**UNIVERSITY OF PIRAEUS**

**DEPARTMENT OF BANKING AND FINANCIAL MANAGEMENT**
**M. Sc** in **FINANCIAL ANALYSIS FOR EXECUTIVES**

# *Option pricing with jump diffusion models*

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**PIRAEUS, FEBRUARY 2013**

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**ABSTRACT**

We discuss about option pricing with jump-diffusion models as well as their parameters effect on option prices through implied volatility figures. After introducing, the reasons about using these models, we discuss two more widely used jump-diffusion models. As amplification, we consider a stochastic volatility model which we compare with them, including their advantages and limitations.

Key words: jump-diffusions models, stochastic volatility, incomplete market, change of measure, model parameters, implied volatility.

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## SECTION 1

## INTRODUCTION

Most of the standard literature in Finance, in particular for pricing and hedging of contingent claims, is based on the assumption that the prices of the underlying assets follow diffusion type process, in particular a geometric Brownian motion (GBM).

Documentation from various empirical studies shows that such models are inadequate, both in relation to their descriptive power, as well as for the mispricing that they might induce. One of the reasons is the total change in the stock price depends on two types of changes. The first one is the fact that we may have fluctuations in the price due to general economic factors such as supply and demand, changes in economic outlook, changes in capitalization rates or other information. These factors cause small movements in the price and are modeled by a geometric Brownian motion with a constant drift term as we have already referred. The instantaneous part of the unanticipated return due to the normal price vibrations is the case of an arrival of important information into the market that will have an abnormal effect on the price. This information could be industry specific or even firm specific. By its very nature important information only arrives at discrete points in time and will be modeled by a jump process. Thus, there will be 'active' times in the underlying asset when the information arrives and 'quiet' when it does not. Both, of course are random.

Furthermore, someone could think, why we derive models with jumps. To begin with, in a model with continuous paths like a diffusion model, the price process follows a Brownian motion and the probability that the stock moves by a large amount over a small period is very small, unless one fixes an unrealistically high value of volatility. Also, in such models the prices of short term out of the money options should be much lower than what one observes in real markets. On the other hand if stock prices are allowed to jump, even when the time to maturity is very short, there is non-negligible probability that after a sudden change in the stock price the option will move in the money. Another reason that using jump diffusion models is very important emerges from the point of view, hedging. In continuous models of stock price behavior generally lead to a complete market, which can be made complete by adding one or two additional instruments, like in stochastic volatility models which we will describe below. In that case, options are redundant since every terminal payoff can be exactly replicated. Unfortunately, in real markets perfect hedging is impossible due to the presence of jumps. So an important case which is provided in our investigation is how the jump risk is eliminated according to the different models.

For these reasons, several jump diffusion models have been introduced to the finance literature in order to explain how they provide an adequate description of stock price fluctuations and market risks. In the last decade, also the research departments of major banks started to accept jump-diffusions as a valuable tool in their day to day modeling. A first approach developing further the basic Black and Scholes model with the inclusion of jumps appears to be that of Merton. Since the introduction of jumps in the Black and Scholes implies that derivative prices are no longer determined by the principle of absence of arbitrage alone, Merton solved the pricing problem, by making some assumptions. One of our purposes, here, is showing these assumptions for Merton’s model and for Kou’s model where the average jumps size follows a different distribution. In essence, each model makes some assumptions, in order to achieve having a riskless portfolio. Otherwise, there is not another way for eliminating the source of randomness which is appeared. The analytical refer is made in other section for each model separately.

Moreover, during the last decade, there has been an increasing interest in modeling the dynamic evolution of the volatility of high frequency series of financial returns using stochastic volatility models. One of the most popular equity option pricing models is Heston’s volatility model. It provides a closed-form solution for the price of a European option when the spot asset is correlated with volatility.

The main purpose of this dissertation is to code the closed form solutions of Merton’s, Kou’s and Heston’s model by Matlab and investigate the effects which are emerged on the European option prices due to the different values of each model parameters through implied volatility figures.

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The reason which we provide implied volatility figures, is that a first requirement for an option pricing model is to capture the state of the options market at a given instant. To achieve this, the parameters of the model are chosen to fit the market prices of options or at least to reproduce the main features of these prices, a procedure known a calibration of the model to the market prices. The notion of model calibration does not exist, since after observing a trajectory of the stock price, the pricing model is completely defined. On the other hand, since the pricing model is defined by a single volatility parameter, the parameter can be reconstructed from a single option price (by inverting the Black-Scholes formula). This value is known as the implied volatility of this option. In essence equate the theoretical price with the market price in Black-Scholes formula.

The Black Scholes formula is not used as a pricing model for options but as a tool for translating market prices into a representation in terms of implied volatility. The implied volatilities are better from historical volatilities for the purpose of option pricing, as historical volatilities may not reflect the current situation. For example suppose an extreme event happens to the Wall Street e.g, a financial crisis or a terrorist attack then it is hard to find similar events in the historical database, thus making historical volatilities unsuitable. A plot of the implied volatility of an option as a function of its strike price is known as a volatility smile. In the Black-Scholes setting, the only model parameter to choose is the volatility $\sigma$, originally defined as the annualized standard deviation of logarithmic stock returns. According to the Black-Scholes model, we should expect options that expire on the same date to have the same implied volatility regardless of the strikes. However empirical studies show that this is not the case: the implied volatility actually varies among the different strike prices. This discrepancy is known as the volatility skew. At-the-money options tend to have lower volatilities that in- or out-of-the-money options Black-Scholes is technically inaccurate because implied volatility should be constant according to this model. Secondly, implied volatility graph should be a horizontal straight line and when implied volatility is graphed, it is presented in volatility skew. Black-Scholes does not consider certain aspects that may alter the price, such as: liquidity or supply and demand. The Black-Scholes model performs a sort of regulation of the market itself, with traders adapting themselves to it which causes the volatility skew.

Stochastic volatility models require a negative correlation between movements in stock and movements in volatility for the presence of a skew. While this can be reasonably interpreted in terms of a leverage effect, it does not explain why in some markets such as options on major foreign exchange rates the skew becomes a smile. Since the instantaneous volatility is unobservable, assertions about its correlation with the returns are difficult to test but it should be clear from these remarks that the explanation of the implied volatility skew offered by stochastic volatility models is no more structural than the explanation offered by local volatility models.

Models with jumps, by contrast, not only lead to a variety of smile or skew patterns but also propose a simple explanation in terms of market anticipations: the presence of a skew is attributed to the fear of large negative jumps by market anticipants. This is clearly consistent with the fact that the skew/smile features in implied volatility patterns have greatly increased since the 1987 crash. Jump processes also allow explaining the distinction between skew and smile in terms of asymmetry of jumps anticipated by the market. For example for index options, the fear of a large downward jump leads to a downward skew while in foreign markets where the market moves are symmetric, jumps are expected to be symmetric thus giving rise to smiles.

There is a large body of literature on skewness across various markets and asset classes. Evidence of skewness in assets has existed for more than three decades to name a few. More recently, suggested Harvey (2000) that investors require payment for negative skew and expected return increases with negative skewness. Their results showed skewness exists in asset prices and that a pricing model incorporating skewness helps explain expected returns in assets beyond beta, size and book to market. As was mentioned found that the majority of developed markets have negative skew. The primary reason skew is important is that analysis based on normal distributions incorrectly estimates expected returns and risk. Moreover, negative skewness can result from the distribution of good and bad news from companies. Companies release more good news than bad news and bad news tends to be released in clumps.

The rest of the dissertation is structured as follows. The following section describes some of the papers which were useful for my investigation. In Section 3, include the dataset. Section 4, explores pricing and hedging in incomplete markets as well as the change of measure. In continuity, Section 5 and 6 examines Merton's and Kou's models, respectively. In Section 7 we describe the stochastic volatility Heston's model. Comparison between such models is appeared in Section 8. The final section concludes.

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**SECTION 2**

**LITERATURE REVIEW:**

In this part, we provide a review of the normative literature. More specifically, all the material that helped us to examine and carry out jump-diffusion models.

First of all, **Merton (1975)**, explains how we can estimate and derive an option pricing formula when the underlying stock returns are generated by a mixture of both continuous and jump processes. Moreover, he shows the features of the original Black-Scholes that his model keeps but has demonstrated that their model of analysis obtains even when the interest rate is stochastic, the stock pays dividends or the option is exercisable before the maturity. Of course, the paper is based on the fact that the total change in the stock price is posited to be normal, which means that produces a marginal change in the price or abnormal which is modeled by a jump process reflecting the non-marginal impact of the information. Jumps follow log-normally distribution and the probability if a jump occurs or not is described from a Poisson process.

Furthermore, as this model has the extra source of randomness through the jumps which cannot be eliminated, this means that we are in incomplete market. So, the return on the portfolio which we have created will not be riskless and equal to the risk free rate. Merton for attaining the elimination of jump risk made some assumptions. To begin with, the first assumption is that the jumps are not correlated with the market, so as the jump component of the stock's return will represent non-systematic risk. The other assumption is that the Capital Asset Pricing Model holds and the expected return on all securities which have zero-beta must be equal to the risk-free rate.

