, we discern the following situations:

1)if the historical low of firm value M_T exceeds or equals the fefault point D, the bond holders will receive their promised payment K and the shareholders will get the remaining V_T –K.

2) if M_T is less than D, the firm defaults. In this case the firm stops operating,bond investors take over its assets D and equity investors receive nothing.

The value B of debt is given by B(V,T) = min (V,D) while equity is equal to max (V-d,0). Thus in this setting pricing a firm's securities is reduced to European style barrier options formula, taking as granted that the default point is constant.

We must make the important remark that the above analysis took as granted that default barrier D is set by debtholders. However in the model of Leland and Toft the decision to default is made by the managers who act to maximize the value of equity. At each moment equity holders are faced with the question if it is worth meeting the promised payments. If the asset value exceeds the default boundary, the firm will continue to meet debt payments even if asset value is less than debt principal value and even if the cash flows are insufficient for debt service. If the asset value lies below the default boundary the firm defaults. If the default barrier is endogenously determined it will affect the equity value. So in such a setting the MM theorem does not hold.

In the first passage approach, a firm defaults if its value falls below the default barrier D_i before maturity. Thus $A_i = \min_{s\leq 1}(m_i s+\sigma_i W_s^i) < B_i = \log(D_i/V_0^i)$, where A_i is the running minimum log-value of firm i at time 1 and B_i is the standardized default barrier. The vector (A_1 … A_n) is inverse Gaussian with mean vector zero and covariance matrix $\Sigma = p_{ij}$ $p_{ij} = \mathrm{Cov}(W_1^i, W_1^j)$ being the asset correlation.

Letting $M_t^i = \min_{s\leq t} V_s^i$ be the running minimum value of firm i at time t, we get for the joint default probability of firm 1 to default before T_1 and the firm 2 to default before T_2 $P(T_1, T_2) = P(M_{T_i}^1 < D_1, M_{t2}^2 < D_2) = \Psi_2\left(p, T_1, T_2; \log\frac{D_1}{V_0^1}, \log\frac{D_2}{V_0^2}\right)$ (20)

where D_i is the constant default barrier of firm i and holding $x,y\leq 0$ fixed, Ψ_2 (r;.,.;x,y)is the bivariate inverse Gaussian distribution function with correlation r.This function is given in closed form in Zhou.(2001)

Estimation techniques

The direct procedure

The first attempt at implementing structural models on corporate bonds was conducted by Jones, Mason, and Rosenfeld (1984) who suggested the following method:

First, estimate the asset value (V) as the sum of the value of equity (E), the observed value of traded debt and the estimated value of non-traded debt (assuming that the book to market ratio of traded and non-traded debt is the same). The volatility of the asset value is then calculated directly from the returns of the estimated asset value. They also proposed refining this by using the following relationship (derived from the equity pricing equation F_E using Ito's lemma):

$$\sigma_E = \sigma_V \frac{V}{E}\frac{\partial F_E}{\partial V} \quad (21)$$

Note that here, F_E depends on the particular structural model to be estimated, of course. An equity volatility is estimated from historical equity returns, and a second estimate of the asset volatility is obtained by plugging this and the first-pass estimate of the asset value into this equation.

The essential feature is that the asset value is estimated by a calculation based on book values and some observed market values of components of the total liabilities. Note that this has no statistical basis, and that it does not involve the assumptions of the model. Although possibly a reasonable educated guess, there is no reason to expect that this method will yield particularly reliable estimates of asset values, asset value volatilities, or to predict bond prices well.

The calibration procedure

The most common approach to implementing structural models to date, sometimes termed `calibration' has been to solve a set of two equations relating the observed price of equity and estimated (i.e. usually historical) equity volatility to asset value and asset value volatility (this method was first used in the context of deposit insurance by Ronn and Verma 1986). The equations used for this are the option-pricing equation describing the value of equity as an option on the underlying asset value (F_E), and the equation describing the relationship between equity volatility and asset value volatility derived from the equity pricing equation via Ito's lemma.

$$\begin{aligned} E &= F_E(V, \sigma_V) \\ \sigma_E &= \sigma_V \frac{V}{E} \frac{\partial F_E}{\partial V}. \end{aligned} \quad (22)$$

Once the equity-implied asset value and asset value volatility have been obtained,we can use them to estimate the default probability etc. This approach is common in the commercial world see e.g. KMV

We have to state that both approaches can be used to translate a time seies of equity values into a time series of asset values. So we can estimate asset correlation.

Comparison of the two approaches

Although the two approaches belong to the same category of models, they have some important differences that must be taken into account. As Zhou (2001) comments:

1) By ignoring the possibility of early default, the Merton approach underestimates both the probability of default of a single party and the probability of joint default.
2) The Merton-type approach as used by the financial industry is inconsistent for multiple horizons. For example to calculate two-year default correlations, the approach does not allow firms to default in the first year.The first-passage approach avoids this inconsistency.
3) The likelihood of early default increases rapidly with time horizon. For this reason, the Merton approach is mainly used by the financial industry to estimate default correlations or default probabilities over a one year horizon.The first-passage approach can be used to estimate them over any horizon.

However both models, classical and first passage posses some of the same problems. In particular they don't allow for contagion effects whose influence is very important now that the markets are integrated more than ever before. Moreover defaults are predictable meaning that they are not surprise events any more. This implies that investors would not demand a default-premium for short-term debt, which is not plausible and also at odds with empirical observations. Furthermore default correlations for non-tradable firms cannot be derived under these models, because a prerequisite for this is the ability to observe market values. These flaws can be faced from reduced –form models. However these models, although good enough for modeling credit spreads, provide us with no accurate estimations of default correlations. The results they generate, when compared with the historical default correlations prove this fact. Moreover, the formulas that reduced form models provide don't always have a closed form solution.

A promising relatively new approach, though not tested much, is given from incomplete information models. However both reduced –form and incomplete information models require for their implementation liquid debt markets so that credit spreads or credit default swap spreads are available. Unfortunately, this is far from reality for Greece.

Another disadvantage of asset value models as they have been presented in this study until now is the assumption of normally distributed assets.

A common fallacy

In Finance dominates the assumption of normality. Even the term asset correlation proves this. The term asset correlation refers to the classical linear correlation. So it will be useful to delve into the properties of this particular dependence measure.

Let $(X,Y)^T$ be a vector of random variables with nonzero finite variances. The linear correlation coefficient for $(X,Y)^T$ is

$$p(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \quad , (23)$$

where Cov(X,Y) = E(XY)-E(X)E(Y) is the covariance of $(X,Y)^T$ and Var(X) and Var(Y) are the variances of X and Y.

Linear correlation is a measure of linear dependence. In the case of perfect linear dependence, i.e. $Y=\alpha X+\beta$ almost surely for $\alpha \in \Re\backslash\{0\}, \beta \in \Re$, we have $|p(X,Y)|=1$.More important is that the converse also holds. Otherwise $-1<p(X,Y)<1$.

Linear correlation is invariant under strictly increasing linear transformations, and is easily manipulated under linear operations. Linear correlation is a popular but also misunderstood measure of dependence. The popularity of linear correlation stems from the ease with which it can be calculated and it is a natural scalar measure of dependence in elliptical distributions. However most random variables are not jointly elliptically distributed and using linear correlation as a measure of dependence in such situations might prove very misleading.

Desired properties of dependence measures.(as they are given in Nelsen [25])

A measure of dependence, like linear correlation, summarizes the dependence structure of two random variables in a single number. We consider the properties that would like to have from this measure. Let δ(.,.) be a dependence measure which assigns a real number to any pair of real valued random variables X and Y. Ideally we desire the following properties:

1) $\delta(X,Y) = \delta(Y,X)$ symmetry
2) $-1 \leq \delta(X,Y) \leq 1$ normalisation
3) $\delta(X,Y)=1$ comonotonic $\delta(X,Y)= -1$ countermonotonic
4) For $T: \Re \rightarrow \Re$ strictly monotonic on the range of X:

$$\delta(T((X),Y) = \begin{cases} \delta(X,Y) & \text{T increasing} \\ -\delta(X,Y) & \text{T decreasing} \end{cases}$$

Linear correlation fulfils properties 1, 2 only.Rank correlation also fulfils 3 and 4 if X,Y are continuous.These obviously represent a selection and the list could be altered or extended in a various ways.

Let X and Y be random variables with distribution functions $F_1$and F_2 and joint distribution function F.Spearman's rank correlation is given by:

$\rho_s(X,Y) = \rho_s(F_1 (X), F_2 (Y))$ (24), where ρ is the usual linear correlation.

Let (X_1, Y_1) and (X_2, Y_2) be two independent pairs of random variables from F, then Kendall's rank correlation is given by

$\rho_\tau(X,Y) = P[(X_1-X_2)(Y_1-Y_2)>0] - P[(X_1-X_2)(Y_1-Y_2)<0]$ (25)

The relationship that connects linear correlation and rank correlation is the following: : $\tau = 2/\pi * \arcsin(p) \Leftrightarrow p = \sin(\tau * \pi/2)$ (26)

The assumption of normally distributed assets can be removed in the framework of asset value models too. In fact we can introduce a more general multivariate normal mixture model for the asset returns W, which accommodate a wide range of more realistic distributions, such as the t-distribution. For some independent random variable U, we set

$W^i_1 = c_i(U) + d(U)(a'_i X + b_i Z_i)$, where $c_i : \Re \rightarrow \Re$,$d : \Re \rightarrow (0,\infty)$.

The distribution of the vector $(W^1,\ldots W^n)$ depends on the the choices for c_i,d and the distribution of U.

Suppose $C_i(u)$ and $d(u) = \sqrt{v/u}$ for some $v>0$ and $U \sim \chi^2(\nu)$.In this case the factor vector $(W^1,\ldots W^n)$ is a multivariate t-distributed with zero mean vector, covariance matrix $\frac{v}{v-2}\Sigma$ and $\nu > 2$ degrees of freedom. It is obvious that the joint default probabilities under these assumption will be t-distributed.

However the assumptions about the marginal distributions don't solve the dependence structure that the joint distribution exhibits, as it could be different from the margins.

For example Roy Mashal and Asaaf Zeevi use a t-dependence model which can a) detect whether the presence of extreme co-movements is statistically significant, while concurrently indicating the extent of this extremal dependence via the DoF parameter and b) test the validity of the Gaussian dependence structure. The main idea that underlies their testing procedure may be described informally as follows. They set the null hypothesis to correspond to some fixed value of the DoF parameter (v_o) in the t-dependence structure, while this is a free parameter in the alternative hypothesis. They vary the null parameter (v_o) in order to ascertain the range of values of the DoF which cannot be rejected, based on the corresponding p-values test.Since the Gaussian dependence structure is nested within the t-family (DoF=∞)they use p-values obtained for arbitrary large values of v_o as a proxy that indicates whether a Gaussian dependence structure is likely to be supported on the basis of the observed empirical asset co-movements. Recall that the multivariate t-distribution is a generalization of the multivariate Normal in the sense that the normal distribution can be considered as a t-distribution with infinite degrees of freedom.

Their findings show that asset returns exhibit extreme co-movements that cannot be captured by normal dependence structure and thus rendering t-dependence structure more appropriate for modeling their dependence. It is worthwhile to stress that they make no assumption about the distribution of the margins.

Thus it will be exciting to make ourselves common with the notion of copula which refers exclusively to the dependence structure without having to make any assumption about the marginal distributions.

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IV. COPULAS

Definition:A function $C:[0,1]^n \rightarrow [0,1]$ is a n-dimensional copula if it satisfies the following properties:

1) For all $u_i \in [0,1]$, $C(1,\dots,1, u_i,1,\dots,1)= u_i$.
2) For all $u \in [0,1]^N$, $C(u_1,\dots, u_N) = 0$ if at least one of the coordinates u_i equals zero.
3) C is grounded and N-increasing, ie, the C-measure of every box whose vertices lie in $[0,1]^N$ is non-negative.

A copula is a function that links univariate marginals to their joint multivariate distribution.If $F_1,\dots F_n$, are univariate distributions functions, $C(F_1(x_1), \dots F_i(x_i) \dots F_N(x_N))$ is a multivatiate distribution function with margins $F_1,\dots F_n$ because $u_i = F_i(x_i)$ is a uniform random variable. Copulas are then an adapted tool to construct multivariate distributions. Sklar established also the opposite which is even more important and is used in all applications of copulas.

Sklar's Theorem: Let F be a N-dimensional distribution function with continuous margins $F_1,\dots F_N$. Then F has a unique copula representation: $F(x_1, \dots x_i \dots x_N) = C(F_1(x_1), \dots F_i(x_i) \dots F_N(x_N))$. (27)

From Sklar's theorem we see that for continuous multivariate distribution functions, the univariate margins and the multivariate dependence structure can be separated and the dependence structure can be represented by a copula. This theorem is very important, because it provides a way to analyse the dependence structure of multivariate distributions without studying marginal distributions. As a direct consequence of Sklar's theorem copula properties are invariant under strictly increasing transformations of the underlying random variables.

Let F be a n N-dimensional distribution function with continuous margins $F_1,\dots F_N$, and copula C. Then for any u in $[0,1]^N$,$C(u_1, \dots u_N)=H(F_1^{-1}(x_1), \dots ,F_N^{-1}(x_N))$. (28)

For example let us have a bivariate distribution function H with continuous margins F_1, F_2. Then we have $C(x,y) = H(F_1^{-1}(x), F_2^{-1}(y))$.With this copula new bivariate distributions with arbitrary margins say K and L can be constructed $H'(u,v) = C(K(x), L(y))$.

Copula invariance theorem: Let $(X_1, \ldots X_n \ldots X_N)^T$ be a vector of continuous random variables with copula C. If $(a_1, \ldots a_n \ldots a_N)$ are strictly increasing on $RanX_1, \ldots RanX_n \ldots RanX_N$ respectively then also $(a_1(X_1), \ldots a_n(X_n) \ldots a_N(X_N))^T$ has copula C.

A very interesting property that stems from copula is **tail dependence.** It is related to the amount of dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a bivariate distribution.It is a concept that is relevant for the study of dependence between extreme values.

Let $(X,Y)^T$ be a vector of continuous random variables with marginal distribution functions F and G. The coefficient of upper tail dependence dependence of $(X,Y)^T$ is : $\lim_{u\to 1} P\{Y>G^{-1}(u)|X>F^{-1}(u)\}=\lambda_U$ (29)

provided that the limit $\lambda_U \in [0,1]$ exists. If $\lambda_U \in (0,1]$, X and Y are said to be asymptotically dependent in the upper tail; if $\lambda_U = 0$ X and Y are said to be asymptotically independent in the upper tail.

If a bivariate copula C is such that

$\lim_{u\to 1}(1-2u+C(u,u))/(1-u)=)=\lambda_U$ (30) exists,

then C has upper tail dependence if $\lambda_U \in [0,1]$ and upper tai independence if $\lambda_U = 0$.

The concept of lower tail dependence can be defined in a similar way.

If the limit $\lim_{u\to 0} C(u,u)/u=\lambda_L$ (32) exists,then C has lower tail dependence if $\lambda_L \in (0,1]$ and lower tail independence if $\lambda_L = 0$.

Copula representation of default dependence

Copulas are a powerful and very useful tool for measuring default dependence. As Giesecke (2004) states if we assume that $H_i(t,D_i)$is continuous in t and let $J_i = H_i^{-1}(.,D_i)$ denote its generalized inverse the copula C^τ of the default times is for all $u_i \in [0,1]$ given by: $C^\tau (u_1,\ldots, u_n)=C^M_{J_1(u_1),\ldots,J_1(u_n)} (u_1,\ldots, u_n)$ (33).

Given the marginal default probabilities $F_0^i(T) = H_i(T,D_i)$, the default dependence structure C^τ is given by the copula C^M of the historical asset low. This means that the statistical properties of C^M determine the likelihood of joint defaults, and in particular the tail of the aggregate default loss distribution, cf. Frey and McNeil (2001).

For example suppose that there has been no default by time t. Assuming that assets V follow a Markov process with stationary increments we get

$F_t(T_1,\ldots,T_n) = C^M_{T_1-t,\ldots,T_n-t}(H_1(D_1-V_t^i, T_1-t),\ldots,H_n(D_n-V_t^n, T_n-t))$ for $T_i > t$.

This formula offers in fact a general multivariate copula representation of Zhou's (2001) result who directly computed $F_0(T,T)$, in case n=2.

It is important to note that as the variable of interest is taken the running minimum asset process $(M_t^i)_{t\geq 0}$ which is defined by $M_t^i = \min\{V_s^i \mid 0\leq s\leq t\}$, so that M_t^i denotes the historical low of the asset value in the period [0,t]. The random default time is thus given by $\tau_i = \min\{t>0 \mid V_t^i \leq D_i\}$. The thresholds D are assumed to be independent of assets V. With the running minimum asset process M^i we obtain immediately $\{\tau_i \leq t\} = \{M_t^i \leq D_i\}$ meaning that the event of default before time t is equivalent to the event that the assets of the firm have been below the default barrier at least once in [0,t]. Thus the problem of modeling dependent defaults is reduced to finding the dependence structure of sectors' processes of assets minimum over a given time period!

We should state that if we replace the process of assets minimum of the various sectors with their asset's process V we can model their dependence according to the classical approach. In this project we will follow Giesecke's approach which both simplifies the dependence structure of defaults between the industry sectors and makes use of the first passage approach.

Now we are ready to focus on the copulas families which we will use to model the dependence of sectors' asset minimum process.

Elliptical Copulas

The class of elliptical distributions provides a rich source of multivariate distributions which share many of the tractable properties of the multivariate normal distribution and enables modeling of multivariate extremes and other forms of non-normal dependences. In the world of elliptical distributions correlation and covariance are natural measures of dependence. Linear combinations, marginal distributions and conditional distributions of elliptical random variables can largely be determined by linear algebra using knowledge of covariance matrix, mean and generator. Elliptical copulas are simply the copulas of elliptical distributions. Simulation from elliptical distributions is easy and as a consequence of Sklar's theorem so is simulation from elliptical copulas. Furthermore rank correlation and tail dependence coefficients can be easily calculated.

If X is a n-dimensional random vector and for some $\mu \in \Re^n$ and some negative n*n nonnegative definite, symmetric matrix Σ, the characteristic function $\varphi_{X-\mu}(t)$ of X-μ is a function of the quadratic form $t^T\Sigma t$, $\varphi_{X-\mu}(t) = \phi(t^T\Sigma t)$ we say that X has an elliptical distribution with parameters μ, Σ and φ and we write $X \sim E_n(\mu,\Sigma,\varphi)$.

Gaussian Copulas

The copula of the n-variate normal distribution with linear correlation matrix R is

$C^{Ga}_R(u) = \Phi^n_R(\Phi^{-1}(u_1),\ldots \Phi^{-1}(u_n))$ (34)

where Φ^n_R denotes the joint distribution function of the n-variable standard normal distribution with linear correlation matrix R, and Φ^{-1} denotes the inverse of the distribution function of the univariate standard normal distribution. Copulas of the above form are called Gaussian copulas.In the bivariate case the copula expression can be written as :

$$C^{Ga}_R(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-R_{12}^2)} \exp\left\{-\frac{s^2 - 2R_{12}st + t^2}{2(1-R_{12}{}^2)}\right\} dsdt \; .(35)$$

Note that R_{12} is simply the usual linear coefficient of the corresponding bivariate normal distribution.

