**UNIVERSITY OF PIRAEUS**
**Department of Banking and Financial Management**
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**MSc. Thesis:**

# Hedging Performance of Stochastic Volatility and Exponential Levy Models: An Empirical Investigation

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*To the holy memory of my father,*

*Christos*

# Abstract

This thesis examines whether exponential Levy models are providing better hedging results than stochastic volatility models. For this reason we include two stochastic volatility models, Heston and SABR. The first can be considered as benchmark for hedging purposes, since Bakshi et al. (1997) found that the inclusion of jumps in this model does not improve its hedging ability. On the other hand we use the pure jump process, CGMY. We consider single instrument and two instrument hedging schemes. In single instrument hedging we use only the underlying as hedging instrument and we apply Delta and minimum variance hedging. We find that minimum variance improves delta-hedging performance. Also CGMY process is better in minimum variance hedging. In two instrument hedging an option is included in the instrument portfolio. In stochastic volatility models the instrument option is used for hedging against volatility changes. In pure jump models the instrument option is used for hedging against market jumps. For this reason we expand minimum variance strategy so that an option can be used as hedging instrument. We find that when a second option is used, stochastic volatility models provide better hedging results than CGMY. Thus the hedging of stochastic volatility effects is more important than the hedging of jumps.

## Acknowledgments

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*"Every Theory Contains as much Truth as its Mathematics"*

*Immanuel Kant*

# Contents

# 1 Introduction

After the seminal paper of Black & Scholes (1973) many different types of option pricing models have been created in order to capture some stylized facts of the stock returns. Among them we find stochastic volatility models (Hull and White 1987), models based in the constant elasticity of variance (Cox 1975), local volatility models (Dupire 1996), Garch-type option pricing (Duan 1994), jump-diffusion models that incorporate stochastic interest rates and stochastic volatility (Bates1994) and infinite activity Levy models (Carr, Geman, Madan and Yor 2002). Although there is enormous literature in the pricing of options, hedging effectiveness for equity derivatives has not been considered seriously but only in Bakshi et al. (1997) and Alexander and Noguera (2007). This thesis addresses three issues.

Which model could be the best for hedging in incomplete markets? This answer seems to be closely related to the option's maturity. Scott (1997) presents excessive evidence that stochastic interest rates do not influence short-term (< 3 months) options. Bakshi et al. (1997) in a thorough investigation of hedging performance among many parameter stochastic volatility, stochastic interest rates and jumps of Gaussian type models, find that interest rate hedging is not required for short term hedging. Bakshi et al. (2000) continue their research on hedging performance for the same type of models in long term options. They find significant differences of hedging performance depending on the time horizon and that the modeling of stochastic interest rates does matter in the long run. Moreover they report that the volatility surface is relatively flat and jump arrivals are decreasing as the time to maturity increases. In this thesis we will focus on short-term options, an area where market anomalies such as jumps play a dominating role.

Looking for a good hedging model for short term options, one has to accept that pricing performance does not necessarily means improvement of hedging performance, as was stated in Bakshi et al. (1997). They found that adding jumps to Heston's (1993) stochastic volatility model, pricing performance increases, but hedging performance decreases in some cases. So we will use Heston's model as an indicative and well examined hedging model.

In the last decade the development of infinite activity Levy models was important for addressing the jump behavior of assets. In this area even jump diffusions proved insufficient, due to their finite number of jumps assumption. Infinite activity Levy models assume no Brownian motion and asset dynamics evolve only through small and large jumps. Carr et al. (2002) mix the CGMY process with Brownian motion. They find the startling result that indices are pure jump processes that have diversified the diffusion risk. Moreover the diffusion component becomes statistically insignificant even in stock returns. Levy models are a modeling tool, which succeeds to explain the jump behavior of financial time series, an area where all the others models fail. Hedging effectiveness of these models through a real market data set has not been examined before. Here we will examine whether CGMY, an indicative infinite activity model, improves the hedging performance of Heston. This is the first time that these models are compared in a real market data set.

Comparing the hedging performance of two models another important question arises. Which hedging strategy could give the best hedging results? Single instrument hedging is the hedging strategy using as hedging instruments only pieces of the underlying. The uniqueness of the hedging instrument makes this strategy the most convenient. Delta hedging has been widely applied in models with continuous trajectories. Delta hedge ratio is the total derivative of the option's value with respect to the underlying. In models that move only through jumps, that kind of hedging violates severely their assumptions because the left limit of this derivative is not equal to the right. The most appropriate way to make a hedging position in these models is by constructing a position trying to minimize the future variance of the extended portfolio, thus minimum variance strategy. For single instrument hedging only minimum variance strategy could be used as an indicative for testing the differences between these models. Here we consider both Delta and minimum variance hedging, while in CGMY we accept as Delta hedge ratio the right derivative of the option's value with respect to the underlying.

Single instrument hedging is not the best strategy to immunize a portfolio against market risk. In the practitioners industry options are used as hedging instruments extensively. In a stochastic volatility setting this is done, by immunizing a portfolio against both market and volatility risk. Volatility hedging is in favor of stochastic volatility models since it includes options as hedging instruments, in a manner that is out of scope of infinite activity Levy models. The known hedging schemes, that include options in the instrument portfolio, are insufficient to examine Levy models performance. This is because the presence of jumps does not allow replication-type hedging. Here we expand minimum variance hedging so that options can be included as hedging instruments. This hedging scheme was introduced in Cont et al. (2005). This is the first time that is examined in a real market data set.

The third issue that is being addressed, is the enrichment of the large class of scale invariant models. A process is scale invariant if the returns distribution is independent of current price state. Recently Alexander and Noguera (2007) proved that these models produce the same Delta and Gamma hedge ratios for plain vanilla options. Their results are applicable in a large category of affine jump diffusions, studied on Dyffie et al. (2000). Here we examine the assumptions under which infinite activity models can be included in this category. For this reason we include to our research SABR, a non-scale invariant model derived by Hagan et al. (2002). This is a stochastic volatility model but its Delta hedging errors are totally different to Heston's. Based on empirical results we examine whether the scale invariant hedge ratios are providing the most efficient hedging.

The rest of this thesis is constructed as follows. In Chapter 2 we describe the stylized facts of market incompleteness and we introduce the hedging strategies. In Chapters 3 and 4 we construct the solutions of stochastic volatility and jump models respectively. In Chapter 5 we present the family of scale invariant models. In Chapter 6 we present the minimum variance hedging allowing options as hedging instruments. In Chapter 7 the calibration procedure is carried out. In Chapters 8 and 9 we present the empirical results and we conclude.

# 2 Hedging in Incomplete Markets

## 2.1 Incomplete Markets and Stylized Facts

When markets are complete every derivative security can be perfectly hedged. There exists a strategy, involving only the trading of the underlying and a risk free asset, which totally replicates the derivatives value at any given time. This replication strategy signifies a unique price for the derivative, since by no arbitrage argument two same products cannot be priced with different values. The arrival from complete markets signals that the trading of derivatives involves risks that cannot be fully hedged. Moreover the derivative prices are not uniquely determined and they are based on an agreement between the market participants about the non-hedged risk that is undertaken. These general ideas are presented in the next two theorems. The first guarantees the absence of arbitrage opportunities. The second suggests the conditions under which a derivative can be perfectly hedged.

Theorem [2.1] First Fundamental Theorem of Asset Pricing:

Assuming that there exists a probability measure $P$ associated with the real states of the world, a market model does not allow arbitrage opportunities if there exists a measure $Q$ locally equivalent to $P$ under which relative asset prices are martingales.

Theorem [2.2] Second Fundamental Theorem of Asset Pricing:

A market is complete if $Q$ is unique. Moreover if the market is complete every derivative security can be replicated from market securities.

In Black & Scholes framework the market is complete. They consider the stock evolution as a Geometrical Brownian motion $S_t = S_0 \exp\left\{(m - \frac{\sigma^2}{2})t + \sigma W_t\right\}$ (2.1). They assume

1. perfect markets
2. constant volatility (σ)
3. constant interest rates
4. short selling is allowed

Under these assumptions the market is complete, so an agent can sell a derivative (i.e. a European call) and then full hedge it by continuous Delta-hedging [1].

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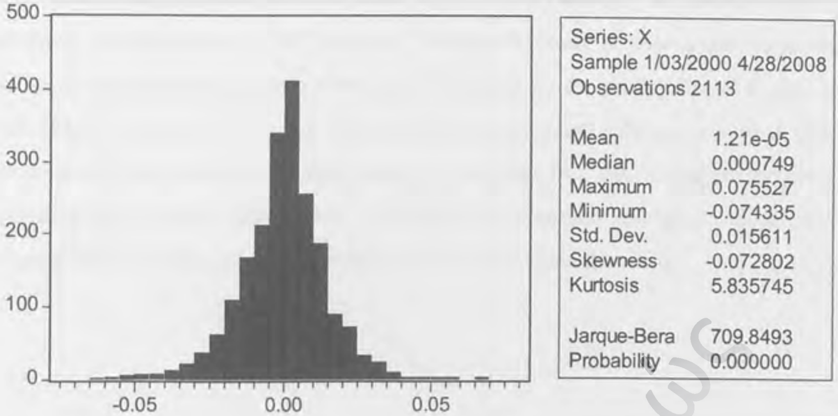
[1] Perfect hedge example: let C be a European option. We form a portfolio P with C plus Δ stocks. Then by Ito's Lemma

$dP = dC + \Delta dS = C_t dt + C_S dS + \frac{1}{2} C_{SS} (dS)^2 + \Delta dS$ Willing to eliminate all sources of risk we select $\Delta = -C_S$ so $dP = C_t dt + \frac{1}{2} C_{SS} \sigma^2 S^2 dt$ thus P contains no risk so this position is fully hedged. By doing this constantly up to options maturity, P would return the yield of zero-coupon bond maturing at T.




Fig

Fig 2.1: Dax daily log returns from 03/01/2000 to 28/04/2008

The most restrictive part of this model seems to be the Geometrical Brownian motion assumption and the constant volatility. The Geometrical Brownian motion implies that the stock returns are drawn from a normal distribution. In fact financial time series produce kurtosis and skewness far from that of the normal distribution. Moreover their tail distribution is heavier than in the normal case. Andersen et al. (2002) as also Aït-Sahalia et al. (2001) find that the level of skewness and kyrtosis for the S&P 500 index, is far more different than that of the normal distribution. In fig. 2.1 we see that excess kurtosis and negative skewness are both present, while Jarque-Beta statistic rejects the case of normality. Both stochastic volatility and jump models deal with this issue with the later being more accurate in the fitting implied of stock returns.

The second restrictive assumption of Black & Scholes model is the constant volatility. However if we use observed options prices for different strikes and calculate their implied volatility the famous picture of volatility smile will be produced (fig 2.2). If the volatility was constant this curve would be flat. In fact this kind of monotonically decreasing implied volatility, which is presented in fig 2.2, is called volatility smirk.

Bates (2000) reports, that after the crash of 1987, this pattern of implied volatility has persisted. Andersen et al. (2002) explain, that the existence of such a pattern means that the implied distribution is more skewed and adjusted for the probability of a large market downside movement. This also can be seen as leverage effect, since in a downside movement in the equity value the financial ratio debt to equity becomes bigger, so the company faces greater risk which is reflected in increased volatility. Both stochastic volatility and exponential Levy models are able to produce the smirk.

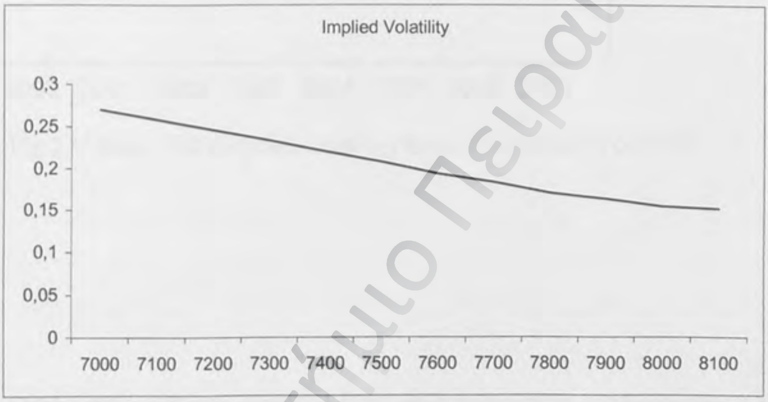



Fig 2.2: Implied volatilities from DAX European call options (3 weeks to maturity) on 31/08/2007.

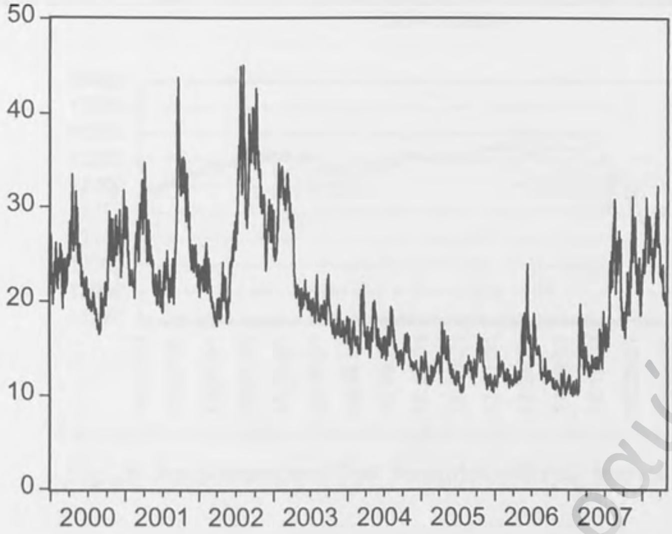



Fig 2.3: Index VIX (implied volatility) from 1/1/2000 to 30/04/2008

Another stylized fact in financial time series is volatility clustering (fig 2.3). This means that implied volatilities from options have spikes and these spikes are gathered. Also from fig.2.3 the mean reversion of volatility is apparent. Volatility exhibits some shocks but returns to some mean level. In this case only stochastic volatility models produce these features while jump models need to incorporate Brownian motion or stochastic time changes.

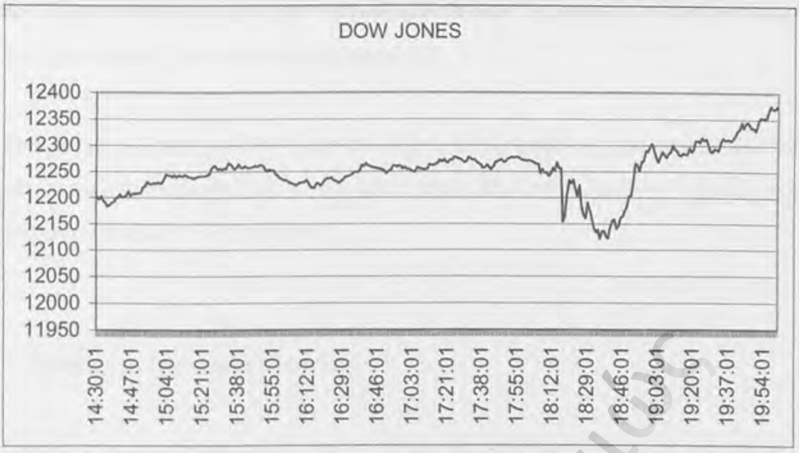



Fig 2.4: Jump behavior of Dow Jones, 18/3/08 FED interest rate decision

The last restriction of Geometrical Brownian motion is the continuous trajectories assumption. Financial time series seem to be continuous in yearly scale, but in smaller scale (i.e. daily) exhibit strong jump behavior. These jumps have financial interpretation as they are connected with important economic events (interest rate decisions, payrolls announcements, corporate profit announcements etc.). Fig. 2.4 presents a case of jump behavior after FED's interest rate decision (03/18/08). Another feature related with non continuous trajectories is that markets often exhibit overreaction and underreaction. Kou (2002) explains that models based on normal jumps (Merton 1976) even in the case that combine stochastic volatility and stochastic interest rates [Bates (1996)], cannot exhibit this feature, in contrast with models that are based in exponential distribution. In fact many stochastic volatility market making models fail to capture this important feature so they are not able to provide prices in the options market but only after the end of this phenomenon (5-10 min). Kou uses a jump-diffusion model where jumps follow double exponential distribution, while Carr et al. (2002) went further assuming no Brownian motion dynamics and stock movement via infinite small jumps following a generalization of hyperbolic distribution. Models drawn from hyperbolic distribution, described in Raible (2000), are found to have explicit fitting in the returns distribution. These models provide good fitting in the case of non-continuous trajectories whereas stochastic volatility models don't produce this feature at all.

Due to all of these stylized facts markets are incomplete. That means that hedging becomes a more complicated issue, thus other strategies beyond buying and selling pieces of underlying should be incorporated.

## 2.2 Hedging in Incomplete Markets

Having discussed the stylized facts which provide clear evidence that the markets are incomplete the main question we address here is how a derivative can be hedged, given such many sources of market incompleteness? Clearly in these types of markets a B&S type perfect hedge would not be feasible at all. So suppose that a bank or a market maker shorts an option. This produces a naked position with possibilities of suffering great losses. Which actions should be taken by the manager in order to cover this naked position? Also this borns another critical issue: What should be hedged? The jump part? The Brownian motion part? The stochastic volatility? Addressing such kind of issues two hedging strategies are available.

*Delta-Vega-Rho Hedging*

This strategy has its origins in the B&S framework. Hedge ratios are used in terms of option sensitivities of the pronounced sources of risk.

Example: Assume that a bank writes an option in a stochastic volatility stochastic interest rates market, where other options exist with different volatility (Vega) and interest rates (Rho) sensitivities. Also assume no jumps. So the bank can hold a portfolio P such that

$$P = -O + aS + bO_1 + cO_2 \tag{2.2}$$

Now wishing to vanish all the possible sources of risk they select a, b, c such that

$$\Delta_P = 0 \Rightarrow -\Delta_O + a + b\Delta_{O_1} + c\Delta_{O_2} = 0$$
$$R_P = 0 \Rightarrow -R_O + bR_{O_1} + cR_{O_2} = 0 \qquad \text{System (1)}$$
$$V_P = 0 \Rightarrow -V_O + bV_{O_1} + cV_{O_2} = 0$$

By choosing $O_1, O_2$ properly so that this 3x3 system has an exact solution, the bank is able to immunize its portfolio against all sources of risk.