To conclude, in this paper is considered the fact that ceteris paribus, an option stock with jumps is more valuable than a stock without jumps. For deep out of the money options, there is relatively little probability that the stock price will exceed the strike price before the maturity if the underlying process is continuous. However, the possibility of a large, finite jump in price significantly increases this probability, and hence makes the option more valuable. Similarly, for the deep in the money options, there is a little probability that the stock would decline below the strike price before the maturity. And in this case the underlying process is continuous and ergo the insurance value of the option would be nil. Then, this need not be the case with jump possibilities. Furthermore, these differences will be larger as one goes to short-maturity options. Analytical review is considered below.

**Kou (2002)** has created a double exponential jump diffusion model. One of the occasions that made Kou doing research for this model, was the leptokurtic feature that the return distribution of assets may have a higher peak and two heavier tails than those of the normal distribution. Another one was the empirical phenomenon called volatility smile in option markets. In the Black and Scholes, the implied volatility should be constant, but in reality is a convex curve of the strike price, like a smile. To illustrate that the model can produce implied volatility smile Steven Kou consider real data set which were used for two year and nine year caplets in the Japanese LIBOR market. Here, the jump sizes are double exponentially distributed and the logarithm of the asset price is assumed to follow a Brownian motion. Moreover, as in Merton's model the probability if a jump occurs follows a Poisson process. Steven Kou explains and proves the change of the probability measure to risk-neutral measure where the endowment process and the price of the underlying asset have the same jump diffusion form, which is very convenient for analytical calculation.

The difficulty with hedging is not applicable in this paper. The riskless hedging is impossible when someone wants to do it in discrete time and Kou follows investigations that have already done as Merton's occasions for hedging.

Lastly, the difficulties of pricing options used this model, except from hedging are the integral that appears in the model and the fact that assumes that the increments are independent. The only way to incorporate the dependence of these increments is to use another point process, replacing Poisson process. In the line of our dissertation, we do not examine this case, so as Steven Kou in his paper.

Moreover, **Ball and Torous** (1985) paper investigate whether Merton's call option pricing model can eliminate the systematic biases which Black and Scholes call option pricing is subjected. These biases have been documented with respect to the call option's strike price, it's time to the maturity and the underlying's stock's volatility. To find if there were differences between Black and Scholes and Merton, Ball and Torous implement the maximum likelihood estimation procedure on the basis of a sample of daily returns to 30 NYSE listed common stock's return process. The result of this research is that there are not operationally significant differences between them. Also, the discrepancies between Black-Scholes and Merton model prices of the call prices do not occur if the underlying stock return process is predominated by large jumps which occur infrequently. However, to the other financial securities, such as foreign exchange may be accurately modeled as a Poisson jump diffusion process characterized by infrequent but very large jumps. Another important fact from this paper is that might not be demonstrable the differences between these two models. That because statistical analysis will have difficulty discriminating between security return data characterized by a jump intensity parameter close to zero and Merton's call formulation when the jump intensity parameter is equal to zero.

To conclude, Ball and Torous have investigated through the statistical estimation of the Poisson Jump diffusion process, the fact of the numerical calculation of the infinite sum for a single Poisson jump diffusion density. The infinite sum shall be truncated at N=10, provided double precision computer accuracy. It must be pointed out that for small values of λ, one could truncate at much smaller value for N and still provide full computer accuracy.

A paper which shows that the jump diffusion models are an essential and easy to learn tool for option pricing and risk management is **Tankov and Voltchkova (2005).** First of all, they describe the two basic buildings of every jump diffusion model which are the Brownian motion, for the diffusion part and the Poisson process, for the jump part. As Brownian motion is a familiar object to every option trader since it appears to the Black-Scholes model, they provide the description of Poisson process. The specific part, of this investigation is hedging jump's risk. Our authors emphasize that the hedging is an approximation problem, as we are in incomplete market. So, instead of replicating an option, one tries to minimize the residual hedging error. In essence, the expected squared residual hedging error is minimizing the squared norm of the market price minus the theoretical price which emerges from the estimation that we have made. Finally, hedging with stock only in the presence of jumps eliminates a large part risk but still leads to an important residual hedging error.

Another published paper which helped my investigation is **Heston (1993)** who derives a closed form solution for a price of European options on an asset with stochastic volatility. This model is one of the most popular option pricing models due to the fact that produces option prices up to an integral that must evaluated numerically. It is versatile enough to describe stock options, bond options and currency options. The Black and Scholes (1973) model shows that the mean spot return does not affect option prices at all, while the variance has a substantial effect. The pricing analysis of this paper controls for the variance when comparing option models with different skewness and kurtosis. The Black Scholes formula produces options prices virtually identical to the stochastic volatility models for at the money options. For at the money options all option models with the same volatility are equivalent. This happen because options are usually traded near the money. Another characteristic of Heston's model is that allows arbitrary correlation between volatility and spot asset returns. This fact captures important skewness effects that arise from such correlation. Skewness in the distribution of spot returns affects the pricing of in the money options relative to out of the money options. Without this correlation, stochastic volatility only changes the kurtosis. Kurtosis affects the pricing of near the money versus far from the money options. To transform to the risk neutral probability measure, Heston uses the characteristic functions because the probabilities are not available immediately in the closed form solution, but this we will analyze it below. Moreover, hedging is possible according to Heston by making zero the sources of randomness which are appeared. Finally, the stochastic volatility can be very flexible and promising description of option prices, in the case which we choose the appropriate parameters.

For stochastic volatility models, **Henderson, Hobson, Howison and Kluge (2003)** have investigated option prices in an incomplete stochastic volatility model with correlation. In general, they analyze the role of the market price of volatility risk and the choice of pricing measure on the prices of options. Also, they compare under various changes of measure option prices in Heston model. The most important observation which comes from this paper is that stochastic volatility models have some typical features. The correlation is negative so the smiles are skewed to the left.

Furthermore, **Bates (2000)** have done a research which is based on the derivation the appropriate characterization of asset market equilibrium when asset prices follow jump diffusion processes. They also provide the problem that stochastic volatility and jump risk option pricing models introduce forms of risk embodied in option prices that are not directly priced by any instrument currently traded in financial markets. One result that Bates considers is that even if the jumps in the underlying asset price are independent, the fact that the jumps occur at the same time implies that jump risk on the asset is no diversifiable and systematic. Only if the jumps are truly firm-specific and do not affect aggregate wealth as is assumed in Merton. A second result from their investigation is that, insofar as jumps in the underlying asset price are positively correlated with jumps in the market, the effect is to bias downwards the mean expected jump would be negative, leading to negative asymmetries relative to Black-Scholes option prices of the sort found in recent years in stock options.

**SECTION 3**

**DATASET**

We are interested in pricing options on S&P 500 with different types of models, like jump diffusion and stochastic volatility models. For this purpose we need first of all to choose a run date for all the options. For our investigation this day was 20/09/2012. The price of the underlying asset that day was taken from BLOOMBERG. Moreover, another data is the interest rate which is constant in both cases which we investigated. The data comes from BLOOMBERG for one, three, nine months and for one year respectively. For this reason, we made an exponential interpolation for fitting these rates with our time to maturity for each case. For the volatility estimation, data comes from BLOOMBERG. We take the 75 previous S&P 500 daily prices from the run date and implement exponential weighted moving average method. For adapting the volatility per year, we multiplied with the root of 252.

For the parameters estimation we made an investigation in the literature which gave us the opportunity to use some values of them. To begin with, in Merton's model we have three parameters. The first one is the jump intensity λ which shows us how many jumps happen per unit of time on average and causes the asset price to jump randomly. Jump intensity can take any positive value. In specific$\lambda \epsilon \mathbb{R}^{+}$. In the case which is equal to zero, we do not have jumps so we follow Black and Scholes model. According to the literature, we have observed that the maximum value which intensity takes was 100. The values which we have used are representative from the literature.

Furthermore, the second Merton's parameter is the percentage change in the asset price caused by the jump. Specifically is the average of the relative jump size μ. When μ is negative, this means that we expect more negative jumps from positive. How much more depends on how far from the zero we are. When the relative jump size is positive, we expect the reverse. In the special case where the jump size of positive and negative jumps is equal, the value of this parameter is zero. In the literature, we have found prices from -1 until 1 which means from -100% until 100%. Realistic, in our investigation we have used the values -0,5, 0, 0,5 respectively for the different figures so as showing what happened to the skew and to the option prices, in its case. To be more specific, when we say that the relative jump size is -0,5, this means that the underlying's price will decrease at the half of its value. The last parameter for Merton's model is the standard deviation of the jump size δ which takes positive values and is measured by percentage. For this parameter, we have taken some representative values from the literature.

Kou's model has some parameters, too. The first one is p which is the probability of upward jumps. As a concomitant q=1-p is the probability of downward jumps. The values which we have taken for these probabilities are from zero up to one. For example when p=0,4 there is 40% probability to happen upward jumps. Moreover, as in Merton's model the average number of jumps which will happen per year are described from the jump intensity λ of the Poisson's distribution. Consequently, takes the same values. Any longer, the average size of the upward jumps $\frac{1}{n_1}$ takes values less from one. This happens because $n_1 > 1$ to ensure that the average upward jump cannot exceed 100%, which is quite reasonable. Respectively, $\frac{1}{n_2}$ is the average size of the downward jumps which is positive. We take three pairs of values, which include the case that the average size for the upward jumps is bigger from the respective for the downward jumps, the conversely and the case where these are equal. The prices that we have found in the literature are fluctuated from 5 until 15.