The following example shows that Gaussian copulas do not have upper tail dependence. Since elliptical distributions are radially symmetric, the coefficient of upper and lower tail dependence are equal. Hence Gaussian copulas do not have lower tail dependence too.

Let $(X,Y)^T$ have the bivariate standard normal distribution function with linear correlation coefficient p. That is $(X,Y)^T \sim C(\Phi(x),\Phi(y))$, where C is a member of the Gaussian family with R_{12} = p. Since copulas in this family are exchangeable, $\lambda_U = 2$

$\lim_{u\to 1}P\{V>u|U=u\}$ and because Φ is a distribution function with infinite fight endpoint,

$\lim_{u\to 1}P\{V>u|U=u\}=\lim_{x\to\infty} P\{\Phi^{-1}(V)>x|\ \Phi^{-1}(U)=x\}= \lim_{x\to\infty} P\{X>x|\ Y=x\}$.

Using the well known fact that $Y|X=x\sim N(px,1-p^2)$ we obtain

$\lambda_U = 2\ \lambda_U = 2\lim_{x\to\infty} \overline{\Phi}((x-px)/\sqrt{1-p^2}) = 2\lim_{x\to\infty} \overline{\Phi}(x\sqrt{1-p}/\sqrt{1+p})$ (36)

from which it follows that $\lambda_U = 0$ for $R_{12} < 1$.

Hence Gaussian copula C with p<1 does not have upper tail dependence.

The problem of random variate generation from the Gaussian copula C^{Ga}_R is now addressed. For our purpose it is sufficient to consider only strictly positive matrices R Write $R= AA^T$ for some n*n matrix A, and if $Z_1,\dots, Z_n,\sim N(0,1)$ are independent, then $\mu+AZ\sim N_n(\mu,R)$. One natural choice of A is the Cholesky decomposition of R. The Cholesky decomposition of R is the unique lower-triangular matrix L with $LL^T = R$.

- Find the Cholesky decomposition A of R.
- Simulate n independent random variates $z_1,\dots, z_n$, from N(0,1)
- Set **x=Az**
- Set $u_i = \Phi(x_i)$, i=1,..,n
- $(u_1,\dots, u_n)^T\sim C^{Ga}_R$

t-copulas

If **X** has the stochastic representation $\mathbf{X} =_d \mu + \frac{\sqrt{v}}{\sqrt{S}}\mathbf{Z}$, where $\mu \in \Re^n$, $S\sim\chi^2_v$ and $\mathbf{Z}\sim N_n(\mathbf{0},\Sigma)$ are independent, then **X** has an n-variate t_v-distribution with mean μ (for v>1) and covariance matrix $\frac{v}{v-2}\Sigma$ (for v>2).If $v\leq 2$ then Cov (**X**) is not defined. In this case we just interpret Σ as being the shape parameter of the distribution of **X.** The copula of **X** can be written as

$C^t_{v,R}(u) = t^n_{v,R}(t_v^{-1}(u_1),\dots t_v^{-1}(u_n))$ (37)

where $R_{ij} = \sum_{ij} \Big/ \sqrt{\sum_{ii}\sum_{jj}}$ for i,j $\in$ {1,…,n) and where $t^n_{v,R}$ denotes the distribution function of $\sqrt{v}Y/\sqrt{S}$ where $S\sim\chi^2_v$ and $\mathbf{Y}\sim N_n(\mathbf{0},R)$ are independent. Here t_v denotes the margins of $t^n_{v,R}$ i.e. the distribution function of $\sqrt{v}Y_1/\sqrt{S}$. In the bivariate case the copula expression can be written as :

$$C^t_{v,R}(u,v) = \int_{-\infty}^{t_v^{-1}(u)} \int_{-\infty}^{t_v^{-1}(u)} \frac{1}{2\pi(1-R_{12}^2)^{1/2}} \left\{1+\frac{s^2-2R_{12}st+t^2}{v(1-R_{12}^2}\right\}^{-(v+2)/2} \text{dsdt.} \quad (38)$$

Note that R_{ij} is simply the usual linear correlation coefficient of the corresponding bivariate t_v-distribution if v>2.

If $(X_1, X_2)^T$ has a standard bivariate t-distribution with v degrees of freedom and linear correlation matrix R, then $X_2| X_1 = x$ is t-distributed with v+1 degrees of freedom and $E(X_2| X_1 = x) = R_{12}\, x$, $Var(X_2| X_1 = x) = (\frac{v+x^2}{v+1})(1-R_{12}^2)$.

This can be used to show that the t-copula has upper (and because of radial symmetry) equal lower tail dependence.

$$\lambda_U = 2 \lim_{x\to\infty} P(X_2 > x \mid X_1 = x) = 2 \lim_{x\to\infty} \bar{t}_{n+1}\left(\left(\frac{v+1}{v+x^2}\right)^{1/2} \frac{x-R_{12}x}{\sqrt{1-p_l^2}}\right) = 2 \lim_{x\to\infty}$$

$$\bar{t}_{n+1}\left(\left(\frac{v+1}{v/x^2+1}\right)^{1/2} \frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}^2}}\right) = 2 \lim \bar{t}_{n+1}\left(\sqrt{v+}\sqrt{1-R_{12}}/\sqrt{1+R_{12}}\right). \quad (39)$$

From this it is also seen that the coefficient of upper tail dependence is increasing in R_{12} and decreasing in v , as one would expect. Furthermore the coefficient of upper (lower) tail dependence tends to zero as the number of degrees of freedom tends to infinity for $R_{12} < 1$.

It follows an algorithm for random variate generation from the t-copula $C^t_{v,R}$

- Find the Cholesky decomposition A of R.
- Simulate n independent random variates $z_1,\dots, z_n$, from N(0,1)
- Set $\mathbf{y}=A\mathbf{z}$
- Set $\mathbf{x} = \frac{\sqrt{v}}{\sqrt{s}}\,\mathbf{y}$
- Set $u_i = t_v(x_i)$, i=1,..,n
- $(u_1,\dots, u_n)^T \sim C^t_{v,R}$

Archimedean Copulas

This class of copulas is worth studying for a number of reasons. Many interesting parametric families of copulas are Archimedean and the class of Archimedean copulas allow for a great variety of different dependence structures. Unlike elliptical copulas they can model asymmetries. This is very useful in finance where it seems reasonable that there is a stronger dependence between big losses than between big gains. Furtermore, in contrast to elliptical copulas are not derived from multivariate distributions functions using Sklar's theorem. A consequence of this is that we need somewhat technical conditions to assert that multivariate extensions of Archimedean 2-copulas are proper n-copulas. A further disadvantage is that multivariate extensions of copulas in general suffer from lack of free parameter choice in the sense that some of the entries in the resulting rank correlation matrix are forced to be equal.

Let φ be a continuous, strictly decreasing function from [0,1] to [0,∞], such that φ(1)=0. The pseudo-inverse of φ is the function $\varphi^{[-1]}$: [0,∞]→[0,1] given by

$$\varphi^{[-1]} = \begin{cases} \varphi^{-1}(t) & 0 \le t \le \varphi(0) \\ 0 & \varphi(0) \le t \le \infty \end{cases} \quad (40)$$

Note that $\varphi^{[-1]}$ is continuous and decreasing on [0,∞] and strictly decreasing on [0, φ(0)]. Furtermore $\varphi^{[-1]}$ (φ(u))= u on [0,1] and

$$\varphi(\varphi^{[-1]}(t)) = \begin{cases} t & 0 \le t \le \varphi(0) \\ \varphi(0) & \varphi(0) \le t \le \infty \end{cases} \quad (41)$$

Let φ be a continuous, strictly decreasing function from [0,1] → [0,∞] such that φ(1) =0, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ.
Let C be the function from $[0,1]^2$→[0,1] given by C(u,v) = $\varphi^{[-1]}$(φ(u)+φ(v)).
Then C is a copula if and only if φ is convex.

Copulas of the above form are called Archimedean copulas. The function φ is called the generator of the copula.If φ=∞ we say that φ is a strict generator. In this case, $\varphi^{[-1]} = \varphi^{-1}$ and C(u,v)= φ^{-1}(φ(u)+φ(v)) is said to be a strict Archimedean copula.
Let C be an Archimedean copula with generator φ. Then

1. C is symmetric, i.e. C(u,v)= C(v,u)for all u,v in [0,1].
2. C is assosiative , i.e. C(C(u,v),w)= C(C(u,(v,w)) for all u,v,w in [0,1].

For example let φ(t) = $(-\ln t)^{\theta}$,where θ≥1.Clearly φ(t) is continuous and φ(1)=0.

$\varphi'(t) = -\theta(-\ln t)^{\theta-1}\frac{1}{t}$, so φ is a strictly decreasing function from [0,1] to [0,∞].$\varphi'' \geq 0$ on [0,1], so φ is convex. Moreover φ(∞) =0 , so φ is a strict generator. This is the Gumbel family of copulas.

Corollary:Let C be an Archimedean copula generated by φ and let

$K_C(t)= V_C(\{(u,v) \in [0,1]^2 \mid C(u.v)\leq t\})$. Then for any t in [0,1] $K_C(t) = t - \frac{\varphi(t)}{\varphi'(t^+)}$ (42)

If (U,V)^T has distribution function C, where C is an Archimedean copula generated by φ, then the function K_C given by the above theorem is the distribution function of the random variable C (U,V).

When C is absolutely continuous its density is given by $\frac{\partial^2}{\partial u \partial v}C(u,v) = -\frac{\varphi''(C(u,v))\varphi'(u)\varphi'(v)}{[\varphi'(C(u,v))]^3}$ (43)

Theorem:Under the hypothesis of the corollary the joint distribution function H(s,t) of the random variables S= φ(u)/[φ(U)+φ(V)] and T= C(U,V)is given by H(s,t)=s $K_C(t)$ for all (s,t) in $[0,1]^2$. Hence S and T are independent and S is uniformly distributed in [0,1].

An application of the above theorem is the following algorithm for generating random variates (u,v)^T whose joint distribution is an Archimedean copula C with generator φ.

- Simulate two independent U(0,1) random variates s and q.
- Set t = $K_C^{-1}(q)$, where K_C is the distribution function of C(U,V)
- Set u =$\varphi^{[-1]}(s\varphi(t))$ and v= $\varphi^{[-1]}((1-s)\varphi(t))$

We note also that Archimedean copulas are related to multivariate distributions generated by mixtures. Consider a latent variable model (X_i, D_i) where X_i is the value of assets and D_i is the equivalent default threshold.Suppose that X has an exchangeable Archimedean copula. The default indicators $X_i = 1_{\{X_x<D_i\}}$ follow an exchangeable Bernouli mixture model. This means that Archimedean copulas generate parsimonious models for relatively homogeneous portfolios.

The Kendall's tau for a copula C can be expressed as a double integral of C.This double integral is in most cases not straightforward to evaluate. However for an Archimedean copula, Kendall's tau can be expreesed as an (one-dimensional) integral of the generator and its derivative.

Let X and Y be random variables with an Archimedean copula C generated by φ.Kendall's tau of X and Y is given by

$$\tau = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt \quad (44)$$

For each Archimedean copula we need to wit:

A) Kendal's τ
B) Theta θ
C) Generator φ(t)
D) Generator's first derivative φ′(t)
E) Generator's Inverse φ^{-1} (t)
F) The distribution function of C (u,v) = $K_{Copula} = t - \frac{\varphi(t)}{\varphi'(t)}$
G) Distribution function inverse K^{-1}_{Copula} (When it has not a closed form solution it can be obtained through the equation $(t - \frac{\varphi(t)}{\varphi'(t)}) - q$ by numerical root finding. For doing this we need the first derivative regard to t of $t - \frac{\varphi(t)}{\varphi'(t)}$).

For the Gumbel copula we have: B) $\theta = \frac{1}{1-\tau}$ C) $(-\ln t)^{\theta}$ D) $-\theta\,(\ln t)^{\theta-1}\,\frac{1}{t}$

E) $e^{\left(-t^{\frac{1}{\theta}}\right)}$ F) $t - \frac{(t\ln t)}{\theta}$ G) $-\frac{\ln(t)}{\theta} - \frac{1}{\theta} + 1$.

The Gumbel Copula can be expressed as $C(u,v) = e^{\left\{-\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{\frac{1}{\theta}}\right\}}$

The Clayton Copula which can be expressed as $C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{\frac{-1}{\theta}}$ is determined if we have knowledge of the following.

B) $\theta = \frac{2\tau}{1-\tau}$ C) $t^{-\theta}-1$ D) $-\theta\, t^{-\theta-1}$ E) $(1+t)^{\frac{-1}{\theta}}$ F) $t - \frac{\left(t^{\theta+1}-t\right)}{\theta}$

G) $-\frac{t^{\theta}(\theta+1)}{\theta} + \frac{1}{\theta} + 1$.

From the above it is prevalent that if we find the Kendal's tau for a distribution we can estimate the parameters of the above Archimedean copulas and thus they are fully determined.

ESTIMATION & SELECTION OF COPULAS

PARAMETRIC ESTIMATION WITHIN A GIVEN FAMILY

The method of maximum likelihood

Let the copula C and the margins F_N be continuous.The density of the joint distribution F is given by the following expression

$$f(x_1, \ldots x_n \ldots x_N) = c(F_1(x_1), \ldots F_n(x_n) \ldots F_N(x_N)) \prod_{n=1}^{N} f_n(x_n) \quad (45)$$

where f_n is the density of the margin F_n and c is the density of the copula

$$c(u_1, \ldots u_n \ldots u_N) = \frac{\partial C(u_1, \ldots, u_n, \ldots u_N)}{\partial u_1 .. \partial u_2 ... \partial u_N}$$

Let X= $\{(x_1^t, \ldots, x_N^t)\}_{t=1}^{T}$ denote a sample.The expression of the log likelihood is also

$$\ell(\theta) = \sum_{t=1}^{T} \ln c(F_1(x_1^t), \ldots F_n(x_n^t) \ldots F_N(x_N^t)) + \sum_{t=1}^{T} \sum_{n=1}^{N} \ln f_n(x_n^t) \quad (46)$$

With θ the K*1 vector of parameters. Let $\overline{\theta_{ML}}$ be the maximum likelihood estimator. Then it verifies the property of asymptotic normality and we have

$\sqrt{T}\,(\theta_{ML} - \theta_o) \rightarrow N(\mathbf{0}, F^{-1}(\theta_o))$ with $F(\theta_o)$ the information matrix of Fisher.

For example we could assume that the marginals follow the normal distribution, while the copula function is Clayton copula. If we assumed that not only the marginals follow the normal distribution but also the copula is gaussian then this method would be equivalent to extracting the maximum likelihood estimator of the multivariate normal distribution. Nevertheless this method which we call the exact maximum likelihood method or EML,could be computational intensive in the case of high dimension, because it requires to estimate jointly the parameters of the margins and the parameters of the dependence structure. However, the copula representation splits the parameters into *specific* parameters for marginal distributions and common parameters for the dependence structure (or the parameters of the copula)The log-likelihood equation could then be written as:

$$\ell(\theta) = \sum_{t=1}^{T} \ln c(F_1(x_1^t), \ldots F_n(x_n^t) \ldots F_N(x_N^t);\alpha) + \sum_{t=1}^{T} \sum_{n=1}^{N} \ln f_n(x_n^t) \quad (47)$$

With $\theta = (\theta_1,\ldots, \theta_N,\alpha)$. Θ_n and α are the vectors of parameters of the parametric marginal distribution F_N and the copula C. We could also perform the estimation of the univariate marginal distributions in a first time $\overline{\theta_n}$ = arg max $\ell^n(\theta_n)$:=arg max $\sum_{t=1}^{T} \ln f_n(x_n^t)$ and then estimate α given the previous estimates

$$\overline{a} = \ell^c(\alpha):=\text{arg max} \sum_{t=1}^{T} \ln c(F_1(x_1^t;\overline{\theta_1}), \ldots F_n(x_n^t;\overline{\theta_n}) \ldots F_N(x_N^t;\overline{\theta_N});\alpha). \quad (48)$$

This two step method is called the method of inference functions for margins or IFM method. The IFM estimator $\overline{\theta_{IMF}}$ is then defined as the vector $(\overline{\theta_1},\ldots,\overline{\theta_N};\overline{a})$. Like the ML estimator, we could show that it verifies the property of asymptotic normality and we have

$\sqrt{T}\,(\overline{\theta_{IMF}} - \theta_o) \rightarrow N(\mathbf{0}, \nu^{-1}(\theta_o))$ with $\nu(\theta_o)$ the information matrix of Godambe.

Let us define a score function in the following way:

$g(\theta) = (\partial_{\theta_1}\ell^1, \ldots, \partial_{\theta_N}\ell^N, \partial_a\ell^c)$.

The Godambe information matrix takes the form :

$\nu(\theta_o) = D^{-1}M(D^{-1})'$ where $D = E[\partial g(\theta)^T/\partial\theta]$ and $M = E[g(\theta)^T g(\theta)]$.

The estimation of the covariance matrix requires to compute many derivatives. Note also that the IFM method can be viewed as a special case of the generalized method of moments with an identity weight matrix.

Using a close idea of the IFM method, the parameter vector α of the copula could be estimated without specifying the marginals. The method consists in transforming the data $(x_1^t,\ldots, x_N^t)$ into uniform variables –using the empirical distributions-and then estimate the parameter in the following way:

$$\bar{a} = \arg\max \sum_{t=1}^{T} \ln c\,(\overline{u_1^t},\ldots,\overline{u_n^t},\ldots\overline{u_N^t};\alpha) \qquad (49)$$

In this case, $\bar{a}$ could be viewed as the ML estimator given the observed margins(without assumptions on the parametric form of the marginal distributions). Because this method is based on the empirical distributions, it is called the canonical maximum likelihood method or CML. Note that the IFM method could be viewed as a CML method with $\overline{u_n^t} = F_n(x_n^t;\overline{\theta_n})$. One of the important issue for the estimation is the existence of analytic solution of the CML estimator because they reduce computational aspects. And as we know this is a key point in Finance Industry.

To estimate the parameter p of the Gaussian copula with the CML method we proceed as follows:

1) Transform the original data into Gaussian data:
   a) Estimate the empirical distribution functions (uniform transformation) using order statistics.
   b) Then generate Gaussian values by applying the inverse of the normal distribution to the empirical distribution functions.
2) Compute the correlation of the transformed data.

To estimate the parameter p of the t-copula we can use the following algorithm:

1) Let $\tilde{p}_0$ be the CML estimate of the p matrix for the Gaussian copula
2) $\tilde{p}_{m+1}$ is obtained using the following equation

$$\tilde{p}_{m+1} = \frac{1}{T}\left(\frac{\nu+N}{\nu}\right)\sum_{t=1}^{T}\frac{\varsigma_t^T\varsigma_t}{1+\frac{1}{\nu}\varsigma_t^T p_m^{-1}\varsigma_t}$$

3) Repeat the second step until convergence $\tilde{p}_{m+1} = \tilde{p}_m$ $(:= \tilde{p}_\infty)$
4) The CML estimate of the p matrix for the Student copula is $\tilde{p}_{CML} = \tilde{p}_\infty$

Non parametric estimation

The Deheuvels or empirical copula

Any copula $\hat{\mathbf{C}} \in$ C defined on the lattice

$$\Im\left\{\left(\frac{t_1}{T},...,\frac{t_N}{T}\right):1 \le n \le N, t_n = 0,...,T\right\} \text{ by } \hat{\mathbf{C}} = \left(\frac{t_1}{T},\dots,\frac{t_N}{T}\right) = \frac{1}{T}\sum_{t=1}^{T}\prod_{n=1}^{N} 1_{[r_n^t \le t_n]} \qquad (50)$$

is an empirical copula.

