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ΔΙΑΤΡΙΒΗ

"Τα Volatility Futures και το μοντέλο των Grűnbichler-Longstaff. Μια εμπειρική ανάλυση"

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Section 1

Introduction and Research Questions

This dissertation examines the properties and the valuation framework of a specialized financial product, volatility futures. Volatility futures belong to the category of volatility derivatives – a new class[1] of derivative instruments whose underlying asset is volatility, specifically an implied volatility index. These products emerged to satisfy the need for instruments and ways to manage volatility risk, the exposure to changes in volatility of an asset. Volatility risk is an important risk behind many investors' positions and it is assumed to have played a major role in several financial debacles (eg. Barings, LTCM)

We are interested in examining the pricing performance of volatility futures under the model of Grűnbichler-Longstaff[2]. The model's accuracy will be assessed further within a value at risk study. These tests will regard the volatility futures contracts of the Chicago Board of Options Exchange which are written on the VIX Implied Volatility Index. Our aim is twofold: testing the model by examining how well it fits market prices and asserting whether a value at risk methodology is effective under the model and the respective process for volatility.

To examine the pricing performance we will implement a technique that is approved by econometric theory and is also common in applied research, calibration. This procedure involves fitting the model to a specific sample of market data in order to get estimated values of the model's unknown parameters and examining the pricing performance of the model in another sample where the deviation between derived model and real prices is calculated (out of sample performance). As for the value at risk study is concerned, we will assume a position in the VIX Index and in a volatility future contract and we will derive daily value at risk figures using a Monte Carlo Simulation approach. Then we will implement a back testing procedure in order to verify the validity of the value at risk model.

In order to have a measure of comparison, the same study will be conducted also within the context of another pricing model – a benchmark model. This model's assumptions are simpler compared to the one of Grűnbichler-Longstaff's and the pricing formula resembles the one of standard (equity) futures. The results of the two models will be compared with the aim to investigate which volatility process is supported the most by the data. The answer to this question is important not only in the context of volatility derivatives but also for the broad area of volatility modeling.

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[1] Various types of volatility derivatives have been introduced in some markets after 1997

[2] Grunbichler-Longstaff (1996), "Valuing Futures and Options on Volatility", *Journal of Banking and Finance 20*

To the best of our knowledge the empirical performance of volatility futures has not been studied thoroughly yet. A main reason for this is probably the fact that volatility futures appeared in the American market last year while their trading behavior in other markets was not good. Hence we are confident that this study will contribute significantly in the relative literature and will be of interest to practitioners as well.

The dissertation is organized as follows. In the next section we refer to certain important issues like volatility risk, stochastic volatility, implied volatility indices and volatility derivatives. In section 3 we describe the valuation framework of volatility futures, the two pricing models and the related processes of stochastic volatility and in section 4 we refer to value at risk. In sections 5 through 7 we present the data and methodology we will follow in order to examine the pricing performance and to conduct the value at risk study. The conclusions drawn are presented in the last section.

In this point I would like to thank my supervisor, Dr. George Skiadopoulos, whose ideas and guidance were crucial to the implementation of this dissertation. Many thanks go also to the other members of my committee, Professor A. Antzoulatos and Assistant Professor A. Benos whose teaching was inspiring. I am finally grateful to the Faculty of the Department of Banking and Financial Management for the fruitful cooperation we had during my under – and postgraduate studies in the University of Piraeus.

Section 2

Background

A. Volatility risk

Volatility risk is one of the main sources of risk an investor faces[3] and the more difficult to deal with. Volatility risk has played a major role in several financial disasters in the past 15 years. Long Term Capital Management is one such example as one of the hedge fund's trading strategies was to sell volatility on the S&P 500 index and other European indexes by selling options on the index. Their exposure was that volatility, as reflected in options premiums, could increase and they did not hedge that exposure. Another pronounce example is the volatility trading done my Nick Leeson in 1994 in the Japanese market. Leeson made big bets on the future direction of the Nikkei 225 using futures and options and his exposure to volatility was the main reason for the demise of Barings Bank.

In the traditional security analysis framework volatility is measured by the standard deviation of an asset or portfolio returns. This is a measure of dispersion, like the variance and it is used along with the mean to classify various alternative investment positions according to their risk-return profile or to identify the efficient frontier in mean variance portfolio theory. In derivatives and especially in Black-Scholes (BS) framework, volatility of a financial asset is defined as the standard deviation per unit of time of the continuously compounded returns and is an important, but also non observable, input to the BS formula. A sensitivity factor that captures the volatility risk exposure of a portfolio containing asset and/or derivatives on the asset is vega (or sigma, one of the so called Greeks). Vega is the rate of change in the price of the portfolio with volatility, as expressed by the partial derivative of the portfolio price with respect to volatility. The analogy to price risk is delta and a position with zero delta and vega is assumed to be delta-vega hedged.

Standard derivative products have the primary use of hedging price risk but they have also been used in various ways to hedge volatility risk. This approach is inefficient since it insures both types of risk and is also more expensive than a direct bet on volatility. The desired instrument should be a hedge against volatility risk only and should have cost less than the other alternatives. Some of the proposed instruments and strategies are an option on a straddle[4] (Brenner, Ou and Zhang [2002]) and a delta hedged position so that the hedging error is connected to volatility risk only (see Carr and Madan [1998] for an examination of trading volatility

[3] Price or directional risk is the other source and regards the investor's exposure to changes in the asset price

[4] A straddle is long position in a call and a put option written on the same underlying asset and with the same strike price.

strategies). Volatility derivatives are the alternative candidate for hedging volatility risk and one of these products is being examined in our study

B. Stochastic volatility

The empirical finding that volatility of financial asset returns tends to change stochastically[5]over time is closely related to the issue of volatility risk and has also implications for option pricing and risk management. There is evidence of this stochastic nature both in realized volatility and in volatility inferred from option prices (implied volatility).[6]

Regarding realized volatility the main findings are mean reversion and the so called clustering and leverage effects.

Mean reversion is a term for the finding (Scott[1987], French [1987], Merville and Pieptea [1989], Stein [1989], Harvey and Whaley [1992]) that volatility oscillates around a constant value and tends to revert to a long run mean.

The clustering effect is attributed to Mandelbrot (1963) who reported that periods of high (low) volatility tend to be followed by periods of high (low) volatility, without one however being able to predict the next change. In the econometrics literature this effect is closely related to autocorrelation and mainly heteroskedasticity of residuals as it was modeled by Engle(1982) and Bollerslev (1986). Their (G)ARCH model intended to capture the dynamics of the conditional variance of a regression's residuals but it appeared to be an important estimator of historical volatility and a good forecaster of realized volatility in financial time series as well (see e.g. Figlewski [1987]).

The leverage effect is about the negative relationship between changes in the price and changes in volatility of an asset. This relationship was first noted by Black (1976) and termed leverage effect by Christie (1982) who attributed the relationship to the fact that a drop in stock prices reflects a drop in the market value of equity so that an increase in the market value of debt or leverage is required to keep the whole market value of a firm stable. An increase in leverage is in turn an increase in a firm's risk as this is measured by volatility[7].

As for implied volatility, the findings that tend to reject the hypothesis of the BS model that volatility is constant through option's life are the implied volatility smiles and the term structure of implied volatilities.

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[5] A deterministic function of volatility is considered in the implied volatility function or implied binomial tree approach of Derman and Kani (1994) and Rubinstein (1994). See Dumas, Fleming and Whaley (1998) for an empirical examination.

[6] For a survey see Psychoyios, Skiadopoulos and Alexakis (2003).

[7] Figlewski and Wang (2000) state that this effect is really a "down market effect" as they observe that this effect is much weaker or nonexistent when positive stock returns reduce leverage and that there is no apparent effect on volatility when a firms leverage changes by a change in its outstanding debt or shares.

Implied volatility is the volatility inferred by the market price of an option and is calculated by inverting the option pricing formula given the market price. Empirical evidence shows that implied volatility is a function of both the strike price and the maturity of the option. The dependence of implied volatility on the strike price is found to exhibit a U-shaped pattern ('smile') mainly in foreign currency options and a downward slopping curve ('skew' or 'smirk') in equity index and futures options. Some reasons for these patterns are the stochastic nature of underlying asset's volatility, the presence of jumps in the underlying asset returns, the leverage effect and the high demand for out of the money put options after the 1987 crash in the U.S. stock market[8].

In addition to the smile, on any given date and for any given strike price the implied volatility varies across different expiries, forming so a term structure of implied volatilities in analogy to the term structure of interest rates. The term structure of implied volatilities can be either upward or downward sloping, mainly depending on the size of short term volatilities. The combination of implied volatility smile and the implied volatility term structure is the implied volatility surface and the above evidence suggests that it is not flat as the BS assumptions suggest. For the issue of implied volatility patterns see Rubinstein (1985), Stein (1989), Taylor and Xu (1994), Tompkins (1998), Skiadopoulos et al.(1999), Derman and Kamal (1997).

An important implication of stochastic volatility evidence is that the BS option pricing model is misspecified and also that the BS Greeks may be no longer valid[9]. Various stochastic volatility models have emerged (Hull and White [1987], Johnson and Shanno [1987], Wiggins [1987], Stein and Stein [19991], Heston [1993]). The main differences between the models are the process assumed to capture the volatility dynamics, the treatment of the issue of the market price of volatility risk and the constraint or not on the correlation between underlying asset returns and volatility.

Characteristic are the models of Hull-White and Heston. The valuation framework is similar (the usual Geometric Brownian Motion assumption about the asset dynamics and a mean reverting diffusion process for the asset's variance). Hull and White assume that volatility has zero systematic or market risk, while Heston accounts for a risk premium proportional to the level of current volatility. Hull and White derive closed form solution for standard European options only when the volatility is uncorrelated with the returns and in the case of correlation they value options numerically; Heston imposes no constraint on the correlation between returns and volatility and derives a closed form solution as well as formulae for the sensitivity factors. The two models results are partially consistent with the implied volatility patterns we have referred to.

An extension to stochastic volatility models is the incorporation of jumps in the underlying asset's returns or even in volatility (see eg. Bates [1996a], Bakshi-Chao-Chen [1997], Duffie-Pan-Singleton [2000]).

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[8] Rubinstein terms the latter phenomenon "crashophobia" as the demand for put options was significant even in periods of bullish markets.

[9] Derman (1999) provides an approximation for the Greeks formulae.

C. Implied Volatility Indices

An implied volatility index is an index that contains the implied volatility inferred from options on a stock index. For example VIX of the Chicago Board of Options Exchange (CBOE) is such an index computed from implied volatilities of S&P 500 index options.[10]

Implied volatility indices have appeared since 1993 with VIX being the first[11]. In 1994 the German Futures and Options Exchange launched an implied volatility index (VDAX), based on DAX index options and in 1997 the French Exchange Market created two indices (VX1 and VX6) that reflect implied volatility of the CAC-40 index options. In 2000 CBOE introduced the Nasdaq Volatility Index (VXN) and recently the Greek implied volatility index was constructed. Regarding the literature about the construction and properties of implied volatility indices see Whaley (1993, 2000), Fleming et al.(1995), Moraux et al.(1999), Wagner and Szimayer (2000, 2004), Skiadopoulos (2004).

A volatility index serves two primary purposes. First it is an "investor's fear gauge"[12] as it provides an up-to-minute indicator of the market consensus estimate of expected future stock market volatility – a measure of stock market risk. The descriptor "fear" arises from the fact that investors are averse to risk and such fears are reflected to stock prices. So if expected stock market volatility increases, investors demand higher rates of return on stocks and stock prices fall. This is an alternative explanation of the negative relationship between returns and volatility. Second, it provides a benchmark upon volatility derivatives can be written.

The volatility index can also be used for Value-at-Risk purposes (Giot [2002b]), to conduct trading strategies in the stock markets using the so called Bollinger bounds and to forecast the future market volatility. The informational content of implied volatility has been found to dominate that of historical information (see Day and Lewis[1992], Canina and Figlewski [1993], Fleming et al[1995], Guo[1996]). Also one can examine the relationship between various implied volatility indices in order to test whether there are implied volatility spillovers among stock markets (eg. Skiadopoulos [2004], Wagner and Szimayer [2004]).

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[10] To be more specific, an implied volatility index reflects usually a synthetical at the money option's implied volatility with short term maturity (eg. VIX has a maturity of 30 calendar days and VDAX of 45 calendar days).

[11] CBOE changed the construction methodology in 2002 and the initially constructed index is now named VXO.

[12] This term is attributed primarily to Whaley (2000).

D. Volatility Derivatives

The need for creating specialized financial instruments with main purpose to hedge volatility risk emerged mainly after the 1987 crash and led to the creation of volatility derivatives, starting from volatility options.

Brenner and Galai (1989, 1993) first suggested options written on a volatility index. Various papers regarding the valuation of volatility options followed, like Whaley (1993), Grűnbichler-Longstaff (1996), Detemple-Osakwe (1999), Heston-Nandi (2000). In 2004 the CBOE introduced volatility options with underlying asset the VIX implied volatility index. The payoff of a volatility option is the difference between the value of (implied) volatility on the expiry (European style) or on any given day (American style) and the strike price which is measured in volatility units.

Volatility futures were the first volatility derivative product that was introduced in an organized market. The OMCX, which is the London based subsidiary of the Swedish Exchange OM, launched volatility futures at the beginning of 1997. in the same year the German Exchange introduced the "VOLAX FUTURE" based on VDAX, but the trading of this contract ceased in 1998. On March 26 2004 the trading in futures on the CBOE volatility index began on the CBOE Futures Exchange (CFE) and this particular product is the focus of our study. Grűnbichler-Longstaff (1996) have created a model to value volatility futures and Locarek-Junge and Roth (1998) have examined the hedging performance of the Volax future. The trading of volatility futures is similar to that of standard futures with the underlying being a volatility rather than an equity index.

There are also volatility or variance swaps as products of over-the-counter markets, mainly in U.S.A. The valuation of these products has been studied among others by Heston and Nandi (2000), Javakeri, Wilmott and Haug (2002), Howison Rafailidis and Rasmussen (2002). A volatility swap contract pays the buyer the difference between the realized volatility and the fixed swap rate determined at the outset of the contract.

Data show that the market for volatility derivatives has not flourished yet. Whaley (1998) considers clientele effects (the possibility that investors will have greater interest in buying than selling, living thus market makers with big short positions) in combination with jump risk and a hesitation from behalf of institutional investors and market makers as possible explanations.

Section 3

Model description

1. Grűnbichler and Longstaff (1996)

Grűnbichler and Longstaff (G-L) derive closed form valuation expressions for a variety of volatility derivatives (volatility futures, volatility options and options on volatility futures) and their model can be extended to other types of volatility apart from stock index volatility like currency or interest rate volatility.

A. The valuation framework

They assume that volatility follows a Mean Reverting Square Root Process (MRSRP):

$$dV_t = (a - kV_t)dt + \sigma\sqrt{V_t}dZ_t \qquad (3.1)$$

under the objective probability measure *P*

where *a,k* and σ are constants and Z_t is a standard Wiener or Brownian Motion process. (3.1) is a stochastic differential equation where the first component is the drift term and th second the diffusion term. V can be either the instantaneous volatility or the volatility implied from option pricing.

This framework is similar to that used by previous papers regarding the pricing of options in stochastic volatility regimes, like Hull and White [1987], Johnson and Shanno [1987], Wiggins [1987], Stein and Stein [19991], Heston [1993]. This process has also been used to capture the dynamics of the short term interest rate (see Cox et al[1985b]).