Finally, in Heston's model we consider four parameters, the correlation between volatility of the asset and the log-returns (ρ), the volatility of the variance (σ), the mean reversion speed for the variance (κ) and the long-run variance (θ). The first one, always takes a value between -1 and 1. This means that a positive indicates a positive association between the volatility of the asset and the log-returns, while a negative correlation indicates a negative association between them. In our investigation we have taken the values -0,5, 0 and 0,5 for showing the effects on option prices in each case, respectively. The volatility of the variance is always positive and is measured as a percentage. According to the literature we have chosen the most representative values like 10% or 20%. Similarly, the long-run variance is positive but because is appertains a big interval of time is smaller than the volatility of the variance.

## SECTION 4

### 4.1 PRICING AND HEDGING IN INCOMPLETE MARKETS

Complete market means that any option can be perfectly replicated by a self-financing strategy involving the underlying and cash. In such markets, options are redundant. For that reason they are perfectly substitutable by trading in the underlying so the very existence of an options market becomes a mystery. Of course, in real markets hedging is not possible and options enable market participants to hedge risks that cannot be hedged by trading in the underlying only. Options thus allow a better allocation and transfer of risk among market participants, which was the purpose for the creation of derivatives market in the first place.

The Black-Scholes arbitrage portfolio under continuous trading has zero-risk due to continuous hedging. However under discrete trading conditions we introduce some risk since the market moves between trades. The portfolio risk has the order of the trading interval length and thus the risk will be to zero as the interval tends to zero. Therefore provided the time interval between trades is not too large, the error between the Black-Scholes price and the realistic discrete trading price will not differ by much.

The Black-Scholes model is not valid though, even if we trade in the continuous limit, if the stock price dynamics do not have a continuous sample path. The Black-Scholes formula is valid if the stock price can only change by a small amount over a small interval of time and if one uses upper bounds for volatility to price and hedge contingent claims. Again empirical studies show that this is not the case and a more sophisticated model of the underlying stock price is required. Market returns are generally leptokurtic meaning the market distribution has heavier tails than a normal distribution. The model should permit large random fluctuations such as crashes or upsurges. The market distribution is generally negatively skewed since downward outliers are usually larger than upward outliers. On the other hand, the upper bound for volatility in a model with jumps is infinity. In other words, the only way to perfectly hedge a call option against jumps is to buy and hold the underlying asset. This remark, shows that when you move from diffusion-based on complete market models to more realistic models, the concept of replication, which is the central in diffusion models, does not provide the right framework for hedging and risk management.

In discontinuous price models, the nonexistence of a perfect hedge is not a market imperfection but an imperfection of complete market models. In model with jumps, riskless replication is an exception rather than the rule. Any hedging strategy has a residual risk which cannot be hedged away to zero and should be taken into account in the exposure of the portfolio. This offers a more realistic picture of risk management of option portfolios.

On the other hand, in stochastic volatility models there exists the impossibility of perfectly hedging options with the underlying. However in diffusion-based stochastic volatility models completeness can be restored by adding a single option to the set of available hedging instruments. After that, stochastic volatility models recommend setting up a perfect hedge by trading dynamically in the underlying and one option.

## 4.2 CHANGE OF MEASURE

To begin with the Girsanov theorem provides the general framework for transforming one probability measure into another equivalent. The theorem covers the case of Brownian motion. Hence, the state space is continuous and the transformations are extended to continuous-time stochastic processes. The probabilities so transformed as equivalent because they assign positive probabilities to the same domains. Although the two probability distributions are different, with appropriate transformation one can always recover one measure from the other. Since such recoveries, are always possible, we may want to use the convenient distribution for our calculations, and then if desired, switch back to the original distribution. Accordingly, if we have to calculate an expectation and if this expectation is easier to calculate with an equivalent measure may not be the one that governs the true states of nature. After all, the purpose is not to make a statement about the odds of various states of nature. The purpose is to calculate a quantity in an convenient fashion. So, the general method can be summarized as follows: (1) We have an expectation to calculate. (2) We transform the original probability measure so that the expectation becomes easier to calculate. (3) We calculate the expectation under the new probability. (4) Once the result is calculated and if desired, we transform this probability back to the original distribution. In our investigation we have observed that in diffusion processes change only the drift. A simple example which we can provide is with a Wiener process in a fixed time horizon T. Let φ is an adapted process and define the process L by

$$\begin{Bmatrix} dL_t = L_t \varphi_t dW_t \\ L_o = 1 \end{Bmatrix} \quad (4.1)$$

Assume that $E^P[L_T] = 1$ and define a new measure Q on $F_T$ by $dQ = L_t dP \ on\ F_t, 0 \leq \mathrm{t} \leq \mathrm{T}$ . Then Q<<P and the process $W^Q$, defined by

$$dX_t = \mu_t dt + \sigma_t \, dW_t \quad (4.2)$$

${W_t}^Q = W_t - \int_0^t \varphi(s) ds$ is Q-Wiener. We can also write this as $W_t = \varphi_t dt + d{W_t}^Q$ . Reliant on this suppose that you have a process X with P dynamics

where μ and σ are adapted and W is P-Wiener. We now do a Girsanov transformation as above, and the question is what the Q-dynamics look like. From Girsanov's theorem we have:

$$dW_t = \varphi_t dt + d{W_t}^Q \quad (4.3)$$

And substituting this into the P dynamics we obtain the Q dynamics as

$$dX_t = \{\mu_t + \sigma_t \varphi_t\} dt + \sigma_t \, d{W_t}^Q \quad (4.4)$$

So, now we can observe, clearly that the drift changed but the diffusion is unaffected. Furthermore, we consider one more example, in order to show how we get from the one probability measure to another. Fix t and consider a normally distributed random variable $z_t \sim N(0{,}1)$. Denote the density of $z_t$ by f($z_t$) and the implied probability measure by P such that

$$dP(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_t)^2} dz_t \quad (4.5)$$

In this example, the state space is continuous, although we are still working with a single random variable, instead of a random process. Next, define the function

$$\xi(z_t) = e^{z_t \mu - \frac{1}{2}\mu^2} \quad (4.6)$$

When we multiply $\xi(z_t)$ by $dP(z_t)$, we obtain a new probability, This can be seen from the following:

$$[dP(z_t)][\xi(z_t)] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_t)^2 + z_t\mu - \frac{1}{2}\mu^2} dz_t \ (4.7)$$

After grouping the terms in the exponent, we obtain the expression

$$d\check{P}(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[z_t - \mu]^2} dz \ (4.8)$$

Clearly, $d\check{P}(z_t)$ is a new probability measure, defined by

$$d\check{P}(z_t) = dP(z_t)\xi(z_t) \ (4.9)$$

By sampling reading from the density, we see that $\check{P}(z_t)$ is the probability associated with a normally distributed random variable mean μ and variance 1. It turns out that by multiplying $dP(z_t)$ by the function $\xi(z_t)$ and then switching to $\check{P}$ , we succeeded in changing the mean of $z_t$. Note that in this particular case the multiplication by $\xi(z_t)$ preserved the shape of the probability measure. In fact

$$d\check{P}(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[z_t - \mu]^2} dz \ (4.10)$$

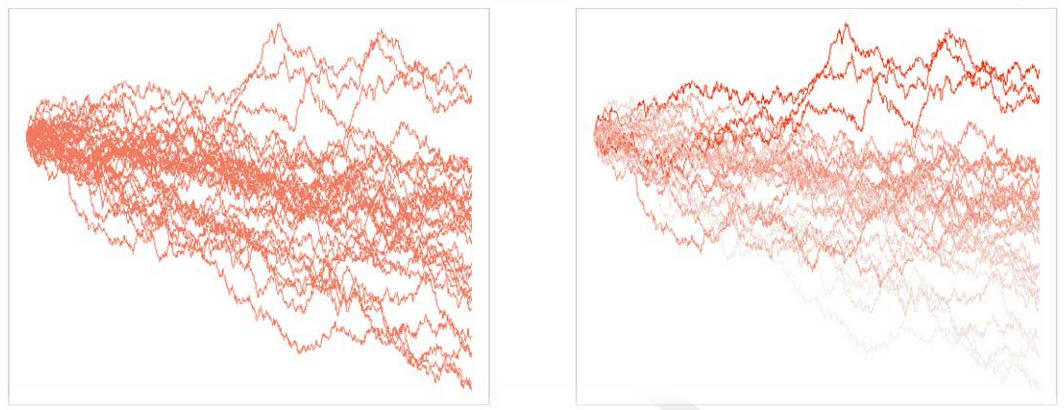
is still a bell-shaped, Gaussian curve with the same variance. But $P(z_t)$ and $\check{P}(z_t)$ are different measures. They have different means and they assign different weights to intervals on the z-axis. Under the measure $P(z_t)$ the random variable $z_t$ has mean zero, $E^P[z_t] = 0$ and the variance, $E^P[{z_t}^2] = 1$. However, under the new probability measure, $\check{P}(z_t)$ , $z_t$ has mean $E^P[z_t] = \mu$. The variance is unchanged. What we have just shown is that there exists a function $\xi(z_t)$ such that if we multiply a probability measure by this function, we get a new probability. The resulting random variable is again normal but has a different mean. Finally the transformations of measures,

$$d\check{P}(z_t) = dP(z_t)\xi(z_t) \ (4.11)$$

Which changed the mean of the random variable $z_t$, is reversible:

$$\xi(z_t)^{-1} d\check{P}(z_t) = dP(z_t) \quad (4.12)$$

The transformation leaves the variance of $z_t$ unchanged and is unique, given μ and σ.