We introduce the notation $\hat{\mathbf{C}}_{(T)}$ in order to define the order of the copula, that is the dimension of the sample used to construct it. Deheuvels obtains then the following conclusions:

1) The empirical measure $\overline{\mu}$ (or the empirical distribution function $\overline{F}$) is uniquely and reciprocally defined by both

(a) the empirical measures of each coordinate $\overline{F_n}$;

(b) the values of an empirical copula $\hat{\mathbf{C}}$ on the set $\Im$.

2) The empirical copula $\hat{\mathbf{C}}$ defined on $\Im$ is in distribution independent of the margins of **F.**

3) If $\hat{\mathbf{C}}_{(T)}$ is any empirical copula of order T, then $\hat{\mathbf{C}}_{(T)} \to \mathbf{C}$. with the topology of C.

Selecting a copula among a given subset of copulas

Let us have now an empirical copula whose values lie on the lattice $\Im$. We can compute its values thanks to the data. We assume that we have a finite subset of copulas $C \in$ C and we are interested in knowing which one of the copulas in C fits best the data (there might be parametric copulas or non parametric copulas). For this reason we consider the distance between each considered copula and the empirical copula. It is suggested to take a distance based on the discrete L^p norm

$$\overline{d_2}\ (\hat{\mathbf{C}}_{(T)}\ \mathbf{C}_k) = \| \hat{\mathbf{C}}_{(T)} - \mathbf{C}_k \|_{L^2} = \left(\sum_{t_1=1}^{T}\dots\sum_{t_n=1}^{T}\dots\sum_{t_T=1}^{T} \left[C_{(T)}\left(\frac{t_1}{T},\dots,\frac{t_n}{T},\dots,\frac{t_n}{T}\right) - C_k\left(\frac{t_1}{T},\dots,\frac{t_n}{T},\dots,\frac{t_N}{T}\right)\right]^2\right) \qquad (51)$$

The best copula in the family C is the copula which minimizes $\overline{d_2}$ ($\hat{\mathbf{C}}_{(T)}$ $\mathbf{C}_k$) The advantage of this method is that it does not depend on the behavior of the empirical copula out of the lattice $\Im = \left\{\left(\frac{t_1}{T},...,\frac{t_N}{T}\right):1 \le n \le N, t_n = 0,...,T\right\}$.

V. RESULTS & FINDINGS

ESTIMATION PROCEDURE

The procedure that we followed for the estimation of default correlations between sectors is the following: First estimate the individual default probabilities of each sector. Then estimate the copula that fits better the dependence structure of assets' minimum processes. After we have specified the dependence structure, we use it in order to compute bivariate joint default probabilities. Finally we turn to the expression that constitutes the definition of default correlation (equation 7) in order to estimate them.

In order to compute individual default probabilities for each sector we used the first-passage approach of asset value models. The estimation technique that we used is the calibration procedure. Thus we estimated asset volatility and mean growth rate of assets and by applying to equation (19) we were able to derive the individual default probabilities.

For the determination of sectors we adopted the ASE industry classification of the various firms that are traded. Each sector was treated as a portfolio. These sectors are: Banks, Insurance, Financial services, Industrial goods-services, Retail, Food-beverage, Basic Resources, Construction-Materials, Oil-Gas, Chemicals, Travel-Leisure, Technology, Telecommunications, Utilities, Health Care. We have not estimated the default correlation of banks with the other sectors due to the fact that if a default takes place in the banking sector the implications about the health of the financial system would have a greater impact in the whole of the economy which cannot be captured only by default correlation. The addition or omission of a firm from a specific sector we don't think that it will cause any problems. Fundamentally more (less) assets mean bigger (smaller) liabilities. Furthermore ASE's Ground rules for the management of indices guarantee that their values remain unaffected from such events.

The period of study starts from 2005 and finishes the March of 2006. The reason is that from 2006 the base for the industry indexes became the 5.000 and the adapted values are publicly available only from the beginning of 2005 for all sectors. Thus the data don't cover a business cycle, as they should. If the time period studied covers the entire ebb and flow of the business cycle, defaults caused by general economic conditions average out over the period, thus lowering default correlation. At the other hand default correlation is maximized when the time period tested most closely approximates the length of economic recession or expansion. This is quite reasonable because during a business cycle the dependence structure of assets is weakened while during a particular phase of the business cycle is increased. As a result we cannot estimate the default correlations among sectors for different time periods and see how they are affected from time, assuming that default probabilities and default correlation are constant through the examined period.

Moreover if we had time series of equity values that cover many years we could see from empirical data how the varying default probability influences default correlation for given sud-periods over the whole examined period.

For each particular sector we formed the unified balance of statements of the firms that belong to it. The balances of statements that we used are the ones that were issued in 2005. The financial structure of each sector was assumed to be constant. The data that we used despite the fact that are older than the desired, are a good proxy for the financial structure of a particular sector, as it cannot be altered dramatically from year to year.

The default point is determined by each sector's liabilities. It is assumed to be constant during the period under study. It was set to be equal to the short-term liabilities plus the half of the long-term.

The risk-free rate is assumed to be constant during the period under study and it was set to be equal to 3,5%. An interesting extension of asset value models concerns the case where the risk-free rate is stochastic. We avoided doing this as it would demand much more effort.

In order to find the dependence structure of joint defaults we need to observe each sector's assets' minimum process. This is necessary because we implement first-passage approach as Giesecke interprets it, i.e. the probability that the minimum of assets over a given sub-period falls below a given default threshold. Thus the whole period that is covered by the data it will be divided to subperiods in order to find the minimum of assets for these sub-periods. The ideal situation would be to have a time series of many years so that we could compute the dependence structure from yearly data. So the sub-periods could be equal to a month or a year.

However, these sub-periods will be by necessity equal to a week as the period that we have available data is relatively small. So there is a kind of "discrepancy" between the default probability which is computed for a year and the data set from which we extract dependence structure. This "discrepancy" would not occur if we computed weekly default probabilities. Nevertheless, it is true that during a week the conditions in the market cannot change dramatically and the computation of weekly default probabilities makes no sense especially to asset value models where default is not a surprise event. We could somehow avoid this discrepancy if existed an organized secondary market where we could observe the value of debt. But now the debt value is computed mainly from the given financial structure which remains relatively the same at least for a year period. However this "discrepancy" is not a problem as we use the same data set both for computing default probabilities and estimating the dependence structure. Thus from the value of the sector index as is given at the end of a day's transaction we will take the lower one that is observed during the week. This will be the data set that it will be used in order to find the parameters of the corresponding copula. This copula will represent the dependence structure of the joint probability to default.

The copula parameters are either dependence measures such as asset correlation as it happens with elliptical copulas either are derived from dependence measures as it happens with Archimedean copulas. Many authors propose to use a factor model in order to estimate asset correlation. Asset correlation is estimated through the sensitivity of assets to certain factors. However this way of estimating has many disadvantages. In particular we cannot know in advance which are the common factors that affect assets. Moreover we cannot estimate different dependence measures which are necessary for other copulas than elliptical. Thus we use the "calibration" procedure to generate a process of asset values and estimate the dependence structure between each pair of assets processes.

In our study we wanted to raise the assumption of normally distributed assets. This could be done by applying a multivariate normal mixture. However even with such a model we would assume that assets follow a particular distribution and the joint distribution is determined by the marginals. So it would be useful to separate the dependence structure from the marginals. We could make various assumptions about the marginal distributions and the underlying dependence structure until we find the appropriate combination which would allow us to model more efficiently the real data. However this would be time consuming and computationally difficult. Thus we decided to make no assumption about the marginal distribution and to focus our attention to the dependence structure. We derived the Gaussian, the t copula and two types of Archimedean copulas, the Gumbel and the Clayton. The estimation method that we adopt is the CML as we don't want to make any particular assumptions about the distributions of the underlying assets. Furthermore the dependence structure that we are looking for will be applied to default probabilities which are binomial events.

We use bivariate copulas to estimate the dependence structure between every possible pair of sectors. As the estimation method of the parameters for each specific copula is the CML we can use the standard algorithms that have been developed for the estimation of normal's and t-studend's cdf but using the CML estimators. The fact that we did not make any assumptions about the distribution that the margins follow allows us to speak about Gaussian and t –copula. The algorithms that we used are Drezner's algorithm about normal cumulative distribution function and Genz's code for t-student cdf. (For more information see Appendix). If we did not use the CML estimator for e.g. the Gaussian copula but the classical linear correlation we would have made the assumption that the marginal distributions follow the normal distribution. Thus we would have spoken about normal copula. This particular distinction it would play an important role in the analysis of the results that follows.

For each pair of sectors we estimated the empirical copula. The appropriate copula is the one that minimizes the distance $\overline{d_2}$ ($\hat{\mathbf{C}}_{(T)}$ $\mathbf{C}_k$) = $\| \hat{\mathbf{C}}_{(T)} - \mathbf{C}_k \|_{L^2}$ = $\left(\sum_{t_1=1}^{T} \cdots \sum_{t_n=1}^{T} \cdots \sum_{t_T=1}^{T} \left[C_{(T)}\left(\frac{t_1}{T},\ldots,\frac{t_n}{T},\ldots,\frac{t_n}{T}\right) - C_k\left(\frac{t_1}{T},\ldots,\frac{t_n}{T},\ldots,\frac{t_N}{T}\right)\right]^2 \right)$ as we have already said. We didn't use any of the statistical tests that are proposed in the literature. Specifically χ^2 and Anderson-Darling tests are proposed. But if the marginal distributions of the univariate time series are unspecified the existing critical values of these tests are no longer valid. As far as GoF tests that are based on the kernel smoothing approach are concerned, more research is needed.

After we have selected the appropriate dependence structure for every possible pair of sectors we can apply it to the equivalent individual default probabilities in order to estimate the joint default probability. So if the appropriate copula of two sectors' asset distributions is C, the joint default probability

$P(A \cap B)$ will be equal to :

$$P(A \cap B) = C(P(A), P(B)).$$

The derivation of default correlation according to equation (7) between every possible pair of sectors is now a simple procedure.

SECTORS' DEFAULT PROBABILITIES

The default probabilities that have been derived are indicative of the financial health and business risk of each sector. They give the probability of the whole sector defaulting. This might make no sense at first as the default of a whole sector not only is very difficult to take place but moreover it would be equivalent to an economic disaster. However these probabilities must be interpreted as we interpret the average cost of capital of a company with many production lines. Each line has its own cost of capital but the average cost of capital is computed for the whole company and to each production line is attributed the average cost of capital. The individual cost of capital is only used for internal control and procedures. This exactly is the way that sectoral default probabilities must be translated. Each firm of the sector has its own default probability and sector's default probability is the average of the firms that comprise the sector. These average default probabilities are going to be implemented in order to compute joint default probability of two sectors and their equivalent default correlation.

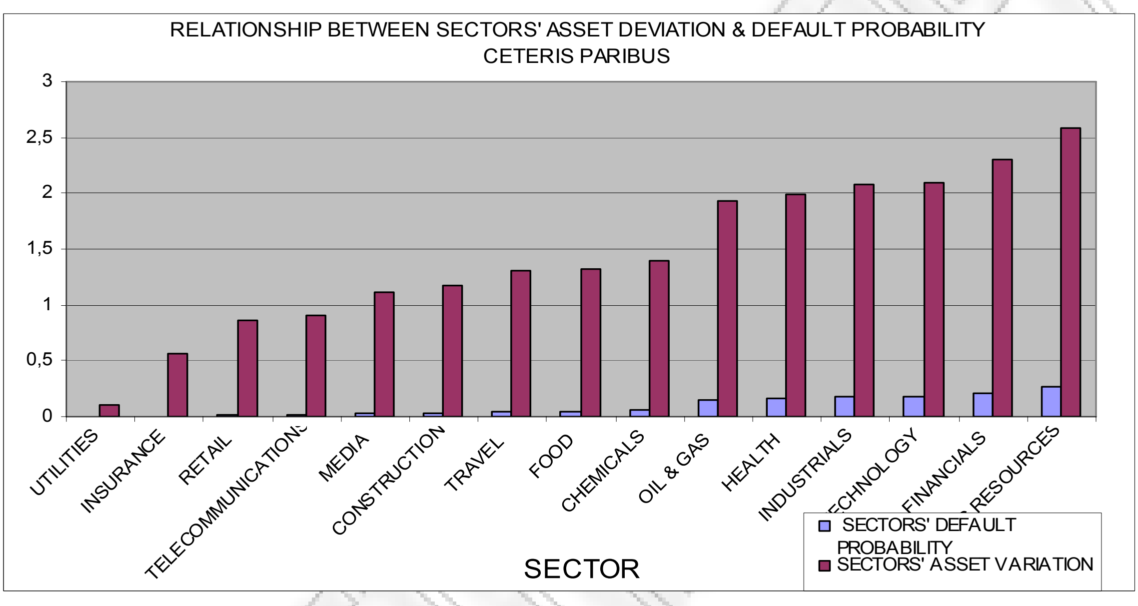
The following table presents each sector's default probability.

SECTOR F.T.S.E./A.S.E.	
OIL & GAS	0,075781
MEDIA	0,029773
CHEMICALS	0,020149
RETAIL	0,00417
INSURANCE	0,841259
HEALTH CARE	0,039659
FINANCIALS	0,137235
BASIC RESOURCES	0,120278
TELECOMMUNICATIONS	0,003478
TECHNOLOGY	0,083204
FOOD & BEVERAGE	0,037431
CONSTRUCTION & MATERIALS	0,008907
TRAVEL & LEISURE	0,019654
UTILITIES	8,32E-16
INDUSTRIAL GOODS & SERVICES	0,150824

The majority of the industry sectors have low enough default probabilities. There are some sectors with default probabilities greater than 10%. But for most sectors the average default probabilities are low enough. We have to stress that asset value and asset volatility are equity implied. This means that sectors' average default probability is also influenced from the enterprising perspectives of each sector. Thus traditional sectors such as basic resources may have a higher default probability if the market is not satisfied from the business initiatives that are taken.

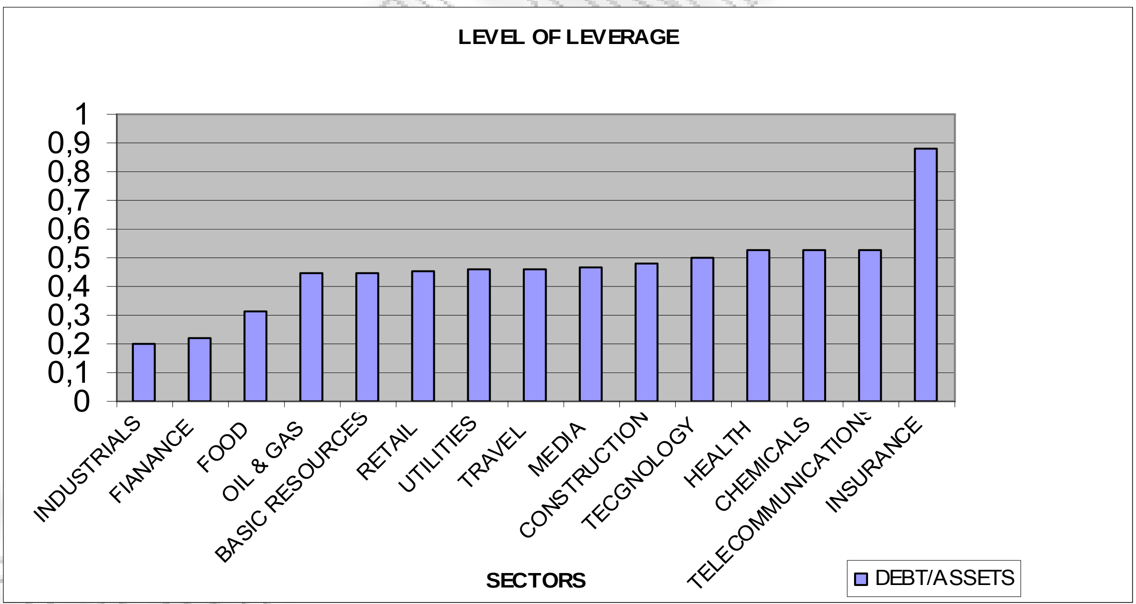
For the insurance sector in particular we have to state that its huge default probability is fascinating. However, it is indicative of the financial health of almost all firms that belong to the sector. We must state here that the financial statements that we used for the derivation of default probabilities were of 2005, a year that international accounting standards have been implemented. Thus the Insurance sector has been forced to include liabilities which they were not included previous years. There has been an increase of equity capital the previous year, which is going to be registered in the financial statements that will be announced this year. The situation has been improved but if we take into account the truly bad financial health of the Insurance sector more actions must be taken.

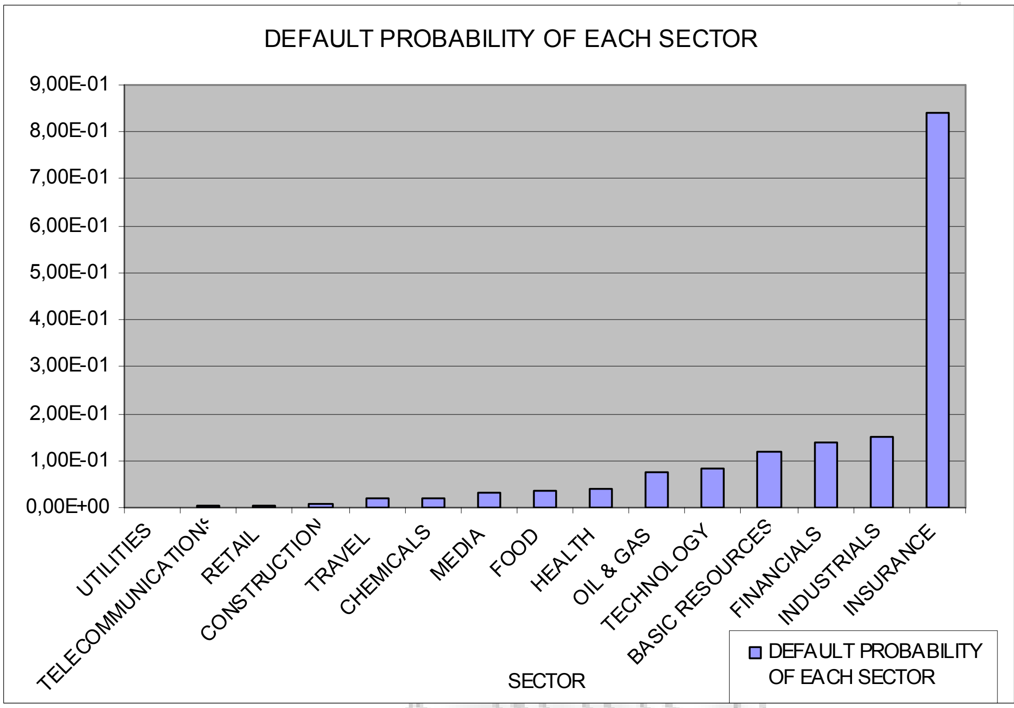
At the following diagram are illustrated the default probabilities of each sector assuming that all of them have the same characteristics e.g. leverage and the only variable which is allowed to vary in each sector is asset volatility.