An important drawback of Delta-hedging strategy is that the options $O_1, O_2$ of the previous example can not be used until maturity. In fact when we'll conduct the next hedging it's possible that we have to substitute these options with some other in order to solve again the system (1). This strategy assumes that trading is continuous. Continuous trading is a restrictive assumption. In industry hedging can be conducted only at fixed times within a day.

Also we must note that we assumed no jumps to obtain this type of strategy. In the case that the stock follows a jump-process this strategy should be enhanced. Merton's argument for jump-diffusion models is that only the Brownian part should be hedged because jump risk is differentiable. But in infinite activity exponential Levy models this is not the case. These models moving continuously by small jumps and they do not engage Brownian motion at all. Moreover their sensitivities could be ill-defined [2]. Although it seems tempting to apply the same Delta-hedging strategy to these models because they share the scale invariance property, as we will show in Chapter 5, this would be essentially wrong.

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[2] In the case continuous trajectories this one can hold $\Delta$ stocks and by selecting $\Delta$ as

$\Delta = \frac{dC}{dS}(t, S_t)$ one can hedge a derivative against market risk for a small time interval.

In non-continuous trajectories this sensitivity is not well defined. Actually

$$\frac{dC}{dS}(t^-, S_{t^-}) \neq \frac{dC}{dS}(t^+, S_{t^+})$$

The right limit is only known at time t. This means that the Delta stocks we hold are providing hedging against market risk, only conditional on no jump arrival.

*Minimum Variance Hedging*

Definition [2.1]: Minimum variance hedge ratio ($\Delta^{mv}$) is the pieces of the underlying that minimizes the instantaneous variance of a portfolio $C-\delta S$ . This is

$$\Delta^{mv}=\delta\in R:[\text{var}(dC-\delta dS)=\min] \tag{2.3}$$

In other words we try to minimize

$$\text{var}(dC-\delta dS)=E[(dC-\delta dS)^2]^3 \tag{2.4}$$

Proposition [2.1]: The optimization problem (2.4) has the exact solution

$$\Delta^{mv}=\frac{\text{cov}(dC,dS)}{\text{var}(dS)} \tag{2.5}$$

Proof:

$$\frac{d}{d\delta}E[(dC-\delta dS)^2]=0\Rightarrow$$

$$\frac{d}{d\delta}E[(dC)^2]+\frac{d}{d\delta}E[(\delta dS)^2]-2\frac{d}{d\delta}\text{cov}(dC,\delta dS)=0\Rightarrow$$

$$2\delta E[(dS)^2]-2\text{cov}(dC,dS)=0\Rightarrow\delta=\frac{\text{cov}(dC,dS)}{\text{var}(dS)}$$

Moreover

$$\frac{d}{d\delta}E[(dC-\delta dS)^2]=E[(dS)^2]>0$$

So this is the infimum.

□

[3] In fact $\text{var}(x)=E(x^2)-(E(x))^2$ but $E(x)$ is always zero, under the equivalent martingale measure of theorem [2.1]

This strategy is something totally different than Delta-hedging. Due to the anomalies of the price trajectories we accept that no replication type strategy can hold. In fact we try to minimize an expectation of a user-defined error function. We accept that profits and losses exist not due to inappropriate hedging, but as a natural outcome of market incompleteness. Hedging is conducted as an approximation problem. In Chapter 6 we expand this strategy allowing options in the instrument portfolio.

# 3 Stochastic Volatility Models

## 3.1 Heston Model

### 3.1.1 Heston Model: Description

Stochastic volatility models expand the geometrical Brownian motion of B&S model by including a second process for the volatility dynamics. Perhaps the most popular among them is Heston's model (1993), with the following dynamics.

$$dS = mSdt + \sqrt{v}Sdw_1 \tag{3.1}$$

$$dv(t) = \kappa[\theta - v(t)]dt + \sigma\sqrt{v}dw_2 \tag{3.2}$$

$$dw_1 dw_2 = \rho dt \tag{3.3}$$

Models like that are eligible enough to exhibit many of the incomplete markets stylized facts that were presented in the previous Chapter. The slope of the volatility smile (fig.2.2) is controlled by $\rho$. In fact when $\rho < 0$ the famous leverage effect feature is produced i.e. the smile is down-sloping, so performance of the underlying is combined with increased volatility. Dumas et al.(1998) find that this leverage effect performance $(\rho < 0)$ is an important prerequisite for a model to fit S&P 500 market. Nandi (1998) finds that when this term is not constrained to zero, Delta hedging performance of Heston's model is significantly improved. Volatility mean reversion is controlled by $\kappa, \theta$. $\theta$ is the long run volatility while $\kappa$ is the rate of mean reversion. Finally the $\sigma$ parameter is the volatility of volatility. Its presence allows the clustering effect to be produced fig.2.3.

### 3.1.2 Heston's PDE

The market under stochastic volatility dynamics is incomplete. That means that all the risks that are associated with the trading of this derivative cannot be hedged. So an agreement for the risk premium must be done. The risk premium has a specific form and can be extracted using the following method.

Suppose that stock price dynamics under real world measure are given by Heston dynamics. By Ito's lemma every option written on S satisfies

$$dO = O_t dt + O_S dS + \frac{O_{SS}}{2}(dS)^2 + O_v dv + \frac{O_{vv}}{2}(dv)^2 + O_{Sv} dSdv \Rightarrow$$

$$dO = O_t dt + O_S S[mdt + \sqrt{v}dw_1] + \frac{O_{SS}}{2} S^2 vdt + O_v[\kappa(\theta - v)dt + \sigma\sqrt{v}dw_2]$$

$$+\frac{O_{vv}}{2}\sigma^2 vdt + O_{Sv} S\sigma v\rho dt \Rightarrow$$

$$dO = [O_t + O_S Sm + \frac{O_{SS}}{2} S^2 v + O_v \kappa(\theta - v) + \frac{O_{vv}}{2}\sigma^2 vdt + O_{Sv} S\sigma v\rho]dt +$$

$$O_S S\sqrt{v}dw_1 + O_v \sigma\sqrt{v}dw_2$$

Now we form a portfolio $P = O - qS - aF$

where $O$ and $F$ are two options written on S.

Applying Ito's lemma in P results

$$dP = a_1 dS + a_2 dv + a_3 dt$$

Where

$$a_1 = O_S - aF_S - q$$

$$a_2 = O_v - aF_v$$

$$\alpha_3 = [O_t + \frac{O_{SS}}{2} S^2 vdt + \frac{O_{vv}}{2}\sigma^2 vdt + O_{Sv} S\sigma v\rho] -$$

$$a[F_t + \frac{F_{SS}}{2} S^2 vdt + \frac{F_{vv}}{2}\sigma^2 vdt + F_{Sv} S\sigma v\rho]$$

Willing to eliminate all sources of randomness we select

$$a = \frac{O_v}{F_v} \quad \text{and} \quad q = O_S - aF_S$$

This results $a_1 = a_2 = 0$.

Hence portfolio P satisfies $dP = a_3 dt$. Thus it contains no sources of risk.

By no arbitrage arguments

$$dP = rPdt.$$

So

$$[O_t + \frac{O_{SS}}{2} S^2 vdt + \frac{O_{vv}}{2} \sigma^2 vdt + O_{Sv} S\sigma v\rho] -$$
$$\frac{O_v}{F_v}[F_t + \frac{F_{SS}}{2} S^2 vdt + \frac{F_{vv}}{2} \sigma^2 vdt + F_{Sv} S\sigma v\rho] = r(O - (O_S - \frac{O_v}{F_v} F_S)S - \frac{O_v}{F_v} F) \quad (3.4)$$

After rearranging the terms in (3.4).

$$[O_t + \frac{O_{SS}}{2} S^2 vdt + \frac{O_{vv}}{2} \sigma^2 vdt + O_{Sv} S\sigma v\rho + rSO_S - rO] / O_v =$$
$$[F_t + \frac{F_{SS}}{2} S^2 vdt + \frac{F_{vv}}{2} \sigma^2 vdt + F_{Sv} S\sigma v\rho + rSF_S - rF] / F_v = \Phi(S, v, t)$$

Where $\Phi$ is depending only on *S, v, t* and not in the type of the derivative. Thus every derivative satisfies

$$O_t + rSO_S + \frac{O_{SS}}{2} S^2 vdt + \frac{O_{vv}}{2} \sigma^2 vdt + O_{Sv} S\sigma v\rho - rO = O_v \Phi(S, v, t) \quad (3.5)$$

Where $\Phi$ equals the volatility drift plus the market price of risk

$$\Phi(S,v,t) = \kappa(\theta - v) + \Lambda(S,v,t) \tag{3.6}$$

So (3.5) becomes

$$O_t + rSO_S + \frac{O_{SS}}{2}S^2 v + [\kappa(\theta - v) + \Lambda(S,v,t)]O_v + \frac{O_{vv}}{2}\sigma^2 v + O_{Sv}S\sigma v\rho - rO = 0 \tag{3.7}$$

$\Lambda(S,v,t)$ is the market price of risk function. It is determined only by the current stock level, the volatility level and the time to maturity. It is independent of the type of the derivative characteristics as far this derivative is of European type and a $C^{2,2,1}(S,v,t)$ function.

### 3.1.3 Characteristic Functions

Heston presents evidence that the market price of volatility risk is analogous to the level of volatility.

So $\Lambda = \lambda v$. Thus (3.7) becomes

$$O_t + rSO_S + \frac{O_{SS}}{2}S^2 v dt + U_v[\kappa(\theta - v) - \lambda v]dt + \frac{O_{vv}}{2}\sigma^2 v dt + O_{Sv}S\sigma v\rho - rO = 0 \tag{3.8}$$

In the rest of this thesis we apply extensively characteristic functions. Here we will present analytically the fundamental results for Heston's model while in the rest of the models this methodology remains the same. This form of solution was first presented in Stein and Stein (1991) and then in Heston (1993). Before this solution the pricing of stochastic volatility models was carried out with Monte Carlo or numerical techniques, Hull and White (1987).

Definition [3.1]: The characteristic function $f(\varphi)$ of a distribution function $F(t)$ is defined as

$$f(\varphi)=\int_{-\infty}^{\infty}\exp(i\varphi x)dF(x) \qquad (3.9)$$

Theorem [3.2]: Characteristic Function Inversion Formula:

$$f(x)=F^{-1}\{F(\varphi)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\varphi x)F(\varphi)d(\varphi) \qquad (3.10)$$

Further by denoting $\tau = T - t_0$ the time to maturity and changing to logarithmic scale [4] (3.8) becomes

$$U_\tau = U_X\left(r-\frac{v}{2}\right)+\frac{U_{XX}}{2}v+U_v[\kappa(\theta-v)-\lambda v]+\frac{U_{vv}}{2}\sigma^2 v+U_{Xv}\sigma v\rho-rU \qquad (3.11)$$

The boundary condition for a European call option is

$$U(0,e^x)=(e^x-K)^+$$

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[4] We used the change of coordinates

$$\frac{dU}{dS}=\frac{dU}{dX}\frac{dX}{dS}=U_X\frac{1}{S}$$

$$\frac{d^2U}{dS^2}=\frac{d(U_X\frac{1}{S})}{dX}\frac{dX}{dS}=U_{XX}\frac{1}{S^2}-\frac{1}{S^2}U_X$$

$$\frac{dU}{dt}=\frac{dU}{d\tau}\frac{d\tau}{dt}=-U_\tau$$

By substituting these in (3.8) we get (3.11)

Now we'll find a solution for (3.11) in the form

$$U(x,v,\tau) = e^x P_1(x,v,\tau) - Ke^{-r\tau} P_2(x,v,\tau) \tag{3.12}$$

$P_1$ is the Delta of the call and $P_2$ is the probability that the call ends in the money. Substituting (3.12) in (3.11) we obtain.

$$\begin{aligned}
&e^x(-P_{1\tau} + (P_{1x} + P_1)(r - \frac{v}{2}) + (P_1 + 2P_{1x} + P_{1xx})0.5v + P_{1v}[\kappa(\theta - v) - \lambda v] + \\
&P_{1vv}0.5\sigma^2 v + (P_{1xv} + P_{1v})\sigma v\rho - rP_1) = \\
&Ke^{-r\tau}(-P_{2\tau} + rP_2 + P_{2x}(r - \frac{v}{2}) + P_{2xx}0.5v + \\
&P_{2v}[\kappa(\theta - v) - \lambda v] + P_{2vv}0.5\sigma^2 v + P_{2xv}\sigma v\rho - rP_2)
\end{aligned} \tag{3.13}$$

By defining the moneyness as $M=\frac{e^x}{Ke^{-r\tau}}$ the PDE (3.13) becomes a first degree polynomial of $M$ which is always zero. So the coefficients

$$-P_{1\tau} + (P_{1x} + P_1)(r - \frac{v}{2}) + (P_1 + 2P_{1x} + P_{1xx})0.5v + P_{1v}[\kappa(\theta - v) - \lambda v] + P_{1vv}0.5\sigma^2 v + (P_{1xv} + P_{1v})\sigma v\rho - rP_1$$

$$-P_{2\tau} + rP_2 + P_{2x}(r - \frac{v}{2}) + P_{2xx}0.5v + P_{2v}[\kappa(\theta - v) - \lambda v] + P_{2vv}0.5\sigma^2 v + P_{2xv}\sigma v\rho - rP_2$$

should be always equal to zero. This produces two PDEs

$$-P_{1\tau} + P_{1x}(r + \frac{v}{2}) + P_{1xx}0.5v + P_{1v}[\kappa\theta - (\kappa + \lambda + \rho\sigma)v] + P_{1vv}0.5\sigma^2 v + P_{1xv}\sigma v\rho = 0$$

$$-P_{2\tau} + P_{2x}(r - \frac{v}{2}) + P_{2xx}0.5v + P_{2v}[\kappa\theta - (\kappa + \lambda)v] + P_{2vv}0.5\sigma^2 v + P_{2xv}\sigma v\rho = 0$$

By stating

$$u_1 = 0.5 \;\; u_2 = -0.5 \;\; \alpha=\kappa\theta \;\; b_1 = \kappa + \lambda - \rho\sigma, b_2 = \kappa + \lambda$$

we only need to solve the PDE

$$-P_{i\tau} + P_{ix}(r + u_i v) + P_{ixx} 0.5v + P_{iv}[\alpha - \beta_i v] + P_{ivv} 0.5\sigma^2 v + P_{ixv}\sigma v\rho = 0 \quad \text{for i=1,2} \tag{3.14}$$

Subject to the boundary condition $P_i(x,0) = 1_{\{x>\ln(K)\}}$

We define the transition densities as

$$p_i(x,\tau) = \text{Prob}\{x(\tau) = x \mid x(0) = x_0\}$$

$p_i$ satisfies the equation (3.14),subject to the boundary condition

$$p_i(x,0) = \Pr ob\{x(0) = x \mid x(0) = x_0\} = \delta(x - x_0)$$

These equations are called Kolmogorov Forward equations. For proof for their existence we refer to Oksendal (2000). $P_i$ is actually the cumulative distribution of these densities conditional that $\{x > \ln(K)\}$.

By multiplying (3.14) with $e^{i\varphi x}$ and taking integrals we conclude that the characteristic functions $f_1(\tau,\varphi;v,x), f_2(\tau,\varphi;v,x)$ of $p_1$ and $p_2$ satisfy the equations

$$-f_{i\tau} + f_{ix}(r + u_i v) + f_{ixx} 0.5v + f_{iv}[\alpha - \beta_i v] + f_{ivv} 0.5\sigma^2 v + f_{ixv}\sigma v\rho = 0 \tag{3.15}$$

for i =1, 2

With boundary condition

$$f_i(0,\varphi;v,x) = \lim_{a\to 0}\int_a^a e^{i\varphi x} p_i(x,0)dx = \lim_{a\to 0}\int_a^a e^{i\varphi x}\delta(x - x_0)dx = e^{i\varphi x_0} \tag{3.16}$$

### 3.1.4 The Formula

Heston was the first that modeled with characteristic functions the risk neutral probabilities $P_1$ and $P_2$. Having this solution is particularly useful for our research, because the determination of hedge ratios is straightforward. The procedure for the determination of the characteristic functions is described in Appendix 1. We obtain that this are

$$f_i = \exp(C_i(\tau) + D_i(\tau)v + i\varphi x_0)) \tag{3.17}$$

Where

$$D_i(\tau) = \frac{b_i - \rho\sigma\varphi i + d_i}{\sigma^2}\left[\frac{1-\exp(d_i\tau)}{1-g_i\exp(d_i\tau)}\right] \tag{3.18}$$

$$C_i = ri\varphi\tau + \frac{\alpha}{\sigma^2}\left\{(b_i - \rho\sigma\varphi i + d_i)\tau - 2\ln\left[\frac{1-g_i\exp(d_i\tau)}{1-g_i}\right]\right\} \tag{3.19}$$

$$\begin{aligned} &\textit{with} \\ &g_i = \frac{b_i - \rho\sigma\varphi i + d_i}{b_i - \rho\sigma\varphi i - d_i} \\ &d_i = ((\rho\sigma\varphi i - b_i)^2 - \sigma^2(2u_i\varphi i - \varphi^2))^{0.5} \end{aligned}$$

To obtain the solution for $p_i$ we need to invert the characteristic functions using(3.10). We follow Lazar (2003) and Pelaez (1952).

$$p_i = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\varphi x)F(\varphi)d(\varphi) \Rightarrow$$

$$P_i = \frac{1}{2} + \frac{1}{2\pi}\int_{0}^{\infty}\frac{\exp(i\varphi x)f_i(-\varphi) - \exp(-i\varphi x)f_i(\varphi)}{i\varphi}d(\varphi)\,[4] \Rightarrow \tag{3.20}$$

$$P_i = \frac{1}{2} - \frac{1}{\pi}\int_{0}^{\infty}\mathrm{Re}\{\frac{\exp(i\varphi x)f_i(\varphi)}{i\varphi}\}d(\varphi) \Rightarrow$$

$$P_i(|\ln(K)) = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty}\mathrm{Re}\{\frac{\exp(-i\varphi\ln(K))f_i(\varphi)}{i\varphi}\}d(\varphi) \tag{3.21}$$

We prove (3.21) in Appendix 2. The last form is the inversion that we use in the rest of the paper. Other derivations for the same formula do exist. We strongly prefer this because we don't have to employ complex integrals. Moreover we know that characteristic functions always exist. The integral in (3.21) can be calculated using a numerical scheme as we will present in Chapter 7. So by substituting(3.17) and (3.21) in (3.12) we obtain the value of a European call option.