The specification of the stochastic volatility dynamics is also consistent with many of the observed properties of stock volatility, especially mean reversion and clustering we referred to in section 2. Regarding the drift parameters, k is the rate or speed of mean reversion of volatility to a long run mean which is $\frac{a}{k} \cong V_L$. If V_t is larger than V_L at any time then (V_L-V_t) is negative so that in the next instant V_t is likely to fall. The opposite is expected to happen if V_t is smaller than V_L. Thus volatility fluctuates around the level V_L. The greater value *k* has the faster *Vt* tends to V_L and on the other hand a small value of *k* indicates autocorrelation and volatility clustering (volatility is persistent). σ is the diffusion coefficient and expresses the volatility of volatility. σ could take any value but the term $\sqrt{V_t}$ ensures that volatility will not take any negative value.

The probability density function of *V* is non-central chi-square χ^2 ($2cV_t$, 2q+2, 2u) with 2q+2 degrees of freedom and a parameter of non-centrality 2u (see Feller [1951], Cox, Ingersoll and Ross [1985b]). We have also (see Skiadopoulos et al. [2003] for a more detailed analysis) that as t approaches infinity, volatility and its expected value converges to the long run mean while the variance of volatility σ^2 converges to a constant positive number.

The authors assume also that security markets are perfect, frictionless and available for continuous trading and that the risk less interest rate r is constant. In this framework, which is typical for contingent claims valuation, they consider the valuation at time zero of a contingent claim with a payoff $B(V_T)$ at time *T* depending only on V_T. The current value of this claim, *A(V,T)*, satisfies the fundamental valuation equation:

$$\frac{\sigma^2}{2} V A_{VV} + (a-b)A_V - rA = A_T \qquad (3.2)$$

subject to boundary condition $A(V_T,0) = B(V_T)$.

(3.2) resembles the Black-Scholes-Merton partial differential equation where V is the underlying asset.

If *D(T)* denotes the current price of a T-maturity risk less unit discount bond, then the solution to (3.2) can be expressed as

$$A(V,T) = D(T)*E_Q[B(V_T)] \qquad (3.3)$$

where *Q* is the risk neutral probability measure.

Deriving a closed form solution via (3.2) is the partial differential equation (p.d.e.) approach and via (3.3) is the equivalent martingale measure (e.m.m.) approach[13]. The two approaches are equivalent.

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[13] p.d.e. approach involves constructing a self financing portfolio which replicates the contigent claim or a hedged portofolio consisting of the claim and the underlying asset. Assuming that each component of this portfolio follow a certain processes leads among with other assumptions to a P.D.E. similar to (2).
e.m.m. approach is based on the fundamental theorem of arbitrage pricing that the lack of arbitrage opportunities is synonymous to the existence of a probability measure under which all assets discounted prices are martingales. Martingales are random variables whose future variations are completely unpredictable given the current information set – the conditional expected value of a martingale is its current value. The concept of martingality is related to that of a "fair" game or market. Thus one can derive expressions like (3) where the expected payoff at maturity is being discounted in the risk less rate as the Martingale Representation Theorem implies. Pricing is made in the risk neutral measure which is equivalent to the objective measure by Girsanov's theorem. (see eg. Neftci[2000] for an introduction to the mathematics that underlie derivatives theory)

Under Q the process for V is now:

$$dV_t = (a - bV_t)dt + \sigma\sqrt{V_t}dZ_t^Q \qquad (3.4)$$

where **b = k+ζ**, $\zeta \in [0,+\infty]$

ζ is a constant parameter and appears in the market price of volatility risk λ(S,V,t). Since volatility is not the price of a traded asset it can not be replicated by a self financing portfolio and thus the market price of volatility risk has to be determined. The authors adopt Heston's (1993) assumption that the volatility risk premium is proportional to the current level of volatility so that $\boldsymbol{\lambda(S,V,t)=\zeta * V_t}$.

This logic is attributed to Breeden's (1979) equilibrium consumption-based model where the risk premium of volatility is

$$\lambda(S,V,t) = \gamma cov(dV, \frac{dC}{C}) \qquad (3.5)$$

where $C(t)$ is the consumption rate and γ is the relative risk aversion of the representative investor. If consumption growth has a constant correlation with the spot asset return, the risk premium can be represented as proportional to *V*.

The same spirit lies in the general equilibrium model of Cox, Ingersoll and Ross (1985a,b) where stochastic consumption follows a MRSRP, like the one in G-L's model for stochastic volatility and has constant correlation with the spot asset return.

B. The volatility futures model

Let *Ft(V,T-t)* denote the price at time t of a future contract on *Vt* with maturity *T-t*. From equations (3.2) and (3.3) we have that

Ft(V,T-t) = (a/b)[1-exp(-b(T-t))] + exp(-b(T-t))Vt (3.6)

(3.6) is also the expected value of the analytic solution of (3.4):

$$V_t = \frac{a}{b}\left(1 - \exp(-bt)\right) + V_0 \exp(-bt) + \sigma \exp(-bt)\int_0^t \exp(-bs)\sqrt{V_s}dW_s \qquad (3.7)$$

So that

$$E(V_t) = \frac{a}{b}\left(1-\exp(-bt)\right) + V_0\exp(-bt) + \sigma\exp(-bt)E\left(\int_0^t \exp(-bs)\sqrt{V_s}\,dW_s\right) \Leftrightarrow$$

$$E(V_t) = \frac{a}{b}\left(1-\exp(-bt)\right) + V_0\exp(-bt)$$

Since the integral in (1,7) is a stochastic (Ito) integral and a martingale so its expected value conditioned at time 0 is zero.

In this model volatility futures prices are exponentially weighted averages of the current value of volatility and the long run mean of the risk adjusted process (4). As we approach expiration, the futures price converges to the current value of *V*, while as time to maturity converges to infinity the futures price convert to the long run mean $\frac{a}{b}$.

C. Properties of volatility futures prices

Volatility futures prices have some interesting properties which differ from those implied by the cost-of-carry model of standard futures.

For example, as volatility converges to zero the volatility futures prices does not converge to zero also, like a standard futures price does when the underlying asset is minimized. This is related to the mean reversion of the volatility process and thus volatility futures prices are bounded above zero.

The basis for volatility futures is given by

$$\text{Ft} - \text{Vt} = \left(\frac{a}{b}-\text{Vt}\right)[1-\exp(-b(T-t))] \qquad (3.8)$$

As *T-t* converges to zero the basis converges to zero and as *T-t* converges to infinity it converges to the first term, the difference between the current value of volatility and its long run mean. This means that the volatility futures basis can be either positive or negative.

Last but not least, the hedging effectiveness of volatility futures is a function of their maturity as the partial derivative of *F(V,T-t)* with respect to *V* (the delta) is

$$\frac{\partial F(V, T-\tau)}{\partial V} = \exp(-b(T-t)). \qquad (3.9)$$

By this and by the fact that when maturity converges to infinity futures prices converge to the long run mean and are unaffected by the current value of volatility, one can conclude that longer-term contracts may not be effective instruments for hedging volatility risk. Thanks to mean reversion property any change in the current value of volatility is expected to be partially reversed prior to the expiration of the contract.

2. Benchmark Model

Whaley[14] (1993) uses the Black's (1976) futures model to price volatility options written on an implied volatility index. He considers volatility futures options with a zero cost of carry. Hence, he assumes implicitly that the volatility index is a traded asset that follows a Geometric Brownian Motion Process.

A. The valuation framework

The volatility process is a GBMP:

$$dV_t = \mu V_t dt + \sigma V_t dW_t \qquad (3.10)$$

under the objective probability measure P

μ and σ in (3.10) are constants and W_t is a standard Brownian Motion. M is the drift parameter and is defined as the expected return on the volatility index per unit of time, while σ is the diffusion parameter and denotes the volatility of the volatility index per unit of time. Both the drift and the diffusion coefficients are proportional to the current level of volatility. Hence, if volatility is currently high it is likely to remain high in the next instant (*dt* later). Similarly, the closer V_t gets to zero, the smaller the increments dV_t. Finaly, volatility under GBMP grows exponentially as a function of time, so that if it starts from a positive number it can never go negative.

Apart from the latter which is a desired property of volatility since in theory is a positive measure, the adoption of this process as the dynamics of instantaneous or implied volatility is a rather simplified assumption as other properties of GBMP show. Volatility is unbounded for large values of μ while for small values of the drift parameter it converges to zero. Both these properties are unrealistic for volatility modeling purposes since they contradict the empirical evidence, such as the mean reversion of volatility. Due to this, volatility should be bounded and it should revert to its long run mean.

Equation (3.10) implies that volatility is log normally distributed:

[14] Whaley R. (1993) "Derivatives on Market Volatility: Hedging Tools Long Overdue." *Journal of Derivatives* 1, pp. 71-84.

$$V_t \sim \ln N\left[V_0 e^{\mu t}, V_0^2 e^{2\mu t}\left(e^{\sigma^2 t - 1}\right)\right] \quad (3.11)$$

while log returns of volatility follow a normal distribution[15]:

$$\ln V_t \sim N\left[\ln V_0 + \left(\mu - \frac{\sigma}{2}\right)t, \sigma\sqrt{t}\right] \quad (3.12)$$

Under the typical Black-Scholes-Merton (BSM) (1973) framework which assumes perfect, frictionless, arbitrage free and continuous trading markets, the current value at time zero *f(V,T)* of a contingent claim with a payoff $X(V_T)$ at time *T* depending only on V_T, satisfies the following partial differential equation:

$$f_t + rVf_V + \frac{\sigma^2}{2}V^2 f_{VV} = rf \quad (3.13)$$

subject to the boundary condition $f(V_T,0) = X(V_T)$

The solution of (3.13) subject to the respective condition at expiration is the value of the volatility derivative , e.g. volatility call or volatility future. Also the value of a derivative written on V_t can be expressed as the discounted value of the expected payoff at maturity:

$$f(V,T) = D(T)*E_Q[X(V_T)] \quad (3.14)$$

where *D(T)* is the current price of a T-maturity risk less unit discount bond and *Q* is the risk neutral probability measure.

In the framework of risk neutral valuation the process for volatility is now:

$$dV_t = bV_t dt + \sigma V_t dW_t^Q \quad (3.15)$$

where *b* is the drift parameter under the risk neutral measure and incorporates the volatility risk premium. Note that as before in the MRSRP of the GL model, only the drift coefficient is altered by the change of probability measure; the diffusion coefficient remains the same. This is a basic principle in changing measures and is implied by the Girsanov's theorem which also states that *P* and *Q* are equivalent probability measures.

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[15] The Normal and the Log Normal distributions are related through the following lemma from statistics: If $Z \sim N(\mu, \sigma^2)$ then $Y = exp(Z) \sim lnN(m,z^2)$, where $m = \exp(\mu + \frac{\sigma^2}{2})$ and $z^2 = \exp\left[\left(2\mu + \sigma^2\right)\left(e^{\sigma^2} - 1\right)\right]$. Here $Z = lnV_t$ and $Y = exp(lnV_t) = V_t$.

B. The volatility futures model

Let *Ft(V,T-t)* denote the price at time t of a future contract on *V* with maturity *T-t*. from equations (3.12) and (3.13) we have that:

$$\mathbf{Ft(V,T) = Vexp(b(T\text{-}t))} \qquad \mathbf{(3.16)}$$

(3.16) is also the expected value of the analytic solution of (3.15):

$$V_t = V_0 \exp\left[\left(b - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] \qquad (3.17)$$

So that

$$E(V_t) = V_0 \exp(bt) E\left[\exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right)\right] \Leftrightarrow$$
$$E(V_t) = V_0 \exp(bt) \exp\left(\sigma W_0 - \frac{\sigma^2}{2}0\right) \Leftrightarrow$$
$$E(V_t) = V_0 \exp(bt) \exp(0) = V_0 \exp(bt)$$

since $\exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right)$ is a martingale, as the solution of a drift less stochastic differential equation ($dV_t = \sigma V_t dW_t$, with V_0=1), and its expected value conditioned at time 0 is one.[16]

(3.16) resembles the pricing formula for standard futures:

$$F_t = S_t \exp[(r-q)(T-t)] \qquad (3.18)$$

where *r* is the risk free rate – the drift parameter under the risk neutral measure – and *q* is zero when *S* denotes an equity, the dividend yield when *S* denotes a stock index and the foreign risk free rate when *S* denotes a foreign currency.

Thus, under the benchmark model volatility future prices are exponentially weighted averages of the current value of volatility, as in standard future prices. As we approach expiration, the volatility futures price converges to the current value of

[16] The mathematical formulation of the martingale property is $E[X_t/F_s] = X_s$ with s<t, where X_t is a stochastic process with time index *t=0,1,…,T* and F_t is a σ-algebra denoting the filtration of information. In the example above we are exploiting the property that $E[X_t] = E[X_t/F_0] = X_0$ since conditioning at time 0 includes the minimum information and is equivalent to not conditioning at all.

volatility, while as the time to maturity converges to infinity, the future price concerts also to infinity.

C. Properties of volatility futures prices

The properties of volatility futures prices are similar to the properties of standard futures prices.

As volatility converges to zero, the volatility futures price also converges to zero, in addition to what happens under the GL model.

The basis for volatility futures is given by

$$Ft - Vt = Vt[\exp(b(T\text{-}t)\text{-}1] \quad (3.19)$$

As T-t converges to zero the basis converges to zero and as T-t converges to infinity it converges to infinity as well. prior to expiration the volatility futures basis can be either positive or negative.

The delta of the volatility futures is exp(b(T-t)) and has positive relationship with the time to maturity. Hence, unlike volatility futures under the GL model, these instruments are effective tools for long term hedging, as the standard futures are.

Section 4

Value at Risk

A. Introduction

Value at Risk (VaR) is an attempt to provide a single number summarizing the total risk in a portfolio of financial assets for senior management. It has become widely used by corporate treasurers and fund managers as well as by financial institutions. VaR is thus a measure for market risk – the risk related to the uncertainty of a financial institution's earnings on a portfolio caused by changes in market conditions such as the price of an asset, interest rates, market volatility and liquidity. Market risk arises whenever a financial institution actively trade assets, liabilities and derivatives rather than holding them for longer term investment, funding or hedging purposes. Nevertheless, with the increasing securitization of bank loans more and more assets have become liquid and tradable (eg. mortgage-backed securities) so that VaR regards investment portfolio as well (for instance the concept of credit value at risk).

Apart from management information, setting limits, resource allocation and other reasons that have to do with a firm's strategy, crucial for the evolution of value at risk was the inclusion of market risk in the determinedness of a financial institution's required level of capital. The Bank of International Settlements capital adequacy regulation, starting in 1988 with the first Basle Accord, requires a financial institution to hold capitals as a proportion (ratio) of the market value of its assets adjusted to the level of total risk the institution faces. This risk-based capital ratio initially took account only credit risk but a revision of the accord in 1998 incorporated market risk as well. In 2001 the New Basle Capital Accord (or Basle II) included operational risk and also allowed banks to use internal models in order to measure their various risks. The Internal Models Approach lead to the development of various models, mainly VaR and its three approaches: the RiskMetrics Approach, the Historical or Back Simulation Approach and the Monte Carlo Simulation Approach.

B. Definition of VaR

VaR is defined as the largest loss that a portfolio is likely to bear if it is left unmanaged during a fixed holding period. In other words VaR asks the simple question "How bad can things get?". The answer it gives is that "We are X% certain that we will not lose more than V amount in a T horizon.".