From our analysis it comes out that the default probability of each sector is influenced mainly from the standard deviation of its assets, i.e. the business risk. For example Financial sector whose firms' business activities are limited to the possession of securities, objectively presents a high business risk. Thus the greater the standard deviation is, the higher is the default probability. This finding confirms that equity value is a monotone function in firm volatility. Equity holders will always benefit from an increase in the business risk. An increase in business risk results in a higher default probability, which leads in a higher risk premium for the debt and debt value is reduced. This fact is known from equity holders and plays an important role in the determination of default threshold as we will see next.

Leverage when it is present in moderate levels does not play such an important role in the determination of default probabilities. This means that for moderate levels of leverage the cost of borrowing remains relatively the same and the MM theorem with taxes obtains. However when leverage becomes too high the default probability becomes too large. This means that exists a level of leverage until which the tax advantages dominate, the cost of borrowing is relatively small, the consequences on the cost of capital negligible and thus the average cost of capital is reduced. The average cost of capital reaches its lowest level at the particular level of leverage and the value of the firm or sector reaches its highest. However, after this particular level of leverage the costs of financial distress become too big. The cost of debt becomes more and more high as it happens with the cost of equity and as a result the average cost of capital increases too. From that particular level of leverage the value of firm or sector begins to reduce. Thus rather the trade-off theory better resembles economic and business reality than the theorems of Modigliani and Miller.





Regardless of how big the leverage is, equity always worth something. This is due to the fact that even if the default probability is too high, the shareholders own the firm and determine its policy. So even if the costs of financial distress are too high there is always the hope that the financial health of the firm can be improved. On the other hand shareholders may take decisions that are against the debtors' interests. They may issue more debt, an action which renders the old one subordinated, or to decide the granting of dividends. But even if these do not happen, the management of the firm has the tendency to undertake risky projects in order to save the company. These risky projects increase the business risk, i.e. the asset volatility and consequently default probability. So, a "battle" takes place between share and debt holders about the determination of the default threshold and so we can talk equivalently about an endogenously and exogenously set default threshold. It is easy to understand that it is for the share-holders best interests the default threshold to be as low as possible, while the debt-holders prefer the default threshold to be as high as possible. The high default threshold, which may be bigger than the present value of debt, guarantees that firm's assets will always be enough to cover firm's liabilities, while the low default threshold, e.g. lower than the present value of firm's debt, means that equity holders gain while debt holders lose as the firm value cannot cover the firm's liabilities. The final result of the "battle" will be determined from the negotiation power of each part.

DETERMINANTS OF DEFAULT CORRELATION

The analysis that follows has as a twofold goal. First to show how individual default probabilities influence joint default probability and consequently default correlation given a dependence structure. And second to stress the importance of selecting the appropriate copula and its parameters as they influence default correlation through the determination of joint default probability.

In order to find default correlation we had to estimate joint default probability between each pair of sectors. And for doing this we should estimate the appropriate dependence structure between each pair of sectors. In 90% of cases the appropriate dependence structure was found to be Gaussian copula.

We have stated that the Gaussian copula is expressed via the following type: $C^{Ga}_R(u) = \Phi^n_R(\Phi^{-1}(u_1),\dots \Phi^{-1}(u_n))$, where Φ^n_R denotes the joint distribution function of the n-variable standard normal distribution with linear correlation matrix R, and Φ^{-1} denotes the inverse of the distribution function of the univariate standard normal distribution. In our study we used the CML estimator of the copula correlation which makes no assumption on the distribution of the margins. However if we assumed that the margins followed the normal distribution we could use the classical linear correlation as an estimate of the underlying correlation. In this case the copula that we would have derived would be quite different than the one that we ended. Particularly in our case classical correlation appears to be greater in absolute value than the absolute value of the CML estimator. So Gaussian dependence with the classical linear correlation results in higher joint default probability and consequently in higher default correlation. This could be justified from the fact that the dependence measure is greater. However we have stated that Gaussian dependence structure does not exhibit tail dependence. Default is a phenomenon that concerns the tail of assets' distribution. Moreover default probabilities are low enough and as Gaussian dependence structure exhibits no tail dependence we would expect joint default probability to be almost zero. However joint default probability not only does not equal zero, but moreover is increased when the dependence measure is increased.

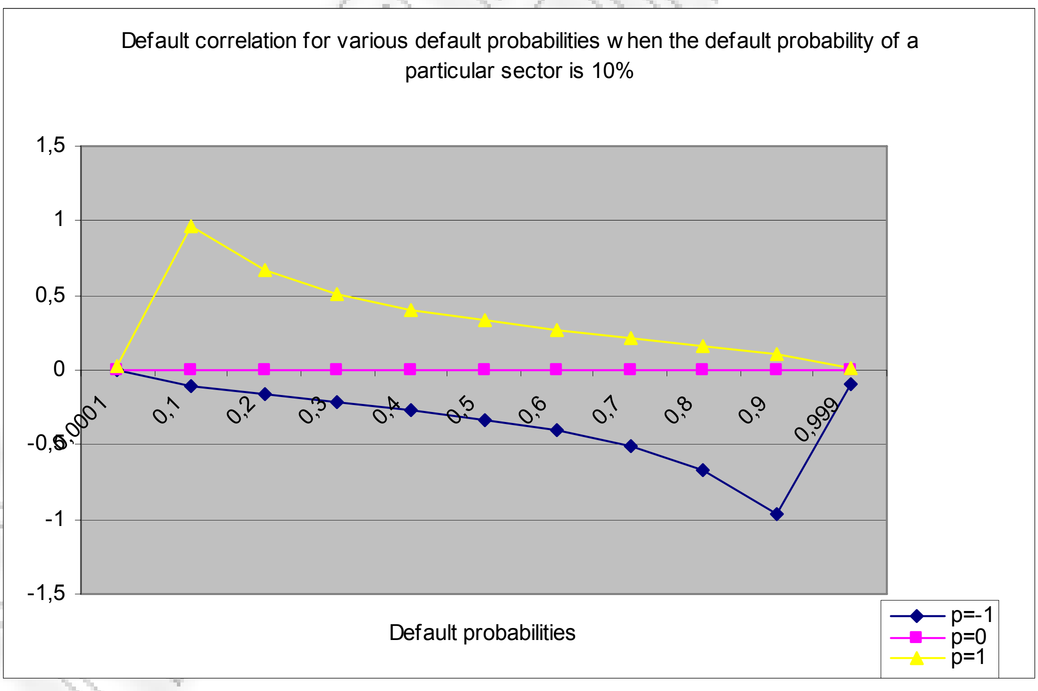
Thus the velocity with which the tail dependence tends to zero appears to depend on the estimate of the correlation that we use. Choosing as estimator of correlation the classical correlation has as impact the joint default probability to be extremely high. The reason is that asset correlation hinges on the correlation of the stock indices as the asset value models by construction treat the indices as options on the industry sectors' assets. However in the ASE is observed great dependence on the movements of the indices. Part of this maybe a consequence of the period that we study which is a part of the economic cycle and particularly the development.

In order to examine the impacts of the copula estimator we compute the lower tail dependence of the Gaussian copula for the two estimators. In particular it is computed for u=0, 00001(see Appendix) The lower tail dependence is of special interest as the default probabilities of most sectors are too low. So it is the lower tail dependence that it will determine how big the joint default probability will be and consequently the default correlation. In order to have a better understanding about what our findings mean we also provide the lower tail dependence for the Gumbel and the Clayton copula(see Appendix) We observe that the Gaussian copula with the CML estimator has lower tail dependence which is like the lower tail dependence of the Gumbel copula. We must state here that Gumbel Copula has no lower tail dependence. Furthermore the lower tail dependence of the Gaussian copula with the classical correlation tries to catch the lower tail dependence of the Clayton copula which exhibits lower tail dependence. This means that for the Gaussian copula with the classical correlation we must minimize u even more in order to become zero.

This of course has tremendous implications for the banking institution as it may have needlessly high reserves.

The above findings show why the Gaussian copula is most times the appropriate one. The industry indices which are seen as an option on the sectors' assets tend to move together especially while the ASE is in development. There are no extreme movements but a steady route of all sectors. The trend, which almost all sectors follow, is strictly increasing and strong. So the extreme values which may be present are very few. In 90% Gaussian copula depicts more accurately the dependence structure. For this 90% Kendal's tau and consequently linear correlation between sectors is extremely high. The rest 10% concerns sectors that Kendal's tau or linear correlation is by far lower. This means that asset distributions exhibit tail dependence which can be captured by normal copula when dependence and consequently the dependence measure are great. However when dependence is weaker and consequently dependence measure smaller this tail dependence cannot be captured by normal copula and thus another copula is the appropriate. Such a strong dependence implies that Stock Exchange has a particular direction. So extreme movements are rarely observed. This may also have the implication that an asset value model may not be the appropriate one for the estimation of default probabilities in the case of Greece where the Stock Exchange is under control.

Let us assume now that the dependence structure between the asset returns of two sectors is Gaussian. The following diagram shows the implications for default correlation. The default probability of one sector is stable while the default probability of the other sector is allowed to vary. The scenarios that are illustrated concern absolute negative dependence, independence and absolute positive dependence, i.e. asset correlation equal to -1, 0 and 1 equivalently.

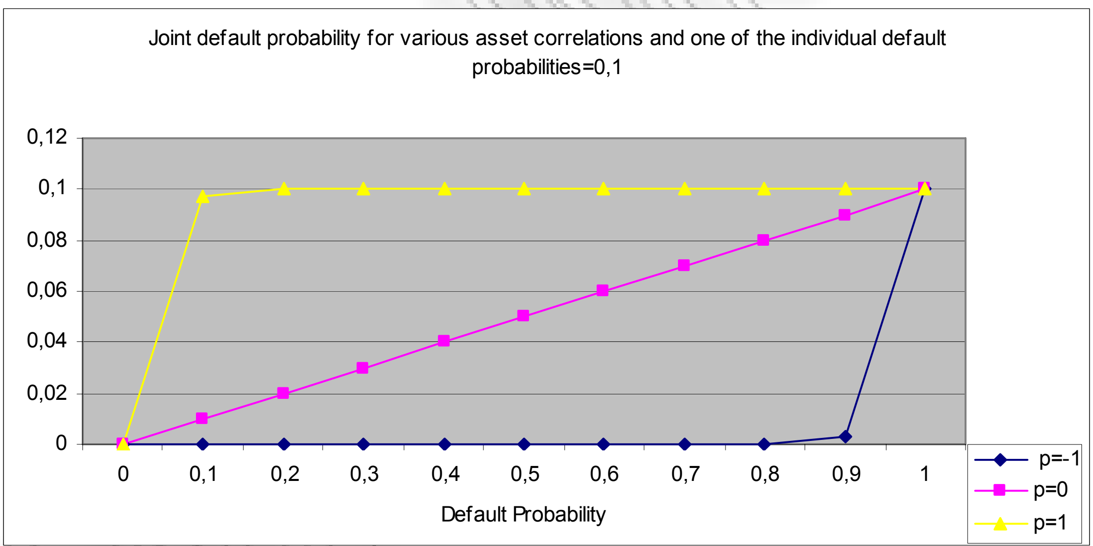


What we can observe is that default correlation has the sign of asset correlation. However its absolute value varies. This is due to the impact that individual default probabilities have. In particular we can see that when asset correlation is 1 default correlation reaches its maximum when both sectors have the same default probability i.e. 10%. When asset correlation is -1 default correlation reaches its minimum when default probabilities of sectors are complementary events i.e. when the one has 10% and the other 90% default probability. This diagram shows that default correlation is extremely sensitive to individual default probabilities. This has the very tremendous implication that default correlation must be estimated as often as default probabilities. When we think that default probability of a particular sector have changed, so has also been done with its default correlation with the other sectors. This is quite reasonable. If the default probability of a particular sector were lower than the default probability of another sector and then it is increased, it is expected that default correlation between these sectors will increase too. The reason is that the number of defaults that are expected to take place will be quite similar for the two sectors. On the other hand when two sectors have almost the same default probability and then the default probability of one from the two sectors is increased default correlation between them is expected to fall as defaults won't be any more connected too much. Defaults in one sector will occur more frequently, i.e. will be more in number than in the other sector.

When dependence between sectors can be characterized as negative, default correlation is reduced till the default probabilities between two sectors are complementary. This is quite reasonable because as default probability of one sector is increased defaults will occur more and more frequently which means that defaults will take place in the other sector more rarely. This happens because economic conditions have the opposite impact on the two sectors. So if default occurs in one sector, there is less chance that it will take place to the other until the default probabilities of the two sectors are complementary events. However when the default probability of one sector is higher than the complementary of the other, default may take place to both of them as the same market conditions may have the same impact to both of them and so there is a connection between the defaults that take place.

The following diagram of joint default probability sheds more light on the impact that individual and joint default probabilities have on default correlations.

Particularly we compare the values that joint default probability can take when the default probability of one sector is 10% and the default probability of the other sector is allowed to vary. When asset correlation equals 1, we can see that from the moment that default probability of the sector whose default probability varies reaches 10%, joint default probability is equal to its maximum i.e. the lower individual default probability. So till the two sectors have the same default probability, joint default probability increases as it happens too with default correlation. But from this point and on, joint default probability remains the same while default probability of one sector increases. So default correlation is reduced. These findings can be generalized for every positive value of the dependence measure, but will be different the value of the default probability for which the joint default probability will reach its maximum. When asset correlation is equal to -1 joint default probability is equal to 0, until the individual default probabilities are complementary events. Thus as the individual default probability of one of the sectors is increased, default correlation it is reduced till it is equal to -1. From the point that the default probability of one sector is 10% and the default probability of the other 90% joint default probability increases and reaches its maximum. This increase of joint default probability has as impact the default correlation to reach zero.



The diagram that follows shows what happens if both sectors have the same default probability for some extremes values of asset correlation. When asset correlation is bigger than, or equal to zero default correlation seems to remain the same independently of default probabilities. Particularly, as we have expected, for a positive dependence measure, default correlation is always at its maximum as default probabilities are the same. When correlataion is zero default correlation is also zero. However when asset correlation is smaller than zero default probabilities play an important role. We observe that when default probabilities are close to zero default correlations tend to be zero too.

In this case, as the occasion where the default probability of one sector was allowed to vary, we can see that when asset correlation is negative, default correlation is negative too. However, its absolute value is very smaller than default correlation's absolute value when asset correlation is positive but has the same absolute value. This has as a result the negative values that default correlation takes to be close to zero. The reason will be illustrated from the diagram that joint default probability is depicted.

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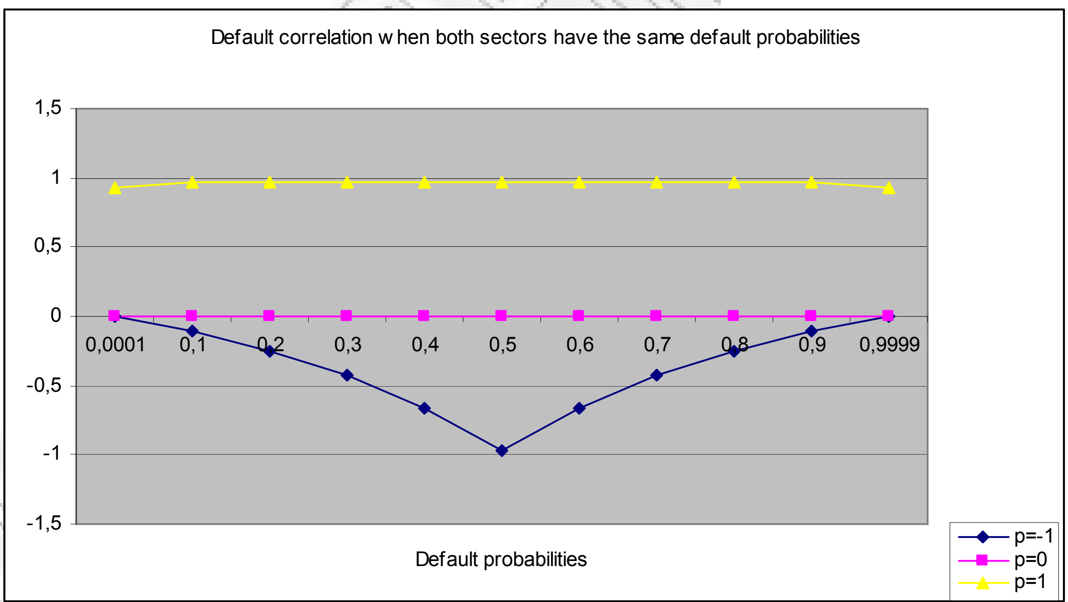
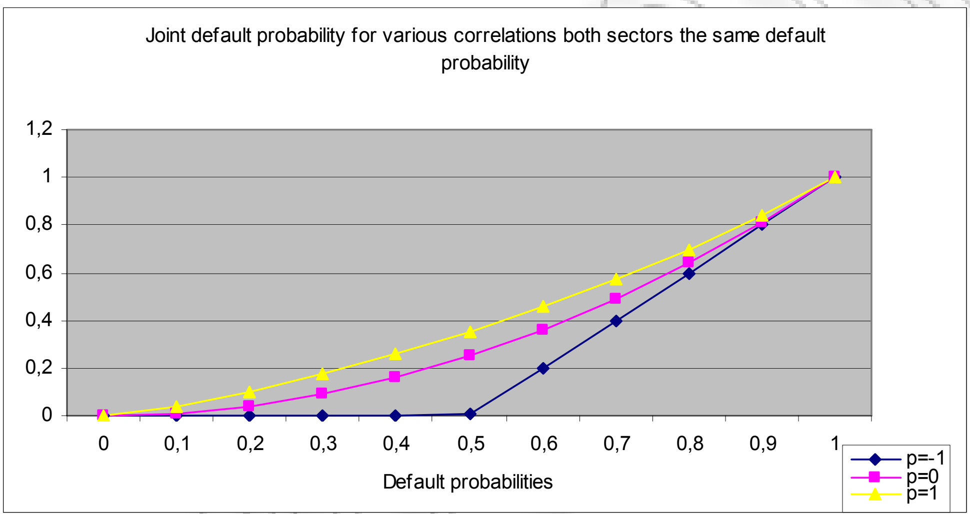
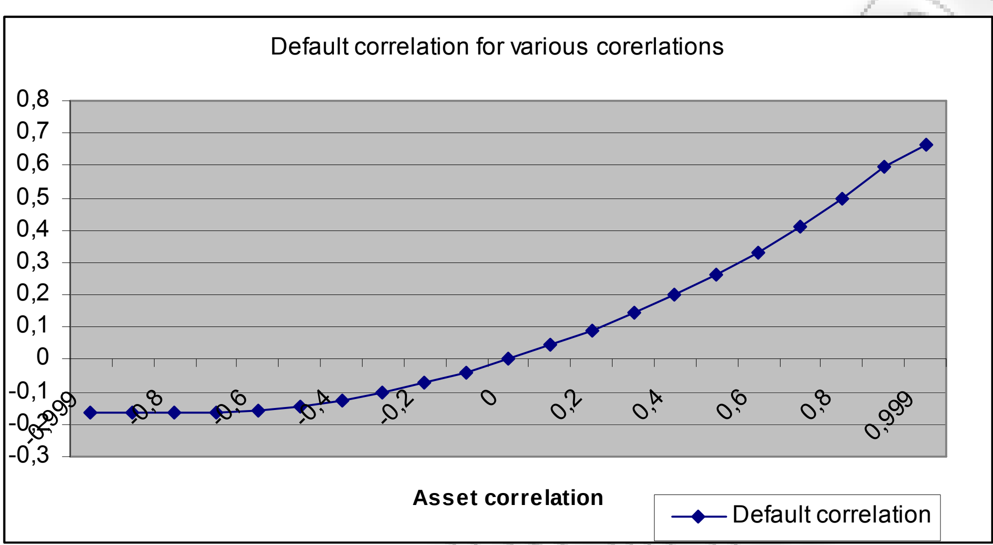