The left side shows a Wiener process with negative drift under a canonical measure $P$. On the right side each path of the process is colored according to its likelihood under the martingale measure $Q$. The density transformation from $P$ to $Q$ is given by the Girsanov theorem.

## SECTION 5 MERTON'S MODEL

### 5.1 THE FORMULA

The first Jump Diffusion model (MJD) was introduced by Robert Merton in 1976. In MJD model, changes in the asset price consist of normal (continuous diffusion) component that is modeled by a Brownian motion with a drift process and abnormal (discontinuous, i.e. jump) component that is modeled by a compound Poisson process. Asset price jumps are assumed to occur independently and identically as we have already reported above. The probability that an asset price jumps during a small time interval can be written using a Poisson process $dN_t$ as:

Pr {an asset price jumps once in dt} = Pr{ $dN_t = 1$}=λdt+O(dt)

Pr {an asset price jumps more than once in dt} = Pr{ $dN_t \geq 2$}=0+O(dt)

Pr {an asset price does not jump in dt} = Pr{ $dN_t = 0$}=1-λdt+O(dt) (5.1)

where the parameter $\lambda \epsilon R^+$ is the intensity of the jump process (the mean number of jumps per unit of time) which is independent of time t and O(dt) is the asymptotic order symbol defined by ψ(dt)=O(dt) if $\lim_{dt \to 0}[\psi(dt)/dt]$.

Suppose in the small time interval dt the asset price jumps from $S_t$to $J_t S_t$ (we call $J_t$as absolute price jump size). So the relative price jump size (i.e. percentage change in the asset price caused by the jump) is

$$\frac{dS_t}{S_t} = \frac{J_t S_t - S_t}{S_t} = J_t - 1 \text{ (5.2)}$$

where Merton assumes that the absolute price jump size $J_t$ is a nonnegative random variables drawn from lognormal distribution, i.e. $\ln(J_t) \sim i.i.d.\, N(\mu, \delta^2)$. This in turn implies that $E(J_t) = e^{\mu + \frac{1}{2}\delta^2}$ and $E[(J_t - E(J_t)^2] = e^{2\mu + \delta^2}(e^{\delta^2} - 1)$ (5.3)

This is because if lnx~N(α,β) then x~Lognormal($e^{a+\frac{1}{2}b^2}, e^{2a+b^2}(e^{b^2}-1)$) (5.4)

Merton Jump Diffusion dynamics of asset price which incorporates the above properties takes the SDE of the form:

$$\frac{dS_t}{S_t} = (a - \lambda k)dt + \sigma dz_t + (J_t - 1)dN_t \quad (5.5)$$

where α is the instantaneous expected return on the asset, σ is the instantaneous volatility of the asset return conditional on that jump does not occur, $dz_t$ is a standard Brownian motion process, and is an Poisson process with intensity $\lambda$. The proof of equation (5.5) is presented in Appendix $A_2$. Standard assumption is $z_t, J_t$ and $N_t$ are independent. The relative price jump size of $S_t$, $J_t - 1$ is lognormally distributed with the mean E$[(J_t - 1)] = e^{\mu+\frac{1}{2}\delta^2} - 1 \equiv \kappa$ and the variance E$[(J_t - 1 - E(J_t - 1)^2)] = e^{2\mu+\delta^2}(e^{\delta^2} - 1)$. Merton assumes that the absolute price jump size $J_t$is a lognormal random variable such that:

$$(J_t) \sim i.i.d.Lognormal\left(e^{\mu+\frac{1}{2}\delta^2}, e^{2\mu+\delta^2}\left(e^{\delta^2} - 1\right)\right)$$

This is equivalent to saying that Merton assumes that the relative price jump size is a $J_t - 1$ lognormal random variable such that:

$$(J_t - 1) \sim i.i.d\ Lognormal\ (\kappa \equiv e^{\mu+\frac{1}{2}\delta^2} - 1, e^{2\mu+\delta^2}\left(e^{\delta^2} - 1\right))$$

This is equivalent to saying that Merton assumes that the log price jump size is a normal random variable such that:

$$\ln(J_t) \sim i.i.d.Normal\ (\mu, \delta^2)$$

This is equivalent to saying that the log-return jump size $\ln(\frac{J_t S_t}{S_t})$ is a normal random variable such that:

$\ln(\frac{J_t S_t}{S_t}) = \ln(J_t) \sim i.i.d.\, Normal\ (\mu, \delta^2)$

It is extremely important to note that:

$E[(J_t - 1)] = e^{\mu + \frac{1}{2}\delta^2} - 1 \equiv \kappa \neq E[(lnJ_t)] = \mu$ (5.6)

because $\ln E[(J_t - 1)] \neq E[ln(J_t - 1)] = E[ln(J_t)]$

The expected relative price change $E[\frac{dS_t}{S_t}]$ from the jump part $dN_t$, in the time interval dt is λκdt since $E[(J_t - 1)dN_t] = E[(J_t - 1)]E[dN_t]$=κλdt. This is the predictable part of the jump. This is why the instantaneous expected return on the asset αdt is adjusted by –λκdt in the drift term of the jump-diffusion process to make the jump part an unpredictable innovation:

$E[\frac{dS_t}{S_t}]$=E[(α-λκ)dt]+E[σd$B_t$] $+ E[(J_t - 1)]dN_t]$ (5.7)

$E[\frac{dS_t}{S_t}]$=(α-λκ)dt+0+$\Lambda\kappa dt = \alpha dt$ (5.8)

If $dN_t$=0, then the return dynamics would be identical to those posited in Black and Scholes(1973). The model described to (1.1) can be rewritten and expands as follows:

$$\frac{dS}{S} = \begin{cases} (a - \lambda k)dt + \sigma dz_t & if\ no\ Poisson\ event\ occurs \\ (a - \lambda k)dt + \sigma dz_t + (J_t - 1) & if\ a\ Poisson\ event\ occurs \end{cases} \quad (5.9)$$

If there is a jump $(dN_t = 1)$ then S immediately goes to the value JS. We can model a sudden 10% fall in the asset price by J=0,9.

## 5.2 PDE Approach by hedging

Merton derived a pricing formula for a European option on the asset S under the Merton jump diffusion model. He used a delta-hedging argument similar to that used by Black and Scholes in the derivation of the call option pricing formula under geometric Brownian motion. When the underlying can jump to any level, the market is not complete, since there are more states than assets. In incomplete market there are many possible choices for a risk-neutral measure, but Merton choose $\mathbb{Q}_M$ obtained as in the Black Scholes model by changing the drift of the Wiener process but leaving the other ingredients unchanged. Also, proposed that the risk-neutral process be determined by two considerations:

i. it has the same volatility and jump statistics

ii. under the risk-neutral $e^{-rT}S$ is a martingale.

A process $(X_t)$ is said to be a martingale if X is nonanticipating (adapted to $F_t$), $\mathrm{E}[X_t]$ is finite for any $t\epsilon[0,T]$ and for any $s> t, \mathrm{E}[X_s/F_t] = X_t$ . In other words, the best prediction of a martingale's future value is its present value.

Merton justified this choice by assuming that jump risk is diversifiable, therefore, no risk premium is attached to it. In other words, assume that the extra randomness due to jumps can be diversified away. If no jump occurs in the asset price then the only risk in the asset evolution comes from the Wiener process $z_t$. Consider a portfolio P of the one long option position $\mathrm{V}(S_t, t)$ on the underlying asset S written at time t and a short position of the underlying asset in quantity Δ to derive option pricing formula in the presence of jumps:

$$P_t = \mathrm{V}(S_t, t) - \Delta S_t \qquad (5.10)$$

Portfolio value changes by in very short period of time:

$$dP_t = d\mathrm{V}(S_t, t) - \varDelta dS_t \quad (5.11)$$

Merton jump diffusion model dynamics of an asset price is given the equation in the differential form as:

$$\frac{\mathrm{dS_t}}{\mathrm{S_t}} = (\mathrm{a} - \lambda \mathrm{k})\mathrm{dt} + \sigma \mathrm{dz_t} + (\mathrm{J_t} - 1)\mathrm{dN_t} \ (5.12)$$

$$\mathrm{dS_t} = (\mathrm{a} - \lambda \mathrm{k})\mathrm{S_t dt} + \sigma \mathrm{S_t dz_t} + (\mathrm{J_t} - 1)\mathrm{S_t dN_t} \quad (5.13)$$

It$\hat{o}$ formula for the jump-diffusion process is given as (Cont and Tankov (2004)):

$$\mathrm{df}(\mathrm{X_t}, \mathrm{t}) = \frac{\vartheta \mathrm{f}(\mathrm{X_t}, \mathrm{t})}{\vartheta \mathrm{t}} \mathrm{dt} + \mathrm{b_t} \frac{\vartheta \mathrm{f}(\mathrm{X_t}, \mathrm{t})}{\vartheta \mathrm{x}} + \frac{\mathrm{\sigma_t}^2}{2} \frac{\vartheta^2 \mathrm{f}(\mathrm{X_t}, \mathrm{t})}{\vartheta \mathrm{x}^2} \mathrm{dt} + \sigma_\mathrm{t} \frac{\vartheta \mathrm{f}(\mathrm{X_t}, \mathrm{t})}{\vartheta \mathrm{x}} \mathrm{dz_t} + [\mathrm{f}(\mathrm{X_{t-}} + \Delta \mathrm{X_t}) - \mathrm{f}(\mathrm{X_{t-}})] \quad (5.14)$$