More specific, if we define α as the significance level of VaR – the associated probability which corresponds to the frequency with which a given level of loss is expected to occur and T as the portfolio holding period, then the α% T-period VaR is the number V such that the probability of losing V or more over T equals α%:

$$Prob(\Delta_T\Pi_T <= -V) = \alpha \Leftrightarrow Prob(\Delta_T\Pi_T <= -VaR\alpha,T) = \alpha \quad (4.1)$$

or equivalently

$$Prob(\Delta_T\Pi_T > -VaR) = 1-\alpha = X \quad (4.2)$$

where $\Delta_T\Pi_T = \Pi_{t+T} - \Pi_t$ is the change in the portfolio's value (Profit/Loss – P/L)

Therefore VaR is the lower α-quantile of the projected distribution of P/L over the target horizon.

C. Methods of Calculation of VaR

There are three major approaches that have been followed:

- The RiskMetrics or the Variance – Covariance Approach
- Historical or Back Simulation Approach
- Monte Carlo Simulation Approach

These methods differ mainly in the distributional assumptions for the returns of the risk factors – the factors that drive the changes in a portfolio's value. Typical risk factors are stock indices, interest and exchange rates. The variance – covariance and the Monte Carlo approaches require specific distributional assumptions. The first is a parametric method which requires a distribution for the factor returns and the second is a semi parametric method, requiring a data generating process for each of the risk factors. On the other hand, the historical simulation approach requires no distributional assumption, rather it projects the historical or empirical distribution of the risk factors returns to the future period over which the VaR measure is calculated.

1) Variance – Covariance Methods

J.P. Morgan first developed RiskMetrics in 1994. the publications "Introduction to RiskMetrics" (1994) and "RiskMetrics Technical Document" (1996) summarize the bank's approach in risk management and essentially in calculating value at risk. The RiskMetrics method, also known as the variance – covariance method, is divided into two analytical approaches to the measurement of VaR: the simple or delta – normal VaR for linear instruments and the delta – gamma VaR for non linear instruments. The terms "linear" and "non linear" describe the relationship between a position's underlying returns and the position's relative change in value.

In the delta – normal VaR approach the assumption being made is that returns on securities follow a multivariate normal distribution[17] and that the relative change in a portfolio's value is a linear function of the underlying return. Such a portfolio could be the one that that is a set of stocks, currencies or commodities.

[17] VaR models assuming other distributions more realistic in comparison to the empirical distribution of asset prices, such as the Student's – t distribution, have also been developed

Assuming that the portfolio's P/L is normally distributed :

$\Delta_T \Pi_T \sim N(\mu_t, \sigma_\tau^2)$ and that $Prob(\Delta_T \Pi_T <= - VaR\alpha,T) = \alpha$

then the standard normal transformation yields that

$$Prob\left(\frac{\Delta_T \Pi_T - \mu_t}{\sigma_t} \leq \frac{-VaR\alpha, T - \mu_t}{\sigma_t}\right) = \alpha$$

If $Z_t = \dfrac{\Delta_T \Pi_T - \mu_t}{\sigma_t} \sim N(0,1)$ then

$$Prob\left(Z_t \leq \frac{-VaR\alpha, T - \mu_t}{\sigma_t}\right) = \alpha \text{ or equivalently } \frac{-VaR\alpha, T - \mu_t}{\sigma_t} = - Z\alpha$$

which gives us the formula for VaR:

$$\boxed{VaR\alpha, T = Z\alpha\sigma_t - \mu_t} \quad (4.3)$$

$Z\alpha$ is the ath percentile of the standard normal density (e.g. for $\alpha = 1\%$ $Z\alpha = 2{,}33$ and for $\alpha = 5\%$ $Z\alpha = 1{,}65$), μ_t and σ_t are the mean or the expected value and the standard deviation of the portfolio P/L respectively:

$$\mu_t = \mathrm{E}(\Delta_T \Pi_T) = \sum_{i=1}^{N} witRit$$

$$\sigma_t = \sqrt{Var(\Delta_T \Pi_T)}$$

$$Var(\Delta_T \Pi_T) = \sigma_t^2 = \sum_{i=1}^{N} wi^2 \sigma^2(Rit) + 2\sum_{i=1}^{N} \sum_{j<1} \rho ijwiwj\sigma(Rit)\sigma(Rjt)$$

where i=1,2, …,N is the number of assets in the portfolio, wi is the amount invested on each asset, Ri is the return of each asset and ρij is the correlation coefficient between the returns of assets i and j.

Usually μ_t is assumed to be zero, especially for 1 day VaR as it is empirically documented that the daily expected return on financial assets is very small, especially in comparison to the standard deviation.

Both μ_t, σ_t should correspond to the T – period of VaR, so either they are calculated directly from a sample with T frequency or more often the daily mean and standard deviation are calculated. In the latter case the 1 day VaR is derived and in order to have the T – period Var (where T is in days) the square root of time rule is applied:[18]

$$VaR\alpha,T = \sqrt{T}VaR\alpha,1$$

The 1 day Var (Varα,1) is also called DEAR (Daily Earnings At Risk).

The delta – normal approach is inappropriate when the relationship between an instrument and its underlying price/rate is non linear. Linear payoffs are characterized by a constant slope, delta – the first derivative of the instrument with respect to the underlying risk factor. Their convexity as measured by the gamma factor – the second derivative with respect to the risk factor – is zero. On the other hand, non linear payoffs have a non zero gamma, thus convexity should be considered as a second order effect.[19] Types of instruments with non linear payoffs are a bond and an option. In the first the relationship between a change in the bond's value and a change in the interest rate – the bond's risk factor – is convex, while in the second the relationship between the option's premium and the underlying asset is also convex.

In such a case a delta – gamma approach should be used. This improves on the delta – normal method by using a second order Taylor series expansion so as to capture the non linearity as it is expressed primary through the gamma factor. In order to conduct the method a pricing model for each of the portfolio's instruments is required, upon which the Taylor series expansion is applied.

The RiskMetrics approach is quick and relative simple, especially in the linear case, to compute. On the other hand, its assumption that the portfolio P&L distribution is normal at any point in time does not always hold. Also very crucial is the accurate estimation of the portfolio's variance – covariance matrix.

2) Historical Simulation

Historical simulation is one popular way of estimating VaR. It involves using past data in a very direct way as a guide to what might happen in the future - historical data are used to build an empirical density for the portfolio P&L.

The essential idea is to take the current market portfolio of assets and revalue them on the basis of the actual prices (returns) that existed on those assets yesterday, the day before that and so on. Frequently financial institutions calculate the market or

[18] This rule is based on the assumption that returns are i.i.d. (independent and identically distributed) which in practice may not hold.

[19] Apart from delta and gamma there may be other sensitivity factors, especially in options where we have vega and theta as well.

value risk of their current portfolios on the basis of prices that existed on each of the last 500 days. Then they calculate the 1 or the 5 percent worst case, that is the portfolio value that has the 5^{th} or the 25^{th} lowest value out of 500. In other words in only 1 or 5 percent of the time would the value of the portfolio fall below this number based on recent historic experience of equity, interest and exchange rate changes.

The procedure can be summarized in the following basic steps:

1. Identify the basic market factors and obtain a pricing formula that relates the factors to the price of the portfolio.
2. Obtain historical values of the market factors over the last N periods. The frequency of the sample should correspond to the T period of VaR.
3. Subject the current portfolio to the changes in the market factors experienced on each one of the N periods and we calculate the P&L per period.
4. The empirical T period P&L density is obtained by building a histogram of $\Delta_T \Pi_T = \Pi_{t+T} - \Pi_t$.
5. The historical VaRα,T is the lower αth percentile of this distribution

Usually a daily frequency is used to collect a sample of N days back and in order to calculate VaR on each of the T days ahead, the N day sample is rolled over forward as we repeat the procedure on every day.

In comparison to the variance – covariance method, this method's obvious benefit is that we do not have to calculate standard deviations and correlations, or to assume normal distributions for asset returns. A second advantage is that since we obtain the entire empirical P&L density we are able to derive the worst case scenario number and not only the α% value which the previous method provides.

The main disadvantage of the historical simulation is the implicit assumption that the pattern that prevailed in the past will also prevail in the future as well. This assumption is highly questionable, especially if the VaR forecast regards a large T period. Also, the result of the procedure is sensitive to the historic (N) period chosen. If the N number is small from a statistical view then there will be a very wide confidence band (or standard error) around the estimated VaR. A very large historic period may however not be appropriate as well since the far away past observations may have little relevance to the current market conditions.

This trade off provides the risk manager with a difficult modeling problem. One approach could be to weight past observations unequally, so that more recent observations have higher weights. Another approach is the Monte Carlo simulation which generates additional observations that are consistent with recent historic experience.

3) Monte Carlo Simulation

Like historical simulation, Monte Carlo simulation is a numerical method which is used to produce different scenarios for the financial asset price on a target day. This time the scenarios are generated in a random fashion rather than from historical data.

More specific, a process in a form of a stochastic differential equation is assumed to be the data generating process of an asset's price and a distribution underlies the stochastic component or the diffusion factor of the process. The process is discretized and random numbers are drawn from the corresponding distribution. For each number a price for the asset is generated through the discretized process or more accurately through the analytic solution of the process. Usually the diffusion factor is a Brownian Motion and the underlying distribution is the standard normal. The following example is typical:

- Assume a Geometric Brownian Motion for the price S of an asset:

$$dS = \mu S dt + vSdW$$

where μ is the drift or the expected return and v the annualized diffusion or standard deviation of the asset's returns, while dt is a very small time increment and dW is a standard Brownian Motion.

- Discretize the process

$$\Delta S_t = S_{t+\Delta t} - S_t = \mu S_t \Delta t + vS_t \varepsilon \sqrt{\Delta t}$$

where S_t is the value of the asset at time t, $\Delta t = \frac{T-t}{n}$ is the time interval in an annual basis and $\varepsilon \sim N(0,1)$ is the random number.

- Get a random number and update the asset price at each time step using the random increments.

- It is more accurate to simulate the path via the solution of the process:

$$S_{t+\Delta t} = S_t x \exp\left[\left(\mu - \frac{v^2}{2}\right)\Delta t + v\varepsilon\sqrt{\Delta t}\right]$$

When more than one process has to be simulated simultaneously, for instance one process for each asset in a portfolio, the correlation between the various asset prices must be taken into account. Usually, this is achieved using the Cholesky decomposition – a method that builds a multivariate set of random variables from simple blocks consisting of i.i.d. variables.[20]

There are some important issues in Monte Carlo simulation. The first is that accuracy requires a large number of scenarios but requires as well more computationally efficiency. This trade off is resolved by the so called variance reduction methods – procedures that can lead to important savings in computation time while preserving accuracy. Such procedures are the Antithetic Variable, the Control Variate, the Importance and the Stratified Sampling. The first two procedures are commonly used in applications of MC simulation in valuing derivatives.

Another important issue is the fact that sometimes the generated random numbers may repeat themselves after a finite number of samples, or may exhibit serial autocorrelation. Thus, the choice of the random number generator is crucial and an alternative that has become popular is the use of a deterministic scheme as researchers have realized that the sequence of points does not have to be chosen randomly. This method is known as Quasi Monte Carlo simulation and some deterministic schemes are the low discrepancy frequencies, the Halton sequences and the Sobol numbers. A drawback of this method is the fact that accuracy cannot be assessed easily since the draws are not independent.

As for the Monte Carlo simulation's application in VaR, the idea is similar to the one of historical simulation where the empirical distribution is replaced by the simulated distribution – the distribution of simulated asset prices, thus simulated portfolio's P&L. The basic steps are the following:

1. Assume a stochastic process for the evolution of each of the portfolio's assets or the risk factors of the assets over time. In the latter case a pricing formula that relates the risk factors to the assets is needed.
2. Apply Cholesky decomposition and perform many simulation runs (usually at least 10000)
3. For each simulation run result calculate the new portfolio value and build the simulated P&L distribution – the histogram of $\Delta_T \Pi_T = \Pi_{t+T} - \Pi_t$, where Π_{t+T} is the simulated value and Π_t the realized value used as a starting point to generate Π_{t+T}.
4. The MC VaRα,T is the lower αth percentile of the simulated distribution.

The main advantages of this approach in calculating value at risk is that it can accommodate various type of processes relative to the empirical features of each risk

[20] There are also the eigenvalue decomposition and the singular value decomposition. In comparison to the Cholesky decomposition the other methods are computationally more intensive but they can be applied in cases where the Cholesky decomposition can not be applied. Such a case is that when *R*, the matrix which is decomposed into its Cholesky factors, is positive semi-definite rather positive definite as the Cholesky decomposition requires.

factor and that it can capture the presence of non normality.[21] Its disadvantages are the requirement for an accurate estimation of the variance – covariance matrix, as in the RiskMetrics approach, and the fact that is computationally expensive, especially when the portfolio under consideration consists of many assets.

Also, the issue of model error arises, as the assumed process should be the one that is the closest to the true data generating process of each risk factor and an appropriate estimation technique should be applied to estimate the parameters of the process. Model error is also present in the variance – covariance method, specifically in the delta – gamma approach when a pricing model is assumed to connect the asset under consideration and the underlying risk factor and the expansion is applied upon this model.

The following table summarizes the pros and cons of each of the three approaches in calculating VaR:

Method	**Advantages**	**Disadvantages**
Variance - Covariance	-it is quick and simple to compute	-it assumes that the portfolio P&L distribution is normal at any point in time
		-it requires accurate estimation of the portfolio variance-covariance matrix
Historical Simulation	-it does not require specific assumption about the analytic form of the portfolio P&L distribution	-it assumes implicitly that the pattern that prevail in the past will continue in the future as well
	-it provides the entire empirical P&L density and not only the a percentile	-it is sensitive to the historic period chosen
Monte Carlo Simulation	-it can accommodate various type of processes	-it requires estimation of the variance-covariance matrix
	-it can capture the presence of non normality	-the issue of model error regarding the specification of the process and the estimation of its parameters appears
	-it can provide multiple scenarios for the risk factors in comparison to historical simulation	-it is computationally intensive

[21] The MC VaR coincides with the variance – covariance VaR in the case that normality is assumed, that is when a Geometric Brownian Motion is adopted for the process of each asset.

D. Back testing of VaR

Since there is a divergence in the VaR methodologies and their application across banks and firms, it is useful to test the performance of VaR models. Moreover this is required by regulators for financial institutions, specifically by the Basel Capital Accord. Such testing is often referred to as 'back testing'. Back testing involves testing how well the VaR estimates would have performed in the past as the procedure requires comparison between the realized losses and the VaR estimates over a specific (past) sample.[22]

Many banks that use VaR models routinely perform simple comparisons of daily profits and losses with model – generated risk measures to gauge the accuracy of their risk measurement systems. However, the development of more sophisticated back testing techniques is just in the beginning and there are considerable differences in type of tests performed. Before presenting the test themselves we note that there are a number of difficulties, with the general approach to back testing which uses realized profit and loss results. The most fundamental of these arises from the fact that such back testing attempts to compare static portfolio risk with a more dynamic revenue flow. In practice bank's portfolios are rarely static. Hence a back test should be based on a comparison of VaR figure against the hypothetical changes in portfolio value that would occur if end-of-day positions were to remain unchanged. Further difficulties in conducting back tests arise because the realized profit and loss figures produced by banks typically include fee income and other income not attributable to position taking. The objective of back testing is however to compare measured position risk taking with pure position taking revenue.

There are several back testing procedures some of which account for unconditional and others account for conditional coverage. The purpose of the back testing measures is twofold. First, to test whether the average number of VaR violations or exceptions (an exception occurs if the predicted VaR is not able to cover realized loss on the given time position) according to a sample period is statistically equal to the expected one. Second, given the fact that an adequate model must wide the VaR forecasts during volatile periods and narrow them otherwise, it is necessary to examine if the violations are also randomly distributed.