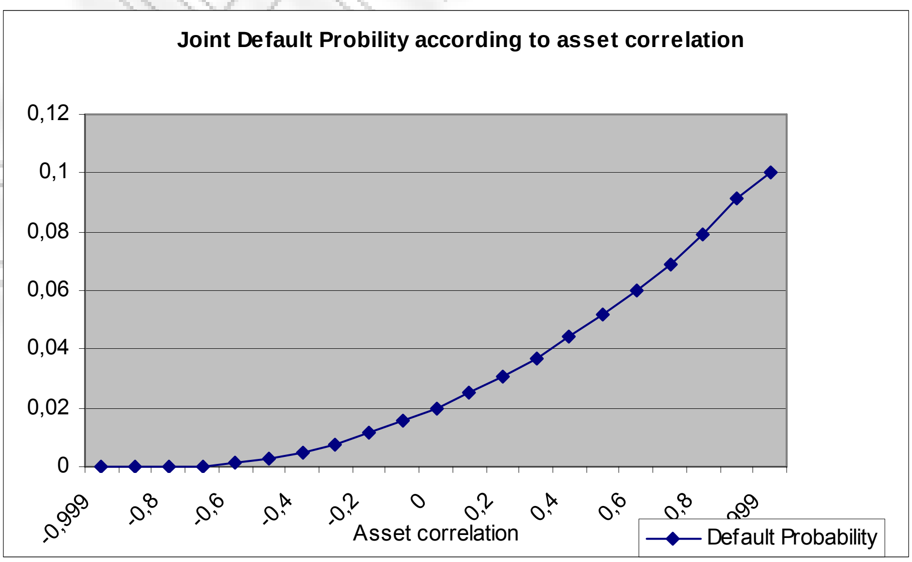
For a positive dependence measure we see that joint default probability is analogue to the individual default probabilities which are the same. When correlation is zero which implies independence joint default probability is just the product of individual default probabilities. However when the dependence measure is negative joint default probability is zero till the individual default probabilities overcome the point that are complementary. This point when individual default probabilities are the same is 50%. Then joint default probability begins to increase. That is why when asset correlation equals -1 default correlation reaches its minimum i.e. -1 when the default probabilities are complementary events i.e. 50% and then it starts to increase and reaches 0 when default probability of both sectors is equal to 1. So the following diagram is quite illustrative about the reasons that default correlation is altered according to individual default probabilities when dependence is negative.



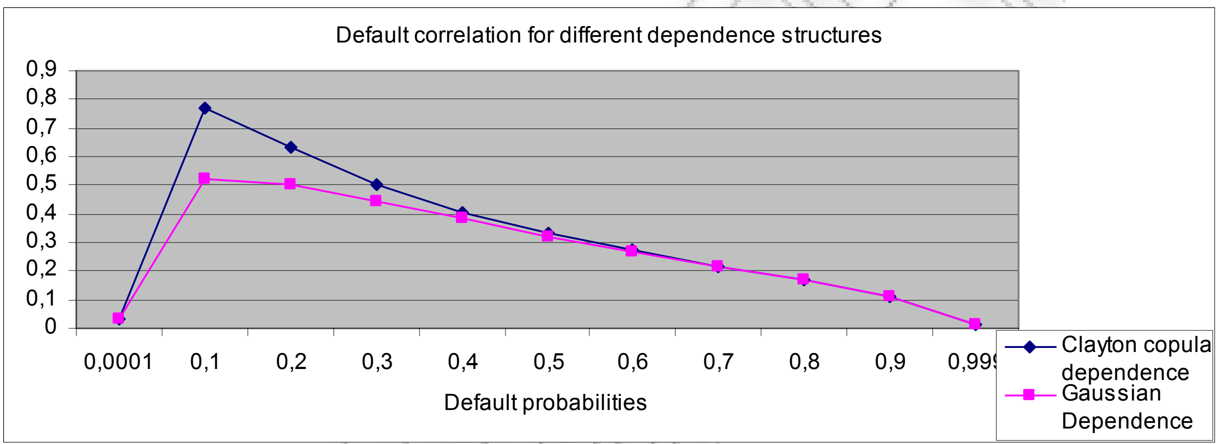
The following diagram shows the values that default correlation takes for various values of asset correlation. The default probabilities of the two sectors are 10% and 20% equivalently. As we can see a greater dependence measure implies a greater default correlation.



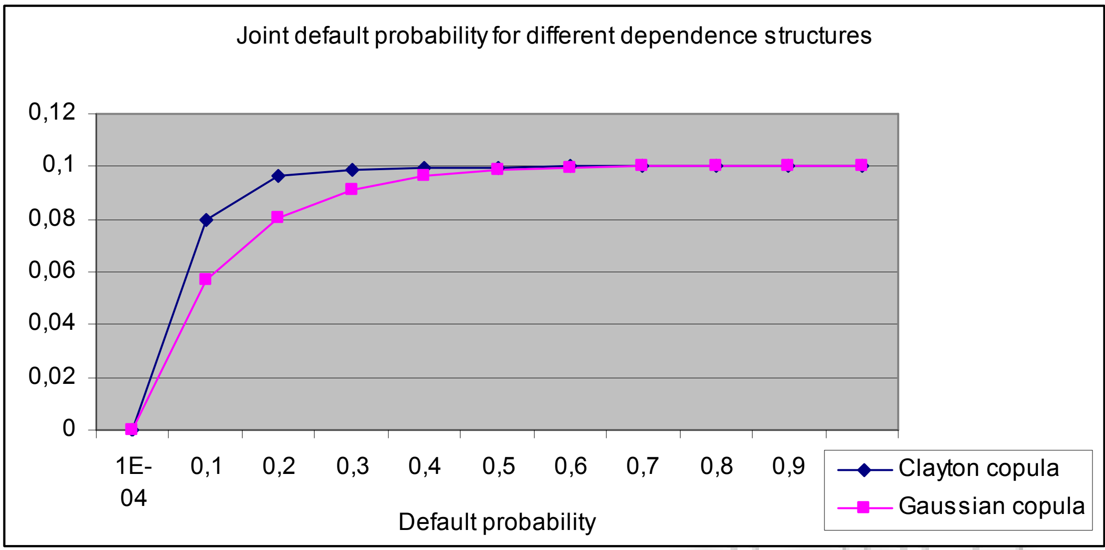
As the individual default probabilities remain the same and only dependence measure is altered, we expect that a higher dependence measure implies a higher joint default probability. Indeed, we can see that joint default probability has a range of values from [0, 0.1] and it is a strictly increasing function of the underlying correlation. A very interesting and important fact is that independently of the underlying asset correlation joint default probability cannot be greater than the lower of the two individual default probabilities.



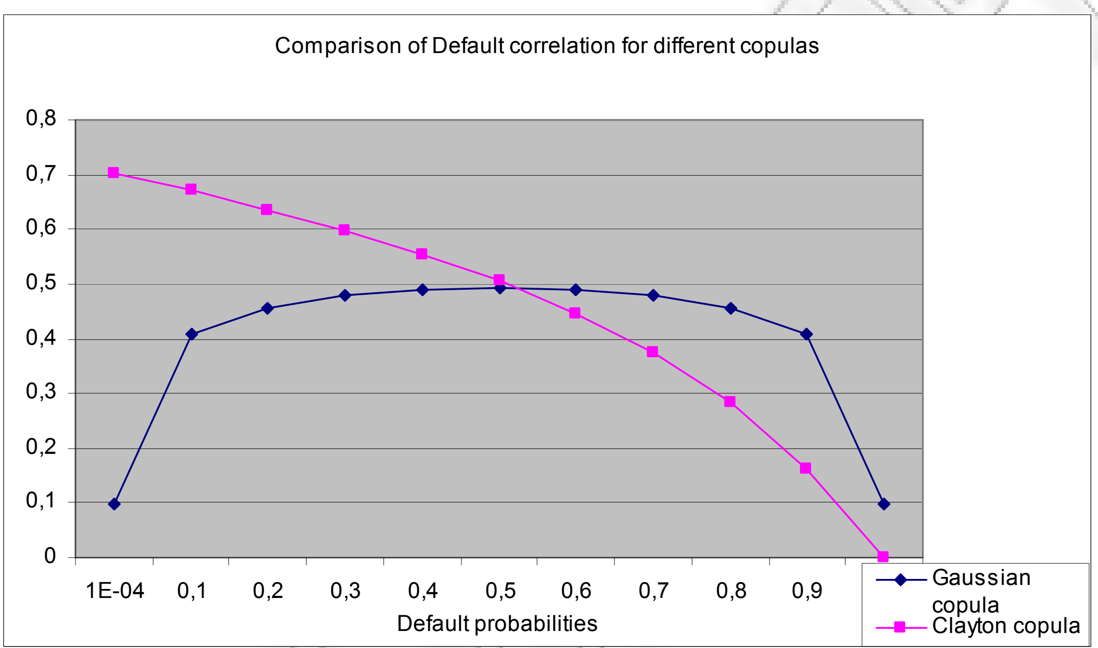
Let us now compare different dependence structures in order to highlight what the impact on default correlations could be. In the diagram that follows we assume that the default probability of one sector is 10% while the default probability of the other sector it is allowed to vary. Kendal's tau is assumed to be 0,5 which according to equation (26) means that asset correlation is 0,7. We can see that Clayton copula means a higher default correlation when the default probability of the other sector has a range of values from 10% to 40%. This means that for those values of default probabilities Clayton copula has as a result a stronger dependence between the defaults in these two sectors.



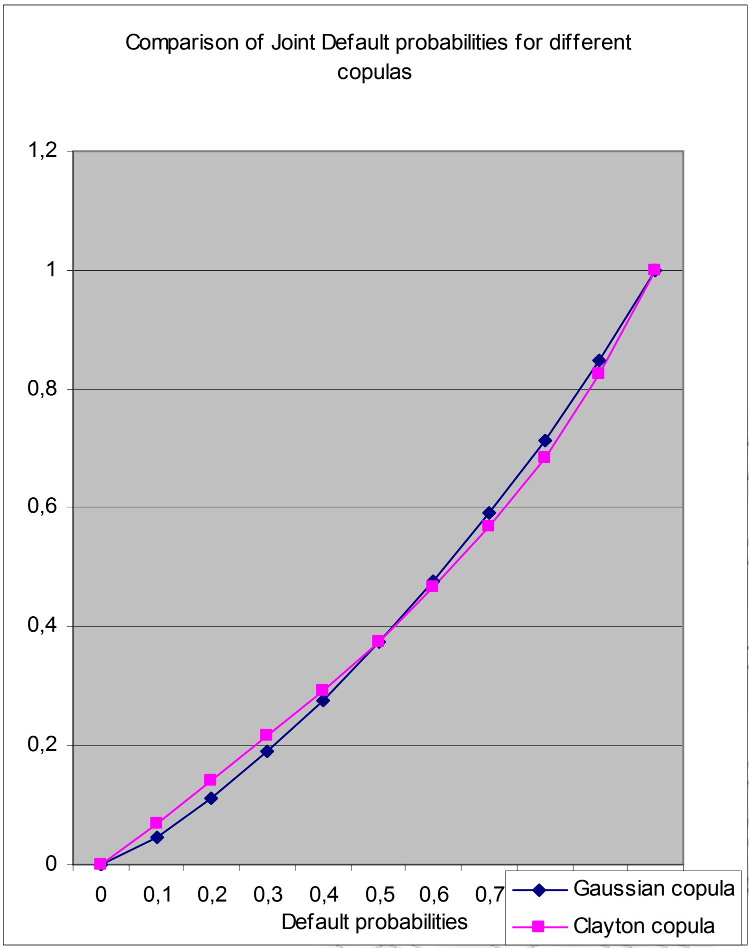
The diagram that follows compares the joint default probability under the assumption that dependence structure of assets is either Gaussian either Clayton Copula. From the diagram it is clear that when the default probability of the sector has a range of values between 0 and 40% Clayton copula implies a higher joint default probability than Gaussian copula. This renders clear why default correlation is higher when the dependence is Clayton copula in the particular range of default probabilities.


Joint default probability for different dependence structures
0,12
0,1
0,08
0,06
0,04
0,02
0
1E-04
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
Default probability
Clayton copula
Gaussian copula

Let us now assume that dependence measures remain the same but both sectors have the same default probability which is allowed to vary from 0,01% to 99,9%. Clayton copula implies bigger default correlations till the sectors have default probability of 50%. Additionally the default correlation increases as the default probability tends to zero. This is of course due to the fact that Clayton copula exhibits lower tail dependence.

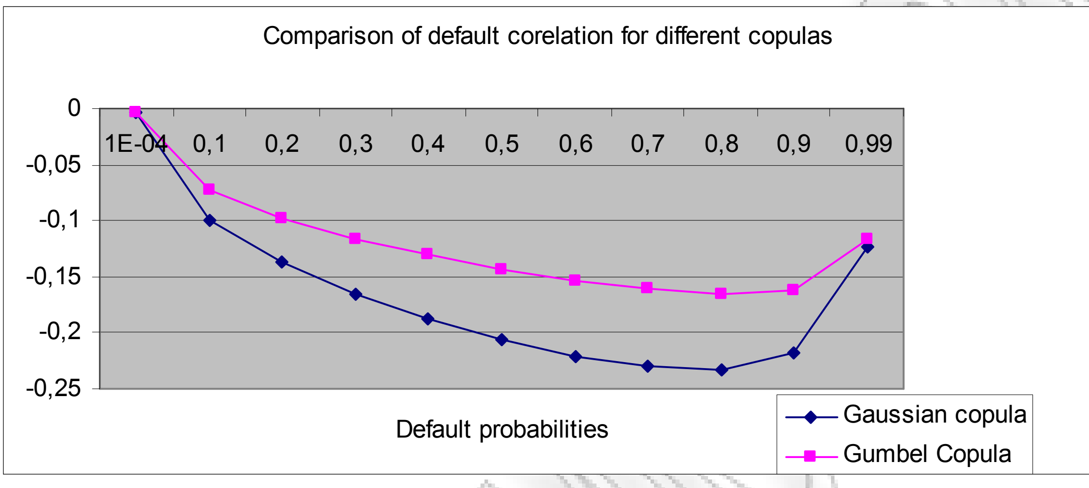


The next diagram is quite illustrative as it shows the values of joint default probability for different dependence measures. Particularly we can observe that joint default probability is greater when the dependence structure is given by Clayton copula till the point that sectors' individual default probabilities are equal to 50%. From that point and on the Gaussian copula implies a higher joint default probability.

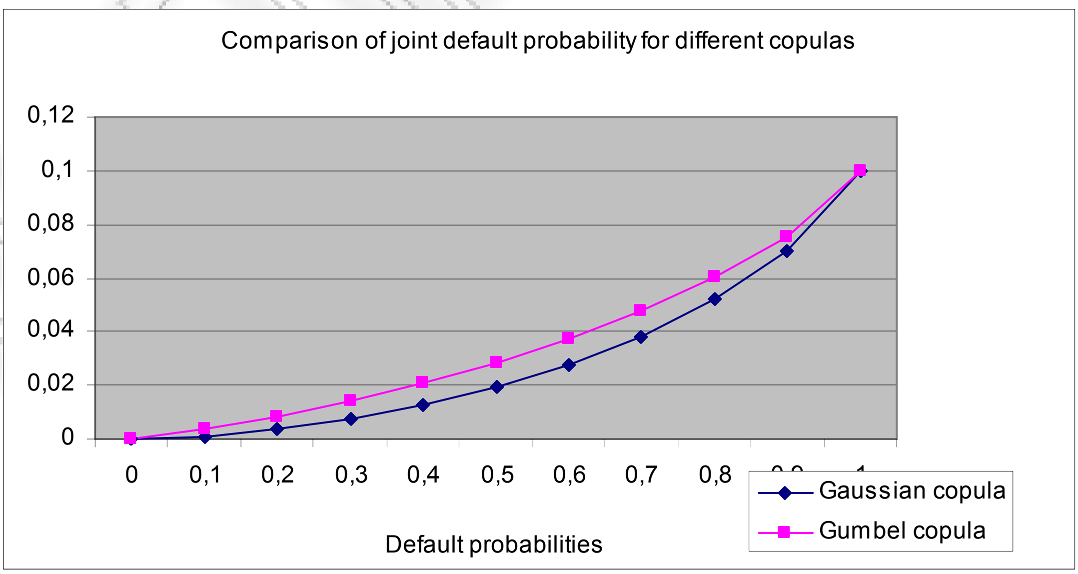


Comparison of Joint Default probabilities for different copulas
1,2
1
0,8
0,6
0,4
0,2
0
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
Default probabilities
Gaussian copula
Clayton copula

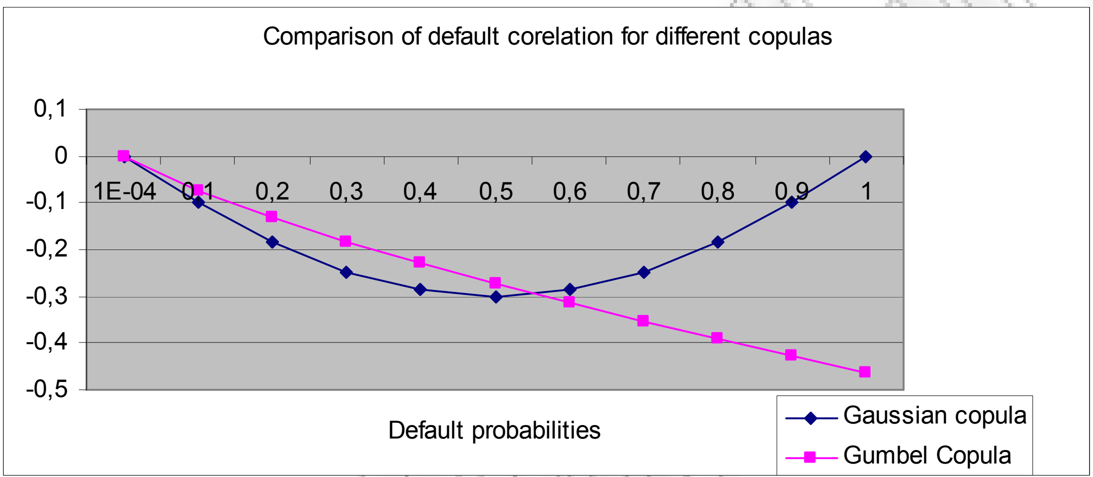
Let us now assume that Kendal's tau of two sectors is -0,3 and the equivalent, according to equation (26), asset correlation is -0,45 . The default probability of one sector is 10% while the default probability of the other sector is allowed to vary. We observe that for all the various default probabilities Gaussian dependence implies lower default correlation.