Thus $$U(S_0, v, \tau) = S_0 P_1(x, v, \tau) - Ke^{-r\tau} P_2(x, v, \tau) \tag{3.22}$$

Where $P_1$ and $P_2$ are given by substituting (3.17) in (3.21).

### 3.1.5 The Hedge Ratios

Another advance of this solution is that the determination of Greeks is straightforward. More specifically Delta is equal to $P_1$ by the construction of this solution.

$$\text{Delta}=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty} \text{Re}\{\frac{\exp(-i\varphi\ln(K))f_1(\varphi)}{i\varphi}\}d(\varphi) \tag{3.23}$$

$$\text{Vega}=\frac{d(S_0P_1 - K\exp(-r\tau)P_2)}{dv} = S_0\frac{dP_1}{dv} - K\exp(-r\tau)\frac{dP_2}{dv} \tag{3.24}$$

Where the derivatives with respect to the volatility are given by

$$\frac{dP_i}{dv} = \frac{1}{\pi}\int_0^{\infty}\text{Re}\left[\frac{\exp(-i\varphi\ln(k)df_i/d(v))}{i\varphi}\right] = \frac{1}{\pi}\int_0^{\infty}\text{Re}\left[\frac{\exp(-i\varphi\ln(k))f_iD_i}{i\varphi}\right]$$

Minimum variance Delta is given by (2.5)

$$\Delta_{mv} = \frac{\mathrm{cov}(dU, dS)}{\mathrm{var}(dS)}^{5}$$

$$= \frac{\langle U_s dS + U_v dv, dS \rangle}{\langle dS, dS \rangle} = \frac{\langle U_s dS, dS \rangle}{\langle dS, dS \rangle} + \frac{\langle U_v dv, dS \rangle}{\langle dS, dS \rangle} =$$

$$U_S + U_v \frac{\langle dv, dS \rangle}{\langle dS, dS \rangle} = U_S + U_v \frac{\rho \sigma S v}{S^2 v} \Rightarrow$$

$$\Delta_{mv} = delta + vega \frac{\rho \sigma}{S} \tag{3.25}$$

[5]For simplicity we use the notation $\langle \ \rangle$ for covariance as in Alexander & Nogueira

## 3.2 SABR

SABR model was derived by Hagan et al. (2002). It is a particular case of stochastic volatility model. The dynamics under the forward measure[6] are assumed to be

$$df = af^{\beta}dW_1 \tag{3.26}$$

$$da = vadW_2 \tag{3.27}$$

$$dW_1 dW_2 = \rho dt \tag{3.28}$$

The authors present the argument that both market risk and volatility risk can be hedged using pieces of underlying and options respectively. So market price of risk is absent here. $a$ is a volatility like parameter and not the actual volatility (except when $\beta = 1$) but they are strongly related. The main advantage of this model is the existence of $\beta \in [0,1]$. As Hagan et al. (2002) and West (2005) show

$$\ln(\sigma(f,f)) = \ln a - (1-\beta)\ln(f) + non\ significant\ terms \tag{3.29}$$

---

[6] Forward measure is a pricing measure absolutely continuous with respect to the risk neutral measure and it uses the bond with maturity T. Let $Q$ and $Q^f$ the risk neutral and the forward measure respectively. Then the Radon-Nikodym derivative is given by

$$\frac{dQ_f}{dQ} = \exp\left\{\int_t^T r(s)ds\right\}$$

$\sigma(f,f)$ is the at the money Black's equivalent volatility, Black (1976). So $\beta$ is the variable that controls the dynamic movement of at the money implied volatility. In the Heston's stochastic volatility models that we presented is $\beta$ always equal to one so at the money implied volatility remains intact in the changes of the underlying in a very short time interval. Its drawback is that here the volatility process is not mean reverting.

The solution of the model is provided with derivation of singular perturbation expansion. The yielding formulas for a European call option are

$$U_{call} = \exp(-rt)\big(fN(d_1) - KN(d_2)\big) \tag{3.30}$$

$$\text{With } d_{1,2} = \frac{\ln\frac{f}{K} \pm \frac{1}{2}\sigma_B^2 t}{\sigma_B\sqrt{t}}$$

N is the cumulative normal distribution t is the time to maturity and the implied volatility $\sigma_B(K,f)$ is given by

$$\sigma_B(K,f) = \frac{a}{(fK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\ln^2(f/K) + \frac{(1-\beta)^4}{1920}\ln^4(f/K) + \dots\right\}}\left(\frac{z}{x(z)}\right)*$$

$$\left\{1 + \frac{(1-\beta)^2}{24}\frac{a^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta v a}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}v^2\right\}t + \dots \tag{3.31}$$

$$\text{Where } z = \frac{v}{a}(fK)^{(1-\beta)/2}\ln(f/K) \text{ and}$$

$$x(z) = \ln\left(\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right)$$

For at the money options implied volatility is simplified to

$$\sigma_B(f,f)=\frac{a}{f^{(1-\beta)}}\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{a^2}{f^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta va}{f^{(1-\beta)}}+\frac{2-3\rho^2}{24}v^2\right]t\right\} \tag{3.32}$$

This solution is the second advantage of SABR. Although it is not closed form, as a series expansion, it is though pretty simple for computational handling since it requires no numerical integration for its implementation.

The derivation of hedge ratios is also simple

$$\Delta_f=\frac{dU}{df}=\frac{dBS}{df}+\frac{dBS}{d\sigma_B}\frac{d\sigma_B}{df} \tag{3.33}$$

Vega hedge ratio can be calculated as

$$Vega=\frac{dU}{da}=\frac{dBS}{d\sigma_B}\frac{d\sigma_B}{da} \tag{3.34}$$

Minimum variance Delta is given by

$$\Delta_f^{mv}=\frac{\operatorname{cov}(dU,df)}{\operatorname{var}(df)}$$

$$=\frac{\langle U_f df+f_a da,df\rangle}{\langle df,df\rangle}=\frac{\langle U_f df,df\rangle}{\langle df,df\rangle}+\frac{\langle U_a da,df\rangle}{\langle df,df\rangle}=$$

$$U_f+U_a\frac{\langle da,df\rangle}{\langle df,df\rangle}=U_f+U_a\frac{\rho a^2 vf^{\beta}}{a^2 f^{2\beta}}=$$

$$delta + vega \frac{\rho v}{f^{\beta}} \tag{3.35}$$

# 4 Exponential Levy Models

## 4.1 Preliminaries

Jump models have been studied extensively in the literature. In most of the cases they are an expansion of Heston's stochastic volatility model by the addition of a periodical jump event arriving from a Poisson distribution. Once a Poisson event occurs the stock price flees from stochastic volatility dynamics and jumps, following at the most of times a normal distribution as described in Merton (1976). Due to their jump behavior they are called affine jump diffusions and their properties have been examined extensively in Dyffie et al. (2000).

While stochastic volatility models and jump diffusion models were created to capture the pattern of implied volatility smile, models that combine these two effects (Bates (1996)) were created to capture the implicit skewness in risk neutral distribution. The most significant evidence against of these models was presented by Bakshi et al. (1997). They found that hedging performance is not increasing comparing with stochastic volatility and additionally in some occasions is poorer than Black & Scholes.

An extension of this model has been examined in Chernov et al. (2003). They enriched stochastic volatility-jump diffusion model by adding a second volatility diffusion as well as jumps in the volatility process. In their paper they used only time series data from the returns of Dow Jones for a 37-year period. They conclude that changes in volatility are a substantial ingredient for the success of a model and that jumps in both the underlying and the volatility process are improving the modeling of equity returns distribution. Pointing the difficulties of the estimation of so large models, they found that the improvement of these models in prediction of equity returns distribution is not statistically significant against the stochastic volatility jump diffusion model.

Based on these results we will present only non-stochastic volatility jump diffusion models and pure jump models. These models have never been tested before for their hedging ability, while the inclusion of more complicated processes such as the process in Chernov et al. (2003) or Brownian motion with infinite activity jumps Carr et al. (2002) proved to be unnecessary.

### 4.2 Jump Diffusion Models (Merton (1976) and Kou (2002))

#### 4.2.1 Introduction to Jump Diffusion Models

Definition Levy Process: A stochastic process $X_t (t \geq 0)$ right continuous with left limits is a Levy process when the following conditions are satisfied

1. *X(0)=0*
2. Independent Increments: for every increasing sequence of times $t_0 ... t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent
3. Stationary Increments: the law of $X_{t+h} - X_t$ depends only on h.
4. $X$ is stochastic continuous : $\forall \varepsilon > 0, \lim_{h \to 0} P(| X_{t+h} - X_t | > \varepsilon) = 0$

Note that the fourth condition does not mean that $X_t$ is continuous. It means that given a time t the jump probability is zero so that jumps arrive only at random times.

Levy-Kinchin Representation: The characteristic function of a levy process $X_t$ is given by $e^{t\psi(\zeta)}$ where

$$\psi(z) = -\frac{1}{2}\sigma^2 z^2 + i\gamma z + \int_{-\infty}^{\infty} \exp(izx) - 1 - izx 1_{|x| \leq 1} \nu(dx). \qquad (4.1)$$

This representation is unique and generates a levy triplet *( $\gamma$ , $\sigma$, $\nu$).*

□

The Levy-Kinchin representation implies that every Levy process can be decomposed in a Brownian Motion with drift and a pure jump process evolving according to the jump measure v. The jump measure v is the expected numbers of jumps per unit time. This representation is uniquely determined, due to the uniqueness of the characteristic function. The jump diffusion models are a special case of exponential Levy models. The usual Brownian process plus a compound Poisson process give the stock price dynamics. In other words the stock follows a geometrical Brownian motion but there exists a probability for the stock to jump, finite times per year, according to the compound Poisson process.

Stock price dynamics are given by

$$\frac{dS}{S} = adt + \sigma dB_t + (y_t - 1)dq_t \tag{4.2}$$

Where $q$ is a Poisson process with parameter $\lambda \in R^+$ such that:

$$\Pr\{dq = 1\} = \lambda dt$$
$$\Pr\{dq = 0\} = 1 - \lambda dt$$
$$\Pr\{dq \geq 2\} = 0$$

In jump diffusions when stock jumps it moves instantaneously from $S$ to $Sy$ according to the probability measure $f(y)$ (in jump diffusions $f(y)$ is actually a probability measure i.e. $\int_R v(dy) = 1$ ). So the Levy measure for the compound jump diffusions is determined by multiplying the Poisson intensity $\lambda$ with the probabity measure $f$. Thus

$$v(y) = \lambda \cdot f(y)$$

Merton (1976) assumes $y$ is Lognormal with $\ln(y) \sim Normal(\mu, \delta)$.

Kou (2002) introduced double exponential distribution where $\ln(y) \sim$ *Double-Exponenential*, which is given by the following density

$$f(y) = p \cdot n_1 \exp(-n_1 y) 1_{y \geq 0} + q \cdot n_2 \exp(n_2 y) 1_{y<0} \text{ where } p+q=1$$

Let $X(t) = \ln(S(t))$ then by the generalized Ito's Lemma [Cont and Tankof (2004)]

$$dX = aS\frac{dX}{dS}dt + \frac{1}{2}\sigma^2 S^2 \frac{d^2X}{dS^2}dt + \sigma S\frac{dX}{dS}dW + [\ln(S^- + \Delta S) - \ln(S^-)]dq_t \Rightarrow$$

$$dX = adt - \frac{1}{2}\sigma^2 dt + \sigma dW + [\ln(S^- + (y_t - 1)S^-) - \ln(S^-)]dq_t \Rightarrow$$

$$dX = adt - \frac{1}{2}\sigma^2 dt + \sigma dW + [\ln(S^- y_t) - \ln(S^-)]dq_t \Rightarrow$$

$$dX = (a - \frac{1}{2}\sigma^2)dt + \sigma dW + \ln(y_t)dq_t \Rightarrow$$

$$S_t = S_0 \exp\{(a - \frac{1}{2}\sigma^2)dt + \sigma dW\}\prod_{i=1}^{q(t)} y_i \tag{4.3}$$

So (4.3) is the solution of (4.2)

Proposition: In Merton's model for every $c \in Z$

$$\int_R e^{cy} f(dy) = \exp(c\mu + \frac{c^2\delta^2}{2}) \tag{4.4}$$

Proof:

$$\int_R e^{cy} f(dy) = \int_R e^{cy} f(y)dy = \frac{1}{\sqrt{2\pi\delta^2}}\int_R \exp(cy - \frac{(y-\mu)^2}{2\delta^2})dy \text{ by setting } z = \frac{y-\mu}{\delta}$$

$$= \frac{1}{\sqrt{2\pi}}\int_R \exp(c(z\delta + \mu) - \frac{z^2}{2})dz = \frac{\exp(c\mu)}{\sqrt{2\pi}}\int_R \exp(cz\delta - \frac{z^2}{2})dz = \exp(c\mu + \frac{c^2\delta^2}{2})$$

□

Proposition: In Kou's model for every $c \in Z$ with $\operatorname{Re}(c) < n_1$ .

$$\int_R e^{cy} f(dy) = p \cdot n_1 \frac{1}{n_1 - c} + q \cdot n_2 \frac{1}{n_2 + c} \tag{4.5}$$

Proof:

$$\int_R e^{cy} f(dy) = \int_R e^{cy} f(y) dy = p \cdot n_1 \int_0^\infty e^{cy} \exp(-n_1 y) + q \cdot n_2 \int_{-\infty}^0 e^{cy} \exp(n_2 y) =$$

$$p \cdot n_1 \frac{1}{n_1 - c} + q \cdot n_2 \frac{1}{n_2 + c}$$

□

Using these propositions we can calculate the expected return from a single jump per time unit as

$$k_{merton} = E[e^y - 1] = \int_R (e^y - 1) f(dy) = \exp(\mu + \frac{\delta^2}{2}) - 1 \tag{4.6}$$

$$k_{kou} = E[e^y - 1] = \int_R (e^y - 1) f(dy) = p \cdot n_1 \frac{1}{n_1 - 1} + q \cdot n_2 \frac{1}{n_2 + 1} - 1 \tag{4.7}$$

### 4.2.2 Partial-Integrodifferential Equation Derivation

Let $U(S,t) \in C^{2,1}$ a derivative written on this stock. Then by Generalized Ito's Lemma

$$dU = U_t dt + aSU_S dt + \frac{\sigma^2}{2} S^2 U_{SS} dt + \sigma U_S dW + [U(S^- + \Delta S) - U(S^-)] \Rightarrow$$

$$dU = U_t dt + aSU_S dt + \frac{\sigma^2}{2} S^2 U_{SS} dt + \sigma U_S dW + [U(Sy) - U(S)]dq \tag{4.8}$$

Now we form a portfolio consisting of the derivative minus $\Delta$ stocks. Then

$$dP = U_t dt + aSU_S dt + \frac{\sigma^2}{2} S^2 U_{SS} dt + \sigma U_S dW + [U(Sy) - U(S)]dq - \Delta[aSdt + \sigma SdB_t + S(y_t - 1)dq_t]$$

By selecting $\Delta = U_s$

$$dP = U_t dt + \frac{\sigma^2}{2} S^2 U_{SS} dt + [U(Sy) - U(S) - U_s(y_t - 1)S]dq$$

Clearly the Brownian driven part of risk has been eliminated. Merton assumes that the jump part of risk is diversifiable so no risk premium should be rewarded for holding this portfolio.

Kou (2002) uses utility maximization approach, to determine the risk neutral parameters. For certain classes of utility functions proofs that the rational expectations equilibrium price of an option is given by taking expectations under an EMM Q under which $S$ remains double exponential jump-diffusion.

Both cases conclude that

$$E[dp] = rPdt \Rightarrow$$

$$U_t dt + \frac{\sigma^2}{2} S^2 U_{SS} dt + E[U(Sy) - U(S) - U_s(y_t - 1)S]E[dq] = r(U - U_s S)dt \Rightarrow$$

$$U_t + \frac{\sigma^2}{2} S^2 U_{SS} dt + (r - \lambda k)SU_s - rU + \lambda \int_R U(Sy) - U(S) f(dy) = 0 \tag{4.9}$$

Change of coordinates

By changing the space variable to $X(t)=\ln(S(t))$ and the time variable to $\tau = T-t$ PIDE (4.9) becomes

$$U_\tau = \frac{\sigma^2}{2}U_{XX} + (r - \frac{\sigma^2}{2} - \lambda k)U_X - rU + \lambda \int_R (U(x+y) - U(x)) f(dy) \tag{4.10}$$

With the boundary condition $U(x,0) = (e^x - K)^+$ for a European call option

### 4.2.3 The Formula

In the Appendix 2 we construct the solution of (4.10). The solution scheme is similar to(3.12). By using the characteristic functions inversion formula (3.10) we obtain that the price of the call is under both jump diffusion models.

The price of a call option is $S_0 P_1 - K\exp(-r\tau)P_2$

We substitute in (3.21)

$$f_i = \exp(C_i + i\varphi x) \tag{4.11}$$

In Merton's model

$$C_1 = \tau(i\varphi(r + \frac{\sigma^2}{2} - \lambda k) - \varphi^2 \frac{\sigma^2}{2} - \lambda - \lambda k + \lambda \exp((1+i\varphi)\mu + \frac{(1+i\varphi)^2 \delta^2}{2})) \tag{4.12}$$

$$C_2 = \tau(i\varphi(r - \frac{\sigma^2}{2} - \lambda k) - \varphi^2 \frac{\sigma^2}{2} - \lambda + \lambda \exp(i\varphi\mu + \frac{(i\varphi)^2 \delta^2}{2})) \tag{4.13}$$

In Kou's model

$$C_1 = \tau(i\varphi(r+\frac{\sigma^2}{2}-\lambda k)-\varphi^2\frac{\sigma^2}{2}-\lambda-\lambda k+\lambda(p\cdot n_1\frac{1}{n_1-(1+i\varphi)}+q\cdot n_2\frac{1}{n_2+(1+i\varphi)})) \tag{4.14}$$

$$C_2 = \tau(i\varphi(r-\frac{\sigma^2}{2}-\lambda k)-\varphi^2\frac{\sigma^2}{2}-\lambda+\lambda(p\cdot n_1\frac{1}{n_1-i\varphi}+q\cdot n_2\frac{1}{n_2+i\varphi})) \tag{4.15}$$

### 4.2.4 The Hedge Ratios

The Delta-hedge ratio is the value of $P_1$ as in Heston's model, while the minimum variance hedge ratio is given by the following proposition.