The most popular back testing procedures are the following:

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[22] Apart from back testing there is also stress testing which involves estimating how the portfolio would have performed under some of the most extreme market moves seen in the last 10 to 20 years. It can be considered as a way of taking into account extreme events that do occur from time to time but are virtually impossible according to the probability distributions assumed for market variables. Thus stress testing 'goes beyond VaR' as does not stop at the (1-α)% confidence level as standard VaR does.

1. The Regulatory Back test.

Since 1996 the Basle Committee on Banking Supervision allows banks and other financial institutions to develop their own risk models in order to evaluate their portfolios risks. This is the Internal Model Approach (IMA) and the most important risk measure is Value at Risk. The Basel Committee's quantitative standards include a horizon of 10 trading days, a 99% confidence interval (α=1%) and an observation period based on at least a year of historical data and updated at least once a quarter. The capital requirement each bank must meet, as far as market risk is concerned, is:

$$\text{Market Risk Charge on day t} = \max\left(k - \frac{1}{60}\sum_{i=1}^{60} VaR_{t-1}, VaR_{t-1}\right) + SRC_t \qquad (4.4)$$

where VaR_{t-1} is the daily VaR in a 99% confidence level, k is a multiplicatice risk factor subject to an absolute floor of 3 and SRC_t is a plus factor – a penalty that should be added to k if the back testing results of the model are poor.

The multiplication factor is determined by the number of times an exception occurred. The minimum floor of 3 is in place to compensate for a number of errors that arise in model implementation, such as simplifying assumptions, analytical approximations, small sample biases and numerical errors. The increase in the multiplication factor is then designed to scale up the confidence level implied by the observed number of exceptions to the 99% confidence level desired by regulators and it is classified into three zones: the green, the yellow and the red zone. The green zone corresponds to back testing results that do not suggest a problem with the quality or accuracy of a bank's model. The yellow zone encompasses results that do raise questions in this regard, but where such conclusion is not definitive. The red zone indicates a back testing result that almost certainly indicates a problem with a bank's risk model.

The three zones have been delineated and their boundaries chosen in order to balance two types of statistical error: (1) the possibility that an accurate risk model would be classified as inaccurate on the basis of its back testing result and (2) the possibility that an inaccurate model would not be classified that way based on its back testing result. Table1 in Appendix II presents the zones and the respective value of the multiplication factor.

This approach, known also as the Basle Traffic Light Approach, although simple to implement, has a major drawback. Since the sample size of daily observations is finite, it is quite probable that the actual number of exceptions may differ from the percentage implied by the model's confidence level even in cases where the model is in fact accurate. Moreover this test neither considers the measure of the exception nor its position in time.

Therefore, the accuracy of the model should be examined by various additional tests which should compensate for the drawbacks of the Basle Approach.

2. Kupiec's (1995) tests

Kupiec[23] presents a more sophisticated approach to the analysis of exceptions based on the observation that a comparison between daily profit or loss outcomes and the corresponding VaR measure gives rise to a binomial experiment. If it can be assumed that a bank's daily VaR estimates are independent, the binomial outcomes of the experiment, that is exception (failure) or not exception (success), represent a sequence of independent Bernoulli trials each with a probability of failure equal to 1 minus the model's specified level of confidence. Hence, testing the accuracy of the model is equivalent to a test of the null hypothesis that the probability of failure on each trial equals the model's specified probability and the appropriate test statistic is a likelihood ratio statistic.

He uses two tests to examine the null hypothesis, which corresponds to an acceptance of the model, the time between failures test and the proportion of failures test.

2.a. Kupiec's Time until First Failure – TUFF Test

This test is based on the number of trading days between failures and is applied each time a failure is observed. This test is most useful in case when a risk manager is monitoring the performance of a VaR model on a daily basis and is focusing on the new information provided by the model. On the other hand it is less well suited to analysis of long runs of ex post data on model performance.

Let v be the observed time (in days) between failures
p be the true (empirical) probability covered by the VaR model
$p*$ be the (nominal) probability specified by the VaR model: 100-confidence interval% = α
$\tilde{p}$ be the maximum likelihood estimator of p, given by 1/v

The likelihood ratio (LR) test which is the most powerful for testing the null hypothesis $H_0 : p = p*$ vs. $H_1: p \neq p*$, is the following:

$$LR_{TUFF} = -2\ln\left(\frac{p^*\left(1-p^*\right)^{v-1}}{\tilde{p}\left(1-\tilde{p}\right)^{v-1}}\right) \sim \chi^2(1) \text{ under } H_0 \qquad (4.5)$$

This statistic is distributed as a chi square distribution with 1 degree of freedom and should its value in a specific back test be larger than the corresponding critical value from a chi square table, the model is rejected by the test.

[23] Kupiec P. (1995) "Techniques for verifying the accuracy of risk measurement models" *Journal of Derivatives*3, pp. 73-84,

2.b. Kupiec's Proportion of Failures – POF Test

The second test is based on the proportion of failures observed over the entire sample period of the back test performed. The associated likelihood ratio statistic is:

$$LR_{POF} = -2\ln\left(\frac{(p^*)^x\left(1-p^*\right)^{n-x}}{\tilde{p}^x\left(1-\tilde{p}\right)^{n-x}}\right) \sim \chi^2(1) \text{ under } H_0 \quad (4.6)$$

where n denotes the total number of outcomes in the sample period, x denotes a Bernoulli random variable representing the total number of observed failures and $\tilde{p}$ is its maximum likelihood estimator given by x/n.

The LR_{POF} is also distributed chi square with 1 degree of freedom and should its value in a specific back test be larger than the corresponding critical value from a chi square table, the model is rejected by the test.

The problem with Kupiec's tests is that they are by themselves not powerful enough to correctly predict errors in the VaR model. The power of the test is very poor, especially for high confidence levels and for small or medium back testing samples, so that the test cannot indicate an inadequate model even if the difference between the observed and the expected failure is significant. In order for the test to have significant power a substantial sample size is required, like 10 years of daily observations.

One improvement on Kupiec's tests is the suggestion of a mixed Kupiec test by Marcus Haas (2001):

$$LR_{mix} = LR_{ind} + LR_{POF} \sim \chi^2(n+1) \text{ under } H_0 \quad (4.7)$$

where LR_{ind} is a statistic for a variation of Kupiec's TUFF test and LR_{POF} the statistic of the standard Kupiec POF test.

3. Crnkovic and Drachman (1996) test

Contrary to other methods the Crnkovic-Drachman[24] (CD) or Kuiper test not only evaluates the exceptions but looks at the entire VaR model instead. Their test uses Kuiper's goodness-of-fit statistic to measure the distance between the entire probability distribution forecast of the portfolio's P&L and the actual P&L distribution. Empirical percentiles are calculated for every portfolio movement:

$$p_i = F(x_i) \quad (4.8)$$

[24] Crnkovitc C. and J. Drachman (1996) "Quality Control", *Risk* Septenber, pp. 138-143

where F represents the modeling distribution. If the model is well calibrated we can expect every percentile value in [0,1] with the same probability. Thus if one looks at a series of values of p_i he should not be able to tell them apart from a series of realizations of uniformly distributed random variables. Thus the testing hypothesis is formulated as

$$p_i \sim U(0,1) \text{ i.id.} \tag{4.9}$$

The smaller the value of Kuiper Statistic, the more accurate the probability distribution forecast is. Crnkovic and Drachman suggest a Q-test for the distributional assumptions (this test compares the maximum distances to the uniform distribution with a benchmark) and a BDS-test for independence.

The CD test is more flexible than the other test mentioned, however it also has some very strong requirements, such as knowledge of the distribution function underlying the model at any given point in time. Also sample sizes of at least 1000 observations should be used to enhance the test's power.

E. Criticism of VaR

VaR may has become the "standard benchmark" for measuring financial risk and is the official risk measure adopted by regulators, but is not a panacea for risk management and has received criticism from time to time.

One aspect is that an attempt to summarize the risk of a portfolio, ore more generally a distribution, in a single number is both strength and a weakness. This simplicity has been the key to the popularity of VaR, particularly as a mean of providing summary information to a bank's senior management. The difficulty with this though is that such a highly aggregate figure may mask imbalances in risk exposure across markets or individual traders, thus VaR should be considered along with other risk measures, traditional or not, like the variance, the semi variance (or downside risk which lie in the same spirit with VaR), the tracking error (the standard deviation of excess portfolio returns compared to those of a benchmark) and others.

Another important drawback of VaR is that it does not fully state "how bad can things get" as its estimated value is based upon a confidence level. Beyond that level the potential loss could be much greater than the VaR estimate. The only thing for sure, assuming the VaR model is accurate, is the probability of exceeding the VaR figure. In reply to this, Artzner et al.[25] (1999) have introduced a new risk measure called conditional value at risk (or expected shortfall). Conditional VaR (CVaR) is the expected loss given that the loss has exceeded the VaR threshold:

[25] Artzner P., F. Delbaen, J. Eber and Heath D. (1999), "Coherent Measures of Risk", *Mathematical Finance* 3, pp. 203-228

$$CVaR = E[X/X > VaR] \tag{4.10}$$

where X is a loss in absolute units

thus $$CVaR \geq VaR.$$

Artzner et al.(1999) in their criticism of risk measures stated certain properties that a risk measure should satisfy and defined such a measure as a “coherent” risk measure. These properties are:

1. Monotonicity of risk. If losses in portfolio X are larger than those in portfolio Y then the risk in portfolio X is higher than the risk in portfolio Y: $\rho(X) \geq \rho(Y)$, where ρ is the risk measure.
2. Homogeneity of risk. If we multiply the size of the portfolio then the magnitude of risk is multiplied as well: $\rho(cX) = c\,\rho(X)$ for $c>0$.
3. Translation invariance of risk. Adding cash to a portfolio decreases its risk by the same amount: $\rho(X+nr) = \rho(X)-n$, where n is the cash amount invested in the risk free rate r.
4. Subadditivity of risk. The risk of the sum of two subportfolios is smaller or equal than the sum of their individual risks:
   $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

According to the authors, VaR does not necessarily satisfy the last property as the VaR of a portfolio with two instruments may be greater than the sum of individual VaRs of these instruments. Therefore, managing risk by VaR may fail to stimulate diversification, probably the most important property of a portfolio. On the other hand, the CVaR measure is a coherent risk measure.

Section 5

Data Specifications

The main data we will need are those of volatility futures prices and the prices of VIX, which is the underlying index of the contracts. Both series are recovered from the Chicago Board of Options Exchange Database. A description of the specifications for these series follows.

A. VIX Implied Volatility Index

In 1993 the Chicago Board of Options Exchange introduced the CBOE Volatility Index VIX, an index which tracks the implied volatility from options on S&P 100 (OEX) and it became the benchmark for stock market volatility. In the ten years following the launch of VIX, theorists and practitioners have changed the way they think about volatility. On 22 September 2003 CBOE updated the construction methodology of VIX in order to ensure that it remains the premier benchmark of U.S. stock market volatility. The changes reflect the latest advances in financial theory and what has become standard industry practice. As far as the old methodology index is concerned, CBOE continues the calculation and dissemination of the original VIX, but under a new ticker symbol – "VXO".

The fundamental features of VIX remain the same. VIX continues to provide a minute-by-minute snapshot of expected stock market volatility over the next 30 calendar days. This volatility is still calculated in real time from stock index option prices and is continuously disseminated throughout each trading day.

The two important changes in the new methodology are the following:

- The most significant change is a new method of calculation. The new VIX estimates expected volatility from the price in stock index options in a wide range of strike prices, not just at-the-money strikes as in the original VIX. Thus it is more robust because it pools the information from option prices over the whole volatility skew, not just from at-the-money options.Also, the new VIX is not calculated from the Black Scholes option pricing model; the calculation is independent of any model. The new VIX uses a newly developed formula to derive expected volatility by averaging the weighted prices of out-of-the money puts and calls.

- The second noteworthy change in that the new VIX calculation will use options on the S&P 500 (SPX) index rather than the S&P 100. while the two indices are well correlated, the S&P 500 is the primary U.S. stock market benchmark as well as the reference point for the performance of many stock funds, with over $800 billion in indexed assets. In addition, the S&P 500 underlies the most active stock index derivatives and also volatility derivatives like volatility and variance swaps.

Details about the calculation of VIX are included in Appendix I.

B. Volatility index Futures

On March 26, 2004 the CBOE Futures Exchange (CFE) introduced a futures contract on the VIX volatility index.[26]

The contract's specifications are the following:

- **Trading hours:** 8:30 a.m. to 3:15 p.m CST (Chicago time)
- **Description-underlying value:** VIX Futures track the level of an "Increased-Value Index" (VBI) which is 10 times the value of VIX.
- **Contract size:** $100 times the Increased Value VIX (VBI) For example, if VIX is 17.5-indicating an implied volatility of 17.5% - the VBI will be 175 and the contract size will be $17,500
- **Minimum tick size:** 10 cents. Therefore, minimum value change will be in $10 intervals
- **Contract months:** Initially, May, June, August and November. Thereafter, two-near term and two additional months on the February quarterly cycle (February, May, August and November)
- **Last trading date:** The Tuesday prior to the third Friday of the expiring month
- **Final settlement date:** The Wednesday prior to the third Friday of the expiring month
- **Final settlement price:** Cash settled. The final settlement price for VIX futures is 10 times a Special Opening Quotation (SOQ) of VIX calculated from the SPX options used to calculate the index on the settlement date (the "Constituent Options"). If there is no opening price for a Constituent Option the average of that option's bid and ask price as determined at the opening of trading is used instead.
- **Margin Requirements:** The minimum speculative margin requirements for VIX futures are: Initial-$3750 and Maintenance-$3000.
- **Position limits:** 5000 contracts

[26] CFE has launched also another specialized product – the S&P 500 3-Month Variance Futures which are futures contracts based on the *realized* variance of the S&P 500 over a three month period.

The following table presents the volatility futures contracts that are traded in CBOE:

Table 5.1: Chicago Board of Options Exchange Volatility Futures Contracts

Contract	Starting Date	Maturity	Duration	# Trading Days
May 2004	26/3/2004	19/5/2004	aprx. 2 M	38
June 2004	26/3/2004	16/6/2004	aprx. 3 M	56
July 2004	21/5/2004	14/7/2004	aprx. 2 M	36
August 2004	26/3/2004	18/8/2004	aprx. 5 M	100
September 2004	19/7/2004	15/9/2004	aprx. 2 M	42
Oktober 2004	20/8/2004	13/10/2004	aprx. 2 M	38
November 2004	26/3/2004	17/11/2004	aprx. 8 M	163
January 2005	21/10/2004	19/1/2005	aprx. 3 M	61
February 2005	18/6/2004	16/2/2005	aprx. 8 M	168
March 2005	24/1/2005	16/3/2005	aprx. 2 M	37
May 2005	20/9/2004	18/5/2005	aprx. 8 M	167
June 2005	21/3/2005	17/6/2005	aprx. 3 M	63
August 2005	19/11/2004	17/8/2005	aprx. 9 M	187
November 2005	22/2/2005	17/11/2005	aprx. 9 M	188
February 2006	23/5/2005	18/2/2006	aprx. 9 M	188

Section 6

Examining the Pricing Performance

A. Methodology

In this part we intend to test the two valuation models of volatility futures as far as their pricing performance is concerned. To achieve this goal we will first estimate the unknown parameters of each model. Given the output of the estimation procedure, we will derive theoretical/model values of volatility futures and we will compute the deviation of these values from those observed in the market (market prices) so that we have a consensus of the pricing bias of each model.

Our estimation method will be **calibration**. Calibration is a procedure for estimating or inferring a model's parameters by fitting the model to market prices. The most usual "goodness-of-fit" measure is the following:

$$\sum_{i=1}^{n}\left(M_i - F_i\right)^2 \tag{6.1}$$

where M_i is the market price and F_i is the theoretical price given by the model for the i^{th} element of the market data series – the calibration instruments.