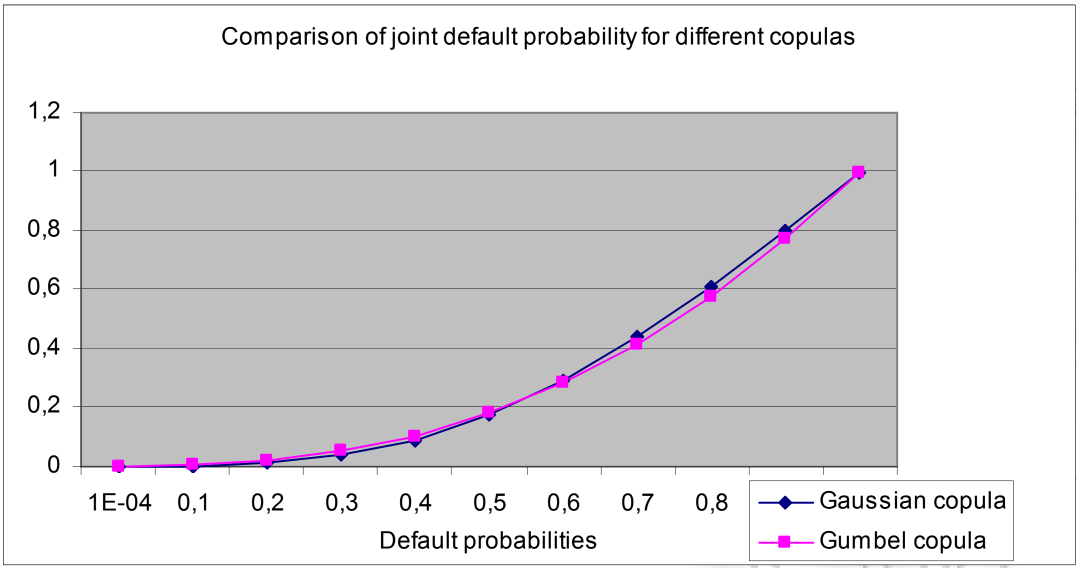
In the next graph we compare joint default probability when dependence structure is Gaussian and Gumbel copula. Gumbel copula implies greater values for the joint default probability independently of the individual default probabilities. This fact justifies that default correlation is higher when dependence structure is given by Gumbel copula.



Again assume that the dependence measures are the same but moreover both sectors have the same default probability which is allowed to vary from 0,001% to 99,9%. Gaussian copula implies lower default correlation until both sectors have default probability of 50% where it reaches its minimum. Then it begins to increase and when both sectors have default probabilities of 100% it is equal to 0. However Gumbel copula follows a decreasing root and reaches its minimum when both sectors have default probability equal to 100%.



From the following diagram that exhibits joint default dependence we can observe that until the point that both individual default probabilities are 60% Gaussian copula implies a little lower joint default probability. So default correlation is lower when dependence structure is given by Gaussian copula. However from that point and on Gaussian copula implies a higher joint default probability and so a higher default correlation. The particular case is indicative of how sensitive is default correlation to assets' underlying structure that determines joint default probability. A joint default probability which is few basis points lower than the expected can result to a a very lower default correlation!


Comparison of joint default probability for different copulas
1,2
1
0,8
0,6
0,4
0,2
0
1E-04
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
Default probabilities
Gaussian copula
Gumbel copula

SECTORS' DEFAULT CORRELATIONS

Now we are ready to proceed to the presentation of the pair-wise default correlations between sectors. These default correlations vary. The vast majority of them is equal or greater than zero. We have to state that many of these default correlations may seem extremely high. However due to the fact that correspond to default events on an aggregated level and not to individual default events are quite reasonable. The main disadvantage is that default correlation is not based anymore to the number of defaults in each sector but on the dependence structure between equity-implied sectors' assets. If market participants believe that the equity value of a sector does not represent rightly the underlying asset value they will alter their beliefs and so equity and asset value will change. In our case they increase and this happens for almost all sectors of ASE. Moreover this increase may be due to other factors such as investors' optimism. Thus the dependence degree will depend on the degree of participation of a particular sector to the development of the whole market. This means that default correlation is not based any more to the normal course of business between sectors, but rather to the beliefs of the market about the right market value of equity. So two sectors with many transactions and a high relationship between them may have a low default correlation. On the other hand, two sectors which theoretically have no relevance but the market believes that their equity value is underestimated and must be increased (or overestimated and so it must be decreased) may have a high default correlation.

The results that we present are very important because it can help whoever posses a portfolio of debt securities to diversify it and eliminate its losses. At the moment the main interested part are the financial institutions in order to estimate more accurately the loss distribution of their loan portfolio and be able to diversify that risk. Our results are also of special interest for the regulators of our country who need default correlation in order to estimate the risk that each financial institution bears.

However the disclosure of information is an important factor for the development of an economy. In that way asymmetry of information is eliminated and innovations can take place .A such innovation could be the creation of a secondary market for the trading of debt where many investors can participate. Moreover big companies would like to know how the sector that they belong could be affected from a wave of defaults in the sectors that their suppliers or their clients belong.

Hence default correlations are known and there is a reliable method for their computation which renders not only the estimation and diversification of a portfolio's risk possible, but moreover it helps the appearance of new instruments for the hedging of credit risk.

The estimation of default correlation is conducive to the establishment in the Greek market of credit derivatives such as default swaps or Collaterized Debt Obligations (CDO). The hedger who participates in a default swap must know which is the default correlation of his counterparty with the party that issues the obligation in order to estimate the risk that he is going to bear. CDO is a way of allocating bond default risk to tranches. Default correlations lie at the heart of pricing CDO's. So knowing default correlation allows us to evaluate CDO's and the first or n^{th} to default swaps. From all these we can deduce that a fast and reliable method for estimating default correlation can lead to the modernization of the Greek financial system.

						Default correlation									
	Chem	Resources	Constr& Mater	Health	Retail	Utilities	Technology	Industrials	Food	Media	Travel	Financials	Telecom	Oil&Gas	Insurance
Chemicals	1,000	0,280	0,253	0,365	0,134	0,000	0,338	0,293	0,187	0,134	0,249	0,285	0,139	0,230	0,060
Resources		1,000	0,235	0,357	0,172	0,000	0,570	0,493	0,410	0,092	0,341	0,452	0,157	0,152	0,159
Constr& Mater			1,000	0,383	0,385	0,000	0,244	0,203	0,302	0,077	0,381	0,217	0,234	0,254	0,041
Health				1,000	0,181	0,000	0,363	0,365	0,329	0,157	0,365	0,405	0,162	0,183	0,089
Retail					1,000	-0,035	0,183	0,137	0,290	0,016	0,377	0,148	0,253	0,204	0,028
Utilities						1,000	0,000	0,000	0,000	0,000	-0,031	0,000	0,000	0,000	0,000
Technology							1,000	0,517	0,353	0,126	0,413	0,359	0,164	0,408	0,131
Industrials								1,000	0,259	0,213	0,271	0,399	0,117	0,343	0,184
Food									1,000	0,018	0,490	0,351	0,210	0,385	0,084
Media										1,000	0,025	0,112	0,012	0,026	0,072
Travel											1,000	0,328	0,194	0,316	0,062
Financials												1,000	0,123	0,393	0,170
Telecom													1,000	0,206	0,026
Oil&Gas														1,000	0,108
Insurance															1,000

The above results that we have presented concern pair-wise default correlations. Thus for their derivation we used the bivariate copula, which most times stems from the bivariate cdf, in order to estimate joint default probability. After we have derived the bivariate joint default probability we proceeded to derive default correlation by implementing the formula that constitutes its definition.

Nevertheless, as Lucas [20] states, unlike the framework of continuous random variables like stock returns, for a binomial variable like default we cannot explain the behavior of the entire portfolio when we know the standard deviation of each variable and the correlation of each pair of variables. If we have three credits A,B and C and assume that the default correlation between each pair of credits is 0, we cannot necessarily conclude that the default correlation between any pair of credits and the third credit is also zero.

Lucas in order to test that higher orders of default correlation are also important for large portfolios computed the probabilities of zero to 100 credits defaulting in a 100-credit portfolio where each credit had 10% probability of default. He considered three correlation scenarios:

- zero pairwise default correlation and zero higher correlations
- zero pairwise default correlation and maximum negative higher correlations.
- zero pairwise default correlation and maximum positive higher default correlations.

His conclusion is that pairwise default correlations do not give all the information is needed to understand the behavior of a portfolio. Intuitively, increasing higher level of default correlation seems logical. Assuming that positive pairwise default correlation exists, the first default in the portfolio will cause us to revise our estimation of the default probability of remaining credits upwards. It seems logical that if a second credit defaults, we would want again to revise our estimation of the default probabilities of the remaining credits upwards.

We can compute also the trivariate default correlation, by treating the default of two sectors as one event and comparing that event to the default of a third sector. The equation that we would use in such an occasion would be:

$$\frac{P(A\cap B\cap C)-P(A\cap B)*P(C)}{\{P(A\cap B)*[1-P(A\cap B)]\}^{1/2}*\{P(C)*[1-P(C)]\}^{1/2}}$$

If we want to extend the computation of default correlation to n sectors we must also derive the joint default probability of n-1 sectors. Thus we must derive the n-1 equivalent copula implementing the algorithms that have been developed for n-variate cdf where n ≥ 2. Moreover in our opinion it seems that the application of the n-copula it is more appropriate for the calculation of the n^{th} to default swap.

IMPLICATIONS FOR A LOAN PORTFOLIO

We will now consider some situations with the aim to shed more light on the consequences that they have on the portfolio. We make use of equations (9)-(12) which are given by Bürgisser et al [6] .

Let us have a portfolio comprised of loans that are given to two sectors. The exposure to each sector is the same and equals 1.000.000 €. Let us assume that each sector has the same default probability which equals 10% and the same default variance which equals 25%. The expected loss and the unexpected loss is the same for both sectors and equivalently are 100.000 and 320.156. We compare five portfolios, Portfolio C, D, E, F and G. In Portfolio C, there is independence of the defaults in each sector which means that joint default probability is 1% and default correlation is 0. In Portfolio D there is dependence between the defaults of each sector, the dependence structure of asset returns is Gaussian and asset correlation equals 0,7. The joint default probability is 4,67% and default correlation is equal to 0,4086. We observe that the two portfolios retain the same default probability and the same Expected Loss (EL). The Unexpected Loss (UL) for portfolio C is 452.760 while in portfolio D equals 455.020. There is also important difference in the default variance of the two portfolios. In the case where we have dependence equals 41,96% whereas in the case that we have independence is 35,35%. We have to highlight here that default variance of the portfolio is the one, together with the assumptions made about the distribution of the portfolio that determines the loss that is possible to occur in extreme conditions

	Sector A	Sector B	Portfolio C	Portfolio D	Portfolio E	Portfolio F	Portfolio G
Exposure	1 m	1 m	2 m	2 m	2 m	2 m	2 m
DP	0,1	0,1	0,1	0,1	0,1	0,1	0,1
Default variance	0,25	0,25	0,3535	0,4196	0,3674	0,4569	0,383
EL	0,1	0,1	0,2	0,2	0,2	0,2	0,2
UL^2	102500 m	102500 m	205000 m	207043 m	205399 m	208350 m	205870 m
UL	0,32015 m	0,32015 m	0,45276 m	0,45502 m	0,45321 m	0,45645 m	0,45373 m
Asset correlation				0,7	0,2	0,7	0,2
Assets' Kendal's tau				0,4939	0,1283	0,4939	0,1283
Dependence Structure			Independence	Gaussian copula	Gaussian copula	Clayton copula	Clayton copula
P(A∩B)			0,01	0,0467	0,0172	0,07	0,0256
Default correlation			0	0,4086	0,0799	0,67	0,174

Portfolio E is characterized from dependence between sectors' defaults. The dependence structure of asset returns is Gaussian while asset correlation equals 0,2. The joint default probability is 1,72 % while default correlation equals 0,0799. Special attention must be paid to the fact that default variance is 36,74%. This means that although the dependence structure is the same the fact that the dependence measure is smaller renders portfolio E not as risky as portfolio D. The lower joint default probability and default correlation are indicative of the former ascertainment.

Portfolio F is characterized from the existence of dependence between the sectors' defaults. Asset correlation is the same as in portfolio D i.e. 0,7. However the dependence structure is given from Gumbel copula. So the appropriate dependence measure is Kendal's tau which equals 0,5. Joint default probability is 7%, default correlation equals 0,67, default variance of the portfolio is 45,7% and unexpected loss 456.454, the highest of all portfolios.

Finally in portfolio G, there is dependence between defaults. The dependence structure of assets is described by Gumbel copula. The appropriate dependence measure is Kendal's tau and is equal to 0,128 when asset correlation is 0,2. Kendal's tau in portfolio G is smaller than that of portfolio F which has as a result joint default probability, default correlation and portfolio's variation of defaults to be smaller than portfolio's F, although they posses the same dependence structure. But simultaneously portfolio's G joint default probability, default correlation and variation of defaults are higher than those of portfolio's E with which they have the same dependence measure, but different copulas.

We study next what happens when we increase the bank's exposure to one of the sectors. Now the first sector has exposure 2.000.000, default probability 10% and default variance 0,25. The expected loss is 200.000 while the unexpected loss is 640.312. Sector B retains the characteristics that it also had previously.

The expected loss is the same to all hypothetical portfolios as it also happens with default probability. However we observe differences in the unexpected loss and the default variance. Both of them are higher in portfolio F which posses the Gumbel copula as dependence structure and a high dependence measure. Next comes portfolio D which has the same dependence measure but its dependence structure is Gaussian. Then it follows portfolio G which is characterized by a lower dependence measure and Gumbel copula. Portfolio E which has a low dependence measure and Gaussian dependence structure has characteristics which are close to that of portfolio C which is characterized by independence. In comparison with the previous case that exposure is the same to all sectors we observe that unexpected loss appears to be a lot higher. This is logical as our portfolio is comprised only from two sectors and the benefits from diversification are relatively small. So the risk of our portfolio is influenced in a great level from the exposure to a particular sector.

	Sector A	Sector B	Portfolio C	Portfolio D	Portfolio E	Portfolio F	Portfolio G
Exposure	2 m	1 m	3 m	3 m	3 m	3 m	3 m
DP	0,1	0,1	0,1	0,1	0,1	0,1	0,1
Default variance	0,25	0,25	0,3727	0,4293	0,3844	0,4619	0,3977
EL	0,2 m	0,1 m	0,3 m	0,3 m	0,3 m	0,3 m	0,3 m
UL^2	410000 m	102500 m	512500 m	516586 m	513299 m	519200 m	514240 m
UL	0,6403 m	0,32015 m	0,7159 m	0,719 m	0,716 m	0,72 m	0,717 m
Asset correlation				0,7	0,2	0,7	0,2
Assets' Kendal's tau				0,4939	0,1283	0,4939	0,1283
Dependence Structure			Independence	Gaussian copula	Gaussian copula	Clayton copula	Clayton copula
P(A∩B)			0,01	0,0467	0,0172	0,07	0,0256
Default correlation			0	0,4086	0,0799	0,67	0,174

The next situation shows what happens if exposure to both sectors is the same but one of them has higher default probability. Let us assume that sector A has default probability 20%. The expected loss is the same with the situation that the exposure is 2.000.000 and default probability 10%. However the unexpected loss is lower for all portfolios in the case that we study now, than the unexpected loss of the equivalent portfolios when default probabilities are the same between sectors and there is greater exposure to one sector.

Default probability to all portfolios has been raised to 15%. The unexpected loss of all portfolios is smaller than in the case that the sectors had the same default probabilities but the exposure to one of them was 2.000.000. In the case that we have independence default variance is the same for both cases (greater exposure and greater default probability) and equal to 37,267%. In comparison with the portfolios that they exhibited greater exposure to one sector, default variance appears to be slightly increased in the case that the dependence structure is Gaussian while it is slightly reduced in the case that the dependence structure is Gumbel Copula.

If we compare the following table with the one that turned up when the two sectors had the same default probability but the exposure was greater to one of them we see that the latter case is riskier. This is quite reasonable because a great exposure implies that if individual default occurs, the loss will be great. Thus the same exposure to sectors with different default probabilities is safer than having a great exposure to one sector when both have the same default probabilities. This happens because of the small number of sectors from which our portfolio is comprised.

	Sector A	Sector B	Portfolio C	Portfolio D	Portfolio E	Portfolio F	Portfolio G
Exposure	1 m	1 m	2 m	2 m	2 m	2 m	2 m
DP	0,2	0,1	0,15	0,15	0,15	0,15	0,15
Default variance	0,25	0,25	0,3727	0,42925	0,3859	0,4506	0,3969
EL	0,2 m	0,1 m	0,3 m	0,3 m	0,3 m	0,3 m	0,3 m
UL^2	210000 m	102500 m	312500 m	316583 m	313407 m	318274 m	314182 m
UL	0,45825 m	0,32015 m	0,559 m	0,5626 m	0,5598 m	0,5641 m	0,5605 m
Asset correlation			0	0,7	0,2	0,7	0,2
Assets' Kendal's tau			0	0,4939	0,1283	0,4939	0,1283
Dependence Structure			Independence	Gaussian copula	Gaussian copula	Clayton copula	Clayton copula
P(A∩B)			0,02	0,0689	0,0308	0,08929	0,04019

Default correlation			0	0,40832	0,0907	0,57744	0,168

We study next what it happens if we have the biggest exposure to the sector with the highest default probability.

	Sector A	Sector B	Portfolio C	Portfolio D	Portfolio E	Portfolio F	Portfolio G
Exposure	2 m	1 m	3 m	3 m	3 m	3 m	3 m
DP	0,2	0,1	0,1667	0,1667	0,1667	0,1667	0,1667
Default variance	0,25	0,25	0,4123	0,45018	0,42102	0,46496	0,42832
EL	0,4	0,1	0,5	0,5	0,5	0,5	0,5
UL2	840000 m	102500 m	942500 m	950666 m	944314 m	954048 m	945865 m
UL	0,91651 m	0,32015 m	0,9708 m	0,975 m	0,9717 m	0,9767 m	0,97255 m
Asset correlation			0	0,7	0,2	0,7	0,2
Assets' Kendal's tau			0	0,4939	0,1283	0,4939	0,1283
Dependence Structure			Independence	Gaussian copula	Gaussian copula	Clayton copula	Clayton copula
P(A∩B)			0,02	0,0689	0,0308	0,08929	0,04019
Default correlation			0	0,40832	0,0907	0,57744	0,168

Sector A which has the higher default probability and the biggest exposure is observed to have four times bigger expected loss and almost three times higher unexpected loss than sector B. The default probability to all hypothetical portfolios is increased and reaches 16,67%. The expected and the unexpected loss and default variance to all portfolios appear to be greater. The unexpected loss and default variance is bigger for portfolio F and next comes portfolio D ,G, E and last C.