*Minimum Variance Delta for Jump Diffusions*: Let $S_t = S_0\exp(L_t)$ a jump diffusion model with Levy triplet *(γ, σ, ν)* then minimum variance Delta (in logarithmic terms) is given by

$$\Delta_{mv} = \frac{\sigma^2 U_S + \frac{1}{S}\int (e^z-1)[U(Se^z)-U(S)]v(dz)}{\sigma^2+\int (e^z-1)^2 v(dz)} \tag{4.16}$$

Proof:

By the change of variable $z=\ln(y)$, (4.2) and (4.8) can be written as:

$$\frac{dS}{S} = adt + \sigma dB_t + (e^z-1)dq_t \tag{4.17}$$

$$dU = U_t dt + aSU_S dt + \frac{\sigma^2}{2}S^2U_{SS}dt + \sigma U_S dW + [U(Se^z)-U(S)]dq \tag{4.18}$$

With the use of (4.17) and (4.18) we have to calculate the two terms of (2.5)

$$\mathrm{var}(dS) = \mathrm{var}(S(adt + \sigma dB_t + (e^z - 1)dq_t)) =$$
$$S^2\,\mathrm{var}(\sigma dB_t) + S^2\,\mathrm{var}((e^z - 1)dv) = S^2\sigma^2 + \int (e^z - 1)^2 \nu(dz) \tag{4.19}$$

From (4.2) and (4.8) we have

$$\mathrm{cov}(dU, dS) = \mathrm{cov}(\sigma U_S dW + [U(Se^z) - U(S)]dq, S(\sigma dB_t + (e^z - 1)dq)) =$$
$$S^2\sigma^2 U_S + S\int (e^z - 1)[U(Se^z) - U(S)]\nu(dz) \tag{4.20}$$

Substituting (4.20) and (4.16)on (2.5) we get (4.16)

□

We derive the formulas for minimum variance Delta for these models in Appendix 3.

## 4.3 CGMY

### 4.3.1 CGMY Model: Description

CGMY model is named by the initials of its developers, Carr et al. (2002). It is a non Brownian Levy model with a levy triplet *( γ , 0, v)* . The Levy measure is expressed as

$$v(x) = \frac{C}{|x|^{1+Y}}\left(\exp(-G|x|)1_{x<0} + \exp(-M|x|)1_{x>0}\right) \tag{4.21}$$

where its parameters must follow the restrictions

$$C > 0, G \geq 0, M \geq 0, Y < 2.$$

The strong difference between this model with the jump diffusions, is that the Levy measure is not always integrable. Due to the infinite number jumps assumption it's possible that:

$$\lim_{a\to\infty} \int_{-a}^{a} v(dx) \to \infty \tag{4.22}$$

The convergence of the integral in (4.22) is controlled by the Y parameter. When Y is lesser than zero this integral converges in $R^{+}$ and the process is of finite activity. Otherwise this integral does not converges and the number of jumps is infinite, thus the process is of infinite activity.

This model is an expansion of Variance Gamma, presented in Madan and Seneta (1991). In Madan et al. (1998) was interpreted as the difference of two independent gamma processes. Variance Gamma process is nested on CGMY since in the first one the Y parameter is always constrained to zero. The name Variance Gamma comes from the fact that it is a Brownian motion with drift where the time is changed by a gamma process. This model as presented by Madan et al. (1998) provides explicit fitting in the volatility smile.

The model's parameters play an important role in capturing various aspects of the stochastic process under study. The parameter C is a measure of the level of market's activity. The parameters *G* and *M*, respectively, control the right and left rate of exponential decay on of the Levy density, leading to skewed distributions when they are unequal. For $G < M$, the left tail of the implied distribution is heavier than the right tail, which is consistent with the risk-neutral distribution typically implied from option prices. Variance Gamma provides explicit fitting in returns distributions and this arises from the fact that is included in the category of generalized hyperbolic distributions described in Raible (2000).

Proposition [4.3]: For every $c \in Z$ with $\mathrm{Re}(c) < \min(G, M)$

$$\int_{R\setminus\{0\}} (exp(cx)-1)v(dx) = C\cdot\Gamma(-Y)\left((M-c)^Y - M^Y + (G+c)^Y - G^Y\right)$$

Proof:

$$\int_{R\setminus\{0\}} (exp(cx)-1)v(dx) =$$

$$\int_{-\infty}^{0} (exp(cx)-1)C\exp(Gx)(-x)^{-1-Y}\,dx + \int_{0}^{\infty} (exp(cx)-1)C\exp(-Mx)(x)^{-1-Y}\,dx =$$

$$\int_{0}^{\infty} (exp(-cz)-1)C\exp(-Gz)(z)^{-1-Y}\,dz + \int_{0}^{\infty} (exp(cx)-1)C\exp(-Mx)(x)^{-1-Y}\,dx =$$

$$C[\int_{0}^{\infty} (exp(-cz)\exp(-Gz)(z)^{-1-Y}\,dz - \int_{0}^{\infty} \exp(-Gz)(z)^{-1-Y}\,dz] +$$

$$C[\int_{0}^{\infty} exp(cx)\exp(-Mx)(x)^{-1-Y}\,dx - \int_{0}^{\infty} \exp(-Mx)(x)^{-1-Y}\,dx] =$$

$$C\Gamma(-Y)\left((G+c)^Y - G^Y + (M-c)^Y - M^Y\right)$$

$\Gamma$ stands for gamma function

$$\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}dt$$

□

So the average jump return per unit of time is

$$k=\int_{R\setminus\{0\}}(exp(x)-1)v(dx)=C\cdot\Gamma(-Y)\left((G+1)^Y-G^Y+(M-1)^Y-M^Y\right)\quad(4.23)$$

### 4.3.2 Esscher Transform

Following Theorem [1.2], an Equivalent Martingale Measure (Q) must discovered under which relative asset prices are martingales. In stochastic volatility markets the choice of Q could be done by using Girsanov's theorem. In Levy markets this measure change is not unique. The most convenient way to be done with the use of Esscher transform, while other measure changes do exist.

*Definition of Esscher Transform*: Let a Levy process $X_t$ defined in a filtered probability space $(\Omega, F, F_t, P)$. We call Esscher transform the change from measure P to a equivalent measure Q with density defined by Radon-Nikodym derivate $Z_t=\left.\frac{dQ}{dP}\right|_{F_t}$ and

$$Q=\frac{\exp(\theta X_t)}{E_P[\exp(\theta X_t)]}$$

Note that $E_P[\exp(\theta X_t)]$ is the Laplace transform of $X_t$, or its moment generating function (mgf).

□

Proposition [4.4]: Let $X_t$ be a Levy process with Levy triplet *(γ, σ, ν)* under P and Q a measure defined by Esscher transform. Then $X_t$ is Levy under Q with Levy triplet *(γ*, σ*, ν*)* and

$$\gamma^* = \gamma + \sigma^2\theta + \int_{-1}^{1}(\exp(\theta x) - 1)\nu(dx) \tag{4.24}$$

$$\sigma^* = \sigma \tag{4.25}$$

$$\nu^* = \exp(\theta x)\nu(dx) \tag{4.26}$$

Proof : Cont & Tankof (2004)

□

The use of Esscher transform is essential for Levy processes. In case that a Brownian part is included, it is left intact as in Girsanov's Theorem. Furthermore the statistical CGMY measure remains CGMY. The rate of activity C and the parameter Y, that controls the structure of asset returns, remain intact. Only the exponential decay parameters G and M, which control the skewness of implied distribution, are changing. The parameter θ is uniquely determined by (4.24).

### 4.3.3 The Formula

The derivation of the main PIDE for CGMY is not as easy as in jump-diffusions. The main reason is that in the case that the process is of infinite variation it includes an exception of Generalized Ito's Lemma.

PIDE for European Options [*Cont& Tankof (2004)*] : let $S_t = S_0 \exp(rt + L_t)$ if the Levy process verifies one of the following

$$\sigma > 0 \text{ or } \exists \beta \in (0,2) : \liminf_{\varepsilon \to 0} \varepsilon^{-\beta} \int_{-\varepsilon}^{\varepsilon} |x|^2 \nu(dx) > 0$$

The value of a European option with terminal payoff $H(S_T)$ is given by

$$C(t,S) = E(H(S_T) / S_t = S)$$

Is continuous and $C^{1,2}$ on $]0,T[\times(0,\infty)$ and verifies the following PIDE

$$C_t + rSC_S + \frac{\sigma^2 S^2}{2} C_{SS} - rC + \int [C(t,Se^y) - C(t,S) - S(e^y - 1)C_S] v(dy) = 0 \tag{4.27}$$

With terminal condition $C(T,S) = H(S)$

□

Having discussed extensively Fourier inversion techniques, here only the results will be presented. Fourier techniques are not well defined for (4.27). The main reason is that Kolmogorov Backward Equations that were used for the transition densities in(3.14), have not been proved for jump models. An effort to this direction has been made in Nualart and Schoutens (2001), but their work is severely criticized in Cont and Voltchkova (2005). We strongly believe that this is theoretical result, which remains to be proved. We validated our results using the traditional Fast Fourier Transform of Carr and Madan (1999). We preferred this solution for the robust derivation of the hedge ratios.

Like (3.22) the price of a European call option is given by

$$S_0 P_1(x,\tau) - Ke^{-r\tau} P_2(x,\tau)$$

To obtain $P_1$ and $P_2$ we substitute in (3.21)

$$f_i = \exp(C_i + i\varphi x) \tag{4.28}$$

Where

$$C_i = \tau * \left((r - \omega_i k) + C \cdot \Gamma(-Y)\left((M - \omega_i)^Y - M^Y + (G + \omega_i)^Y - G^Y\right)\right) \text{ for i=1,2}$$

And

$$\omega_1 = 1 + i\varphi \ , \ \omega_2 = i\varphi$$

### 4.3.4 The Hedge Ratios

The Delta-hedge ratio (as a right limit) is the value of $P_1$.

Under the assumptions of (4.27) minimum variance hedge ratio is given by (4.16) and following the same procedure as in Appendix 3 it can be shown that

$$\Delta_{mv} = \frac{-\frac{1}{S}U(S) + \frac{1}{S}\left( S\{\frac{1}{2}(B+k) + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp(-i\varphi \ln(\mathrm{K})) f_1(x) m_1(\varphi)}{i\varphi}\right] d\varphi\} \right)}{B} - \frac{1}{S}\frac{Ke^{-rt}\{\frac{1}{2}k + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp(-i\varphi \ln(\mathrm{K})) f_2(x) m_2(\varphi)}{i\varphi}\right] d\varphi\}}{B}$$

Where

$$B = C \cdot \Gamma(-Y)\left((M-2)^Y - (M-1)^Y + (G)^Y - (G+1)^Y\right) - k$$

$$m_\iota = C \cdot \Gamma(-Y)\left((M-\omega_\iota-1)^Y - (M-\omega_\iota)^Y + (G+\omega_\iota+1)^Y - (G+\omega_\iota)^Y\right)$$

# 5 Scale Invariant Models

The work of Alexander and Noguera (2007) is of highly importance for Delta-hedging strategy. They prove that under daily calibration all scale invariant models yield the same sensitivities. The family of scale invariant models includes a large range of models as will be explained below. Their theorems for Scale Invariant Models are presented below.

Definition: A process $S_t$ is scale invariant iff the marginal distribution of $X_t = \frac{S_t}{S_0}$ is independent of $S_0$ for every t>0.

$$\frac{df(x)}{dS_0} = 0 \qquad \forall t > 0$$

Where $$f(x) = \frac{d}{dx} prob\{X_t < x\}$$

Theorem 5.1[*Alexander and Nogueira 2007*]: A process $S_t$ is scale invariant if it is a semi-martingale and can be written in the form:

$$\frac{dS}{S} = \Theta' d\Lambda$$

Where $\Theta$ is a vector of deterministic coefficients and $\Lambda$ is a semi-martingale[7] measure.

---

[7] A process S is a semi-martingale if it is right continuous with left limits and can be decomposed as the sum of a local martingale and an *F*-adapted right continuous with left limits finite-variation process.

Theorem [5.2]: Suppose a claim-payoff is homogenous of degree k and the process $S_t$ is scale invariant. Then all partial derivatives of the claim with respect to S at any time $t<T$ are given by linear combinations of g = g(T,K;t,S) and its partial derivatives with respect to K, and in particular

$$g_S = S^{-1}(kg - K' g_K)$$

Proof:

$$\begin{aligned} & g(T,uK;t,uS) = u^k g(T,K;t,S) \Rightarrow \\ & \frac{d(g(T,uK;t,uS))}{du} = \frac{d(u^k g(T,K;t,S))}{du} \Rightarrow \\ & uSg_S(T,K;t,S) + uK' g_K(T,K;t,S) = ku^{k-1} g(T,K;t,S) \end{aligned}$$

By setting u=1 this produces $g_S = S^{-1}(kg - K' g_K)$

□

Theorem [*Alexander and Nogueira 2007*]: Let $\theta(T,K;t,S)$ the implied volatility of a standard European option. If the process for $S_t$ is scale invariant then

$$\theta(T,K;t,S) = \theta(T,uK;t,uS)$$

And $$\theta_S(T,K;t,S) = -(\frac{K}{S})\theta_K(T,K;t,S)$$

□

These results are of great importance for the Delta-hedging strategy. If two scale invariant models are calibrated to the same market these models will produce the same sensitivities for plain vanilla European options, so they will provide the same Delta-hedging strategy. The potential differences among these models will exist only due to calibration and parameterization misspecifications.

There are two ways to identify a scale invariant model.

The first is the theorem (5.2). This theorem is particularly useful for stochastic volatility models.
Heston's model can be written as

$$\frac{dS}{S} = rdt + \sqrt{v}(\rho dW_1 + \sqrt{1-\rho^2}\, dW_2) \text{ with } \operatorname{cov}(dW_1, dW_2) = 0$$

This directly means that it is scale invariant. This identification includes the Kou and Merton model.

This identification is not directly applicable in infinite activity models. In these models due to the right continuous property of the process we have that $S^+ \neq S^-$ so the martingale representation of the model is written as

$$\frac{dS}{S^-} = \gamma dt + \int (e^y - 1)\tilde{\mathcal{J}}(dy, dt)$$

Where $\tilde{\mathcal{J}}$ denotes the compensated ump measure of the Levy process (Cont and Tankov [2004]). This slight modification allows an infinite activity model to be regarded as scale invariant, as far one can accept a right derivative as a hedge ratio.

A second more robust approach is by the returns ($x_t = \ln(S_t / S_0)$) characteristic function[8] .If the characteristic function is independent of the initial state ($S_0$) then the model is scale invariant. The proof for this arises from the uniqueness of characteristic function. Using (3.10) and the fact that the characteristic function is independent of $S_0$ we obtain a returns function independent of the current price state. This identification includes CGMY and a large class of Levy processes (see Schoutens (2003) for a review).

The only non-scale invariant model that we present is SABR. This can be seen with two ways. The first is the theorem (5.2). Since SABR is written as

$$\frac{dS}{S} = rdt + aS^{\beta-1}e^{\beta \cdot r(T-t)}dW_1$$

so $\frac{dS}{S}$ is not independent of S.

The second is its implied volatility behavior. Combining the result of theorem (5.3), we obtain that in scale invariant models the sensitivities of at the money implied volatilities with respect to the underlying, are independent of the level of the underlying. This doesn't hold for SABR and this is directly observed by the relationship(3.29).

Delta hedging of market and volatility risk with scale invariant models will produce the same P&L results as far as the models can be calibrated. This is a very powerful result for such kind of hedging. As it is argued in Alexander and Nogueira, replication type hedging is not the most robust when sources of market incompleteness such as stochastic volatility and jumps exist. Hedging should become an approximation problem and minimum variance hedging out performs Delta-hedging. Unfortunately these results cannot be extended in minimum variance hedging. The main reason is the denominator of(2.5). This is a risk neutral variance which is model dependent under daily calibration.

8

This characteristic function is slightly different than the one we derived earlier. It is the characteristic function of returns distribution under the forward measure.

$$x = \ln\left(\frac{S_t \exp(-r\cdot(T-t))}{S_0}\right)$$

It satisfies the same PDE with $P_2$ excluding the drift term, because they are based on a future time exercise probability, but here the boundary condition is

$$\lim_{a\to 0}\int_a^a e^{i\varphi x}\delta(x-x_0)dx = e^{i\varphi x_0} = 1 \text{ because } \ln(x_0) = 1$$

So the relation between this characteristic function $\varphi$ and these we derived earlier $f_2$ for $P_2$ is

$$\varphi = f_2 * \exp\{-i\varphi[\ln(S_0) + r(T-t)]\}$$

Applying this relation to the scale invariant models we presented, we conclude to characteristic functions independent of $S_0$.

# 6 Minimum Variance Hedging with Options

## 6.1 The Use of Options as Hedging Instruments

Hedging with the underlying is a common tactic in both practitioner's applications and academic researches. This tactic is popular because it can be carried out through direct trading of the underlying or the relative future's contract, which are both the most liquid assets. This can be done by forming Delta neutral portfolios or by using the minimum variance hedge ratio that we can obtain from(2.5).

Another approach is hedging using options. Options are trading and liquid assets so they can be included in a hedging scheme. Moreover they are characterized by interesting features such as convexity and volatility dependence, so their incorporation should improve hedging performance, as has been verified in Bakshi et al. (1997). Actually in stochastic volatility models the trading of options is a fundamental assumption under which the models are priced. In these models continuous trading, in pieces of underlying and options respectively creates a hedging strategy, by which we extracted the market price of risk in section 3.1.2.

Continuous trading is an assumption that holds as a limiting case for the pricing of options. Unfortunately it is very demanding for hedging purposes. Continuous trading has significant costs and cannot be carried out except of fixed times within a trading day. This is a fundamental shortcoming for stochastic volatility models, the pricing of which is based on this assumption.