Where the $b_t$ corresponds to the drift term and $\sigma_t$ corresponds to the volatility term of a jump-diffusion process $X_t = X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_s dz_s + \sum_{i=1}^{N_t} \varDelta X_\iota$ Apply this to our case of option price function $\mathrm{V}(\mathrm{S}, t)$:

$$\mathrm{d\,V}(\mathrm{S_t}, \mathrm{t}) = \frac{\vartheta \mathrm{V}}{\vartheta \mathrm{t}} \mathrm{dt} + (\mathrm{a} - \lambda \mathrm{k})\mathrm{S_t} \frac{\vartheta \mathrm{V}}{\vartheta \mathrm{S_t}} \mathrm{dt} + \frac{\mathrm{\sigma_t}^2 \mathrm{S_t}^2}{2} \frac{\vartheta^2 \mathrm{V}}{\vartheta \mathrm{S_t}^2} \mathrm{dt} + \sigma \mathrm{S_t} \frac{\vartheta \mathrm{V}}{\vartheta \mathrm{S_t}} \mathrm{dz_t} + [\mathrm{V}(\mathrm{J_t S_t}, \mathrm{t}) - \mathrm{V}(\mathrm{S_t}, \mathrm{t})]\mathrm{dN_t} \quad (5.15)$$

The term $[\mathrm{V}(J_t S_t, t) - V(S_t, t)]dN_t$ describes the difference in the option value when a jump occurs. Now the change in the portfolio value can be expressed as by substituting (1.5) and (1.6) into (1.4):

$$dP_t = d\mathrm{V}(S_t, t) - \varDelta dS_t \qquad (5.16)$$

$$\mathrm{dP_t} = \frac{\vartheta \mathrm{V}}{\vartheta \mathrm{t}} \mathrm{dt} + (\mathrm{a} - \lambda \mathrm{k})\mathrm{S_t} \frac{\vartheta \mathrm{V}}{\vartheta \mathrm{S_t}} \mathrm{dt} + \frac{\mathrm{\sigma_t}^2 \mathrm{S_t}^2}{2} \frac{\vartheta^2 \mathrm{V}}{\vartheta \mathrm{S_t}^2} \mathrm{dt} + \sigma \mathrm{S_t} \frac{\vartheta \mathrm{V}}{\vartheta \mathrm{S_t}} \mathrm{dz_t} + [\mathrm{V}(\mathrm{J_t S_t}, \mathrm{t}) - \mathrm{V}(\mathrm{S_t}, \mathrm{t})]\mathrm{dN_t} - \Delta\{(\mathrm{a} - \lambda \mathrm{k})\mathrm{S_t dt} + \sigma \mathrm{S_t dz_t} + (\mathrm{J_t} - 1)\mathrm{S_t dN_t}\} \quad (5.17)$$

$$\mathrm{dP_t} = \left\{\frac{\vartheta \mathrm{V}}{\vartheta \mathrm{t}} + (\mathrm{a} - \lambda \mathrm{k})\mathrm{S_t} \frac{\vartheta \mathrm{V}}{\vartheta \mathrm{S_t}} + \frac{\mathrm{\sigma_t}^2 \mathrm{S_t}^2}{2} \frac{\vartheta^2 \mathrm{V}}{\vartheta \mathrm{S_t}^2} - \Delta(\mathrm{a} - \lambda \mathrm{k})\mathrm{S_t}\right\} \mathrm{dt} + \left(\sigma \mathrm{S_t} \frac{\vartheta \mathrm{V}}{\vartheta \mathrm{S_t}} - \Delta \sigma \mathrm{S_t}\right) \mathrm{dz_t} + \{\mathrm{V}(\mathrm{J_t S_t}, \mathrm{t}) - \mathrm{V}(\mathrm{S_t}, \mathrm{t}) - \Delta(\mathrm{J_t} - 1)\mathrm{S_t}\}\mathrm{dN_t} \qquad (5.18)$$

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If there is no jump between time 0 and t (i.e. $dN_t = 0$), the problem reduces to Black-Scholes case in which setting $\Delta=\frac{\vartheta V}{\vartheta S_t}$ makes the portfolio risk-free leading to the following (i.e. the randomness $dz_t$ has been eliminated):

$$dP_t = \left\{\frac{\vartheta V}{\vartheta t} + (a-\lambda k)S_t\frac{\vartheta V}{\vartheta S_t} + \frac{\sigma_t{}^2 S_t{}^2}{2}\frac{\vartheta^2 V}{\vartheta S_t{}^2} - \frac{\vartheta V}{\vartheta S_t}(a-\lambda k)S_t\right\}dt + \left(\sigma S_t\frac{\vartheta V}{\vartheta S_t} - \frac{\vartheta V}{\vartheta S_t}\sigma S_t\right)dz_t \qquad (5.19)$$

$$dP_t = \left\{\frac{\vartheta V}{\vartheta t} + \frac{\sigma_t{}^2 S_t{}^2}{2}\frac{\vartheta^2 V}{\vartheta S_t{}^2}\right\}dt \quad (5.20)$$

This in turn means that if there is a jump between time 0 and t (i.e. $dN_t \neq 0$), setting $\Delta=\frac{\vartheta V}{\vartheta S_t}$, does not eliminate the risk. Suppose we decided to hedge the randomness by diffusion part $z_t$ , in the underlying asset price and not to hedge the randomness caused by jumps $dN_t$(which occur infrequently) by setting $\Delta=\frac{\vartheta V}{\vartheta S_t}$ . Then the change in the value of the portfolio is given from equation:

$$dP_t = \left\{\frac{\vartheta V}{\vartheta t} + (a-\lambda k)S_t\frac{\vartheta V}{\vartheta S_t} + \frac{\sigma_t{}^2 S_t{}^2}{2}\frac{\vartheta^2 V}{\vartheta S_t{}^2} - \frac{\vartheta V}{\vartheta S_t}(a-\lambda k)S_t\right\}dt + \left(\sigma S_t\frac{\vartheta V}{\vartheta S_t} - \frac{\vartheta V}{\vartheta S_t}\sigma S_t\right)dz_t + \left\{V(J_t S_t, t) - V(S_t, t) - \frac{\vartheta V}{\vartheta S_t}(J_t - 1)S_t\right\}dN_t \qquad (5.21)$$

$$dP_t = \left\{\frac{\vartheta V}{\vartheta t} + \frac{\sigma_t{}^2 S_t{}^2}{2}\frac{\vartheta^2 V}{\vartheta S_t{}^2}\right\}dt + + \left\{V(J_t S_t, t) - V(S_t, t) - \frac{\vartheta V}{\vartheta S_t}(J_t - 1)S_t\right\}dN_t \qquad (5.22)$$

In Merton’s model an approach that is presented is that if the source of jumps is such information, then the jump component of the stock’s return will represent non-systematic risk. I.e. the jump component will be uncorrelated with the market. Merton makes the assumption that the risk associated with the jumps in the asset is diversifiable since the jumps in the individual asset price are uncorrelated with the market as a whole, in other words the risk is diversifiable (non-systematic) therefore the beta of this portfolio is zero. Hence, Merton’s model has not risk premium. Furthermore, another approach to the pricing problem in Merton’s model is that we assume that the Capital Asset Pricing model was a valid description of equilibrium security returns. For that reason if the Capital Asset Pricing model holds, then the expected return on all zero-beta securities must equal the riskless rate. Merton's formula was deduced from the twin assumptions that we have already depicted. Although, there have been no empirical studies of the correlation between the jump component of stocks returns and the market return. So one can hardly claim strong empirical evidence to support these assumptions. If this is the case then the Capital Asset Pricing Model (CAPM) says the jump terms offer no risk premium and the asset still grows at the risk free rate:

$$\mathrm{E}[dP_t]=\mathrm{r}P_t dt \quad (5.23)$$

After substitution by setting $\Delta=\frac{\vartheta V}{\vartheta S_t}$:

$$\mathrm{E}[\left\{\frac{\vartheta V}{\vartheta t}+\frac{{\sigma_t}^2{S_t}^2}{2}\frac{\vartheta^2 V}{\vartheta {S_t}^2}\right\}dt+\left\{V(J_t S_t,t)-V(S_t,t)-\frac{\vartheta V}{\vartheta S_t}(J_t-1)S_t\right\}dN_t]=r\{V(S_t,t)-\Delta S_t\}dt$$

$$(\frac{\vartheta V}{\vartheta t}+\frac{{\sigma_t}^2{S_t}^2}{2}\frac{\vartheta^2 V}{\vartheta {S_t}^2})dt++E[V(J_t S_t,t)-V(S_t,t)-\frac{\vartheta V}{\vartheta S_t}(J_t-1)S_t]E[dN_t]=$$

$$r\left\{V(S_t,t)-\frac{\vartheta V}{\vartheta S_t}S_t\right\}dt$$

$$(\frac{\vartheta V}{\vartheta t}+\frac{{\sigma_t}^2{S_t}^2}{2}\frac{\vartheta^2 V}{\vartheta {S_t}^2})dt++E[V(J_t S_t,t)-V(S_t,t)-\frac{\vartheta V}{\vartheta S_t}(J_t-1)S_t]\lambda dt=$$

$$r\left\{V(S_t,t)-\frac{\vartheta V}{\vartheta S_t}S_t\right\}dt$$

$$\frac{\vartheta V}{\vartheta t}+\frac{{\sigma_t}^2{S_t}^2}{2}\frac{\vartheta^2 V}{\vartheta {S_t}^2}++\lambda E[V(J_t S_t,t)-V(S_t,t)-\frac{\vartheta V}{\vartheta S_t}(J_t-1)S_t]=$$

$$r\left\{V(S_t,t)-\frac{\vartheta V}{\vartheta S_t}S_t\right\}$$

Thus, the Merton's jump-diffusion model counterpart of Black-Scholes PDE is:

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$$\frac{\vartheta V}{\vartheta t} + \frac{\sigma_t{}^2 S_t{}^2}{2}\frac{\vartheta^2 V}{\vartheta S_t{}^2} + \lambda E[V(J_t S_t, t) - V(S_t, t)] - \lambda \frac{\vartheta V}{\vartheta S_t} E(J_t - 1)S_t - rV(S_t, t) + \frac{\vartheta V}{\vartheta S_t} rS_t = 0 \ (5.24)$$

where the term $E[V(J_t S_t, t) - V(S_t, t)]$ involves the expectation operator and $E[(J_t - 1)] = e^{\mu + \frac{1}{2}\delta^2} - 1 \equiv \kappa$ (which is the mean of relative asset price jump size). Obviously, if jump is not expected to occur (i.e. λ=0), this reduces to Black-Scholes PDE:

$$\frac{\vartheta V}{\vartheta t} + \frac{\sigma_t{}^2 S_t{}^2}{2}\frac{\vartheta^2 V}{\vartheta S_t{}^2} + -r\,V(S_t, t) + \frac{\vartheta V}{\vartheta S_t} rS_t = 0 \ (5.25)$$

The assumption of diversifiability of jump risk that Merton's does is not justifiable if we are pricing index options, because a jump in the index is not diversifiable.

All these, show that in models with jumps, contrarily to diffusion models, a pricing measure cannot be simply obtained by adjusting the drift coefficient μ. This choice means that we are not pricing the risk due to jumps and has implications that are not difficult to justify in terms of risk premia and hedging.

Merton's simple assumption that the absolute price jump size is log normally distributed makes it possible to solve the jump-diffusion PDE to obtain the following price function of European options as a quickly converging series of the form:

$$\sum_{n=0}^{\infty} \frac{1}{n!} e^{-\lambda'(T-t)} \left(\lambda'(T-t)\right)^n V_{BS}(S, t; \sigma_n r_n) \ (5.26)$$

where

$$\lambda' = \lambda \exp(\alpha + (1/2)\delta^2) \qquad (5.27)$$

$$\sigma_n{}^2 = \sigma^2 + \frac{n\delta^2}{T-t} \qquad (5.28)$$

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$$r_n = r - \lambda(\exp(\alpha + (1/2)\delta^2) - 1) + \frac{n(\alpha + \left(\frac{1}{2}\right)\delta^2)}{T - t} \quad (5.29)$$

and $V_{BS}$ is the Black-Scholes formula for the option value in the absence of jumps. Thus, Merton's jump-diffusion model can be interpreted as the weighted average of the Black-Scholes price conditional on that the underlying asset price jumps i times to the expiry with weights being the probability that the underlying jumps i times to the expiry. To perform Merton's jump-diffusion model we have to make a numerical issue. The infinite sum in equation is truncated to j=10 following standard practice. Ball and Torous (1985) have found that this truncation provides accurate maximum-likelihood estimates.

## 5.3 EFFECT OF PARAMETERS

As we have already mentioned jump diffusion models provide an explanation of the implied volatility smile phenomenon since in these models the implied volatility is both different from the historical volatility and changes as a function of strike and maturity. Below figures, show possible implied volatility patterns (as a function of strike) in the Merton jump diffusion model.

**Figure 1:** implied volatilities of call options on S&P 500 as a function of their strikes for different values of the mean jump size in Merton jump diffusion model. Other parameters: volatility=0,22%, jump intensity λ=0,09 jump standard deviation δ=0,7, option maturity=1 month.

- The mean jump is the expected average value of jump's size and as we can observe determines the sign of skewness. When the mean jump is negative, the skewness is negative.
- When the mean jump size is positive the pitch of the straight's line is positive.
- The mean jump size is zero, then we have a volatility smile. This happens because in that case we expect that the size of the positive jumps will be equal with the size of negative jumps.
- In the money option's price decrease as jump size increases, because in this case the volatility decreases.
- For at the money options when jump size is -0,5 or zero, the volatility does not change, so as does not affect and the call option's price. But, as the jump size increases from zero the call price increases too, for the at the money options.
- Finally, for out of the money options as the average jump size increases, out of the money options become more expensive.

The figure below show the effects of jump's intensity different prices to the options pricing with implied volatility graph.

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**Figure 2:** implied volatilities of call options on S&P 500 as a function of their strikes for different values of the jump intensity in Merton jump diffusion model. Other parameters: volatility=11%, jump size μ=-0,5, jump standard deviation δ=0,7, option maturity=1 month.

- In figure 2, because the mean jump size is negative, we have negatively skewness as we can observe. Also, the in the money call options depict that they have large sensitivity to the jump intensity. As the probability to occur a jump is bigger the volatility increases. This happens because as the probability increases, the underlying's variance gets bigger, so the volatility increases.
- The large volatility for in the money options increases their price.
- At the money options have smaller sensitivity to jump intensity and at out of the money options we expect big volatility, so the impact of the jump intensity's it is not important and does not change the effect.

The figure below show the impact of jump's size different prices to the options pricing with implied volatility graph.

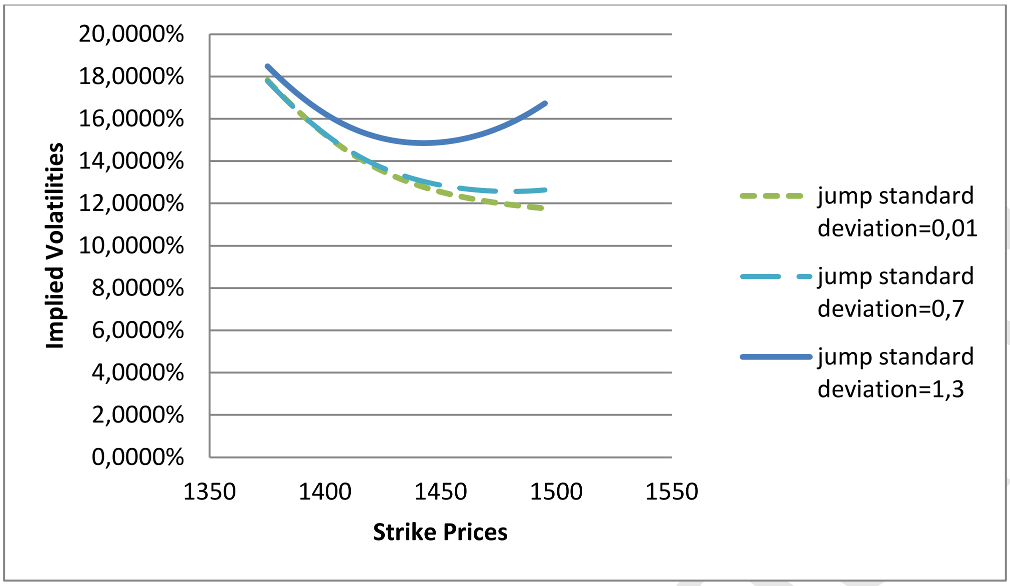


**Figure 3:** implied volatilities of call options on S&P 500 as a function of their strikes for different values of the jump standard deviation in Merton jump diffusion model. Other parameters: volatility=11%, jump intensity=0,09, jump size μ=-0,5, option maturity=1 month.

- Jump standard deviation does not affect neither of the straight line's pitch and skewness. It changes the smile's deepness.
- As jump standard deviation increases, the figure is closer to volatility smile.
- In the money and out of the money options are both more expensive when jump standard deviation increases due to the fact that implied volatility increases. The effect at the out of the money options is bigger.

# SECTION 6 KOU MODEL

## 6.1 THE FORMULA

We have already seen how are changed the prices on Merton's model with the different values of the parameters, where jumps size are normally distributed. Now we consider the case where jumps size is double exponential distributed with Steven's Kou model. All the models in finance world are rough approximations of reality. We can refer that they are all wrong. For these reasons, we will consider the cause for choosing Steven's Kou model. In general, a model must be internally self-consistent. This means that a model must be arbitrage-free and can be embedded in an equilibrium setting. Some alternative models may have arbitrage opportunities, and thus are not self-consistent. The double exponential jump-diffusion model can be embedded in a rational expectations equilibrium setting. Moreover, Steven's Kou model is able to reproduce the leptokurtic feature of the return distribution and the volatility smile observed in option prices. In addition, the empirical tests performed in Ramezani and Zeng (1999) suggest that the double exponential jump-diffusion model fits better than the normal jump-diffusion model. However, we should emphasize that empirical tests should not be used as the only criterion to judge a model good or bad. Empirical tests tend to favor models with more parameters. However, models with many parameters tend to make calibration more difficult and tend to have less tractability. This is a part of the reason why practitioners still like the simplicity of the Black-Scholes model. A model must be simple enough to be amenable to computation. Like the Black-Scholes model, the double exponential jump-diffusion model not only yields closed-form solutions for standard call and puts options, but also leads to a variety of closed-form solutions for path-dependent options, such as barrier options, lookback options, and perpetual American options. At last, a model must have some interpretation. One motivation for the double exponential jump-diffusion model comes from behavioral finance. It has been suggested from extensive empirical studies that the markets tend to have both overreaction and underreaction to various good or bad news. One may construe the jump part of the model as the market response to outside news. More precisely, in the absence of outside news the asset price simply follows a geometric Brownian motion. Good or bad news arrives according to a Poisson process, and the asset price changes in response according to