Finally we see what happens if the greater exposure is to the sector with the lower default probability. Both sectors have the same expected loss while the unexpected loss is greater to the sector with the bigger exposure. Default probability to all constructed portfolios is lower than the case where the greater exposure was to the sector with the higher default probability. It is also lower than the case that the exposure was the same but the one sector had higher default probability than the other. The expected loss of the portfolios is the same. When compared with the case where the bigger exposure was to the sector with the higher default probability, we observe that expected loss is lower in the scenario that we study now. Default variance of the portfolios is very low and almost equal with those of the scenario that both sectors had the same exposure and default probability. However unexpected loss is lower only than the case that the greater exposure is on the sector with the higher default probability.

This is rather expected because we have two sources that influence unexpected loss. The one is the great exposure to one sector which means that if default occurs the loss will be great while the other is the second's sector great default probability which means that is more possible for a loss to occur. This is due to the fact that our portfolio is comprised from only two sectors. However we observe that default variance which is used in CreditRisk+ in order to compute the overall loss distribution appears to lessen.

	Sector A	Sector B	Portfolio C	Portfolio D	Portfolio E	Portfolio F	Portfolio G
Exposure	2 m	1 m	3 m	3 m	3 m	3 m	3 m
DP	0,1	0,2	0,1333	0,1333	0,1333	0,1333	0,1333
Default variance	0,25	0,25	0,35355	0,41957	0,36924	0,444	0,3821
EL	0,2	0,2	0,4	0,4	0,4	0,4	0,4
UL^2	410000 m	210000 m	620000 m	628166 m	621814 m	631548 m	623365 m
UL	0,64031	0,4582	0,7874	0,7925	0,78855	0,7947	0,7895
Asset correlation			0	0,7	0,2	0,7	0,2
Assets' Kendal's tau			0	0,4939	0,1283	0,4939	0,1283
Dependence Structure			Independence	Gaussian copula	Gaussian copula	Clayton copula	Clayton copula
P(A∩B)			0,02	0,0689	0,0308	0,08929	0,04019
Default correlation			0	0,40832	0,0907	0,57744	0,168

In the situations that we examined, each portfolio included only two sectors. Thus the effects from the diversification did not show up and the risk due to statistical nature of default events, i.e. a crisis in a particular sector, plays an important role. This means that unexpected loss is mainly influenced from the exposure to a particular sector and the sector's default probability. However, even in a portfolio with small diversification it is prevalent the role that default correlation plays. So in order to estimate the loss distribution of a loan portfolio we must always take into account default correlation which is defined from sectors' default probabilities and from the underlying dependence structure of assets.

In a diversified portfolio the statistical nature of default events is limited and the systematic change in industries as they are reflected from default correlation retain the crucial role. And the more diversified the portfolio is, the greater the role that default correlation plays…

In order to illustrate this last statement we will study some more portfolios. We make the assumption that all sectors have the same characteristics. The exposure to each sector is equal to 1 million monetary units, their default probability is equal to 10%, their default variance is 0,25 , their expected loss is 100.000 monetary units and the unexpected loss 320.156.

Portfolios A, B and C are comprised of loans to only two sectors. Their one and only difference concerns the value of default correlation between the sectors. In portfolio A there is independence of defaults and so default correlation is equal to 0, in portfolio B default correlation is equal to 0,2 and in portfolio D default correlation is equal to 0,4.

When we compare these three portfolios we observe that default variance and unexpected loss depend on the default correlation. So Portfolio C which has the higher default correlation it has also the greater default variance and unexpected loss. If default correlation was equal to 0,2 but we ignored it, the underestimation of default variance and unexpected loss would be equal to 9,5% and 0,2% equivalently.

If default correlation was equal to 0,4 but we ignored it, the underestimation of default variance and unexpected loss would be equal to 18,3% and 0,4% equivalently. We observe that the underestimation is twice grater in portfolio C which has a twice greater default correlation.

Portfolios D, E and F are comprised of loans to three sectors. Their one and only difference concerns the value of default correlation between the sectors which we assume that is the same for every possible pair-wise combination. In portfolio D there is independence of defaults and so default correlation is equal to 0, in portfolio E default correlation is equal to 0,2 and in portfolio F default correlation is equal to 0,4. Again we observe that default variance and unexpected loss is greater to the portfolio which has the greater default correlation.

If default correlation was equal to 0,2 but we ignored it, the underestimation of default variance and unexpected loss would be equal to 26,5% and 0,4% equivalently.

If default correlation was equal to 0,4 but we ignored it, the underestimation of default variance and unexpected loss would be equal to 48,3% and 0,98% equivalently. Again we observe that the underestimation is almost twice greater in portfolio C which has a twice greater default correlation. These findings confirm the intuition that the more diversified a portfolio is, the more attention must be paid on default correlation. And the higher the default correlation is, the greater the underestimation of potential loss would be.

Portfolios A, B and C have greater default variance than portfolios D, E and F. Thus, we see that the more diversified a portfolio is, the lower its default variance is.

Portfolios D, E and F have a greater unexpected loss than portfolios A, B and C. This is due to the fact that exposure is greater and so expected and unexpected losses are greater too.

Let us now assume that a bank's total exposure is 3.000.000 as in the case that the financial institution had granted loans to three sectors, but the total exposure in this case is divided equally to only two sectors. Thus portfolios G, H and K are constructed. We observe that default variance remains the same with those that portfolios A, B and C have. So we can deduce that default variance of a portfolio that determines the tail of loss distribution in CreditRisk+ is affected mainly from the number of sectors and the default correlation among them.

Portfolios G, H and K have an unexpected loss which not only is greater than the equivalent ones of portfolios A, B and C, but moreover they are greater than the equivalent ones of portfolios D, E and F. This means that diversification reduces unexpected loss which is affected very much from the exposure to each sector and default correlation.

	Sector A	Portfolio A	Portfolio B	Portfolio C	Portfolio D	Portfolio E	Portfolio F	Portfolio G	Portfolio H	Portfolio K
Exposure	1000000	2000000	2000000	2000000	3000000	3000000	3000000	3000000	3000000	3000000
DP	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1
Default variance	0,5	0,353553	0,38729833	0,41833	0,235703	0,2981429	0,3496033	0,353553	0,387298	0,41833
EL	100000	200000	200000	200000	300000	300000	300000	300000	300000	300000
UL	320156,2	452769,3	453872,229	454972,5	552268,1	554977,5	557673,76	679153,9	680808,3	682458,8
Default correlation		0	0,2	0,4	0	0,2	0,4	0	0,2	0,4
Underestimation of default variance			0,09544512	0,183216		0,2649099	0,4832377		0,095445	0,183216
Underestimation of UL			0,00243606	0,004866		0,004906	0,0097882		0,002436	0,004866

CONCLUSION

Our aim was to derive estimates of default correlation. For doing this, first we derived sectors' default probabilities. These are determined mainly from the business risk of the sector. Our findings show that financial structure does not play such an important role as far as financial leverage is moderate. Thus for moderate levels of financial leverage the MM theorem with taxes obtains. But as financial leverage is increased the costs of financial distress are greater than the tax advantages.

Then we estimated the dependence structure between each pair of sectors in order to be able to derive the joint default probability. We used the notion of copula, without making assumptions about the marginal distributions. For 90% of sectors' pairs the appropriate copula was found to be the Gaussian. We showed the implications of using different estimators of Gaussian copula in particular and the impact of a different dependence structure.

Knowing the individual default probabilities of sectors and their joint we were able to estimate default correlation. The method that we followed allows us to take into account the change in default probabilities when we estimate default correlations. The latter must be estimated in regular periods so loan loss distribution and credit derivatives' price is correctly estimated.

Concluding, although asset value models allow us to estimate default probabilities and default correlation, they are based on the dependence of equity implied asset values. Thus, it is market participants' beliefs about sectors' perspectives and the future value of sectors' equity, rather than normal course of business, that determine default correlation. This fact renders the pair-wise default correlations that have been derived with this method questionable.

However these default correlations constitute good estimates that can be used in a number of financial applications such as the evaluation of credit derivatives. Moreover they are necessary for the estimation of a loan portfolio loss distribution. If default correlations are neglected, the loss that can take place in extreme conditions will be underestimated with catastrophic consequences for the survival of the financial institution.

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APPENDIX

			TAIL DEPENDENCE GAUSSIAN												
	Chemicals	Resources	Constr & Mater	Health	Retail	Utilities	Technology	Industrials	Food	Media	Travel	Financials	Telecom	Oil& Gas	Insurance
Chemicals															
Resources	0,107														
Constr& Mater	0,076	0,180													
Health	0,139	0,070	0,322												
Retail	0,023	0,171	0,289	0,041											
Utilities	0,000	0,000	0,000	0,000	0,000										
Technology	0,138	0,210	0,113	0,066	0,080	0,000									
Industrials	0,150	0,105	0,123	0,091	0,054	0,000	0,160								
Food	0,019	0,128	0,139	0,054	0,193	0,000	0,063	0,024							
Media	0,008	0,001	0,003	0,007	0,000	0,000	0,003	0,015	0,218						
Travel	0,053	0,158	0,223	0,096	0,244	0,000	0,218	0,071	0,341	0,000					
Financials	0,110	0,076	0,154	0,129	0,071	0,000	0,039	0,044	0,076	0,002	0,170				
Telecom	0,024	0,157	0,101	0,035	0,086	0,000	0,067	0,033	0,076	0,000	0,864	0,036			
Oil&Gas	0,107	0,061	0,119	0,042	0,119	0,000	0,487	0,036	0,082	0,000	0,073	0,059	0,069		
Insurance	0,193	0,059	0,199	0,133	0,035	0,000	0,098	0,263	0,022	0,005	0,074	0,048	0,060	0,062	

				NORMAL TAIL DEPENDENCE											
	Chemicals	Resources	Constr & Mater	Health	Retail	Utilities	Technology	Industrials	Food	Media	Travel	Financials	Telecom	Oil&Gas	Insurance
Chemicals															
Resources	0,456														
Constr& Mater	0,464	0,529													
Health	0,578	0,548	0,492												
Retail	0,294	0,428	0,549	0,340											
Utilities	0,000	0,000	0,000	0,000	0,000										
Technology	0,307	0,476	0,367	0,360	0,262	0,000									
Industrials	0,445	0,522	0,459	0,416	0,331	0,000	0,327								
Food	0,088	0,206	0,205	0,156	0,252	0,000	0,209	0,077							
Media	0,036	0,011	0,014	0,018	0,006	0,000	0,006	0,059	0,339						
Travel	0,276	0,437	0,514	0,348	0,456	0,000	0,346	0,297	0,339	0,002					
Financials	0,607	0,569	0,514	0,663	0,375	0,000	0,306	0,464	0,136	0,022	0,342				
Telecom	0,103	0,227	0,170	0,141	0,147	0,000	0,289	0,088	0,383	0,000	0,464	0,113			
Oil&Gas	0,132	0,456	0,150	0,578	0,115	0,000	0,195	0,099	0,239	0,000	0,228	0,142	0,374		
Insurance	0,638	0,554	0,499	0,542	0,334	0,000	0,407	0,545	0,115	0,032	0,330	0,562	0,148	0,228	

				TAIL DEPENDENCE OF GUMBEL COPULA											
	Chemicals	Resources	Constr & Mater	Health	Retail	Utilities	Technology	Industrials	Food	Media	Travel	Financials	Telecom	Oil&Gas	Insurance
Chemicals															
Resources	0,072														
Constr& Mater	0,056	0,153													
Health	0,089	0,106	0,125												
Retail	0,007	0,073	0,101	0,023											
Utilities	0,002	0,012	0,000	0,004	0,002										
Technology	0,124	0,165	0,093	0,107	0,023	0,004									
Industrials	0,121	0,049	0,051	0,030	0,010	0,000	0,051								
Food	0,009	0,090	0,066	0,032	0,086	0,016	0,030	0,007							
Media	0,001	0,000	0,000	0,000	0,000	0,000	0,000	0,003	0,118						
Travel	0,027	0,100	0,157	0,088	0,130	0,000	0,027	0,028	0,118	0,000					
Financials	0,068	0,119	0,071	0,148	0,023	0,002	0,043	0,044	0,034	0,000	0,062				
Telecom	0,103	0,188	0,072	0,053	0,062	0,019	0,072	0,020	0,120	0,000	0,044	0,047			
Oil&Gas	0,030	0,072	0,071	0,089	0,079	0,013	0,454	0,018	0,100	0,000	0,090	0,070	0,080		
Insurance	0,228	0,107	0,071	0,079	0,016	0,002	0,119	0,126	0,016	0,001	0,036	0,064	0,057	0,638	

					TAIL DEPENDENCE OF CLAYTON COPULA										
	Chemicals	Resources	Constr & Mater	Health	Retail	Utilities	Technology	Industrials	Food	Media	Travel	Financials	Telecom	Oil& Gas	Insurance
Chemicals															
Resources	0,864														
Constr& Mater	0,848	0,908													
Health	0,877	0,887	0,897												
Retail	0,696	0,864	0,884	0,788											
Utilities	-	-	-	-	-										
Technology	0,896	0,912	0,879	0,888	0,786	-									
Industrials	0,895	0,840	0,842	0,806	0,725	-	0,842								
Food	0,716	0,878	0,859	0,811	0,875	-	0,806	0,695							
Media	0,511	0,127	0,242	0,229	0,001	0,107	0,251	0,609	0,893						
Travel	0,800	0,884	0,909	0,876	0,899	-	0,799	0,802	0,700	0,037					
Financials	0,861	0,894	0,863	0,906	0,786	-	0,831	0,832	0,814	0,294	0,855				
Telecom	0,890	0,919	0,864	0,845	0,855	-	0,864	0,778	0,894	0,011	0,832	0,836			
Oil&Gas	0,806	0,864	0,863	0,877	0,869	-	0,921	0,769	0,884	0,001	0,878	0,862	0,871		
Insurance	0,930	0,888	0,863	0,869	0,757	-	0,894	0,897	0,760	0,430	0,819	0,857	0,849	0,93	

Kendal's t	Chemicals	Resources	Constr & Mater	Health	Retail	Utilities	Technology	Industrials	Food	Media	Travel	Financials	Telecom	Oil&Gas	Insurance
Chemicals	1,000														
Resources	0,703	1,000													
Constr& Mater	0,677	0,782	1,000												
Health	0,725	0,743	0,760	1,000											
Retail	0,489	0,704	0,738	0,592	1,000										
Utilities	-0,377	-0,533	-0,385	-0,438	-0,395	1,000									
Technology	0,759	0,790	0,730	0,744	0,590	-0,428	1,000								
Industrials	0,757	0,665	0,668	0,617	0,519	-0,241	0,668	1,000							
Food	0,509	0,727	0,695	0,623	0,721	-0,555	0,617	0,488	1,000						
Media	0,341	0,142	0,196	0,190	0,022	0,172	0,200	0,411	-0,046	1,000					
Travel	0,608	0,737	0,785	0,724	0,764	-0,432	0,754	0,611	0,754	0,079	1,000				
Financials	0,698	0,755	0,701	0,779	0,590	-0,378	0,652	0,654	0,627	0,221	0,689	1,000			
Telecom	0,602	0,804	0,703	0,672	0,689	-0,571	0,703	0,580	0,756	0,059	0,703	0,660	1,000		
Oil&Gas	0,617	0,703	0,702	0,725	0,712	-0,541	0,714	0,569	0,737	0,031	0,727	0,700	0,808	1,000	
Insurance	0,826	0,744	0,702	0,712	0,555	-0,394	0,755	0,761	0,559	0,190	0,724	0,692	0,679	0,826	1,000

Drezner (1978) developed a computationally efficient method for the bivariate normal integral.
The probability distribution of the normalized normal distribution is :

$$\Phi(h, k, \rho) = \Pr\{(x_1 < h) \cap (x_2 < k)\},$$

$$\Phi(h, k, \rho) = (2\pi\sqrt{1-\rho^2})^{-1} \int_{-\infty}^{h} \int_{-\infty}^{k} \exp\left[-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right] dx_1\, dx_2 .$$

If we substitute

$$u_1 = \frac{h - x_1}{[2(1-\rho^2)]^{1/2}}; \quad u_2 = \frac{k - x_2}{[2(1-\rho^2)]^{1/2}} .$$

And define

$$h_1 = \frac{h}{[2(1-\rho^2)]^{1/2}}; \quad k_1 = \frac{k}{[2(1-\rho^2)]^{1/2}}.$$

We have that

$$\Phi(h, k, \rho) = \frac{(1-\rho^2)^{1/2}}{\pi} \int_0^{\infty} \int_0^{\infty} \exp[-u_1^2]\exp[-u_2^2]$$

$$\exp[h_1(2u_1 - h_1) + k_1(2u_2 - k_1) + 2\rho(u_1 - h_1)(u_2 - k_1)]\, du_1\, du_2 .$$

By Gauss Quadrature :

$$\Phi(h, k, \rho) \doteq \frac{(1-\rho^2)^{1/2}}{\pi} \sum_{i,j=1}^{k} A_i A_j f(x_i, x_j),$$

Where

$$f(x, y) = \exp[h_1(2x - h_1) + k_1(2y - k_1) + 2\rho(x - h_1)(y - k_1)] .$$

The method for calculating the double integral is the following:

$$\Phi(h, k, \rho) = \phi(h) + \phi(k) - 1 + \Phi(-h, -k, \rho) \quad (1)$$

$$\Phi(h, k, \rho) = \phi(k) - \Phi(-h, k, -\rho) \quad (2)$$

$$\Phi(h, k, \rho) = \phi(h) - \Phi(h, -k, -\rho) \quad (3)$$

Where

$$\phi(h) = \frac{1}{2\pi}\int_{-\infty}^{h} \exp\left[-\frac{x^2}{2}\right] dx.$$

For h,k $\neq$ 0

$$\Phi(h, k, \rho) = \Phi(h, 0, \rho(h, k)) + \Phi(k, 0, \rho(k, h)) - \delta_{hk}, \qquad (4)$$

Where

$$\rho(h, k) = \frac{(\rho h - k)\mathrm{Sgn}(h)}{\sqrt{h^2 - 2\rho hk + k^2}}, \quad \delta_{hk} = \frac{1 + \mathrm{Sgn}(h) \cdot \mathrm{Sgn}(k)}{4}$$

And

$$\mathrm{Sgn}(x) = \begin{Bmatrix} 1, & x \geqslant 0 \\ -1, & x < 0 \end{Bmatrix}.$$

Now we can follow the following algorithm:

i) If $h \leq 0$, $k \leq 0$, $p \leq 0$ compute directly.
ii) If $h \leq 0$, $k \geq 0$, $p \geq 0$ use 3.
iii) If $h \geq 0$, $k \leq 0$, $p \geq 0$ use 2.
iv) If $h \geq 0$, $k \geq 0$, $p \leq 0$ use 1.
v) If $h*k*p > 0$ use 4.