From a first view, this procedure resembles the classical least squares estimation procedure, as it involves the minimization of the sum of squared error between market and model prices[27]. In our case it would be a non-linear least squares regression.

To implement this technique we create four series of daily volatility futures market settlement prices, one with contracts having the shortest maturity per trading day, the other with the second shortest maturity and so on. We create the series this way because these contracts are generally short term and we are not able to construct a series of a single contract with an adequate number of observations[28]. So these contemporaneous series represent hypothetical contracts while in fact contain prices of consecutive contracts rolled over. We also reject the prices of the last 5 trading days prior to expiration and prices of trading days with volume less than 5 contracts.

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[27] Calibration is in fact estimation. (see eg. E.Balistreri-R.Hillberry [2004] "Estibration: An illustration of structural estimation as calibration")

[28] We cannot increase the sample by using panel data, a common procedure in testing options models where also cross-sectional data are available since on each day various option prices exist for different strike prices.

The other set of data series is the VIX values and the calculated VBI values for the same time interval with the volatility futures price series. VBI is ten times the value of VIX and is the underlying asset of volatility futures according to the contracts specifications. Hence this series should correspond to the *V* variable in the model. Finally, *T-t* in the model is the annualized time to maturity for each contract in trading days (assuming there are 252 trading days per year).

The procedure includes calibrating each model using the shortest maturity series and examining the pricing bias in the other three maturity series. The pricing bias is the difference between the market and model price. Specifically, we compute the Average Absolute Percentage Pricing Bias (AAPPB) per maturity series and per pricing model:

$$\frac{1}{N}\sum_{i=1}^{N}\left|\frac{F_i - M_i}{M_i}\right| \qquad (6.2)$$

where F_i and M_i are the model and market price respectively

We obtained data from 26 March 2004 to 17 June 2005. The four maturity series are presented in the following table:

Table 6.1: Volatility Futures Series

Series	Starting Date	Ending Date	Duration	# Observations
Shortest maturity	26/3/2004	17/6/2005	aprx. 14 M	308
Second shortest maturity	26/3/2004	17/6/2005	aprx. 14 M	306
Third shortest maturity	26/3/2004	17/6/2005	aprx. 14 M	288
Fourth shortest maturity	26/3/2004	9/6/2005	aprx. 9 M	180

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Descriptive statistics for the VBI and the four volatility futures maturity series are provided in table 6.2. As we can observe, all series exhibit non-normality, positive skewness (except the fourth shortest maturity series) and are platykurtic. Figures 6.1 and 6.2 contain a diagram of the VBI Index and the four maturity series we have created respectively.

Table 6.2: Summary statistics of the VBI Index and the four series of Volatility Futures

	VBI	shortest maturity series	second shortest maturity series.	third shortest maturity series	fourth shortest maturity series
Mean	144,41	154,73	165,941	174,3	182,58
Median	142,80	151,55	160,35	168,35	178,52
Maximum	199,6	203,20	210,50	217,3	219,5
Minimum	111	114,8	122	131,6	106
Std. Deviation	18,99	21,33	23,29	23,90	24,02
Skewness	0.484	0,351	0,283	0,228	-0,006
Kyrtosis	2,818	2,23	1,913	1,742	2,019
Jarque-Berra (p-value)	12,43 (0,002)	13,85 (0,001)	19,15 (0,000)	21,49 (0,000)	7,323 (0,004)
Observations	308	308	306	288	180

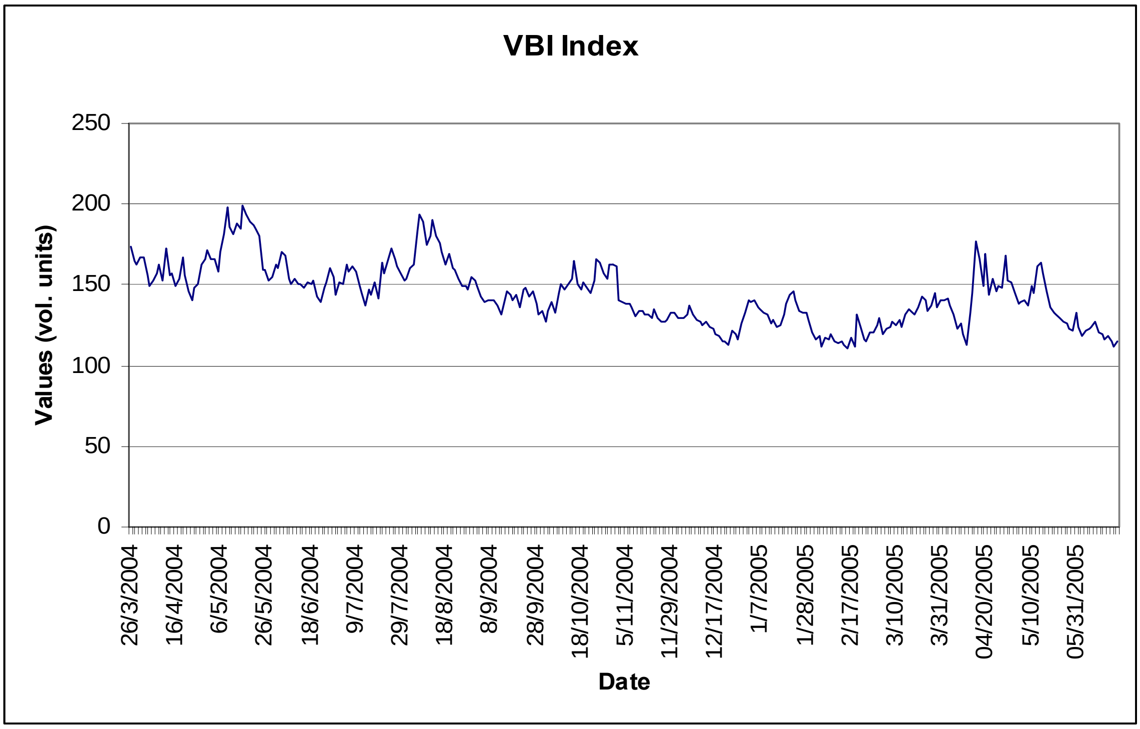

Figure 6.1 : Evolution of VBI Implied Volatility Index

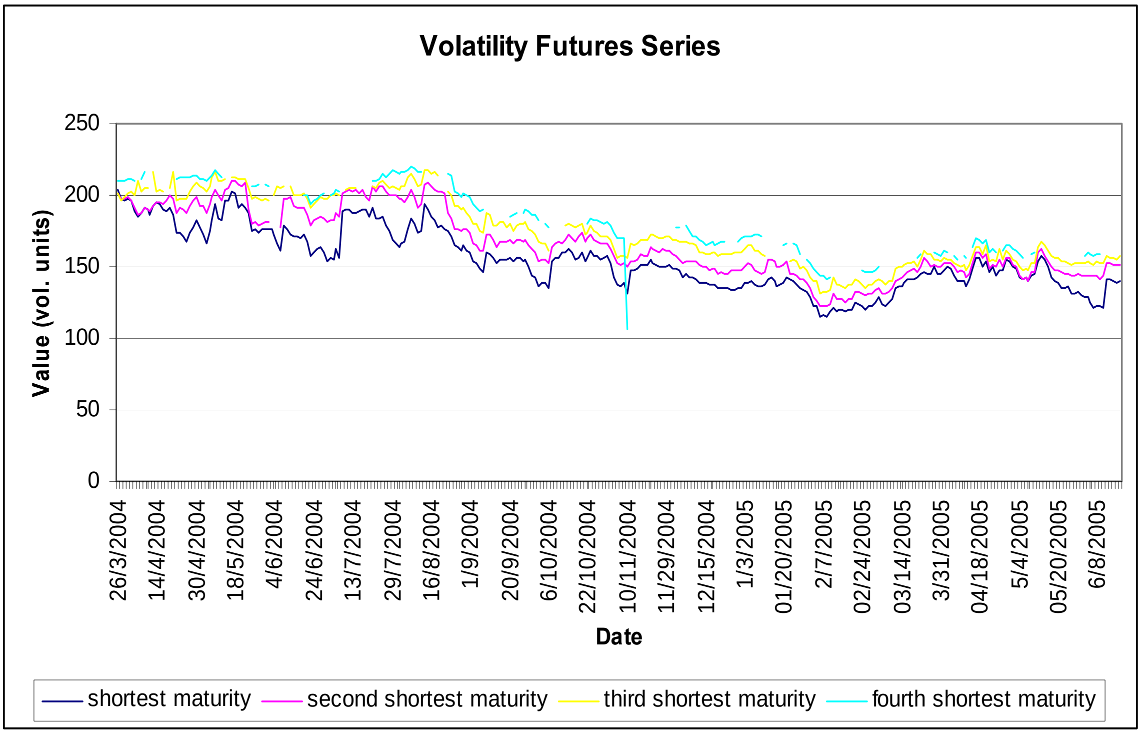

Figure 6.2 : Evolution of the four Volatility Futures series

We can observe that the four series of volatility futures are almost perfectly correlated as they move very closely and they also verify the positive relationship with the volatility index, their underlying asset.

B. <u>Results</u>

In this section we present the empirical results of the procedure which we followed in order to examine the pricing performance of the two volatility futures models.

In both of the models the econometric procedure which was followed to estimate the unknown parameters is non linear regression. A model is nonlinear in parameters if the model's partial derivatives of 1^{st} order with respect to parameters are themselves functions of the parameters. Nonlinear least squares minimize the sum-of-squared residuals with respect the choice of parameters, as in the linear regression. The difference is that in the related first order conditions of this minimization problem instead of the regressors their partial derivatives of 1^{st} order with respect to parameters take place.

Also in the non linear model there is in general no explicit for the non linear least squares estimator as in the linear model and thus an iterative method must be applied. Usually basic optimization methods like the Gauss-Newton and the Newton-Raphson are adequate to solve the minimization problem. Otherwise special procedures may be applied, like the Quadratic Hill Climbing or the BHHH algorithm. In general, the usual procedures regarding estimation and hypothesis testing that stand in the linear case are valid also in the non linear case as the work mainly from T. Amemiya has shown.

The calibration procedure results for each model are:

1. Grűnbichler-Longstaff model

The first attempt to estimate the parameters with non-linear ordinary least squares (NL-OLS) gave unsatisfactory results. The parameters values were bizarre and non statistically significant, while there was strong evidence for autocorrelation both form the Durbin-Watson statistic and the correlogramm which displays the autocorrelation function. The correlogramm of squared residuals and the ARCH Lagrange Multiplier (LM) test showed no evidence for heteroskedasticity. Correcting autocorrelation with the Newey-West did not work. This problem is also reported in various papers with similar estimation procedure in testing option pricing models (Bates [1996a], Nandi [1998], Guo [1998]) and it is mainly attributed to serial correlation of residuals. Autocorrelation can arise from misspecification of the model, such as the omission of a relevant state variable.

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Bates (1996a) estimates the parameters using non-linear generalized least squares methodology modeled on Engle and Mustafa (1992), while Nandi (1998) uses a two stage procedure in a framework of non-linear seemingly unrelated regressions and assuming a AR(1) process for the autocorrelation of residuals. Guo (1998) also adopts a two step iteration procedure.

We overcame the problem using non-linear generalized least squares (NL-GLS). This is a usual procedure for handling autocorrelation or/and heteroskedasticity in a regression's residuals – a problem called non–spherical disturbances and consists of a non identifiable residual variance – covariance matrix which makes the ordinary least squares estimator (OLS) inefficient. The procedure includes transforming the model so that the OLS estimator can be used for inference and hypothesis testing, while the variance – covariance matrix is being restricted to few unknown parameters which can be easily estimated. The latter is known as feasible generalized least squares (FGLS) and is a parametric correction of the non – spherical disturbances problem.[29]

[29] A non parametric correction is the one that uses the White or the Newey-West estimator in order to identify the residual variance – covariance matrix, which we already have tried.

In our case autocorrelation is parameterized assuming that the residuals follow a first order autoregressive model - AR(1)[30] so that the estimation framework is as follows:

$$F_t = \frac{a}{b}\left[1-\exp\left(-bT_t\right)\right]+\exp\left(-bT_t\right)V_t + \varepsilon_t \tag{6.3}$$

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t \tag{6.4}$$

$$E\left(\varepsilon_t \varepsilon_t'\right) = \Omega = \Omega(\rho) \tag{6.5}$$

The first equation is our basic regression and provides the OLS estimators $\hat{a}, \hat{b}$ and the residuals ε_t which are the inputs to the second equation, the AR(1) scheme for the residuals. The latter provides the estimator $\hat{\rho}$ so that Ω, the residual variance – covariance matrix, becomes identifiable and the estimated values $\hat{a}, \hat{b}$ are updated. This is an iterative procedure and is continued until convergence is achieved.

The FGLS in this framework is known as the Prais-Winsten or the Cochrane-Orcutt estimator and was first introduced in linear models.

We came up with the following results:

Table 6.3: Calibration results in GL model

Parameters	Estimated values	t - stastistic
$\frac{\hat{a}}{\hat{b}} = \hat{V}_L$	**165,93**	79,393 (***)
$\hat{b}$	**10,2**	12,207 (***)
$\hat{\rho}$	0,956	63,134 (***)
R-squared	0,9703	
Akaike info criterion	5,4436	
Durbin-Watson statistic	2,17	

*** indicates statistical significance at 5% level

[30] Higher order models were rejected.

From the table we observe that the long run mean of the VBI series is $\frac{\hat{a}}{\hat{b}}=$ 165,93 which implies that the corresponding value for VIX is 16,59 (indicating an implied volatility long run mean of 16,59% from the options on S&P500). Also the speed or rate of mean reversion of this series to its long run mean is (under the risk neutral measure) $\hat{b}=$ 10,2 indicating a half-life of 0,82 calendar month.[31] In an other interpretation of the rate of mean reversion, VIX returns to its long run mean over the course of $1/\hat{b}=$ 0,098 years or 1,18 months. This value provides evidence for a fast mean reversion and a high volatility of implied volatility and is consistent with the findings of Taylor and Xu (1994) and Guo (1996).

2. Benchmark model

In the estimation results from non-linear ordinary least squares (NL-OLS) there was evidence for non-spherical disturbances. The Durbin-Watson statistic and the correlogram of residuals showed significant evidence of autocorrelation in residuals, while the correlogram of squared residuals and the ARCH LM rejected the null hypothesis that residuals are homoskedastic. Thus a non-linear generalized least squares (NL-GLS) methodology was again employed.

Specifically, an autoregressive model was assumed to capture the dynamics of autocorrelation and a GARCH model was assumed in order to model the heteroskedasticity in the OLS residuals. Data supported an AR(2) model and a GARCH(1,1) model respectively.

Thus, the estimation framework is the following:

$$F_t = \exp(bT_t)V_t + \varepsilon_t \quad (6.6)$$

$$\varepsilon_t = \rho_1\varepsilon_{t-1} + \rho_2\varepsilon_{t-2} + v_t \quad (6.7)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2 \quad (6.8)$$

The first equation is our basic regression, the second is the AR(2) model of the residuals and the last equation is the GARCH (1,1) model where σ_t^2 is the conditional variance of the residuals ε_t, α_0, α_1, $\beta_1 \geq 0$ and $\alpha_1 + \beta_1 < 1$. OLS residuals are

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[31] The half-life is a concept from physics and is calculated as 12*ln(2)/ $\hat{b}$ in months (see Guo [1996])

computed from equation (5.4) and are further modelled through equations (5.5) and (5.6) so that the problem of non – spherical disturbances may be overcome and the estimated value $\hat{b}$ become trustworthy.