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the jump size distribution. The double exponential has a high peak and heavy tails. For that reason, it can be used to model both the overreaction (attributed to the heavy tails) and underreaction (attributed to the high peak) to outside news. We can also refer, that the model suggests that the fact of markets having both overreaction and underreaction to outside news can lead to the leptokurtic feature of asset return distribution. The model is simple. The logarithm of the asset price is assumed to follow a Brownian motion plus a compound Poisson process with jump sizes double exponentially distributed. For that reason the parameters in the model can be easily interpreted, and the analytical solutions for option pricing can be obtained. The behavior of the asset $S_t$ under the risk-neutral probability is modeled as follows:

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)}(V_i - 1)\right) \quad (6.1)$$

where W(t) is a standard Brownian motion, N(t) is a Poisson process with rate λ, and $\{V_i\}$ is a sequence of independent identically distributed (i.i.d) nonnegative random variables such that Y=log(V) has an asymmetric double exponential distribution with the density

$$f_y(y) = p * n_1 e^{-n_1 y} 1_{\{y \geq 0\}} + q * n_2 e^{n_2 y} 1_{\{y<0\}}, \quad n_1 > 1, n_2 > 0 \ (6.2)$$

where p,q$\geq 0, p + q = 1$, represent the probabilities of upward and downward jumps. In other words,

$$\log(V)=Y= \begin{Bmatrix} \xi^+, \text{with probability p} \\ -\xi^-, with\ probability\ q \end{Bmatrix}$$

where $\xi^+$ and $\xi^-$ are exponential random variables with means $1/n_1$ and $1/n_2$, respectively, and the notation = means equal in distribution. In the model, all sources of randomness, N(t), W(t), and Y are assumed to be independent. For notational simplicity and in order to get analytical solutions for various option-pricing problems, the drift μ and the volatility σ are assumed to be constants and the Brownian motion and jumps are assumed to be one dimensional. If we solve the stochastic differential equation gives the dynamics of the asset price:

$$S(t) = S(0)exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma W(t)\right\}\prod_{i=1}^{N(t)} V_i (6.3)$$

Note that E(Y)=$\frac{p}{n_1} - \frac{q}{n_2}$, Var(Y)=pq($\frac{1}{n_1} + \frac{1}{n_2})^2 + \left(\frac{p}{{n_1}^2} + \frac{q}{{n_2}^2}\right)$ (6.4) $and$

E(V)=E($e^Y$)=q$\frac{n_2}{n_2+1} + p\frac{n_1}{n_1-1}$, $n_1 > 1,\ n_2 > 0$ (6.5)

The requirement $n_1 > 1$ is needed to ensure that E(V)<∞ and E(S(t))<.∞ It essentially means that average upward jump cannot exceed 100% which is quite reasonable. There are two interesting properties of the double exponential distribution that are crucial for the model. First, it has the leptokurtic feature of the jump size distribution which is inherited by the return distribution. Also, as we have already referred before this distribution has a unique feature of the double exponential distribution. This special property explains why the closed-form solutions for various option-pricing problems, including barrier, lookback, and perpetual American options, are feasible under the double exponential jump-diffusion model while it seems impossible for many other models, including the normal jump-diffusion model (Merton's model).

The expected price of a European option, with the payment $S_T$ at the maturity T, is given by

$$S_t = e^{-r(T-t)}E(S_T/F_t), \quad for\ t\epsilon[0,T]\ (6.6)$$

To compute one has to know the sum of the double exponential random variables. Fortunately, this distribution can be obtained in closed form in terms of the Hh function, a special function of mathematical physics.

For every n≥0, the Hh function is a no increasing function defined by:

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$$Hh_n(x) = \int_x^{\infty} Hh_{n-1}(y)dy = \frac{1}{n!}\int_x^{\infty} \int (t-x)^n e^{-\frac{t^2}{2}}dt \geq 0,\ n = 0,1,2,\dots \ (6.7)$$

$$Hh_{-1}(x) = e^{-\frac{x^2}{2}} = \sqrt{2\pi}\varphi(x)\ (6.8) \qquad Hh_0(x) = \sqrt{\pi}\Phi(-x)\ (6.9)$$

The Hh function can be viewed as a generalization of the cumulative normal distribution function. The integral in the above relation can be evaluated very fast by many software packages as Matlab.

A three-term recursion is also available for the Hh function as below:

$$nHh_n(x) = Hh_{n-2}(x) - xHh_{n-1}(x), \qquad n \geq 1 \quad (6.10)$$

Therefore, one can compute all $Hh_n(x), n \geq 1$, by using the normal density function and normal distribution function. For any given probability P, define:

$$Y(\mu,\sigma,\lambda,p,n_1,n_2;a,T) \coloneqq P\{Z(T) \geq a\} \qquad (6.11)$$

where $(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i$ , Y has a double exponential distribution with density $f_Y(y) \sim p \cdot n_1 e^{-n_1 y} 1_{\{y\geq 0\}} + q \cdot n_2 e^{y n_2} 1_{\{y<0\}}$, and N(t) is a Poisson process with rate λ. The pricing formula of the call option will be expressed in terms of Y, which in turn can be derived as a sum of Hh functions. An explicit formula for Y is

$$P\{Z(T) \geq a\} = Y(\mu,\sigma,\lambda,p,n_1,n_2;a,T) = \frac{e^{(\sigma n_1)^2 \frac{T}{2}}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n \sum_{\kappa=1}^{n} p_{n,\kappa} (\sigma\sqrt{T} n_1)^k \cdot$$

$$I_{\kappa-1}\left(\alpha - \mu T; -n_1, -\frac{1}{\sigma\sqrt{T}}, -\sigma\sqrt{T}n_1\right) + \frac{e^{(\sigma n_2)^2 \frac{T}{2}}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n \sum_{\kappa=1}^{n} Q_{n,\kappa} (\sigma\sqrt{T} n_2)^k \cdot$$

$$I_{\kappa-1}\left(\alpha - \mu T; -n_2, -\frac{1}{\sigma\sqrt{T}}, -\sigma\sqrt{T}n_2\right) + \pi_0 \Phi\left(-\frac{\alpha-\mu T}{\sigma\sqrt{T}}\right) \qquad (6.12)$$

where $p_{n,\kappa} = \sum_{i=\kappa}^{n-1} \binom{n-k-1}{i-k}\binom{n}{i} \cdot \left(\frac{n_1}{n_1+n_2}\right)^{i-k} \left(\frac{n_2}{n_1+n_2}\right)^{n-i} p^i q^{n-i}$ (6.13)

$$Q_{n,\kappa} = \sum_{i=\kappa}^{n-1} \binom{n-k-1}{i-k}\binom{n}{i} \cdot \left(\frac{n_1}{n_1+n_2}\right)^{n-i} \left(\frac{n_2}{n_1+n_2}\right)^{i-k} p^{n-i} q^i \ (6.14)$$

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$$\pi_n = \frac{e^{-\lambda T}\lambda^n}{n!} \text{ (6.15)}$$

$1\leq k \leq n-1, p_{n,n} = p^n, q_{n,n} = q^n$ and $\binom{0}{0}$ is defined to be one.

Moreover, for option pricing it is important to evaluate the integral

$I_n(c;a,\beta,\delta) = \int_c^\infty e^{ax} Hh_n(\beta\chi - \delta)dx,\ n \geq 0$, for arbitrary constants a, c and β.

We have two cases. The first is if β$> 0\ \alpha\nu\ \alpha \neq 0$, then for all n$\geq -1$,

$$I_n(c;a,\beta,\delta) - \frac{e^{ac}}{a}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i} Hh_i(\beta c - \delta) + \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\Phi(-\beta c + \delta + \frac{\alpha}{\beta}) \text{ (6.16)}$$

If β$< 0\ and\ \alpha < 0$, then for all n$\geq -1$,

$$I_n(c;a,\beta,\delta) - \frac{e^{ac}}{a}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i} Hh_i(\beta c - \delta) - \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\Phi(\beta c - \delta - \frac{\alpha}{\beta}) \text{(6.17)}$$

So a European call price is given from the equation above:

$$C = S(0)Y\left(r + \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \tilde{\lambda}, \tilde{p}, \widetilde{n_1}, \widetilde{n_2}; \log\left(\frac{K}{S(0)}\right), T\right) - Ke^{-rT}\cdot Y\left(r - \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \lambda, p, n_1, n_2; \log\left(\frac{K}{S(0)}\right), T\right)$$
(6.18)

where $\tilde{p} = \frac{p}{1+\zeta}\cdot\frac{n_1}{n_1-1}$ , $\widetilde{n_1} = n_1 - 1,\quad \widetilde{n_2} = n_2 + 1$

$\tilde{\lambda} = \lambda(\zeta + 1),\ \zeta = \frac{pn_1}{n_1-1} + \frac{qn_2}{n_2+1} - 1$ (6.19)

The price of the corresponding put option can be obtained by the put-call parity:

$P - C = Ke^{-rT} - S(0)$ under risk-neutral measure.