The above algorithm has been translated into VBA code by Mario Melchiori.

```
Function ND2(az As Single, bz As Single, rhoz As Single) As Single
   If az * bz * rhoz > 0 Then
      ND2 = biv2(az, bz, rhoz)
   Else
      ND2 = biv1(az, bz, rhoz)
   End If

End Function
 Function biv1(az As Single, bz As Single, rhoz As Single) As Single
   With Application
      If rhoz = 0 Then
         biv1 = .NormSDist(az) * .NormSDist(bz)
      ElseIf az <= 0 And bz <= 0 And rhoz <= 0 Then
         biv1 = phiz(az, bz, rhoz)
      ElseIf az <= 0 And bz >= 0 And rhoz >= 0 Then
         biv1 = .NormSDist(az) - phiz(az, -bz, -rhoz)
      ElseIf az >= 0 And bz <= 0 And rhoz >= 0 Then
         biv1 = .NormSDist(bz) - phiz(-az, bz, -rhoz)
      ElseIf az >= 0 And bz >= 0 And rhoz <= 0 Then
         biv1 = .NormSDist(az) + .NormSDist(bz) - 1 + phiz(-az, -bz, rhoz)
      End If
```

```
    End With
End Function
Function biv2(az As Single, bz As Single, rhoz As Single) As Single
    Dim signa As Integer
    Dim signb As Integer
    Dim rhoAB As Single
    Dim rhoBA As Single
    Dim delta As Single

    If az >= 0 Then signa = 1 Else signa = -1
    If bz >= 0 Then signb = 1 Else signb = -1
    rhoAB = (rhoz * az - bz) * signa / Sqr(az ^ 2 - 2 * rhoz * az * bz + bz ^ 2)
    rhoBA = (rhoz * bz - az) * signb / Sqr(az ^ 2 - 2 * rhoz * az * bz + bz ^ 2)
    delta = (1 - signa * signb) / 4

    biv2 = biv1(az, 0, rhoAB) + biv1(bz, 0, rhoBA) - delta
End Function
Function phiz(aa As Single, bb As Single, rhoo As Single) As Single
    Dim w(1 To 5) As Single
    Dim x(1 To 5) As Single
    Dim a1 As Single
    Dim b1 As Single
    Dim fsum As Single
    Dim f As Single
    Dim i As Integer
    Dim j As Integer

    w(1) = 0.24840615
    x(1) = 0.10024215
    w(2) = 0.39233107
    x(2) = 0.48281397
    w(3) = 0.21141819
    x(3) = 1.0609498
    w(4) = 0.03324666
    x(4) = 1.7797294
    w(5) = 0.00082485334
    x(5) = 2.6697604
    a1 = aa / Sqr(2 * (1 - rhoo ^ 2))
    b1 = bb / Sqr(2 * (1 - rhoo ^ 2))

        For i = 1 To 5
            For j = 1 To 5
                f = Exp(a1 * (2 * x(i) - a1) + b1 * (2 * x(j) - b1) + 2 * rhoo * (x(i) - a1) *
(x(j) - b1))
                fsum = fsum + w(i) * w(j) * f
            Next j, i
            phiz = 0.31830989 * Sqr(1 - rhoo ^ 2) * fsum
    End Function
```

Dunnet and Sobel were the first who developed an algorithm for the calculation of bivariate t –distribution. They started to evaluate the following integral:

$$P_n(h,k;\rho) = \int_{-\infty}^{k}\int_{-\infty}^{h} g_n(u,v;\rho)\,du\,dv.$$

At first made the transformation:

$$\left.\begin{aligned} r\cos\theta &= \frac{u-\rho v}{\sqrt{(1-\rho^2)}}, \\ r\sin\theta &= v, \end{aligned}\right\}$$

And they obtained the expression

$$P_n(h,k;\rho) = \tfrac{1}{4}(1+\operatorname{sgn} h)(1+\operatorname{sgn} k) - \int_{C(h,k;\rho)}^{\pi}\int_{k\csc\theta}^{\infty}\phi_n(r)\,dr\,d\theta - \int_{C(k,h;\rho)}^{\pi}\int_{h\csc\theta}^{\infty}\phi_n(r)\,dr\,d\theta$$

Where

$$\phi_n(r) = \frac{r}{2\pi}\left(1+\frac{r^2}{n}\right)^{-\frac{1}{2}(n+2)}, \quad C(h,k;\rho) = \arctan\frac{k\sqrt{(1-\rho^2)}}{h-\rho k},$$

And

$$\operatorname{sgn} m = \begin{cases} +1 & \text{if} \quad m \geqq 0, \\ -1 & \text{if} \quad m < 0. \end{cases}$$

They observvwd that the function $\phi_n(r)$ is immediately integrable and they simultaneously defined the function

$$Q_{\frac{1}{2}n}(m,h,k) = \frac{1}{2\pi}\int_{C(h,k;\rho)}^{\pi}\left(1+\frac{k^2\csc^2\theta}{m}\right)^{-\frac{1}{2}n} d\theta$$

And they expressed the result of the integration as

$$P_n(h,k;\rho) = \tfrac{1}{4}(1+\operatorname{sgn} h)(1+\operatorname{sgn} k) - Q_{\frac{1}{2}n}(n,h,k) - Q_{\frac{1}{2}n}(n,k,h).$$

In order to reduce the above expression they used the recursion formula

$$Q_{\frac{1}{2}n}(m,h,k) = Q_{\frac{1}{2}n-1}(m,h,k) - \frac{k}{4\sqrt{(m\pi)}}\frac{1}{(1+k^2/m)^{\frac{1}{2}(n-1)}}\frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma(\frac{1}{2}n)}\{1+\operatorname{sgn}(h-\rho k)\,I_{x(m,h,k)}[\tfrac{1}{2},\tfrac{1}{2}(n-1)]\}$$

Where

$$x(m,h,k) = \frac{(h-\rho k)^2}{(h-\rho k)^2 + (1-\rho^2)(m+k^2)}$$

and

$$I_{x(m,h,k)}(p,q) = \int_0^{x(m,h,k)} \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)} y^{p-1}(1-y)^{q-1}\,dy$$

They ended to make the distinction between even and odd number of freedom degrees in order to present an outcome. Thus if the degrees of freedom is an even number the expression of the cdf is

$$\begin{aligned} P_n(h,k;\rho) &= \frac{1}{2\pi}\arctan\frac{\sqrt{(1-\rho^2)}}{-\rho} \\ &+ \frac{k}{4\sqrt{(n\pi)}}\sum_{j=1}^{\frac{1}{2}n}\frac{\Gamma(j-\frac{1}{2})}{\Gamma(j)}\frac{1}{(1+k^2/n)^{j-\frac{1}{2}}}[1+\operatorname{sgn}(h-\rho k)\,I_{x(n,h,k)}(\tfrac{1}{2},j-\tfrac{1}{2})] \\ &+ \frac{h}{4\sqrt{(n\pi)}}\sum_{j=1}^{\frac{1}{2}n}\frac{\Gamma(j-\frac{1}{2})}{\Gamma(j)}\frac{1}{(1+h^2/n)^{j-\frac{1}{2}}}[1+\operatorname{sgn}(k-\rho h)\,I_{x(n,k,h)}(\tfrac{1}{2},j-\tfrac{1}{2})], \end{aligned}$$

While the for odd number of freedom degrees the equinalent expression is

$$\begin{aligned} P_n(h,k;\rho) &= \frac{1}{2\pi}\arctan\left\{\sqrt{n}\left[\frac{-(h+k)(hk+\rho n)-(hk-n)\sqrt{(h^2-2\rho hk+k^2+n[1-\rho^2])}}{(hk-n)(hk+\rho n)-n(h+k)\sqrt{(h^2-2\rho hk+k^2+n[1-\rho^2])}}\right]\right\} \\ &+ \frac{k}{4\sqrt{(n\pi)}}\sum_{j=1}^{\frac{1}{2}(n-1)}\frac{\Gamma(j)}{\Gamma(j+\frac{1}{2})}\frac{1}{(1+k^2/n)^{j}}[1+\operatorname{sgn}(h-\rho k)\,I_{x(n,h,k)}(\tfrac{1}{2},j)] \\ &+ \frac{h}{4\sqrt{(n\pi)}}\sum_{j=1}^{\frac{1}{2}(n-1)}\frac{\Gamma(j)}{\Gamma(j+\frac{1}{2})}\frac{1}{(1+h^2/n)^{j}}[1+\operatorname{sgn}(k-\rho h)\,I_{x(n,k,h)}(\tfrac{1}{2},j)]. \end{aligned}$$

Finally they used the reduction formula

$$I_x(p,q) = I_x(p-1,q) - \frac{\Gamma(p+q-1)}{\Gamma(p)\,\Gamma(q)} x^{p-1}(1-x)^q,$$

And they obtained the expressions

$$I_x(\tfrac{1}{2},j-\tfrac{1}{2}) = \frac{2}{\pi}\arctan\sqrt{\left(\frac{x}{1-x}\right)} + \frac{2}{\pi}\sqrt{\{x(1-x)\}}\sum_{i=0}^{j-2}\frac{4^i(i!)^2}{(2i+1)!}(1-x)^i$$

$$I_x(\tfrac{1}{2},j) = \sqrt{x}\sum_{i=0}^{j-1}\frac{(2i)!}{4^i(i!)^2}(1-x)^i.$$

Which they recommended to use for computational purposes.

Alan Genz developed an algorithm in FORTRAN which is an application of Dunnet's and Sobel's results. The code is the following

DOUBLE PRECISION FUNCTION BVTL(NU, DH, DK, R) which calculates the probability that X < DH and Y < DK.

```
parameters
*
*   NU number of degrees of freedom
*   DH 1st lower integration limit
*   DK 2nd lower integration limit
*   R   correlation coefficient
*
      INTEGER NU, J, HS, KS
      DOUBLE PRECISION DH, DK, R
      DOUBLE PRECISION TPI, PI, ORS, HRK, KRH, BVT, SNU, BVND, STUDNT
      DOUBLE PRECISION GMPH, GMPK, XNKH, XNHK, QHRK, HKN, HPK,
HKRN
      DOUBLE PRECISION BTNCKH, BTNCHK, BTPDKH, BTPDHK, ONE, EPS
      PARAMETER ( ONE = 1, EPS = 1D-15 )
      IF ( NU .LT. 1 ) THEN
        BVTL = BVND( -DH, -DK, R )
      ELSE IF ( 1 - R .LE. EPS ) THEN
          BVTL = STUDNT( NU, MIN( DH, DK ) )
      ELSE IF ( R + 1  .LE. EPS ) THEN
        IF ( DH .GT. -DK )  THEN
          BVTL = STUDNT( NU, DH ) - STUDNT( NU, -DK )
        ELSE
          BVTL = 0
        END IF
      ELSE
        PI = ACOS(-ONE)
        TPI = 2*PI
        SNU = NU
        SNU = SQRT(SNU)
        ORS = 1 - R*R
        HRK = DH - R*DK
```

```
KRH = DK - R*DH
IF ( ABS(HRK) + ORS .GT. 0 ) THEN
  XNHK = HRK**2/( HRK**2 + ORS*( NU + DK**2 ) )
  XNKH = KRH**2/( KRH**2 + ORS*( NU + DH**2 ) )
ELSE
  XNHK = 0
  XNKH = 0
END IF
HS = SIGN( ONE, DH - R*DK )
KS = SIGN( ONE, DK - R*DH )
IF ( MOD( NU, 2 ) .EQ. 0 ) THEN
  BVT = ATAN2( SQRT(ORS), -R )/TPI
  GMPH = DH/SQRT( 16*( NU + DH**2 ) )
  GMPK = DK/SQRT( 16*( NU + DK**2 ) )
  BTNCKH = 2*ATAN2( SQRT( XNKH ), SQRT( 1 - XNKH ) )/PI
  BTPDKH = 2*SQRT( XNKH*( 1 - XNKH ) )/PI
  BTNCHK = 2*ATAN2( SQRT( XNHK ), SQRT( 1 - XNHK ) )/PI
  BTPDHK = 2*SQRT( XNHK*( 1 - XNHK ) )/PI
  DO J = 1, NU/2
    BVT = BVT + GMPH*( 1 + KS*BTNCKH )
    BVT = BVT + GMPK*( 1 + HS*BTNCHK )
    BTNCKH = BTNCKH + BTPDKH
    BTPDKH = 2*J*BTPDKH*( 1 - XNKH )/( 2*J + 1 )
    BTNCHK = BTNCHK + BTPDHK
    BTPDHK = 2*J*BTPDHK*( 1 - XNHK )/( 2*J + 1 )
    GMPH = GMPH*( 2*J - 1 )/( 2*J*( 1 + DH**2/NU ) )
    GMPK = GMPK*( 2*J - 1 )/( 2*J*( 1 + DK**2/NU ) )
  END DO
ELSE
  QHRK = SQRT( DH**2 + DK**2 - 2*R*DH*DK + NU*ORS )
  HKRN = DH*DK + R*NU
  HKN = DH*DK - NU
  HPK = DH + DK
  BVT = ATAN2( -SNU*( HKN*QHRK + HPK*HKRN ),
```

```
&                         HKN*HKRN-NU*HPK*QHRK )/TPI
          IF ( BVT .LT. -EPS ) BVT = BVT + 1
          GMPH = DH/( TPI*SNU*( 1 + DH**2/NU ) )
          GMPK = DK/( TPI*SNU*( 1 + DK**2/NU ) )
          BTNCKH = SQRT( XNKH )
          BTPDKH = BTNCKH
          BTNCHK = SQRT( XNHK )
          BTPDHK = BTNCHK
          DO J = 1, ( NU - 1 )/2
            BVT = BVT + GMPH*( 1 + KS*BTNCKH )
            BVT = BVT + GMPK*( 1 + HS*BTNCHK )
            BTPDKH = ( 2*J - 1 )*BTPDKH*( 1 - XNKH )/( 2*J )
            BTNCKH = BTNCKH + BTPDKH
            BTPDHK = ( 2*J - 1 )*BTPDHK*( 1 - XNHK )/( 2*J )
            BTNCHK = BTNCHK + BTPDHK
            GMPH = 2*J*GMPH/( ( 2*J + 1 )*( 1 + DH**2/NU ) )
            GMPK = 2*J*GMPK/( ( 2*J + 1 )*( 1 + DK**2/NU ) )
          END DO
        END IF
        BVTL = BVT
      END IF
*      END BVTL
      END
```

Student t Distribution Function

```
*
*                T
*         STUDNT = C   I  ( 1 + y*y/NU )**( -(NU+1)/2 ) dy
*               NU -INF
*
      INTEGER NU, J
      DOUBLE PRECISION T, ZRO, ONE, PI, PHID
      DOUBLE PRECISION CSSTHE, SNTHE, POLYN, TT, TS, RN
      PARAMETER ( ZRO = 0, ONE = 1 )
```

```
PI = ACOS(-ONE)
IF ( NU .LT. 1 ) THEN
  STUDNT = PHID( T )
ELSE IF ( NU .EQ. 1 ) THEN
  STUDNT = ( 1 + 2*ATAN(T)/PI )/2
ELSE IF ( NU .EQ. 2 ) THEN
  STUDNT = ( 1 + T/SQRT( 2 + T*T ))/2
ELSE
  TT = T*T
  CSSTHE = 1/( 1 + TT/NU )
  POLYN = 1
  DO J = NU-2, 2, -2
    POLYN = 1 + ( J - 1 )*CSSTHE*POLYN/J
  END DO
  IF ( MOD( NU, 2 ) .EQ. 1 ) THEN
    RN = NU
    TS = T/SQRT(RN)
    STUDNT = ( 1 + 2*( ATAN(TS) + TS*CSSTHE*POLYN )/PI )/2
  ELSE
    SNTHE = T/SQRT( NU + TT )
    STUDNT = ( 1 + SNTHE*POLYN )/2
  END IF
  STUDNT = MAX( ZRO, MIN( STUDNT, ONE ) )
ENDIF
END
```

The above code was translated into VBA while taking for granted the needs of our application.

```
Function tsc(dh As Double, dk As Double, r As Double, nu As Double, inte As Double) As Double
tpi = 2 * Application.WorksheetFunction.Pi
opi = Application.WorksheetFunction.Pi
```

```
ors = 1 - r * r
hrk = dh - r * dk
krh = dk - r * dh
If Abs(hrk) + ors > 0 Then
   xnhk = hrk ^ 2 / (hrk ^ 2 + ors * (nu + dk ^ 2))
   xnkh = krh ^ 2 / (krh ^ 2 + ors * (nu + dh ^ 2))
Else
   xnhk = 0
   xnkh = 0
End If
If hrk >= 0 Then hs = 1 Else hs = -1
If krh >= 0 Then ks = 1 Else ks = -1
If nu / 2 = inte Then
   bvt = Application.WorksheetFunction.Atan2(ors ^ 0.5, -r) / tpi
   gmph = dh / ((16 * (nu + dh ^ 2)) ^ 0.5)
   gmpk = dk / ((16 * (nu + dk ^ 2)) ^ 0.5)
   btnckh = 2 * Application.WorksheetFunction.Atan2((xnkh ^ 0.5), ((1 - xnkh) ^
0.5)) / opi
   btpdkh = 2 * ((xnkh * (1 - xnkh)) ^ 0.5) / opi
   btnchk = 2 * Application.WorksheetFunction.Atan2((xnhk) ^ 0.5, ((1 - xnhk) ^
0.5)) / opi
   btpdhk = 2 * ((xnhk * (1 - xnhk)) ^ 0.5) / opi
   For j = 1 To nu / 2
      bvt = bvt + gmph * (1 + ks * btnckh)
      bvt = bvt + gmpk * (1 + hs * btnchk)
      btnckh = btnckh + btpdkh
      btpdkh = 2 * j * btpdkh * (1 - xnkh) / (2 * j + 1)
      btnchk = btnchk + btpdhk
      btpdhk = 2 * j * btpdhk * (1 - xnhk) / (2 * j + 1)
      gmph = gmph * (j - 1 / 2) / (j * (1 + dh ^ 2 / nu))
      gmpk = gmpk * (j - 1 / 2) / (j * (1 + dk ^ 2 / nu))
    Next j
Else
   qhrk = (dh ^ 2 + dk ^ 2 - 2 * r * dh * dk + nu * ors) ^ 0.5
```

```
   hkrn = dh * dk + r * nu
   hkn = dh * dk - nu
   hpk = dh + dk
   bvt = Application.WorksheetFunction.Atan2(-(nu ^ 0.5) * (hkn * qhrk + hpk * hkrn), hkn * hkrn - nu * hpk * qhrk) / tpi
   If bvt < -0.00000000001 Then
      bvt = bvt + 1
   End If
    gmph = dh / (tpi * (nu ^ 0.5) * (1 + dh ^ 2 / nu))
    gmpk = dk / (tpi * (nu ^ 0.5) * (1 + dk ^ 2 / nu))
    btnckh = xnkh ^ 0.5
    btpdkh = btnckh
    btnchk = xnhk ^ 0.5
    btpdhk = btnchk
    For j = 1 To (nu - 1) / 2
      bvt = bvt + gmph * (1 + ks * btnckh)
      bvt = bvt + gmpk * (1 + hs * btnchk)
      btpdkh = (2 * j - 1) * btpdkh * (1 - xnkh) / (2 * j)
      btnckh = btnckh + btpdkh
      btpdhk = (2 * j - 1) * btpdhk * (1 - xnhk) / (2 * j)
      btnchk = btnchk + btpdhk
      gmph = gmph * j / ((j + 1 / 2) * (1 + dh ^ 2 / nu))
      gmpk = gmpk * j / ((j + 1 / 2) * (1 + dk ^ 2 / nu))
    Next j
End If
tsc = bvt

End Function
```