The estimation procedure came up with the following results:

Table 6.3: Calibration results in Benchmark model

Parameters	Estimated values	t - stastistic
$\hat{b}$	**0,699**	16,709 (***)
$\hat{\rho}_1$	0,695	8,831 (***)
$\hat{\rho}_2$	0,182	2,34 (***)
$\hat{\alpha}_0$	6,183	3,01 (***)
$\hat{\alpha}_1$	0,241	4,142 (***)
$\hat{\beta}_1$	0,549	5,90 (***)
R-squared	0,934	
Akaike info criterion	6,12	
Durbin-Watson statistic	1,98	

*** indicates statistical significance at 5% level

Thus, the estimated parameter of *b* is 0,699. This is the value of the volatility drift parameter under the risk neutral objective measure. The values of the other estimated parameters are normal and the ones of the GARCH process satisfy the stability property of the model as $\hat{\alpha}_1 + \hat{\beta}_1 = 0{,}79 < 1$.

3. Pricing performance of the two models

Given the estimation output in each model we computed model prices of volatility futures regarding the other three maturity series and we calculated each model's pricing bias. The following table and figure summarize the pricing performance of the two models:

Table 6.4 Average Absolute Percentage Pricing Bias (AAPPB)

Maturity Series	Grűnbichler-Longstaff model	Benchmark model
Second shortest	10,16%	6,69%
Third shortest	11,04%	10,94%
Fourth shortest	11,90%	20,33%

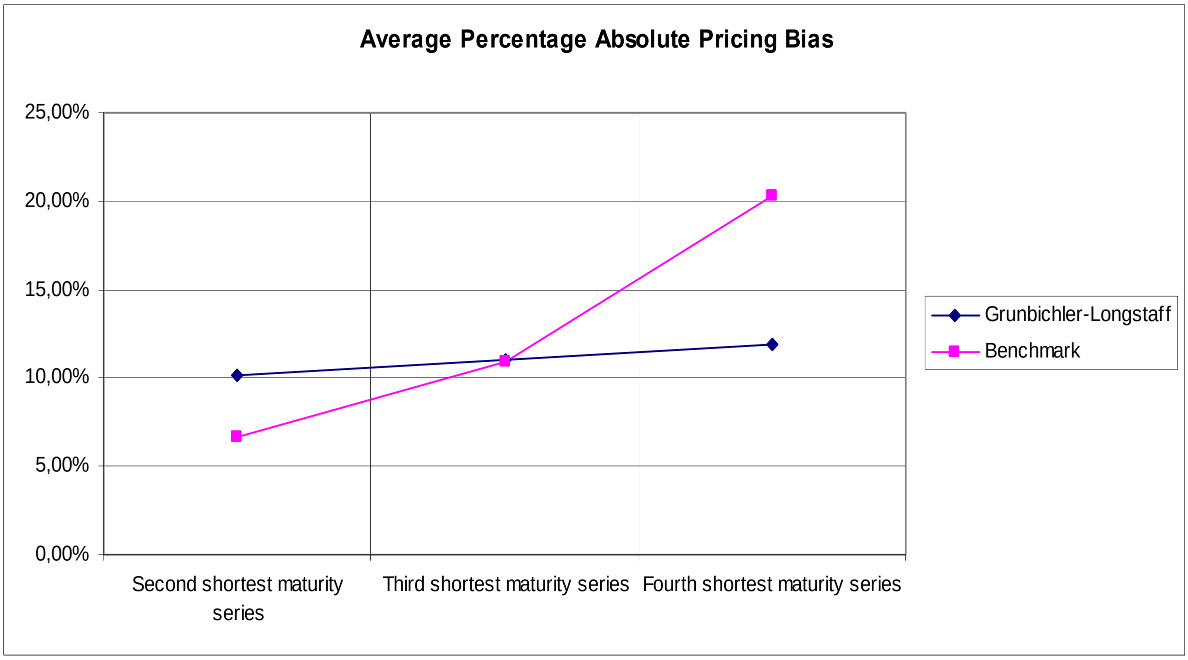

Figure 6.3 Pricing biases of the two models

We observe that the pricing bias is significant in both models and in all maturity series. The pattern for the Grűnbichler-Longstaff model is more balanced as the average absolute percentage pricing bias is around 11%. On the other hand, the respective figure for the Benchmark model exhibits an upward trend as it grows from 6,69% in the second shortest maturity series to 20,33% in the fourth shortest series. Comparing the results between the two models yields a mixed picture. The Grűnbichler-Longstaff model performs worse than the Benchmark model in the second shortest maturity series but has substantially smaller pricing bias in the fourth series. In the third maturity series the figure is roughly the same for the two models, with the Benchmark model performing slightly better (by 1%).

Thus, contrary to the belief that a more sophisticated assumption about the evolution of the underlying asset yields a more accurate pricing formula, the results of our study support the opinion "simpler is better". Although Geometric Brownian Motion is considered as a relative simplified assumption for the dynamics of volatility because it does not account for volatility clustering and mean reversion like the Mean Reverting Square Root Process does, the Benchmark model is apparent to price volatility futures better, at least in the first two maturity series..

In the standard derivatives literature, mainly options literature, an analogous empirical result is documented by Bakshi, Cao and Chen (1997) and Dumas, Fleming and Whaley (1998). The former authors in a comparison of various schemes from three perspectives (misspecification, pricing and hedging error) find that models that incorporate stochastic interest rates and jumps in the underlying asset's returns do not significantly improve the performance of the Black-Scholes model and a stochastic volatility model like Hull & White's (1987) or Heston's (1993) respectively. Dumas, Fleming and Whaley examine the predictive and hedging performance of the deterministic volatility function option valuation model of Derman-Kani (1994), Dupire(1994) and Rubinstein (1994). They document that the latter is not better than an ad hoc procedure that merely smoothes Black-Scholes implied volatilities across prices and time to expiration.

On the other hand, Daouk and Guo (2003) in a Monte Carlo study of Grűnbichler-Longstaff model regarding volatility options find a negative pricing bias[32] while an alternative valuation model stemming from a more sophisticated process for volatility fits the data better. They state that the MRSRP fails to account for volatility asymmetry and for regime switching shifts – other properties of volatility that are also empirically documented. Hence, they support that research should turn to a more realistic, though more complicated, assumption about the data generating process of the underlying asset. Which direction is right is yet to be determined by additional tests that would possibly account for other volatility models as well.

[32] Using the metric of Daouk & Guo, which accounts for the sign of the pricing bias, we also document a negative average pricing bias in the GL model: -1,03%, -3,61% and -7,51% for the three series respectively.

Section 7

Value at Risk Study

A. Methodology

In this part of our work we intend to test the two valuation models of volatility futures and the related volatility processes within a value at risk study. More specific we are interested in the back testing results of a VaR model regarding a position in VIX Index and in a volatility futures contract, in order to see under which model the VaR measure is more effective as a risk management tool as well as a measure accepted by regulators. The approach we will follow in order to calculate the VaR figures is Monte Carlo Simulation and as far as the back testing is concerned we will implement the Basle Traffic Light Approach and the Kupiec Proportion of Failures test.

We will simulate daily prices for the VIX implied volatility index for the period of 29 March 2004 to 17 June 2005 and calculate first the daily VaR of the index. Then with these prices as inputs and given the estimates of the formula's unknown parameters from the calibration procedure we will compute (simulated) volatility future prices so as to estimate the VaR of this position. The number of simulations per day will be 10000 and for each day the simulated profit and loss distribution of both positions will be derived and the 1 day VaR will be calculated for a 99% and a 95% confidence level. The study will regard the first three maturity series of volatility futures we created to examine the pricing performance (the fourth maturity series will not be considered because it does not contain adequate observations for back testing procedure). We will conduct the value at risk study separate for each series and for both pricing models, the Grűnbichler-Longstaff (GL) and the Benchmark model. We choose the same sample period and the same data in both the examination of the pricing performance and the value at risk study so that a direct comparison may be made between the pattern of pricing bias and the one of VaR figure per series.

In order to simulate prices for VIX we will use the respective volatility process for each model: the Mean Reverting Square Root Process (MRSRP) for the GL model and the Geometric Brownian Motion Process (GBMP) for the Benchmark model. Both processes are under the objective probability measure:

$$dV_t = k(\mu - V_t)dt + \sigma\sqrt{V_t}dW_t \tag{7.1}$$

$$dV_t = vV_t dW_t \tag{7.2}$$

(7.2) is a drift less GBMP as in the typical value at risk framework the drift parameter – the expected rate of return is assumed to be zero, at least for daily VaR.

This part of the procedure requires as a first step the estimation of the parameters in each process; k, μ and σ in the MRSRP and v in the GBMP.

Regarding the first process (4.13) the estimation procedure we will follow to get values for the speed of mean reversion k, the long run mean μ and the volatility σ of the VIX index, is the Parametric Method of Moments. We prefer to follow a parametric estimation method since it is the most efficient as it exploits fully the information that data provide. For this reason parametric methods are more demanding as well as they require that the density function of the probability model that generates the data is known. The valuation framework we have meets this requirement; under MRSRP the marginal and conditional density of volatility is the gamma and the non central chi square respectively.

Ideally, a maximum likelihood method would be applied in the conditional density as it is in theory the most efficient estimation method. Unfortunately the complexity of the conditional density formula did not allow us to conduct this method. Thus we will apply the parametric method of moments in the marginal density which is the asymptotical limit of the conditional one. Therefore, any estimation strategy or specification test that exploits the conditional density naturally nests the ones that rely on the marginal density. The difficulty in applying the maximum likelihood method in the non central chi square distribution is documented from statistics (see e.g. Johnson & Kotz [1970] and F.Lopez-Blazaar [2000]) and from the interest rate literature regarding empirical tests of the Cox, Ingersoll and Ross model of the short rate as a very small part of the related literature chooses maximum likelihood as the estimation method.

The Parametric Method of Moments (PMM) was developed in 1895 by K. Pearson in the context of descriptive statistics. The original method was proposed as both a specification and an estimation method but was later adapted as just an estimation method in modern statistical inference. The idea behind the method is, given a probability model and the respective density function, to match the population or theoretical moments, which are a function of the unknown parameters, with the corresponding sample moments and solve the resulting system of equations for the unknown parameters.

PMM should not be confused neither with the moment matching principle, where *distribution* moments are matched with sample moments and no system is solved for unknown parameters, nor with the Generalized Method of Moments (GMM). The latter lies in the same spirit but it exploits no distributional assumption and is thus categorized as a semi parametric method – a less efficient method in compared to PMM.

In our case the unknown parameters k, $\mu = \frac{a}{k}$ and σ appear in the formulae of the mean, the variance and the first order serial correlation of the marginal Gamma function:

$$E[V_t] = \frac{a}{k} \tag{7.3}$$

$$Var[V_t] = \frac{a\sigma^2}{2k^2} \tag{7.4}$$

$$\rho(1) = \exp(-k\Delta t) \tag{7.5}$$

where Δt = 1/252 since the sample of VIX is obtained with a daily frequency

By obtaining estimates of these three moments from the sample period from 1 January 2003 to 25 March 2004 we will solve the system of equations (7.3) to (7.5) for a, k and σ.

The next step is to simulate 10000 VIX values per day. This can be done via the discretized MRSRP:

$$V_{t+\Delta t} = V_t + k(\mu - V_t)\Delta t + \sigma\sqrt{V_t}\,\varepsilon\sqrt{\Delta t} \tag{7.6}$$

or via the discretized analytic solution of the process:

$$V_{t+\Delta t} = \mu\left(1 - e^{-k\Delta t}\right) + V_t e^{-k\Delta t} + \sigma e^{-2k\Delta t}\sqrt{V_t}\,\varepsilon\sqrt{\Delta t} \tag{7.7}$$

where $\varepsilon \sim N(0,1)$ is the random number generator.

Regarding the Geometric Brownian Motion Process the only parameter to be estimated is v, the volatility of VIX per unit of time:

$$v = \sigma\sqrt{252} \tag{7.8}$$

where σ is the standard deviation of the daily log returns of VIX.

A historical moving average of 100 and 300 days will be used to estimate the variance σ^2. We believe this size is adequate for a one day forecasting horizon neither it goes too back in the past and includes observations whose size is far different from the current period.[33] The Moving Average (MA) estimator is:

[33] An improvement over the moving average estimator is the Exponentially Weighted Moving Average (EWMA) model where the weights given in each observation are not equal as in the

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-1}^2 \qquad (7.9)$$

where m = 100 and 300 in our case and $u_i = \ln\left(\frac{VIX_i}{VIX_{i-1}}\right)$ is the log return of VIX. Thus, σ in (7.8) will be the square root of (7.9) on each day.

Given the estimate of v we will simulate 10000 VIX values per day via the discretized analytic solution of the process:

$$V_{t+\Delta t} = V_t \exp\left[-\frac{v^2}{2}\Delta t + v\varepsilon\sqrt{\Delta t}\right] \qquad (7.10)$$

where $\varepsilon \sim N(0,1)$ is the random number generator.

In both (7.7) and (7.10) Δt is chosen to be 0,00004 so that each day is partitioned in 100 steps. More specific $\Delta t = \frac{T-t}{n}$ and in our case *T-t* is one day, thus 1/252 = 0,004 on an annual basis and n =100. The final (100^{th}) step is considered to represent the group of 10000 simulated values per day.

The simulation procedure will be conducted for each day in the period 29 March 2004 to 17 June 2005 which corresponds to a sample of 307 days (at least for VIX and the shortest maturity series of volatility futures). After the VaR figure is obtained we will compare the realized one day losses with this figure and count the number of exceptions in order to implement the back testing.

B. Results

The results[34] of the back testing procedure are summarized in the following table:

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moving average rather they decrease exponentially as we move back through time. Also an alternative estimator would be the GARCH(p,q) estimator, in which the EWMA estimator nests as a special case.

[34] The VaR figures are in volatility units. That does not affect the validity of the results as we are interested in the comparison (and not in the absolute figure) between the real loss and the VaR estimate which gives the same result (exception or not) in a given day regardless if both these two numbers are in volatility units or dollars. Thus our study holds for any nominal amount behind the position.

Table 7.1: Back testing results for the Monte Carlo VaR model of VIX and of volatility futures maturity series

VIX Index	**MRSRP**	**GBMP - MA 100**	**GBMP - MA 300**
99% VaR			
No. of exceptions (%)	3 (0,98%)	3 (0,98%)	4 (1,30%)
Kupiec POF test	0,0016	0,0016	0,2598
95% VaR			
No. of exceptions (%)	8 (2,61%)	10 (3,26%)	9 (2,93%)
Kupiec POF test	4,4569*	2,2269	3,2272
Shortest series	**Grűnbichler - Longstaff Model**	**Benchmark Model - MA 100**	**Benchmark Model - MA 300**
99% VaR			
No. of exceptions (%)	65 (21,17%)	5 (1,63%)	4 (1,30%)
Kupiec POF test	286,56817*	1,0299	0,2598
95% VaR			
No. of exceptions (%)	79 (25,73%)	21 (6,84%)	16 (5,21%)
Kupiec POF test	146,5852*	1,9733	0,0286
Second shortest series	**Grűnbichler - Longstaff Model**	**Benchmark Model - MA 100**	**Benchmark Model - MA 300**
99% VaR			
No. of exceptions (%)	78 (25,74%)	14 (4,62%)	15 (4,95%)
Kupiec POF test	377,2988*	21,3200*	24,5287*
95% VaR			
No. of exceptions (%)	81 (26,73%)	22 (7,26%)	24 (7,92%)
Kupiec POF test	156,2502*	2,8782	4,6574*
Third shortest series	**Grűnbichler - Longstaff Model**	**Benchmark Model - MA 100**	**Benchmark Model - MA 300**
99% VaR			
No. of exceptions (%)	70 (25,55%)	51 (18,61%)	59 (21,53%)
Kupiec POF test	337,4124*	210,8604*	262,2612*
95% VaR			
No. of exceptions (%)	70 (26,73%)	64 (23,36%)	71 (25,91%)
Kupiec POF test	128,9182*	107,1249*	132,6870*

The critical value for Kupiec test is 3,841 indicating a 95% confidence level. (*) denotes a rejection of the VaR model. The back testing sample is consisted of 307 observations in the VIX and in the shortest maturity series model and of 303 and 274 observations in the other two series respectively.