In models that include jumps other assumptions have been carried out. Merton assumes that jump risk is differentiable and can be absorbed in a well diversified portfolio. This statement doesn't hold in equity markets because even index portfolios (Dow Jones fig. 2.4.) produce strong jump behavior. Another approach is the utility maximization for Kou's model and the Esscher transform for CGMY. Under these two approaches an agreement for the risk premium is made. All of these assumptions leave jump risk uncovered. A realization of a rare jump event could generate great losses for the writer of the option.

In models with jumps, current literature has been narrowed only in risk premium investigation or single instrument minimum variance hedging like (2.5). This is mainly because jumps don't allow replication type hedging. They arrive at random times so the usual Delta hedge ratio represents the right limit of the derivative with respect to the underlying, while the left limit is unknown. This makes the traditional Delta hedging a good hedging scheme, only conditional on no jump arrival. Naturally this hedging fails totally in models that move essentially through jumps like infinity activity Levy and it is highly risky for jump diffusions.

## 6.2 The Extended Formula for Minimum Variance Hedging

The main idea of this chapter is to enrich single instrument minimum variance hedging by allowing options as hedging instruments. As far as we know this idea has been tested only in Cont et al. (2005). Following the discussion in Chapter 2 by adopting the following strategy we adopt the following assumptions. Market incompleteness by definition does not allow perfect replication. Continuous hedging is a limited and theoretical case that actually is very costly to be carried out. Trading of derivatives involves sources of uncertainty that cannot be fully hedged. Instead of this we accept that an error is natural to be occurred as a consequence of market incompleteness, so we try to minimize the variance of this error.

Similarly to (2.4) here we will minimize:

$$E[(dC-\delta^0 dS-\delta^i dO^i)^2] \tag{6.1}$$

Where $C$ is the target option and $O^i$ $i=1,n$ are tradable market options. It is easy to verify that the optimal hedging strategy $\delta:[\delta^0,\delta^1,...,\delta^n]$ is given by

$$\delta = A^{-1}b \tag{6.2}$$

$$A = \begin{bmatrix} \mathrm{var}(dS) & \mathrm{cov}(dS, dO^1) \ldots \mathrm{cov}(dS, dO^n) \\ \mathrm{cov}(dO^1, dS) & \mathrm{var}(dO^1) \quad \ldots \mathrm{cov}(dO^1, dO^n) \\ . & . \quad\quad\quad . \\ . & . \quad\quad\quad . \\ . & . \quad\quad\quad . \\ \mathrm{cov}(dO^n, dS) & \mathrm{cov}(dO^n, dO^2) \ldots \mathrm{var}(dO^n) \end{bmatrix}$$

$$b = \begin{bmatrix} \mathrm{cov}(dC, dS) \\ \mathrm{cov}(dC, dO^1) \\ . \\ . \\ . \\ \mathrm{cov}(dC, dO^n) \end{bmatrix}$$

For conducting this strategy a decision about the measure, under which the mean in (6.1) will be calculated, has to be made. Under the real measure (the one obtained from time series estimation) this variance is more realistic as a probabilistic representation. But since this strategy involves trading of options, this measure should reflect the preferences of market participants about future uncertainty. As pointed out in Ait-Sahalia and Lo (1998), this can be achieved only by selecting the risk neutral parameters (those obtained from option calibration), because they reflect the market risk premiums. The drawback using this measure is that time stable parameters are required, an issue that is strongly questioned in Hagan et al.(2002) and Wuang(2007).

### 6.3 How Many Options Should Be Used?

We will use this strategy allowing only one option as a hedging instrument. This is because under the risk neutral measure of a pure jump model and accepting the assumptions of(4.27), the option price evolves as:

$$dO = O_t + (r-k)O_S + \int v(dy)[O(t, Se^y) - O(t,S)] \Rightarrow$$

$$dO = O_t + (r-k)O_S + SO_S \int (e^y - 1)v(dy) \quad (6.3)$$

The last equation implies that the martingale component of an option is uniquely determined by its Delta. Using the fact that (6.3) is linear for Delta and by the additive property of an option's Delta, we conclude that adding a second instrument option in (6.1) one can create only a linear transformation between the Deltas, leaving the variance process intact. Since we have included minimum variance hedging with options only to improve the performance of pure jump models, we will test this strategy using only one option.

The result that adding a second option in (6.1) leaves its hedging ability intact, steams from the following result. In pure jump models only market risk exist, while assuming time stable parameters there is not an indicative measure about future market variance, which is modeled with stochastic volatility in Brownian markets. Suppose that one wishes to construct volatility hedging with(6.1). For volatility hedging an at the money straddle instrument could be used (long call and put with the same strike). This synthetic product (SR) is Delta neutral and is constructed by the most liquid options in the market, so is the most plausible candidate for a volatility derivative. Under pure jump dynamics and using the fact that this is a zero delta product (6.3) is written as:

$$d(SR) = (SR)_t \tag{6.4}$$

So the straddle product evolves deterministically for a short time range. By adding this product in (6.1), would not result to any variance reduction. This result is significant for the hedging behavior of pure jump models as will be discussed in Chapter 8.

# 7 Data and Calibration Procedure

## 7.1 Data Description

Our data consisted from daily closing prices of S&P 100 European style options for the time interval 16/10/2007 to 18/4/2008. Following Linaras and Skiadopoulos (2005) we used as option price the bid-ask average. In this analysis we used only third-Friday options. Options with maturities greater than three months or less than five trading days were discarded as well as options with premiums less than 0.5 units. Moreover we removed options that exhibit unrealistic implied volatility patterns. US Libor rates were obtained from DataStream and maturity fitting was done by linear interpolation. Daily estimated dividend yield was obtained by inverting put-call parity using at the money options. Standard arbitrage bounds were checked and the options that violated them were discarded. Following Alexander and Nogueira (2007), only options within the range of 10% from the price of S&P 100 were calibrated, approximately 23 per day.

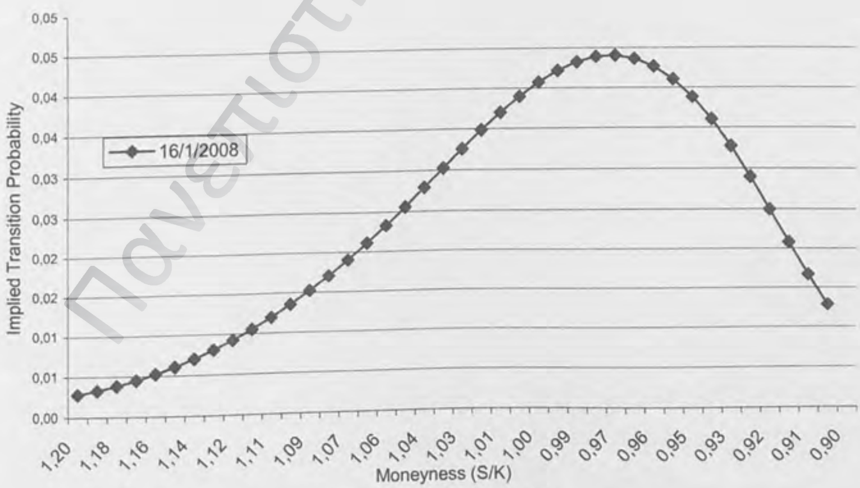


fig 7.1 Implied Distribution obtained by 1 month call options

Contrary to other researches that used data sets from the early 90's, we apply our results to whole new data set. We selected October 2007 to March 2008. This selection was done because this market was highly volatile and many crash days are included. Since these days are the most valuable for testing the hedging performance of a model or scheme, selecting this period would focus this problem, while for other data sets a crash day would be considered just as an extreme value. In this period S&P 100 average daily log return was -0.067% and their standard deviation was 1.34%, in contrast with the previous six months that these numbers were 0.045% and 0.89% respectively. The dominated effect of this data set was the implied skewness. This feature is presented clearly in the figure (7.1), where is drawn the implied distribution from one month to maturity call options at and 16/01/2008. The shape of implied distribution signals the effect of negative skewness in a high volatile market, due to the increase of risk premiums.

The probabilities in the figure (7.1) were extracted using the calibrated distribution of an option to become in the money ($P_2$), as obtained with CGMY model. The model free representation of Breeden and Litzenberger (1978) failed to produce realistic results.

### 7.2 Calibration Procedure

Having established each model's hedge ratios and the options data set, the last step is to obtain each model's parameters. For this, two approaches are widely acceptable. The first is to use the parameter's that are inferred from the options, from now on the calibration procedure. The second is to obtain these parameters from the historical data set of the underlying, which is the only observable process. This is called the estimation procedure and is carried out through maximizing the maximum likelihood estimation function i.e:

$$\Theta = \arg\max_{\theta}\left(L(S_0, S_1 \ldots S_N; \theta)\right)$$

These two approaches provide quite different results. More specifically the implied skewness and kyrtosis of the returns distribution are quite different and they are stronger in the distribution obtained from option prices. Aït-Sahalia and Lo (1998) explain that this is due to the fact that option prices reflect the investors risk premiums. Bates (2000) explains that maximum likelihood estimation needs a pretty large amount of data, including intraday observations to be carried out correctly. Honore (1998) finds that the standard ML procedure is invalid for jump-diffusion models because the log-return is equivalent to a discrete mixture of N normally distributed variables, with N goes to infinity. Moreover he reports small amounts of standard deviation. Javaheri(2005) using Monte Carlo simulated data finds that for small time interval observations MLE function does not converge to right parameter set. He also created trading strategies of the higher moments wishing to speculate in any arbitrage opportunities that could be created from the inconsistency between the statistical and the calibrated measure. His findings are conclude that no strategy creates systematic profits. The same findings have been presented in Aït-Sahalia et al.(2001). Moreover market incompleteness implies that the statistical measure is insufficient for option pricing, which is based upon a determination of a risk premium. Due to these findings we selected the calibration approach.

Following Bakshi et al (1997) we selected to minimize the mean square error of market option prices and model option prices for each given day of the sample.

$$e_t(\theta) = C'(\theta; t, \mathrm{K}) - C(t, \mathrm{K})$$
$$\Phi = \arg\min_\theta \left( \sum (e_t(\theta))^2 \right) \tag{7.1}$$

Christoffersen and Jacobs (2004) test different error functions for the calibration of options. They conclude that an error function should meet evaluation criteria. We selected this error function because it gives more weight to in the money options that contain the most information due to their tight spreads. Following Kilin (2006) for minimizing the sum of squares error we used non linear least squares (Levenberg-Marquardt). The days that Levenberg-Marquardt converged to a local minimum, Differential Evolution[9] (Price, Storn and Lampinen (2007)) gave a new initial value. Next we describe the pricing performance of each model.

In the above table we present the pricing performance for all the models as well as the average parameters obtained for each model.

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[9] Differential Evolution is a global minimization algorithm that was created in Berkeley 1997. An open source code can be downloaded from:

http://www.icsi.berkeley.edu/~storn/code.html

**Table 7.1**

For each day in the sample we minimize (7.1) to obtain a parameter set for each model. The average values of the parameters as well as the average sum of square errors and parameters for each model are reported. In parenthesis we report the standard deviation of each variable.

| Model | | | | | | **SSE** (sum of square error) |
|---|---|---|---|---|---|---|
| **HESTON** | κ | θ | σ | ρ | V | |
| | 7 (0,72) | 0,09 (0,045) | 1,45 (0.88) | -0,83 (0,135) | 0,05 (0,026) | 0,37 (0,3) |
| **CGMY** | C | G | M | Y | | |
| | 34,106 (3,55) | 26,7 (8,1) | 132,5 (58.) | -0,02 (0,2) | | 0,48 (0,45) |
| **SABR** | a | ρ | v | | | |
| | 1,03 (0,15) | -0,85 (0,144) | 2,13 (1.) | | | 0,47 (0,37) |

In figure 7.2 we present the sum of square errors for each day in the sample.

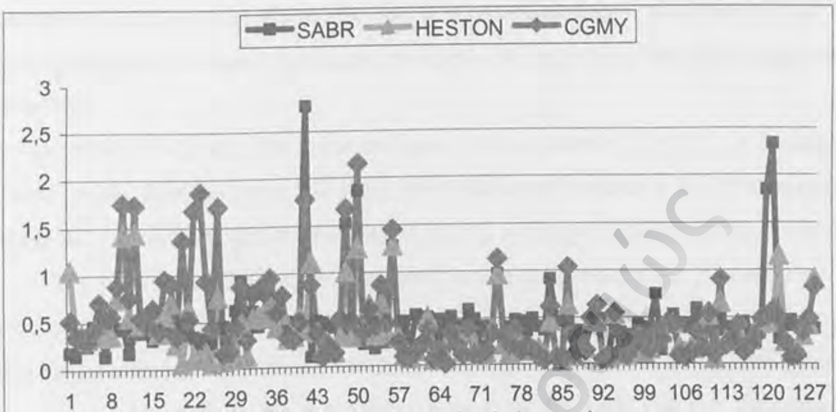


fig 7.2 Sum of Square Errors in the calibration procedure for each day in the sample

The jump diffusions were by far the worst performers. This could be caused by the fact that contrary to previous researches we selected a rough market with negative trend, increased risk premiums and significantly negatively skewed implied distributions so that their constant volatility parameter was enough to explain any possible upward market movement. For Kou's model running several different calibrations, with both gradient based and pure evolutionary algorithms, we observed that they stopped in the upper bound of the parameter $\lambda$ (number of jumps per year). Also the upward jump probability has never found significantly different from zero. Searching the bibliography we did not find reported calibrated parameters. In Merton's model we had to make a selection between a standard deviation parameter (d) none significantly different than zero (0.01) and a mean log return jump parameter (m) with pretty large negative value (-780). Even with these relaxed parameters jump diffusions produced three times bigger pricing errors. These parameters are violating the model assumptions, so we preferred to discard these models, rather than drawing results upon spurious parameters.

For SABR, following Hagan et al. (2002) and West (2005), we fixed $\beta$-parameter a-priori using the regression scheme (3.29). As they suggest, fitting $\beta$ in a single smile is like trying to fit the noise, because this is a parameter that captures the dynamic movement of the smile. Of course this is a time dependent parameter but as West (2005) suggests it remains the same for short-term maturities. Our data suggested that $\beta$=0,7633.

As expected for CGMY, the lower tail decay rate parameter G is 26.7 on average and much smaller than the upper tail decay rate parameter M which is 132.5. Less decayed lower tail means the negative skewness of the log return probability density function.

The parameters C, G and M, that we found, are quite large but they are consistent with parameters obtained from Carr et al. (2002). Also consistent with their findings, is that the parameter Y was always less than one, signalling that finite variation processes should be used for modelling indices. From the table 7.1 is obvious that the parameters G, M are highly volatile. We didn't find in the bibliography reported standard deviations for these parameters, so we believe that this is due to the absence of a Brownian component.

In the next two sections we present a time consuming procedure, required for a fast calibration.

### 7.3 Attari Transformation

The method we apply here is based on Attari (2004). Attari (2004) derived an alternative pricing formula to(3.12), which is preferred because the convergence of the integral is pretty faster. Suppose we need to calculate the price of a European call option.

$$C(t,S_t;T,K)=e^{-r(T-t)}E_t^Q[(S_T-K)^+]=$$
$$e^{-r(T-t)}E_t^Q[S_T\mid S_T\geq K]-Ke^{-r(T-t)}E_t^Q[1\mid S_T\geq K]=$$
$$e^{-r(T-t)}S_tE_t^Q[e^{x(t,T)}\mid x(t,T)\geq \ln K]-Ke^{-r(T-t)}E_t^Q[1\mid x(t,T)\geq \ln K]=$$
$$e^{-r(T-t)}S_t\int_l^{\infty}[e^{x(t,T)}q(x(t,T))]dx(t,T)-Ke^{-r(T-t)}\int_l^{\infty}[q(x(t,T))]dx(t,T)\Rightarrow$$

$$C(t,S_t;T,K)=e^{-r(T-t)}S_tP^1-Ke^{-r(T-t)}P^2 \tag{7.2}$$

We have defined $l=\ln\left(\dfrac{K\exp(-r(T-t))}{S_t}\right)$

So $$P^1=\int_l^{\infty}e^xq(x)dx \tag{7.3}$$

And $$P^2=\int_l^{\infty}q(x)dx \tag{7.4}$$

Let $f_i(\varphi)$ the characteristic function of $P^i$

Now we note that

$$q(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f_2(\varphi)\exp(-i\varphi x)d\varphi$$

So $$P^1=\int_l^{\infty}e^xq(x)dx=\int_l^{\infty}e^x\frac{1}{2\pi}\int_{-\infty}^{\infty}f_2(\varphi)\exp(-i\varphi x)d\varphi dx$$

We should note that the characteristic function $f_2$ is not the same as the one we used in the main body of this text. Due to the change in the forward measure in (7.2) this is the characteristic function of the returns distribution and it the same as the one we derived in footnote 7 of Chapter 5.

Using Fubini's Theorem (*Marsden & Tromba (2000)*) $P^1$ can be written as

$$P^1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(\varphi)\int_{l}^{\infty}\exp(-i(\varphi+i)x)dxd\varphi$$

Now we use the fact that a zero strike option has Delta equal to one.

$$\lim_{x\to-\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(\varphi)\int_{x}^{\infty}\exp(-i(\varphi+i)x)dxd\varphi = 1$$

So

$$P^1 = \frac{1}{2}+\frac{1}{4\pi}\int_{-\infty}^{\infty} f_2(\varphi)\int_{l}^{\infty}\exp(-i(\varphi+i)x)dxd\varphi - \frac{1}{4\pi}\int_{-\infty}^{\infty} f_2(\varphi)\int_{-\infty}^{l}\exp(-i(\varphi+i)x)dxd\varphi \Rightarrow$$

$$P^1 = \frac{1}{2}+\frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(\varphi)\frac{\exp(-i(\varphi+i)l)}{i(\varphi+i)}d\varphi$$

$$-\frac{1}{4\pi}\lim_{r\to\infty}\int_{-\infty}^{\infty} f_2(\varphi)\frac{\exp(i(\varphi+i)r)+\exp(-i(\varphi+i)r)}{i(\varphi+i)}d\varphi \qquad (7.5)$$

Now by the substitution $u=-\varphi$ the last term is of (7.5) is written

$$\frac{1}{4\pi}\lim_{r\to\infty}\int_{-\infty}^{\infty} f_2(\varphi)\frac{\exp(i(\varphi+i)r)}{i(\varphi+i)}d\varphi - \frac{1}{4\pi}\lim_{r\to\infty}\int_{-\infty}^{\infty} f_2(-u)\frac{\exp(i(u-i)r)}{i(u-i)}du \qquad (7.6)$$

For further simplification of the last term Cauchy's integral theorem from complex analysis is needed.