Regarding the VaR model of a position in VIX Index both processes assumed for volatility yield good back test's results, with the exception of the 95% VaR under the MRSRP which was rejected by Kupiec test. In the 99% case the number of exceptions are almost the same among the two processes: 3 in the MC VaR under MRSRP, 3 in the MC VaR under GBMP with MA 100 and 4 with MA 300. According to the Basle Traffic Light Approach they all fall in the green BIS zone and are all accepted by the Kupiec test. The same picture is in the 95% case where the number of exceptions corresponds to an empirical size of 2,61%, 3,26% and 2,93% respectively versus the nominal size of 5%. Hence, both processes for the evolution of VIX perform similarly in the value at risk study, with the Geometric Brownian Motion Process exhibiting slightly more exceptions.

In the case of a position in volatility futures the Mc VaR of the Grűnbichler-Longstaff model is rejected in all maturity series and in both levels of confidence. In all circumstances the number of exceptions was big enough to correspond about to 25% of the back testing sample – a number indicating that the multiplicative factor increases to 4 (red zone) and also a penalty would be added by the supervisor. When using the Benchmark model to calculate the VaR estimate of volatility futures the results are mixed. In the shortest maturity series the VaR model is accepted by both criteria. In the second shortest series the back test results are worse, mainly for the 99% case where the number of exceptions is almost tripled. Kupiec POF test rejects all models apart from the 95% MA 100 model. In the third series the results are bad in all cases and resemble those of the Grűnbichler-Longstaff model's.

Thus, we observe that the pattern of the back testing performance of both models is analogous to that of the pricing performance. In both studies the performance of the Grűnbichler-Longstaff model is relative stable across maturities while the Benchmarks model's performance deteriorates as the time to maturity increases and we move from the first to the last maturity series.

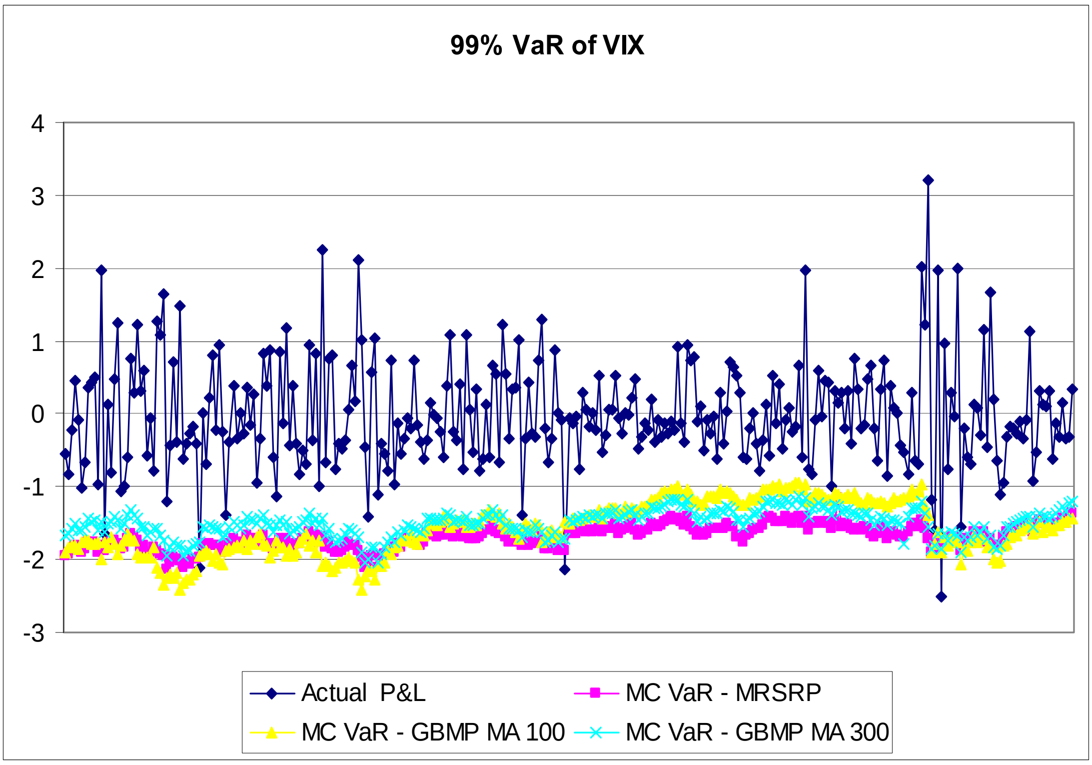
The main reason for the surprisingly high number of exceptions and the resulting failure of VaR models is the presence of positive VaR figures among the sample – that is, the model projected a profit rather than a loss as it should. This is attributed to the bias of each model's pricing formula which within a series is both positive and negative.[35] More specific, in the sub period where the pricing bias was big and positive (negative) the simulated P&L distribution was positively (negatively) skewed and the corresponding VaR estimate was positive (negative). Hence, the pricing bias of each model affects greatly the performance of the respective VaR model.

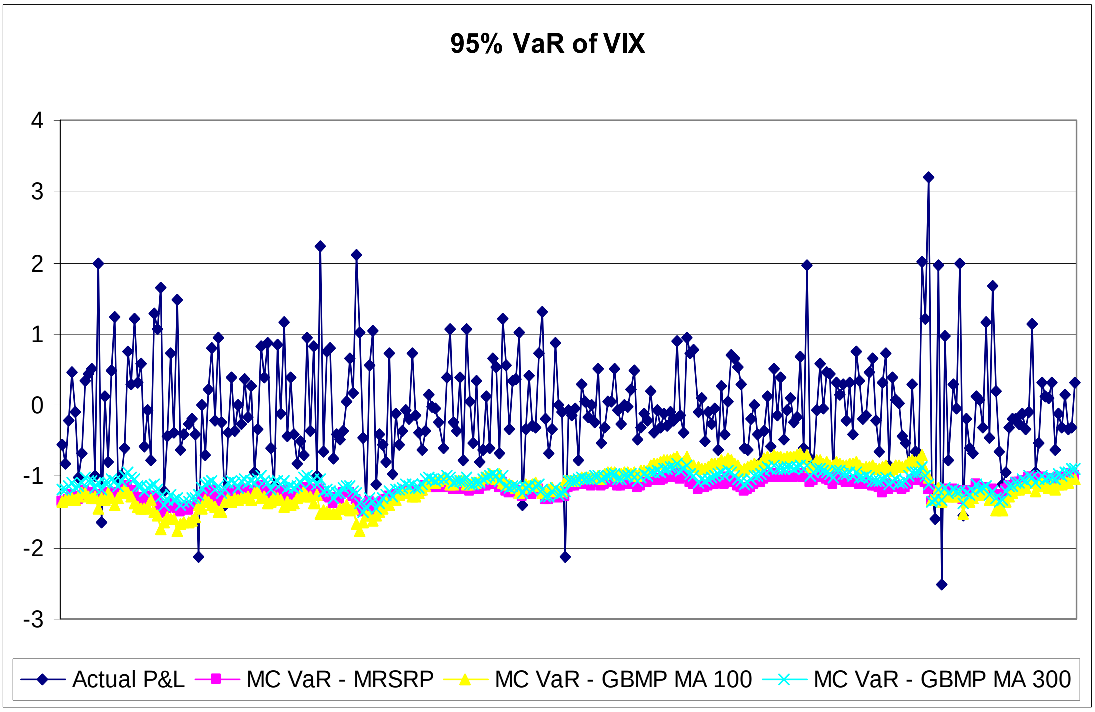
We conclude that as in the pricing performance examination the results are again mixed. Both valuation models and their respective volatility processes perform good regarding a position in VIX , but perform poorly in the case of a position in a volatility futures contract. The back testing indicates that the Monte Carlo VaR would be misleading a risk manager in the daily management of a bank's position in these

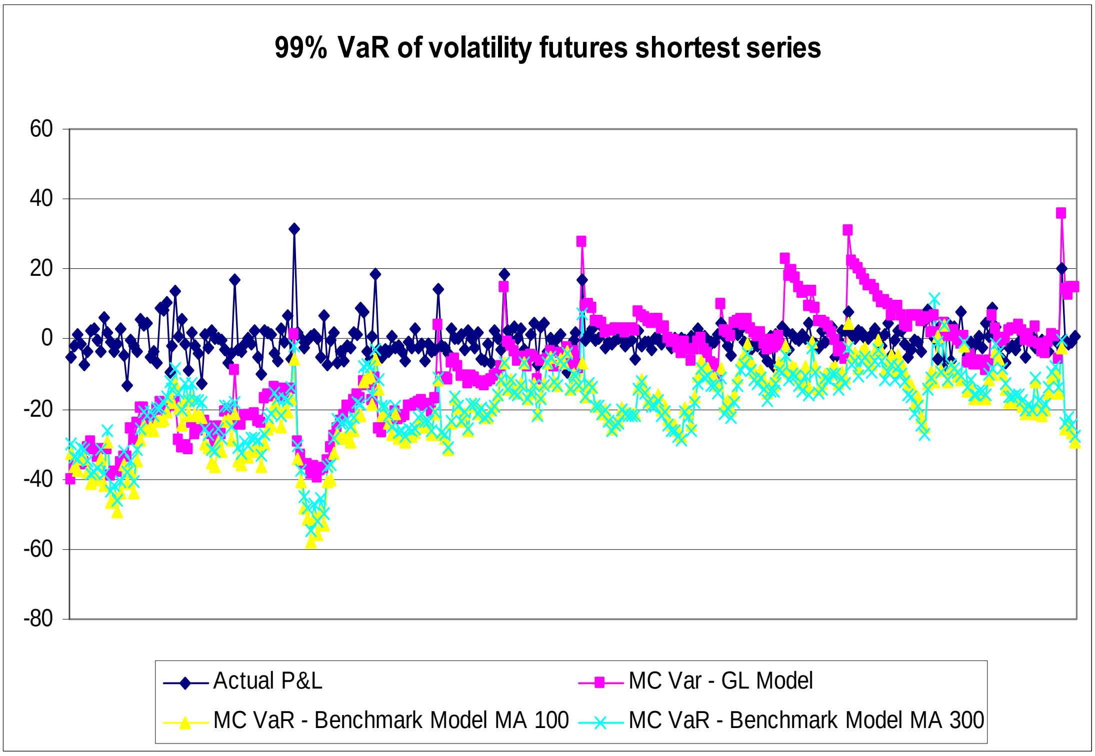
[35] That is way we chose the *absolute* average percentage pricing bias as a metric in the examination of the pricing performance; so as to avoid the canceling out among observations.

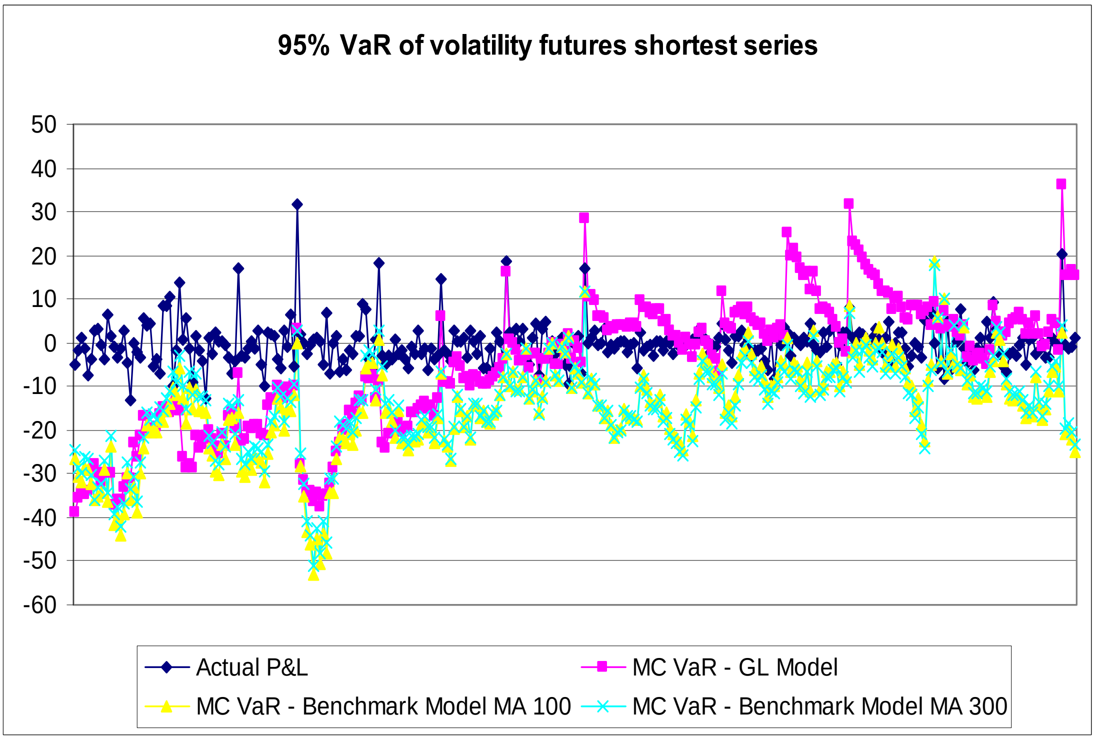
instruments; moreover the bank would be penalized by regulators as it would be obliged to tie excessive capital. Only the use of the Benchmark model to calculate volatility futures which are close to expiration would yield satisfactory results, especially when using a moving average of 300 days to calculate volatility of the underlying index.