Cauchy's Integral Theorem (*Churchill & Brown 2001*):

Let f an integrable function in a simple and closed contour C. If $z_0 \in C$ then

$$\int_C \frac{f(z)dz}{z-z_0} = 2\pi i f(z_0)$$

□

Using Cauchy's Integral Theorem the last term of (7.6) is

$$\frac{1}{4\pi}\lim_{r\to\infty}\int_{-\infty}^{\infty} f_2(-u)\frac{\exp(i(u-i)r)}{i(u-i)}du = 2\pi i\frac{1}{4\pi}\lim_{r\to\infty}\left\{f_2(-i)\frac{\exp(i(i-i)r)}{i}\right\} = -\frac{f_2(-i)}{2}$$

Moreover by the Fourier inversion of (7.3)

$f_2(-i) = \int_{-\infty}^{\infty} q(x)\exp(\varphi x)d\varphi = 1$ (Because a zero strike option has Delta equal to one)

So $P^1$ is written as

$$P^1 = 1 + \frac{e^l}{2\pi}\int_{-\infty}^{\infty} f_2(\varphi)\frac{\exp(-i\varphi l)}{i(\varphi+i)}d\varphi$$

So the pricing equation (7.2) becomes

$$C(t,S_t;T,K) = S_t\left[1 + \frac{e^l}{2\pi}\int_{-\infty}^{\infty} f_2(\varphi)\frac{\exp(-i\varphi l)}{i(\varphi+i)}d\varphi\right] -$$

$$\exp(-r(T-t))K\left[\frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(\varphi)\frac{\exp(-i\varphi l)}{i\varphi}d\varphi\right]$$

We have to calculate

$$\int_{-\infty}^{\infty} f_2(\varphi)\frac{\exp(-i\varphi l)}{i(\varphi+i)}d\varphi - \int_{-\infty}^{\infty} f_2(\varphi)\frac{\exp(-i\varphi l)}{i\varphi}d\varphi = \int_{-\infty}^{\infty} f_2(\varphi)\exp(-i\varphi l)\left(\frac{1}{i(\varphi+i)} - \frac{1}{i\varphi}\right)d\varphi$$

$$\int_{-\infty}^{\infty} f_2(\varphi)\exp(-i\varphi l)\left(\frac{1}{i(\varphi+i)} - \frac{1}{i\varphi}\right)d\varphi = -\int_{-\infty}^{\infty} f_2(\varphi)\exp(-i\varphi l)\left(\frac{1}{\varphi^2 + i\varphi}\right)d\varphi$$

$$= -\int_{-\infty}^{\infty} f_2(\varphi)\exp(-i\varphi l)\left(\frac{\varphi^2}{\varphi^4+\varphi^2} + \frac{-i\varphi}{\varphi^4+\varphi^2}\right)d\varphi$$

$$= -\int_{-\infty}^{\infty}\left(\frac{\mathrm{Re}[f_2]\cos(-\varphi l) - \mathrm{Im}[f_2]\sin(-\varphi l) + i\,\mathrm{Re}[f_2]\sin(-\varphi l) + i\,\mathrm{Im}[f_2]\cos(-\varphi l)}{1+\varphi^2}d\varphi\right)$$

$$-\int_{-\infty}^{\infty}\left(\frac{i\,\mathrm{Re}[f_2]\cos(-\varphi l) - i\,\mathrm{Im}[f_2]\sin(-\varphi l) - \mathrm{Re}[f_2]\sin(-\varphi l) - \mathrm{Im}[f_2]\cos(-\varphi l)}{(1+\varphi^2)\varphi}d\varphi\right) \quad (7.7)$$

Using the fact that $\mathrm{Re}[f_2], \cos(\varphi l)$ are even and $\mathrm{Im}[f_2]\sin(\varphi l)$ are odd, (7.7)becomes

$$2\int_0^{\infty}\left\{\left(R_2(\varphi) + \frac{I_2(\varphi)}{\varphi}\right)\cos(\varphi l) + \left(I_2(\varphi) - \frac{R_2(\varphi)}{\varphi}\right)\sin(\varphi l)\right\}/(1+\varphi^2)d\varphi$$

So the pricing formula is

$$C(t,S_t;T,K) = S_t - \frac{\exp(-r(T-t))K}{2} -$$

$$\frac{\exp(-r(T-t))K}{\pi}\int_0^{\infty}\left\{\left(R_2(\varphi)+\frac{I_2(\varphi)}{\varphi}\right)\cos(\varphi l)+\left(I_2(\varphi)-\frac{R_2(\varphi)}{\varphi}\right)\sin(\varphi l)\right\}/(1+\varphi^2)d\varphi \quad (7.8)$$

The main issue for a fast calibration is the avoidance of many evaluations of the characteristic functions, because due to their complex form their evaluation is the most time demanding part. Also many evaluations produce cumulative rounding errors. This formula requires two evaluations of the characteristic function like (3.12) with the use of (3.21), but the main advantage of this formula is the quadratic rate of decay. The integral's convergence here is pretty faster than (3.21). Also the integral in (7.8) is independent of the strike price, so for a series of options with the same maturity this is computed only once and cached. Kilin (2004) compares the speed of this formula for Heston's model with fast Fourier and fractional fast Fourier. His results indicate that the former is a much faster approach. For the calculation of the integral we use Gauss-Laguerre integration, as we will show below.

### 7.4 Gauss-Laguerre Integration

Another important advantage of this formula is that the quadratic rate of decay is so strong that the oscillating nature of the integrals is not as apparent as in formula(3.21). The two integrals that have to be calculated are

$$\int_0^{\infty} A(\varphi)d\varphi=\int_0^{\infty}\left\{\left(R_2(\varphi)+\frac{I_2(\varphi)}{\varphi}\right)\cos(\varphi l)\right\}/(1+\varphi^2)d\varphi$$

And $$\int_0^{\infty} B(\varphi)d\varphi=\int_0^{\infty}\left\{\left(I_2(\varphi)-\frac{R_2(\varphi)}{\varphi}\right)\sin(\varphi l)\right\}/(1+\varphi^2)d\varphi$$

In the next two figures we present these integrals for sample parameters of Heston and CGMY.

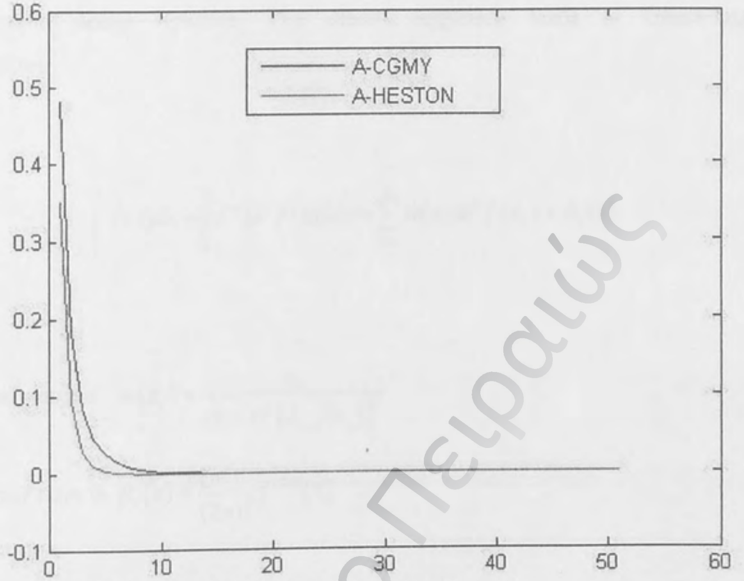


fig 7.3 A-function for CGMY and Heston Model

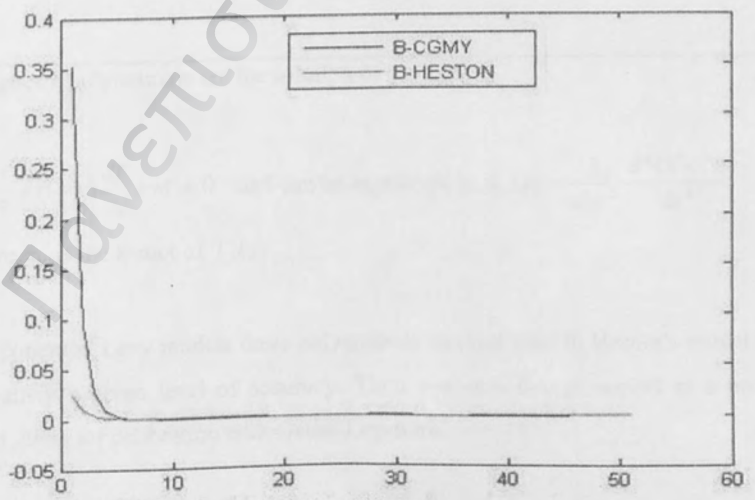


fig 7.4 B-function for CGMY and Heston

These results clearly suggest that the functions in the integral are not oscillating. So Gauss-Chebyshev integration that is required for the integral in (3.21) is not necessary. Actually their behavior is pretty much like a hyperbolic cotangent or an exponential decay function. The second approach leads to Gauss-Laguerre[10] integration.

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} e^{-x}(e^{x} f(x))dx = \sum_{k=1}^{n} w(x_k)e^{x} f(x_k) + R_n(x)$$

The weights are $w(x_k) = \dfrac{x_k}{(n+1)^2\left[L_{n+1}(x_k)\right]^2}$

The error term is $R_n(x) = \dfrac{(n!)^2}{(2n)!} f^{(2n)}(x)$

[10] Laguerre polynomials are the solution of the ODE

$x\dfrac{d^2 f}{dx^2} + (1-x)\dfrac{df}{dx} + nf = 0$ and can be expressed as $L_n(x) = \dfrac{1}{n!e^{-x}} \dfrac{d^n(x^n e^{-x})}{dx^n}$

Where $x_k$ is the k-root of $L_n(x)$

In exponential Levy models these polynomials worked fine. In Heston's model they do not satisfy a given level of accuracy. Their outcome though served as a pretty fast initial guess for calibration with Gauss-Legendre.

For further information about integration schemes we refer to Akrivis and Dougalis (2004)

# 8 Results

## 8.1 Properties of Delta Hedge Ratios

In Chapter 4 we have successfully used the Fourier transform techniques, that were established by Dyffie et. al (2000) for the pricing of affine models, in the pricing of infinity activity models. Here we go one step further by showing that infinite activity models can be included in the large family of scale invariant models. In figure 8.1 we present the cumulative density function of all Delta hedge ratios that were obtained for all options in the sample. In figure 8.2 we plot the Delta hedge ratios by strike price for a given day in the sample.

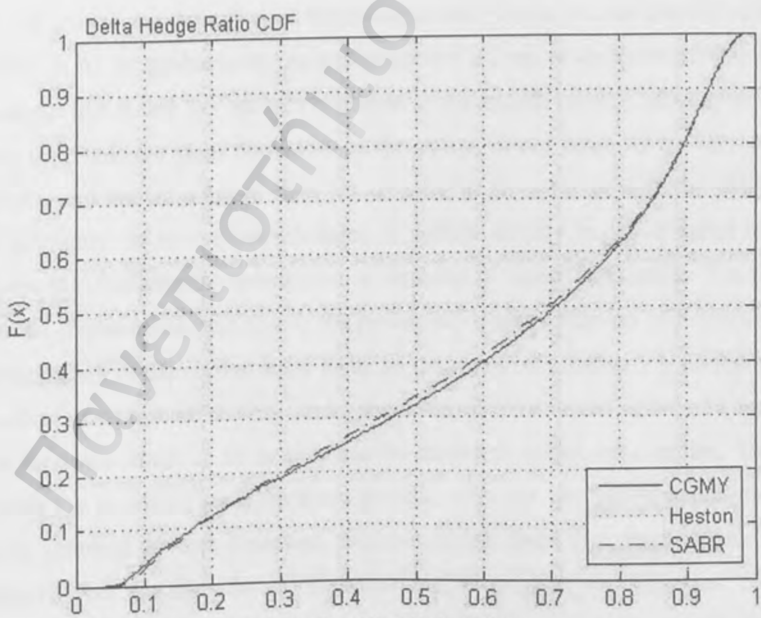


fig8.1 Cumulative Density Function of Delta Hedge Ratios

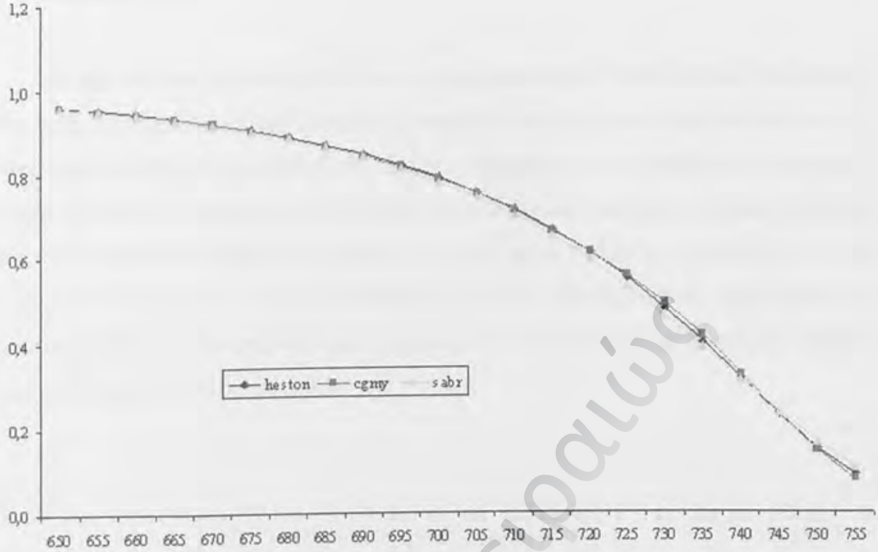

fig. 8.2 Delta Hedge Ratios obtained at 18/10/2007. S&P 100 was 718,31

Of course the definition of Delta hedge ratio for an infinite activity model, like CGMY, is by far problematic. As explained in Cont and Voltchkova (2005) the main reason for this is that the option's evolution is not smooth upon S. More specifically at every time unit the right derivative of the option is not equal to the left, while the second is not known a-priori. Here we accepted as Delta hedge ratio the right limit of this derivative. Of course Delta hedging of infinite activity models is sinful because it violates the fundamental assumption of equality of lateral derivatives. On the other hand the characteristic function of the probability density function of the stock returns is independent of the current stock level, so the results of Chapter 5 yield that the right derivative of an infinite activity model should equal the total derivative of a continuous scale invariant model as far as they can be calibrated to the same market. This result expands the family of scale invariant models, since all the known exponential Levy models (Normal Inverse Gaussian, Meixner etch.) share this characteristic function property. The non-smoothness of option value remains an important theoretical obstacle, but given the theoretical assumption that a class of models can fit the market prices, then the infinite activity models will give the same Delta-hedging errors with the continuous scale invariant models.

## 8.2 Hedging Errors

In this section we present the average hedging errors of each model. We suppose that both the option and the instrument portfolio are liquidated after one day or one week respectively. We consider both single instrument and two instrument hedging. In single instrument hedging we use the traditional Delta and minimum variance hedging. In two instrument hedging we apply the Delta-Vega hedging, as described in the example of Chapter 2, and the minimum variance hedging using one option, as described by (6.1). We used as target options only the calibrated options in the sample. So the hedging errors are given by:

$$\left\{-(O_{t+n}^{T}-O_{t}^{T})+\Delta(S_{t+n}-S_{t})\right\}\cdot\exp\left(r\cdot\frac{n}{252}\right) \qquad (8.1)$$

$$\left\{-(O_{t+n}^{T}-O_{t}^{T})+\Delta(S_{t+n}-S_{t})+(O_{t+n}^{I}-O_{t}^{I})\right\}\cdot\exp\left(r\cdot\frac{n}{252}\right) \qquad (8.2)$$

Where $O^{T}$ is the target option, S is the S&P 100 value, $O^{I}$ is the instrument option, Δ and V are the pieces of the underlying and the pieces of instrument options required for every strategy. $n$ is the number of days that passed until the liquidation and is set to 1 for daily hedging or 5 for weekly. $r$ is the one week US Libor rate. The instrument option was always selected to be an at the money option.

For testing the differences among distributions we used the Kolmogorov-Smirnov statistic, described in Massey (1951) and Stephens (1970). This statistic calculates the cumulative distributions of two populations. Then it calculates the maximum difference of the cumulative distributions for every value x i.e.

$$d=\max(|F_{1}(x)-F_{2}(x)|)$$

$d$ has predetermined distribution, which is described in Massey (1951).

In table 8.1 we present the average daily hedging errors for each model and each strategy. Single instrument hedging is used to immunize a portfolio's market risk for short-term positions. Searching for comparisons among the models ability to immunize portfolio's risk, in table 8.2 we test the absolute prices of daily hedging errors using the Kolmogorov-Smirnov statistic.

**Table 8.1: Single Instrument Hedging Errors**

Average Sum of Hedging Errors obtained from(8.1). For daily errors n is set to one, while for weekly errors n is set to five. For each day in the sample every calibrated option is set as target option. Then the hedging error is calculated with(8.1). The sum of hedging errors is calculated for every day in the sample. The average sum of hedging errors for each strategy is reported in the above table.

| Model | 1 Day Hedging Error | | 5 Days Hedging Error | |
|---|---|---|---|---|
| | Delta | Minimum-Variance | Delta | Minimum-Variance |
| Heston | -0.393 | -0.39221 | -1.47781 | -1.47193 |
| CGMY | -0.39299 | -0.32628 | -1.44649 | -0.89921 |
| SABR | -0.1491 | -0.15663 | -1.15769 | -0.8689 |

**Table 8.2: Kolmogorov-Smirnov Test for the single instrument daily hedging errors**

The null hypothesis is that the distribution in the row is equal, greater or lesser (according to symbol in the parenthesis) than the distribution in the column. The reported value is the significance level of acceptance of the null hypothesis.
For testing the superiority of a distribution we used the absolute prices of the hedging errors.