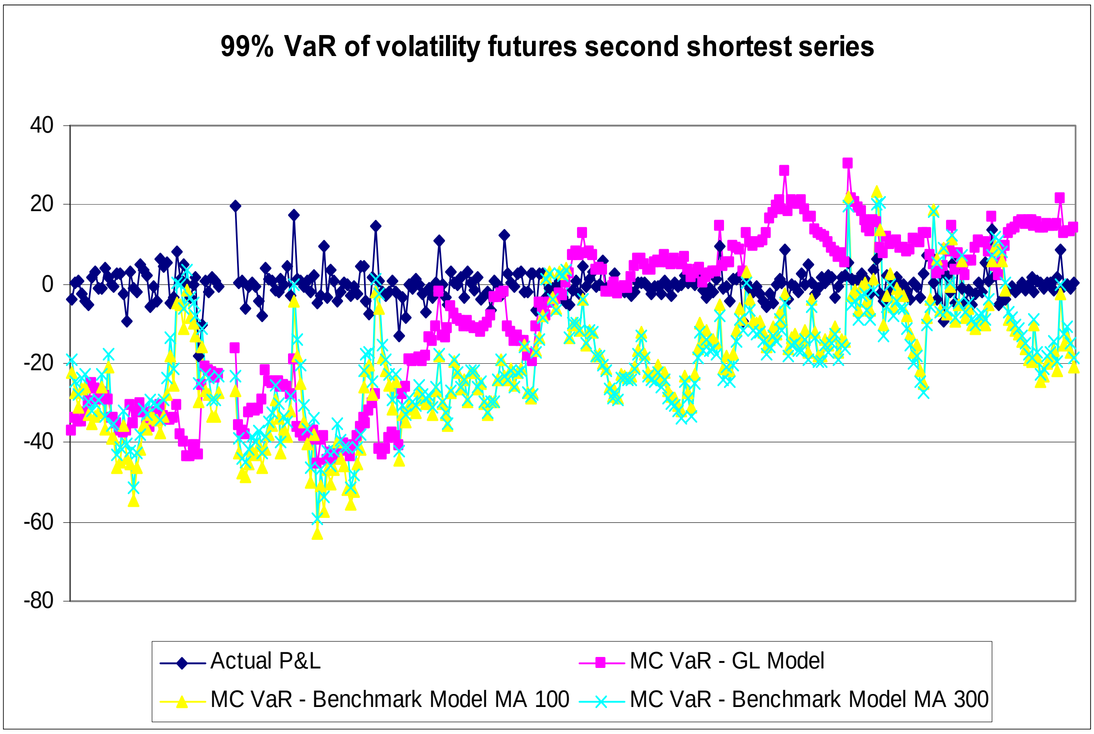
The following figures show the calculated 99% and 95% VaR and the realized profits and losses over the back testing sample for all positions:

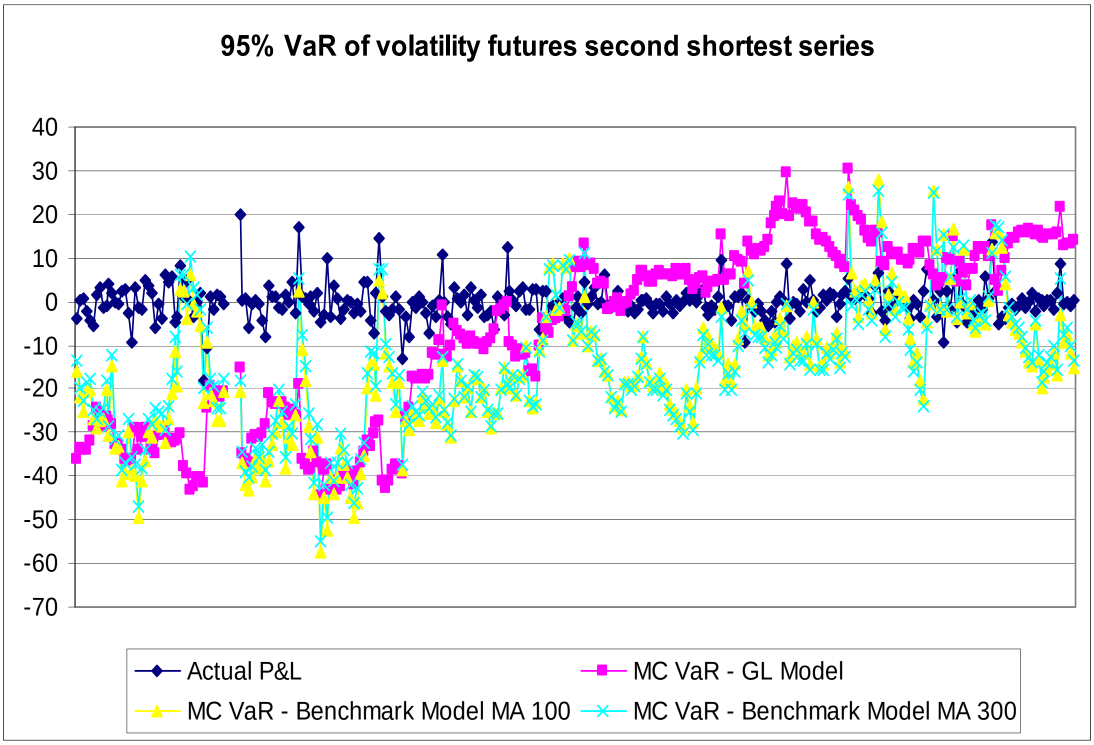




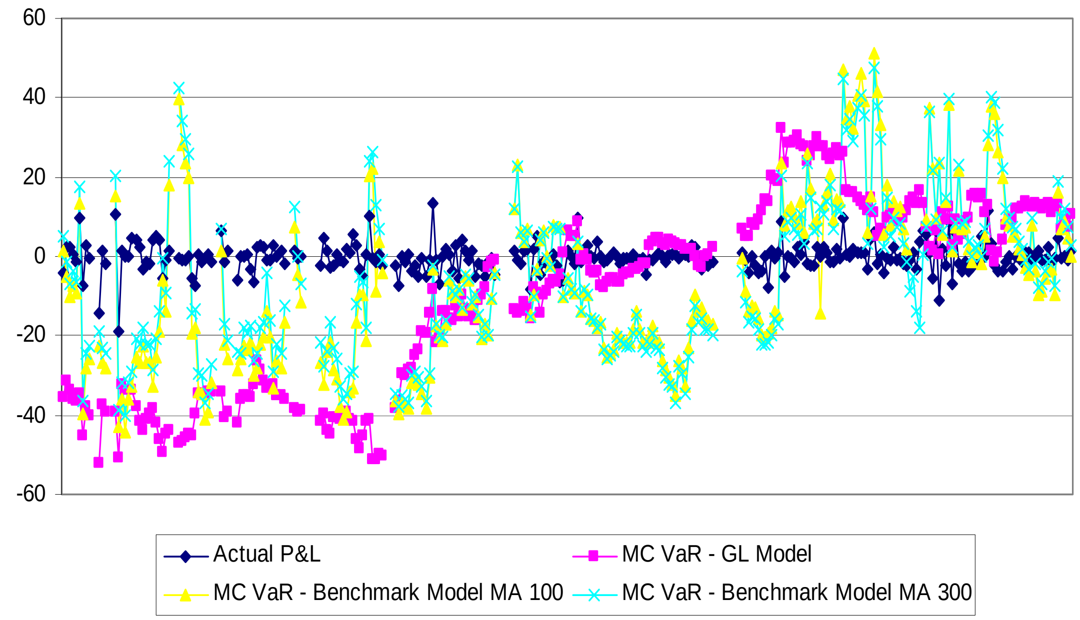

99% VaR of volatility futures shortest series
60
40
20
0
-20
-40
-60
-80
Actual P&L
MC Var - GL Model
MC VaR - Benchmark Model MA 100
MC VaR - Benchmark Model MA 300


95% VaR of volatility futures shortest series
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
Actual P&L
MC VaR - GL Model
MC VaR - Benchmark Model MA 100
MC VaR - Benchmark Model MA 300

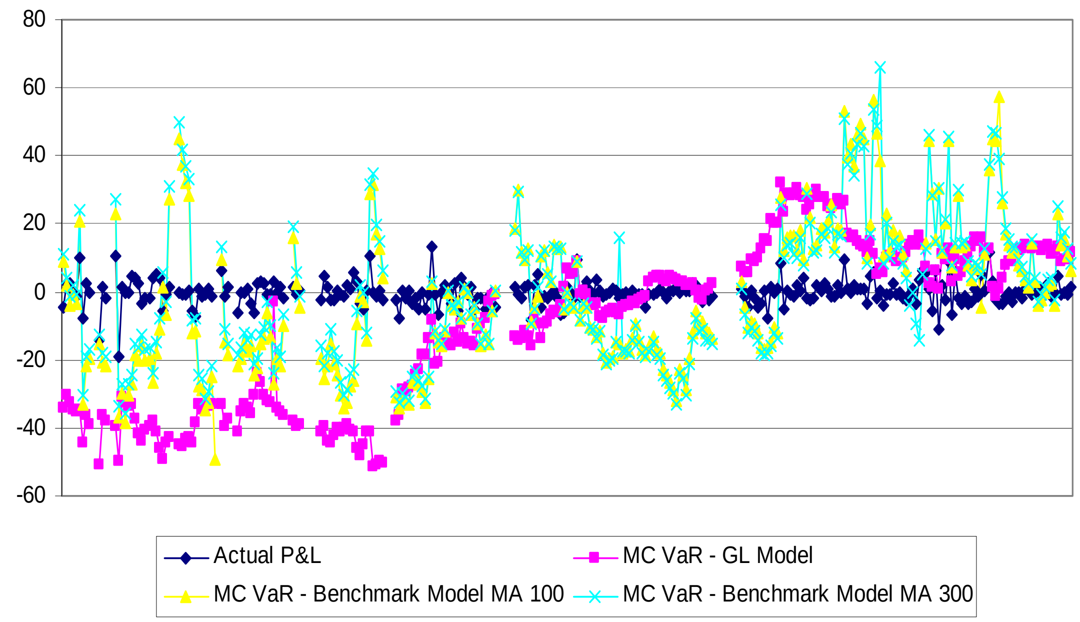

99% VaR of volatility futures second shortest series
40
20
0
-20
-40
-60
-80
Actual P&L
MC VaR - GL Model
MC VaR - Benchmark Model MA 100
MC VaR - Benchmark Model MA 300


95% VaR of volatility futures second shortest series
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-70
Actual P&L
MC VaR - GL Model
MC VaR - Benchmark Model MA 100
MC VaR - Benchmark Model MA 300

99% VaR of volatility futures third shortest series



95% VaR of volatility futures third shortest series



Section 8

Conclusions

We have examined empirically two valuation models of futures on volatility – an implied volatility index – and their respective processes of volatility. The first model is the Grűnbichler-Longstaff Model which assumes a Mean Reverting Square Root Process (MRSRP) for volatility and the second is a Benchmark Model which makes a more simplified assumption and under which volatility follows a Geometric Brownian Motion Process (GBMP). The examination included the assessment of their pricing performance and the verification of the accuracy of a value at risk measure regarding a position in both the underlying index and the volatility derivative.

The first procedure's goal was to document which model exhibits the better data fit and so to verify which is the less misspecified. We calibrated each model in a specific sample (the shortest maturity series of vol. futures contracts) and calculated the average pricing bias in another sample (longer maturity series). The value at risk study was conducted via Monte Carlo Simulation. We simulated the VIX index according to the assumed process and derived 99% and 95% VaR estimates of a position in the index itself and a position in a volatility future contract, one for each maturity series. A back testing procedure consisting of the regulatory and the Kupiec proportion of failures test was employed in order to examine under which volatility futures model and under which volatility process the value at risk methodology is more effective.

The results of the two testing procedures are mixed and depend mainly on the respective sample, that is, the maturity series of the volatility futures contracts. In the shortest and second shortest maturity series the Benchmark model exhibits a better data fit and is also the only model that passes the back testing criteria, mainly in the first series. The Grűnbichler-Longstaff model has a lower pricing bias in the fourth shortest maturity series and its pricing performance is more stable throughout the sample but in the value at risk study fails to passes the back tests in all series. Also, the performance of the models had a common pattern among the two examinations- the calibration procedure and the VaR methodology. Finally, the VaR model of a position in the VIX index is accurate whatever the process for volatility is assumed.

The documentation that the Benchmark model seems to have an overall better performance is important since it is an example in favour of the opinion that a more complex model with sophisticated assumptions is not necessary the one that it is most supported by the data. As far as the pricing behaviour is concerned, a simpler valuation framework yields better results also in the studies of Bakshi, Cao and Chen (1997) and Dumas, Fleming and Whaley (1998) which regard standard options.

In order to reach to solid conclusions and determine the direction of future research about volatility derivatives and stochastic volatility modeling, a number of additional tests and methodologies should be conducted. The value at risk methodology should be supplemented and compared to the results of a historical simulation or the variance-covariance approach. The latter can be implemented with various volatility estimators. Also, apart from the pricing performance the effectiveness of volatility futures as hedging instruments regarding volatility risk should be investigated.

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Appendix I

VIX Calculation

The generalized formula of VIX is VIX = σ x 100

where σ is the weighted implied volatility as a result of interpolation between the implied volatilities of a series from options. For every option, σ is the square root of

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K} - 1\right]^2$$

T is time to expiration in minutes

F is forward index level derived from index option prices

K_i is strike price for the i[th] out-of-the money option; a call if $K_i > F$ and a put if $K_i < F$

ΔK_i is the interval between strike prices – half the distance between the strike on either side of K_i : $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$

K_0 is the first strike price below the forward index level F

R is the risk-free interest rate to expiration

$Q(K_i)$ is midpoint of the bid-ask spread for each option with strike K_i

The options used are put and call options in the two nearest-term expiration months in order to bracket a 30-day calendar period. However, with 8 days left to expiration, the new VIX "rolls" to the second and third contract months in order to minimize pricing anomalies that might occur close to expiration.

The time of the VIX calculation is assumed to be 8:30 a.m. (Chicago time). The new VIX calculation measures the time to expiration T in minutes rather than days in order to replicate the precision that is commonly used by professional option and volatility traders. The time to expiration is given by the following expression:

$$T = \{M_{\text{Current day}} + M_{\text{Settlement day}} + M_{\text{Other days}}\}/\text{Minutes in a year}$$

where $M_{\text{Current day}}$ is the number of minutes remaining until midnight of the current day

$M_{\text{Settlement day}}$ is the number of minutes from midnight until 8:30 a.m. on SPX settlement day

$M_{\text{Other days}}$ is the total number of minutes in the days between current day and settlement day

The basic steps in the calculation of (1) are the following:

Step 1. Select the options to be used in the formula. For each contract month:

- Determine the forward index level F, based on at-the-money option prices. The at-the-money strike is the strike price at which the difference between the call and put prices is smallest. The formula used to calculate the forward index level is F = Strike price + e^{RT} x (Call price – Put price).
- Determine K_0, the strike price immediately bellow the forward index level F.
- Sort all of the options in ascending order by strike price. Select call options that have strike prices greater than K_0 and a non-zero bid price. After encountering two consecutive calls with a bid price of zero, do not select any other calls. Next, select put options that have strike prices less than K_0 and a non-zero bid price. After encountering two consecutive puts with a bid price of zero, do not select any other puts. Select both the put and a call with strike price K_0. Then average the quoted bid-ask prices for each option.
  Notice that two options are selected at K_0, while a single option is used for every other strike price. This is done to center the strip of options around K_0. in order to avoid double counting the put and call prices are averaged tom arrive at a single value.

Step 2. Calculate implied volatility for both near term and next term options by applying the formula (1).
The new VIX is an amalgam of the information reflected in prices of all of the options used. The contribution of a single option to the new VIX value is proportional to the price of that option and inversely proportional to the option's strike price.

Step 3. Interpolate between the $\sigma^2{}_{\iota s}$ to arrive at a single value with a constant maturity of 30 days to expiration. Then take the square root of that value and multiply by 100 to get VIX. If $\sigma^2{}_1$ and $\sigma^2{}_2$ are the values for the near term and next term options respectively, then the interpolation formula is

$$\sigma = \sqrt{\left\{ T_1 \sigma_1^2 \left[\frac{N_{T2} - N_{30}}{N_{T2} - N_{T1}} \right] + T_2 \sigma_2^2 \left[\frac{N_{30} - N_{T1}}{N_{T2} - N_{T1}} \right] \right\}} \times \sqrt{\frac{N_{365}}{N_{30}}}$$

where N_{T1} = number of minutes to expiration of the near term options

N_{T2} = number of minutes to expiration of the next term options

N_{30} = number of minutes in 30 days (30 x 1440 = 43200)

N_{365} = number of minutes in a 365-day year (365 x 1440 = 525600)

Appendix II

The Basle Traffic Light Approach

Table1: The Basle Penalty Zones: 99% VaR over a year (250 observations)

Zone	Number of Exceptions	k
Green	0 to 4	3
Yellow	5	3,4
	6	3,5
	7	3,65
	8	3,75
	9	3,85
Red	10	4