N.S. stands for nothing statistically significant.

| | Heston Δ | Heston MV | CGMY Δ | CGMY MV | SABR Δ | SABR MV |
|---|---|---|---|---|---|---|
| Heston Δ | - | | | | | |
| Heston MV | 100% (=) | - | | | | |
| CGMY Δ | 100% (=) | 100% (=) | - | | | |
| CGMY MV | 99.65% (<) | 100% (<) | 99.65% (<) | - | | |
| SABR Δ | N.S. | N.S. | N.S. | 85.75% (>) | - | |
| SABR MV | 99.85% (<) | 100% (<) | 100% (<) | N.S. | 99.38% (<) | - |

The most obvious result from these tables is that the Delta hedging with Heston and CGMY has given the same hedging results. Surprisingly we found that Heston's model minimum variance hedging performed the same with Delta hedging. This is confirmed by the results in table 8.1 and table 8.2. This is opposite to (3.25) according to which the two models are expected to perform the same only when the correlation coefficient is restricted to zero. This is because the Vegas of short-term options are small (except of that of at the money money options). Moreover the denominator of (3.25) is the price of S&P 100 which is multiple times larger than the numerator. So the contribution of the last term in (3.25) is rather insignificant. The fact that minimum variance hedging in Heston's model is strongly dependent to the level of the underlying leads to the conclusion that we cannot find a model free property for minimum variance hedge ratios in scale-invariant models, because in a typical scale-invariant model like Heston's, minimum variance hedge ratio is strongly related to the level of the underlying.

In both CGMY and SABR, minimum variance hedging improved the performance of Delta hedging. These two models have captured the increased levels of negative skewness, a feature that dominated in the period we examined. CGMY is a pure jump model drawn from the generalized hyperbolic distribution, that is described in Raible (2000). SABR model due to the presence of $\beta$-parameter, with it's additional leverage effect influence, succeeded to capture the dynamic movement of implied skewness. Among CGMY and SABR no comparison was statistically significant except that both Delta and minimum variance hedging with SABR was significantly better than Delta hedging with CGMY.

Based on the findings of Bakshi et al. (1997) which showed that including in Heston's model stochastic interest rates and jumps of lognormal distribution (Merton) does not increase it's hedging performance, we conclude that CGMY and SABR are better hedging models in single instrument minimum variance hedging. This finding leads us to investigate for a good hedging model beyond a very large and well studied family of stochastic volatility-jump diffusion models (Bakshi et al.(1997), Bates (1996), Dyffie et al.(2000)).

In table 8.3 we present the average daily hedging errors for each model and each strategy. Two instrument hedging is strategy to immunize the portfolio risk for longer term positions avoiding daily rebalancing. So for this strategy we will focus our attention on weekly hedging errors.

**Table 8.3: Two Instrument Hedging Errors**

Average Hedging Errors obtained from(8.2). For daily errors n is set to one, while for weekly errors n is set to five. For each day in the sample every calibrated option is set as target option. Then the hedging error is calculated with(8.2). The sum of hedging errors is calculated for every day in the sample. The average sum of hedging errors for each strategy is reported in the above table.

| Model | 1 Day Hedging Error | | 5 Days Hedging Error | |
|---|---|---|---|---|
| | Delta-Vega | Minimum-Variance | Delta-Vega | Minimum-Variance |
| Heston | -0.4165 | -0.1829 | -0.7474 | -0.9773 |
| CGMY | - | -0.2030 | - | -0.3782 |
| SABR | -0.2096 | * | -0.4836 | * |

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*

Minimum Variance hedging with SABR totally failed. We believe that this is due to the cumulative numerical techniques that required for the derivation of hedge ratios.

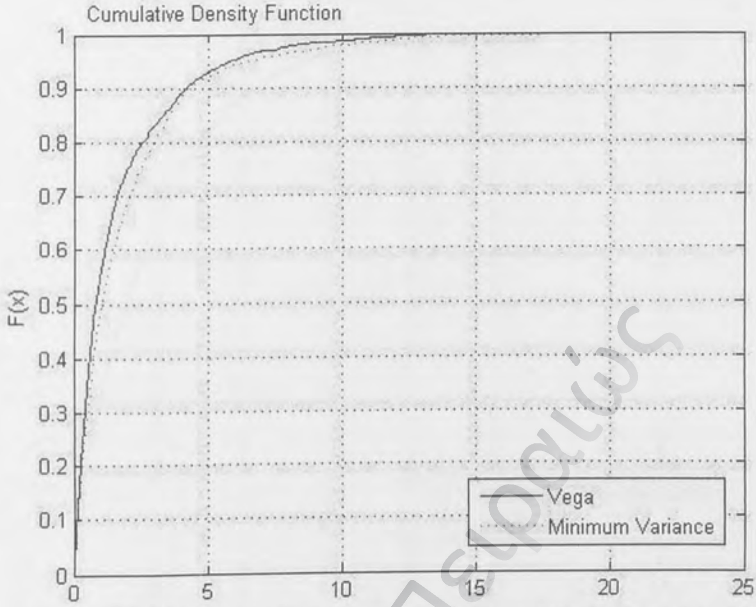


fig 8.3 CDF for Heston two instrument absolute weekly hedging errors

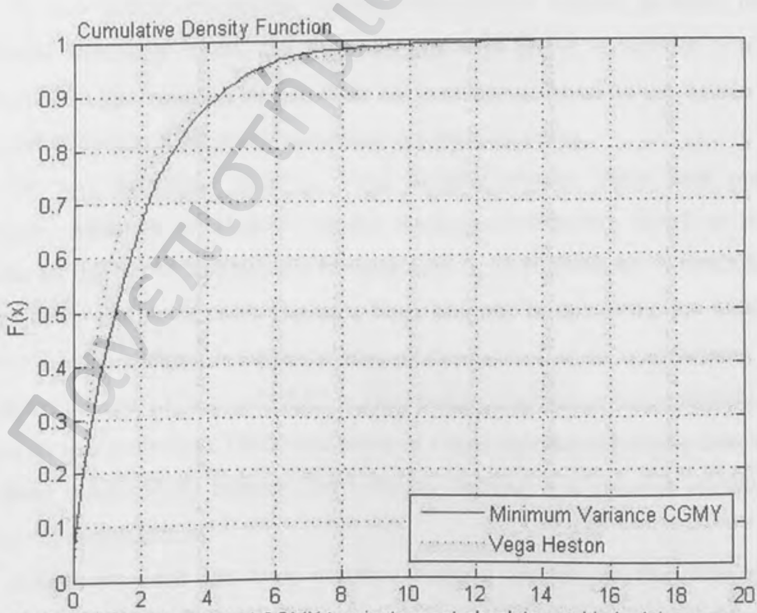


fig 8.4 CDF for Heston and CGMY two instrument absolute weekly hedging errors

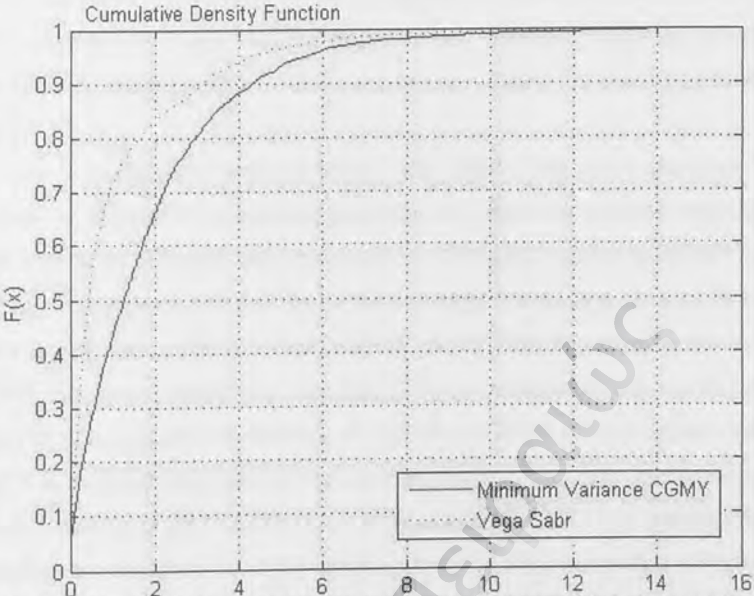


fig 8.5 CDF for SABR and CGMY two instrument absolute weekly hedging errors

In two instrument-hedging, Kolmogorov-Smirnov statistic provided only the statistical significant result, that Vega hedging with SABR is superior to all other strategies. In this occasion we can draw our conclusions based on the figures for the cumulative density functions of absolute 5-day hedging errors.

In two instrument hedging, Vega hedging worked better than minimum variance. Although in Heston's model Kolmogorov-Smirnov failed to return a significant difference, this result is obvious from fig. 8.3. Minimum variance hedging with CGMY did not improve Heston's Vega hedging. In fact in fig. 8.4 we see that Heston's Vega hedging is better in 90% of the most close to zero hedging errors. Moreover the rest 10 %, that CGMY worked better, is subject to computational errors. Based on the result that CGMY was better in single instrument hedging than Heston, the latest result clearly indicate that volatility hedging is a valuable ingredient for long-term immunization.

SABR provided the most accurate hedging against all the other models. Contrary to Heston's model SABR was more successful in capturing the skewness of implied distribution, while the relative advantage of Heston, the mean reverting volatility process, seemed to be rather insignificant for short term positions. The second success of SABR against Heston makes us to conclude that the first is superior hedger against the large family of stochastic volatility jump diffusions. This conclusion is straightforward with the use of the evidence presented by Bakshi et al. (1997).

For a comparison between SABR and CGMY the conclusions are not so obvious. In Heston Vega hedging outperformed minimum variance hedging. This could lead us to the result that our experiment is biased against minimum variance hedging. This is due to the fact that in this strategy we used as an instrument only the at the money option. In fact Vega hedging can be done by choosing any option at random. In contrast we are not sure that (6.1) is optimized by the use of an at the money option. Despite the findings of Carr et al. (2002), which claimed that once CGMY is modeled Brownian motion is not necessary, the absence of a measure for market volatility is penalized by the extreme volatility in CGMY parameters. This parameter inconsistency could be another cause for the performance of this model, since (6.1) is based on time stable parameters. On way to remedy this inconsistency, is to consider minimum variance hedging under a different measure than the risk-neutral that penalizes both, bad pricing performance as also parameter time inconsistency. A possible candidate for this measure could be the relative entropy measure, Cont & Tankof (2004).

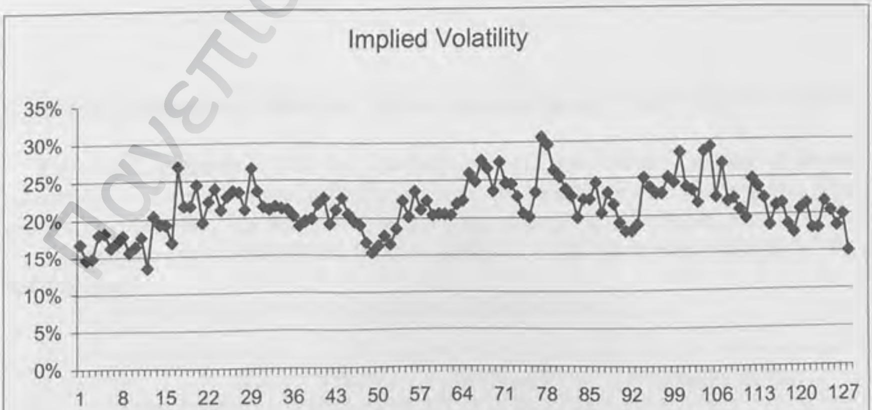

fig.8.6 Implied Volatility of shortest maturity at the money option for each day in the sample

Another important reason for CGMY's failure, comparing with Vega hedging for weekly horizon is that an important market ingredient, the volatility is left unhedged,. As we show in fig 8.6 implied volatilities of at the money call options exhibit a strong clustering persistence. Running a AR(1) regression for implied volatilities we found that the lag coefficient was 60% and the relative t-statistic was 8.41. Thus implied volatility exhibited strong autocorrelation, but this feature was totally beyond the scope of CGMY construction. This effect is more persistent in our data, because we considered a negative trend market. In Chapter 6 we explained that volatility hedging with CGMY is totally non feasible. The inclusion of an option in the instrument portfolio is only for hedging against jump movements. For this reason CGMY is been penalized in two instrument hedging. This result is augmented by the results of Bakshi et al. (1997), who found that the inclusion of jumps in stochastic volatility models does not improve their hedging performance. One possible way to remedy CGMY as to explain this feature is to consider a larger category of new developed models, Levy processes with stochastic arrivals Carr et al. (2003). In these models the time change is a stochastic process, where the selection of a mean reverting process produces the clustering effect.

To explain whether the time instable parameters or the absence of volatility hedging was the main reason that CGMY performed poorer than SABR in two-instrument hedging, we perform Kolmogorov-Smirnov test for 1 day hedging errors.

**Table 8.4: Kolmogorov-Smirnov Test for two instrument daily hedging errors**

The null hypothesis is that the distribution in the row is equal, greater or lesser (according to symbol in the parenthesis) than the distribution in the column. The reported value is the significance level of acceptance of the null hypothesis.

For testing the superiority of a distribution we used the absolute prices of the hedging errors.

N.S. stands for nothing statistically significant.

| | Heston VEGA | CGMY MV | SABR VEGA |
|---|---|---|---|
| Heston VEGA | - | | |
| CGMY MV | N.S. | - | |
| SABR VEGA | 98.63%(<) | 85.82%(<) | - |

From Table (8.4) we obtain that Vega hedging with SABR outperformed Heston and CGMY in one day two-instrument hedging. These results indicate that volatility hedging is important when a second is included in the instrument portfolio. In single instrument minimum variance hedging CGMY performed as well as SABR. Combining this with the results of Table (8.4), we conclude the absence of hedging against market volatility is the main reason that CGMY failed in two instrument hedging. The time inconsistency of CGMY parameters is probably the second in order reason for its failure against SABR in weekly horizon.

# 9 Conclusions and further Research

We showed that exponential Levy models can be included in the family of scale invariant models. This was straightforward because their characteristic function is independent of the current price state. These models will provide the same Delta hedging errors with scale invariant models as far one can accept a right derivative as a hedge ratio. In exponential Levy models the right derivative of the option with respect to the underlying is only observable in the market, while the left is not known. This inequality of the derivatives is an important theoretical frontier, since Delta hedging strategy is based on the assumption of equality of these derivatives. Here we subsidized this drawback by the assumption of equal good fitting of pure jump and continuous models in market option prices.

Searching for differences among the models and the strategies we found that in single instrument hedging for daily covering, minimum variance hedging works much better than Delta hedging. We found that the models in the current literature were insufficient to describe the implicit skewness in a negative trend market. Based on the results of Bakshi et al. (1997) we concluded that both CGMY and SABR are better hedging models than the large category of stochastic volatility jump diffusions, which are thoroughly studied in the literature. For pure jump models and SABR, this is the first time that their hedging ability is tested. We believe their superiority is because of the nature of these models. Pure jump CGMY arises from the generalized hyperbolic distribution and SABR provides additional skewness persistence through its β-parameter.

For longer-term immunization we included options as hedging instruments. To achieve this in a pure jump model we expanded minimum variance hedging to include options as hedging instruments. To the best of our knowledge this is the first time that this strategy was tested in a real market data set. We constrained this strategy allowing only one at the money option as an instrument. At this form traditional volatility hedging proved to be more accurate. Vega hedging with SABR and Heston outperformed minimum variance hedging with Heston and CGMY. Again SABR was superior hedger than Heston. This was because it was better to describe the negative skewness of options implied distribution. The fact that Vega hedging outperformed CGMY means that volatility hedging is an important ingredient for successful hedging. We showed that implied volatility produced a strong clustering pattern, which CGMY was unable to explain. As we showed in Section 6.3 the dynamics of volatility is beyond the scope of CGMY's construction. This important drawback in CGMY has been also addressed in Carr et al (2003), who construct a new family of processes based on time changes with stochastic arrivals, to include this stylized fact in exponential Levy models. Based on the result that CGMY performed well in single instrument hedging, we conclude that volatility hedging is important for long-term immunization, so its absence in CGMY setting caused SABR's superiority.

The inclusion of an option in the instrument portfolio is necessary for a successful hedging in incomplete markets. In stochastic volatility models the instrument option provides hedging against volatility. In pure jump models, the jumps can be controlled only in a minimum variance portfolio, according to (6.1). Pure jump models are severely violating the assumptions of (6.1) by their high volatile parameters. Hedging under a measure that penalizes this important inconsistency could be considered.

We considered a market with negative trend and increased volatility. In our data many of the financial time series stylized facts were present such as jumps and volatility clustering. While the Levy models were created to explain the jump behavior of assets, the hedging against market volatility proved to be more important for portfolio immunization. Our results in this area are consistent with the results of Bakshi et al. (1997), who found that the inclusion of jumps in stochastic volatility models does not improve their hedging performance. Recently developed models, where time changes are stochastic arrivals, explain both the financial time series jump behavior and the stochastic volatility and should be included in a later research.

## APPENDIXES

**1.**

We guess a solution for (3.15) in the form $f=exp(C(\tau)+D(\tau)v+i\varphi x_0)$.

With C(0)=D(0)=0. Thus

$$\begin{aligned} f_x &= i\varphi x f \\ f_{xx} &= -\varphi^2 x f \\ f_v &= Df \\ f_{vv} &= D^2 f \\ f_{xv} &= i\varphi D f \\ f_\tau &= (\dot{C}+v\dot{D})f \end{aligned}$$

Whereas ( $\dot{}$ ) denotes the first derivative according to the time variable.

Thus (3.15) becomes

$$-\dot{C}+aD+ri\varphi+v(-\dot{D}-0.5\varphi^2+\rho\sigma\varphi iD+0.5\sigma^2D^2+u_i\varphi i-b_iD)=0 \tag{1.1}$$

(1.1) is a first degree polynomial of v so a system of ODEs is obtained.

$$-\dot{C}+aD+ri\varphi=0 \tag{1.2}$$

$$-\dot{D}-0.5\varphi^2+\rho\sigma\varphi iD+0.5\sigma^2D^2+u_i\varphi i-b_iD=0 \tag{1.3}$$

With C(0)=D(0)=0

(1.3) is a Ricatti type ODE with exact solution

$$D_i(\tau) = \frac{b_i - \rho\sigma\varphi i + d_i}{\sigma^2}\left[\frac{1-\exp(d_i\tau)}{1-g_i\exp(d_i\tau)}\right]$$

*with*

$$g_i = \frac{b_i - \rho\sigma\varphi i + d_i}{b_i - \rho\sigma\varphi i - d_i}$$

$$d_i = ((\rho\sigma\varphi i - b_i)^2 - \sigma^2(2u_i\varphi i - \varphi^2))^{0.5}$$

Then we can solve (1.3) so we obtain

$$C_i = ri\varphi\tau + \frac{\alpha}{\sigma^2}\left\{(b_i - \rho\sigma\varphi i + d_i)\tau - 2\ln\left[\frac{1-g_i\exp(d_i\tau)}{1-g_i}\right]\right\}$$

**2**

(3.20) is proved in Pelaez (1952) as an extension of Levy inversion theorem. For the next step we must note that $f(-\varphi)$ and $f(\varphi)$ are conjugate functions. Indeed

$$RE[f(\varphi)] = RE[\int_{-\infty}^{\infty}\exp(i\varphi x)f(x)dx] = \int_{-\infty}^{\infty}\cos(\varphi x)f(x)dx = \int_{-\infty}^{\infty}\cos(-\varphi x)f(x)dx = RE[f(-\varphi)]$$

$$\mathrm{Im}[f(\varphi)] = \mathrm{Im}[\int_{-\infty}^{\infty}\exp(i\varphi x)f(x)dx] = \int_{-\infty}^{\infty}\sin(\varphi x)f(x)dx = -\int_{-\infty}^{\infty}sin(-\varphi x)f(x)dx = -\mathrm{Im}[f(-\varphi)]$$

So

$$\exp(i\varphi x)f(-\varphi) - \exp(-i\varphi x)f(\varphi) = (\cos(\varphi x) + i\,sin(\varphi x))f(-\varphi) - (\cos(-\varphi x) + i\,sin(-\varphi x))f(\varphi)$$

$$= \cos(\varphi x)f(-\varphi) + i\,sin(\varphi x)f(-\varphi) - \cos(\varphi x)f(\varphi) + i\,sin(\varphi x)f(\varphi)$$

$$= \cos(\varphi x)(f(-\varphi) - f(\varphi)) + i\,sin(\varphi x)(f(-\varphi) + f(\varphi)) \Rightarrow$$

$$\frac{\exp(i\varphi x)f(-\varphi) - \exp(-i\varphi x)f(\varphi)}{i\varphi} =$$

$$\frac{-i\cos(\varphi x)(f(-\varphi) - f(\varphi)) + sin(\varphi x)(f(-\varphi) + f(\varphi))}{\varphi} =$$

$$= 2\,\mathrm{Im}\left[\frac{f(\varphi)}{\varphi}\right]\mathrm{Re}\left[\exp(i\varphi x)\right] + 2\,\mathrm{Re}\left[\frac{f(\varphi)}{\varphi}\right]\mathrm{Im}\left[\exp(i\varphi x)\right]$$

$$= -2\,\mathrm{Im}\left[\frac{f(\varphi)\exp(i\varphi x)}{\varphi}\right]$$

This was used in Bates (1996) and Duffie, Pan and Singleton (2000)

Moreover

$$2\frac{i}{i}\mathrm{Im}\left[\frac{f(\varphi)}{\varphi}\right]\mathrm{Re}\left[\exp(i\varphi x)\right] + 2\frac{i}{i}\mathrm{Re}\left[\frac{f(\varphi)}{\varphi}\right]\mathrm{Im}\left[\exp(i\varphi x)\right] = -2\,\mathrm{Re}\left[\frac{f(\varphi)\exp(i\varphi x)}{i\varphi}\right]$$

3

For a call option we need to solve

$$U_\tau = +\frac{\sigma^2}{2}U_{xx} + (r - \frac{\sigma^2}{2} - \lambda k)U_x - rU + \lambda\int_R U(x+y) - U(x) f(dy)$$

With initial condition $U(x,0) = (e^x - K)^+$

We'll try to find a solution in the form $e^x P_1 - Ke^{-r\tau} P_2$

This produces two ordinary integrodifferential equations

$$P_{1\tau} = (P_{1x} + P_1)(r - \frac{\sigma^2}{2} - \lambda k) + (P_1 + 2P_{1x} + P_{1xx})\frac{\sigma^2}{2} - rP_1 + \lambda\int_R e^y P_1(x+y) - P_1(x) f(dy)$$

$$P_{2\tau} - rP_2 = P_{2x}(r - \frac{\sigma^2}{2} - \lambda k) + P_{2xx}\frac{\sigma^2}{2} - rP_2 + \lambda\int_R P_2(x+y) - P_2(x) f(dy)$$

$$\Rightarrow \begin{array}{l} P_{1\tau} = P_{1x}(r + \frac{\sigma^2}{2} - \lambda k) + P_{1xx}\frac{\sigma^2}{2} - \lambda P_1 - \lambda k P_1 + \lambda\int_R e^y P_1(x+y) f(dy) \\ P_{2\tau} = P_{2x}(r - \frac{\sigma^2}{2} - \lambda k) + P_{2xx}\frac{\sigma^2}{2} - \lambda P_2(x) + \lambda\int_R P_2(x+y) f(dy) \end{array}$$

We guess a solution of the form $f_i(\varphi) = \exp\{C_i(\tau) + i\varphi x\}$

Characteristic functions solve the same system respectively

$$f_{1\tau} = f_{1x}(r + \frac{\sigma^2}{2} - \lambda k) + f_{1xx}\frac{\sigma^2}{2} - \lambda f_1 - \lambda k f_1 + \lambda\int_R e^y f_1(x+y) f(dy)$$

$$f_{2\tau} = f_{2x}(r - \frac{\sigma^2}{2} - \lambda k) + f_{2xx}\frac{\sigma^2}{2} - \lambda f_2(x) + \lambda\int_R f_2(x+y) f(dy)$$

$$\Rightarrow \begin{array}{l} \dot{C}_1 = i\varphi(r + \frac{\sigma^2}{2} - \lambda k) - \varphi^2\frac{\sigma^2}{2} - \lambda - \lambda k + \lambda\int_R e^{y+i\varphi y} f(dy) \\ \dot{C}_2 = i\varphi(r - \frac{\sigma^2}{2} - \lambda k) - \varphi^2\frac{\sigma^2}{2} - \lambda + \lambda\int_R e^{i\varphi y} f(dy) \end{array}$$

In Merton's model (using(4.4) )

$$C_1 = \tau(i\varphi(r + \frac{\sigma^2}{2} - \lambda k) - \varphi^2 \frac{\sigma^2}{2} - \lambda - \lambda k + \lambda \exp((1+i\varphi)\mu + \frac{(1+i\varphi)^2 \delta^2}{2})) \quad (3.1)$$

$$C_2 = \tau(i\varphi(r - \frac{\sigma^2}{2} - \lambda k) - \varphi^2 \frac{\sigma^2}{2} - \lambda + \lambda \exp(i\varphi\mu + \frac{(i\varphi)^2 \delta^2}{2})) \quad (3.2)$$

In Kou's model (using(4.5) )

$$C_1 = \tau(i\varphi(r + \frac{\sigma^2}{2} - \lambda k) - \varphi^2 \frac{\sigma^2}{2} - \lambda - \lambda k + \lambda(p \cdot n_1 \frac{1}{n_1 - (1+i\varphi)} + q \cdot n_2 \frac{1}{n_2 + (1+i\varphi)})) \quad (3.3)$$

$$C_2 = \tau(i\varphi(r - \frac{\sigma^2}{2} - \lambda k) - \varphi^2 \frac{\sigma^2}{2} - \lambda + \lambda(p \cdot n_1 \frac{1}{n_1 - i\varphi} + q \cdot n_2 \frac{1}{n_2 + i\varphi})) \quad (3.4)$$

**4**

Having established the formula (4.16) for minimum variance Delta two integrals have to be computed

$$A = \int (e^y - 1)[U(Se^y) - U(S)]\lambda f(dy)$$
$$B = \int (e^y - 1)^2 \lambda f(dy)$$

$$B = \int (e^y - 1)^2 \lambda f(dy) = \int e^{2y} \lambda f(dy) - 2\int e^y \lambda f(dy) + \int \lambda f(dy)$$

Using (4.4) and (4.5) respectively

In Merton's model

$$B = \lambda(\exp(2\mu + \frac{4\delta^2}{2}) - 2\exp(\mu + \frac{\delta^2}{2}) + 1)$$

In Kou's model

$$B=\lambda\left( p\cdot n_1\frac{1}{n_1-2}+q\cdot n_2\frac{1}{n_2+2}-2(p\cdot n_1\frac{1}{n_1-1}+q\cdot n_2\frac{1}{n_2+1})+1\right)$$

Now

$$A=\int(e^y-1)U(Se^y)\lambda f(dy)-\int U(S)(e^y-1)\lambda f(dy)$$
$$=\int(e^y-1)(e^{x+y}P_1(x+y)-Ke^{-r\tau}P_2(x+y))\lambda f(dx)-U(S)\int\ (e^y-1)\lambda f(dy)\Rightarrow$$
$$A=e^x\int(e^y-1)e^yP_1(x+y)\lambda f(dx)-Ke^{-r\tau}\int\ P_2(x+y)\lambda f(dx)-U(S)\lambda k \tag{4.1}$$

Now by using the characteristic functions inversion formula

$$P_i=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_i)}{i\varphi}\right]d\varphi$$

The first term of (4.1) becomes

$$\int(e^y-1)e^yP_1(x+y)\lambda f(dx)=\int(e^y-1)e^y\{\frac{1}{2}+\frac{1}{\pi}(\int_0^\infty \mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_1(x+y)}{i\varphi}\right]d\varphi)\}\lambda f(dx)=$$
$$\int(e^y-1)e^y\frac{1}{2}\lambda f(dx)+\frac{\lambda}{\pi}\int(e^y-1)e^y\int_0^\infty \mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_1(x+y)}{i\varphi}\right]d\varphi f(dx) \tag{4.2}$$

Now we note that the first term of (4.2) is

$$\int(e^y-1)e^y\frac{1}{2}\lambda f(dy)=\frac{1}{2}\int(e^{2y}-e^y)\lambda f(dy)=\frac{\lambda}{2}\int e^{2y}v(dy)-\frac{\lambda}{2}\int e^y f(dy)=$$
$$\frac{\lambda}{2}\int(e^{2y}-e^y-e^y+e^y+1-1)f(dy)=\frac{\lambda}{2}\int(e^{2y}-2e^y+1)f(dy)+\frac{\lambda}{2}\int(e^y-1)f(dy)\Rightarrow$$
$$\int(e^y-1)e^y\frac{1}{2}\lambda f(dy)=\frac{1}{2}(B+\lambda k) \tag{4.3}$$

Now we have to calculate the second term of (4.2)

$$\frac{\lambda}{\pi}\int (e^{y}-1)e^{y}\int_{0}^{\infty}\operatorname{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_{1}(x+y)}{i\varphi}\right]d\varphi f(dx) \tag{4.4}$$

Using Fubini's theorem (4.4) is written as

$$\frac{\lambda}{\pi}\int_{0}^{\infty}\int \operatorname{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_{1}(x+y)(e^{y}-1)e^{y}}{i\varphi}\right]f(dy)d\varphi =$$

$$\frac{\lambda}{\pi}\int_{0}^{\infty}\operatorname{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))}{i\varphi}\int f_{1}(x+y)(e^{y}-1)e^{y}f(dy)\right]d\varphi =$$

$$\frac{\lambda}{\pi}\int_{0}^{\infty}\operatorname{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))}{i\varphi}f_{1}(x)\int e^{i\varphi y}(e^{y}-1)e^{y}f(dy)\right]d\varphi$$

We define

$$m_{1}(\varphi)=\int e^{i\varphi y}(e^{y}-1)e^{y}f(dy)=\int e^{1+i\varphi y}(e^{y}-1)f(dy)=\int e^{2+i\varphi y}f(dy)-\int e^{1+i\varphi y}f(dy)$$

In Merton's model

$$m_{1}(\varphi)=\exp((2+i\varphi)\mu+\frac{(2+i\varphi)^{2}\delta^{2}}{2})-\exp((1+i\varphi)\mu+\frac{(1+i\varphi)^{2}\delta^{2}}{2})$$

In Kou's model

$$m_1(\varphi)=p\cdot n_1\frac{1}{n_1-(2+i\varphi)}+q\cdot n_2\frac{1}{n_2+(2+i\varphi)}-p\cdot n_1\frac{1}{n_1-(1+i\varphi)}-q\cdot n_2\frac{1}{n_2+(1+i\varphi)}$$

So we have that

$$\frac{\lambda}{\pi}\int (e^y-1)e^y\int_0^\infty \operatorname{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_1(x+y)}{i\varphi}\right]d\varphi f(dx)=\frac{\lambda}{\pi}\int_0^\infty \operatorname{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_1(x)m_1(\varphi)}{i\varphi}\right]d$$

Now we have to calculate the second term of (4.1)

$$Ke^{-r\tau}\int P_2(x+y)\lambda f(dx)$$

By applying the same technique we conclude that

$$\int P_2(x+y)\lambda f(dx)=\frac{1}{2}\lambda k+\frac{\lambda}{\pi}\int_0^\infty \operatorname{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_2(x)m_2(\varphi)}{i\varphi}\right]d\varphi$$

In Merton's model

$$m_2(\varphi)=\exp((1+i\varphi)\mu+\frac{(1+i\varphi)^2\delta^2}{2})-\exp(i\varphi\mu+\frac{(i\varphi)^2\delta^2}{2})$$

In Kou's model

$$m_2(\varphi)=p\cdot n_1\frac{1}{n_1-(1+i\varphi)}+q\cdot n_2\frac{1}{n_2+(1+i\varphi)}-p\cdot n_1\frac{1}{n_1-i\varphi}-q\cdot n_2\frac{1}{n_2+i\varphi}$$

**5**

Following Appendix 4 two integrals have to be computed

$$A=\int(e^{y}-1)[U(Se^{y})-U(S)]v(dy)$$
$$B=\int(e^{y}-1)^{2}v(dy)$$

$$B=\int(e^{y}-1)^{2}v(dy)=\int e^{y}(e^{y}-1)v(dy)-\int(e^{y}-1)v(dy)$$

Using the Proposition 4.3 and the relationship (4.23)

$$B=C\cdot\Gamma(-Y)\left((M-2)^{Y}-(M-1)^{Y}+(G)^{Y}-(G+1)^{Y}\right)-k$$

Now for the A integral we have that

$$A=\int(e^{y}-1)U(Se^{y})v(dy)-\int U(S)(e^{y}-1)v(dy)$$
$$=\int(e^{y}-1)(e^{x+y}P_{1}(x+y)-Ke^{-rt}P_{2}(x+y))v(dx)-U(S)\int\ (e^{y}-1)v(dy)\Rightarrow$$
$$A=e^{x}\int(e^{y}-1)e^{y}P_{1}(x+y)v(dx)-Ke^{-rt}\int(e^{y}-1)P_{2}(x+y)v(dx)-U(S)k \qquad (5.1)$$

Now by using the characteristic functions inversion formula

$$P_{i}=\frac{1}{2}+\frac{1}{\pi}\int_{0}^{\infty}\mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_{i})}{i\varphi}\right]d\varphi$$

The first term of (5.1) becomes

$$\int(e^{y}-1)e^{y}P_{1}(x+y)v(dx)=\int(e^{y}-1)e^{y}\{\frac{1}{2}+\frac{1}{\pi}(\int_{0}^{\infty}\mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_{1}(x+y)}{i\varphi}\right]d\varphi)\}v(dx)=$$
$$\int(e^{y}-1)e^{y}\frac{1}{2}v(dx)+\frac{1}{\pi}\int\ (e^{y}-1)e^{y}\left(\int_{0}^{\infty}\mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_{1}(x+y)}{i\varphi}\right]d\varphi\right)v(dx) \qquad (5.2)$$

Now we note that the first term of (5.2) is

$$\int (e^{y}-1)e^{y}\frac{1}{2}v(dx)=\frac{1}{2}C\cdot\Gamma(-Y)\left((M-2)^{Y}-(M-1)^{Y}+(G)^{Y}-(G+1)^{Y}\right)$$

Now we have to calculate the second term of (5.2)

$$\frac{1}{\pi}\int (e^{y}-1)e^{y}\int_{0}^{\infty}\mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_{1}(x+y)}{i\varphi}\right]d\varphi v(dx) \qquad (5.3)$$

Using Fubini's theorem (5.3) is written as

$$\frac{1}{\pi}\int_{0}^{\infty}\int \mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_{1}(x+y)(e^{y}-1)e^{y}}{i\varphi}\right]v(dy)d\varphi=$$

$$\frac{1}{\pi}\int_{0}^{\infty}\mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))}{i\varphi}\int f_{1}(x+y)(e^{y}-1)e^{y}v(dy)\right]d\varphi=$$

$$\frac{1}{\pi}\int_{0}^{\infty}\mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))}{i\varphi}f_{1}(x)\int e^{i\varphi y}(e^{y}-1)e^{y}v(dy)\right]d\varphi$$

We define

$$m_{1}(\varphi)=\int e^{i\varphi y}(e^{y}-1)e^{y}f(dy)=\int e^{1+i\varphi y}(e^{y}-1)v(dy)$$

Using the Proposition 4.3

$$m_{1}=C\cdot\Gamma(-Y)\left((M-\omega_{1}-1)^{Y}-(M-\omega_{1})^{Y}+(G+\omega_{1}+1)^{Y}-(G+\omega_{1})^{Y}\right)$$

Where

$$\omega_{1}=1+i\varphi$$

Now we have to calculate the second term of (5.1)

$$Ke^{-r\tau}\int(e^{y}-1)P_2(x+y)v(dx)$$

By applying the same technique we conclude that

$$\int(e^{y}-1)P_2(x+y)v(dx)=\frac{1}{2}k+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left[\frac{\exp(-i\varphi\ln(\mathrm{K}))f_2(x)m_2(\varphi)}{i\varphi}\right]d\varphi$$

Using the Proposition 4.3

$$m_2=C\cdot\Gamma(-Y)\left((M-\omega_2-1)^{Y}-(M-\omega_2)^{Y}+(G+\omega_2+1)^{Y}-(G+\omega_2)^{Y}\right)$$

Where

$$\omega_2=i\varphi$$

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