## 6.2 CHANGE OF MEASURE

As we have already referred, Girsanov theorem helps us to transform one probability measure into another probability measure in more complicated issues. In general, a representative investor tries to solve a utility maximization problem $max_c E\left[\int_0^\infty U(c(t), t)dt\right]$, where $U(c(t), t)$ is the utility function of the consumption process c(t). The investor has the opportunity to invest in a security which pays no dividends. Moreover, there is an exogenous endowment process, denoted by δ(t), which is available to the investor. If δ(t) follows Markovian process, the rational expectation equilibrium price p(t) must satisfy the Euler equation $p(t) = \frac{E[U_c(\delta(T),T)p(T)/F_t]}{U_c(\delta(t),t)}$ (1), for any $T\epsilon[t, T_o]$ (8) where $U_c$ is the partial derivative of U with respect to c. At this price p(t), the investor will never change his/her current holdings to invest in (either long or short) the security, even though he/she has the opportunity to do so. The fact that we are interested in is when the endowment process δ (t) follows a general jump-diffusion process under the physical measure P:

$$\frac{d\delta(t)}{\delta(t-)} = \mu_1 dt + \sigma_1 dW_1(t) + d\left[\sum_{i=1}^{N(t)} (\breve{V}_i - 1)\right] \quad (6.20)$$

where the $\breve{V}_i \geq 0$ any independent identically distributed, nonnegative random variables. In addition, all three sources of randomness, the Poisson process N(t), the standard Brownian motion $W_1(t)$, and the jump sizes $\breve{V}$ are assumed to be independent. The most important part of this situation is that the asset price p(t) and the δ(t) must follow the same jump-diffusion process and that because may not have the similar jump dynamics. Steven Kou thinks that the asset pays no dividends and there is an outside endowment process contrary with Naik and Lee who require that the asset pays continuous dividends and there is not an outside endowment process. Moreover, Steven Kou for showing the rational equilibrium price of a European option and the change of probability measure used a simple utility function of the special forms $U(c, t) = e^{-\theta t}\frac{c^\alpha}{a}$ if 0<α<1, and $U(c, t) = e^{-\theta t}\log(c)$ if α=0, where θ>0. Even though Kou chose a simple utility function, most of the results below hold for more general utility functions. So, in this case the rational expectations equilibrium price of (1) becomes

$$p(t) = \frac{E(e^{-\theta t}(\delta(T))^{\alpha-1}p(T)/F_t)}{e^{-\theta t}(\delta(T))^{\alpha-1}} \quad (6.21)$$

An assumption that Kou makes is that the discount rate θ should be large enough so that

$$\theta > -(1-\alpha)\mu_1 + \frac{1}{2}{\sigma_1}^2(1-\alpha)(2-\alpha) + \lambda{\zeta_1}^{(\alpha-1)} (6.22)$$

where the notation ${\zeta_1}^{(a)}$ means ${\zeta_1}^{(a)} := E\left[(\breve{V})^{\alpha} - 1\right]$. This assumption guarantees that in equilibrium the term structure of interest rates is positive. Below we consider the reason which this happen. Suppose that ${\zeta_1}^{(\alpha-1)} < \infty$. In this economy the risk-free interest rate is constant. B(t,T) be the price of a zero coupon bond with maturity T, the yield r:=-(1/(T-t))log(B(t,T)) is a constant independent of T,

$$r = \theta + (1-\alpha)\mu_1 - \frac{1}{2}{\sigma_1}^2(1-\alpha)(2-\alpha) - \lambda{\zeta_1}^{(\alpha-1)} > 0 \ (6.23)$$

Also, let $Z(t) := e^{rt}U_c(\delta(t),t) = e^{(r-\theta)t}(\delta(t))^{a-1}$. Then $Z(t)$is a martingale under P,

$$\frac{dZ(t)}{Z(t-)} = -\lambda{\zeta_1}^{(\alpha-1)}dt + \sigma_1(\alpha-1)dW_1(t) + d\left[\sum_{i=1}^{N(t)}\left({\breve{V}_1}^{\alpha-1} - 1\right)\right] (6.24)$$

Using $Z(t)$, one can define a new probability measure $P^*$: $\frac{dP^*}{dP} := Z(t)/Z(0)$. Under $P^*$, the Euler equation holds if and only if the asset price satisfies

$S(t) = e^{-r(T-t)}E^*(S(T)/F_t)$ (6.25) for any $T\epsilon[t,T_o]$

Furthermore, the rational expectations equilibrium price of a European option, with the payoff $y_s$(T) at the maturity T, is given by

$$y_s(t) = e^{-r(T-t)}E^*(S(T)/F_t) \text{ for any } t\epsilon[0,T] \ (6.26)$$

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Now, if we have the endowment process δ(t), it must be decided what stochastic processes are suitable for the asset price S(t) to satisfy the equilibrium requirement . According to Steven Kou we take a special jump-diffusion form for S(t)

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma\left\{\rho dW_1(t) + \sqrt{1-\rho^2}\, dW_2(t)\right\} + d\left[\sum_{i=1}^{N(t)}(\breve{V}_i - 1)\right], V_i = \breve{V}^{\beta}_i \quad (6.27)$$

Where $W_2(t)$is a Brownian motion independent of $W_1(t)$. In other words, the same Poisson process affects both the endowment δ(t) and the asset price S(t), and the jump sizes are related through a power function, where the power $\beta \epsilon (-\infty, \infty)$ is an arbitrary constant. The diffusion coefficients and the Brownian motion part of δ(t) and S(t), though, are totally different. It remains to determine what constraints should be imposed on this model so that the jump-diffusion model can be embedded in the rational expectations equilibrium requirement (6.23) or (6.25).

Now, suppose that ${\zeta_1}^{(\alpha+\beta-1)} < \infty$ $and$ ${\zeta_1}^{(\alpha-1)} < \infty$. The model (6.27) satisfies the equilibrium requirement (6.26) if and only if

$$\mu = r + \sigma_1\sigma\rho(1-\alpha) - \lambda\left({\zeta_1}^{(\alpha+\beta-1)} - {\zeta_1}^{(\alpha-1)}\right) = \theta + (1-\alpha)\left\{\mu_1 - \frac{1}{2}{\sigma_1}^2(2-\alpha) + \sigma_1\sigma\rho\right\} - \lambda{\zeta_1}^{(\alpha+\beta-1)}$$
(6.28)

If (14) is satisfied, then under $P^*$

$$\frac{dS(t)}{S(t-)} = rdt - \lambda^* E^*({\breve{V}_i}^{\beta} - 1)dt + \sigma dW^*(t) + d\left[\sum_{i=1}^{N(t)}\left({\breve{V}_i}^{\beta} - 1\right)\right] \quad (6.29)$$

Here, under $P^*$, $W^*(t)$ is a new Brownian motion, N(t) is a new Poisson process with jump rate $\lambda^* = \lambda E\left({\breve{V}_1}^{\alpha-1}\right) = \lambda\left({\zeta_1}^{(\alpha-1)} + 1\right)$, and $\{\breve{V}_i\}$ are independent identically distributed random variables with a new density under $P^*$:

$f_{\breve{V}}{}^*(x) = \frac{1}{{\zeta_1}^{(\alpha-1)}+1} x^{a-1} f_{\breve{V}}(x)$. For the proof see Appendix A. □

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### 6.3 EFFECT OF PARAMETERS


**Figure 4:** implied volatilities of call options on S&P 500 as a function of their strikes for different values of the jump intensity in Kou's jump diffusion model. Other parameters: volatility=11%, p=0,4 $n_1 = 10, n_2 = 5$ and option maturity=1 month.

As we can observe the implied volatility graph is very close to the implied volatility graph of Merton's model. This is true and expected, as both models are jump diffusion models and λ is the jump intensity of Poisson's distribution.

- As λ increases, we can observe that is assigned the kurtosis. The implied volatility increases in all cases but at out of the money the effect is smaller than in the money options. However, since the implied volatility increases, the price of the options increases, too.


**Figure 5:** implied volatilities of call options on S&P 500 as a function of their strikes for different values of the jump intensity in Kou's jump diffusion model. Other parameters: volatility=11%, p=0,4 λ=1, and option maturity=1 month.

- These parameters change the skewness similarly with the average jump size which have already considered in Merton's model.
- When $n_1 > n_2$ the average upward jumps size seems to be smaller than the average downward jumps size. So, this case corresponds negative skewness like as μ<0 in Merton's model.
- When $n_1 < n_2$ the average upward jumps size seems to be bigger than the average downward jumps size. So, this case corresponds positive skewness like as μ>0 in Merton's model.
- When $n_1 = n_2$ the average upward jumps size is equal to the average downward jumps size. So, this case corresponds volatility smile like as μ=0 in Merton's model.

## SECTION 7 HESTON MODEL

### 7.1 THE FORMULA

In finance, the Heston model, named after Steven Heston is a mathematical model describing the evolution of the volatility of an underlying asset. It is a stochastic volatility model which means that the volatility of the asset is not constant, nor ever deterministic, but follows a random process. Today is one of the most widely used stochastic volatility models.

The basic Heston model assumes that $S_t$ , the price of the asset is determined by a stochastic process:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t{}^1 (7.1)$$

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t{}^2 (7.2)$$

$$dW_t{}^1 dW_t{}^2 = \rho dt (7.3)$$

where $\{S_t\}_{t\geq 0}$ and $\{V_t\}_{t\geq 0}$ are the price and volatility processes, respectively, and $\{W_t{}^1\}_{t\geq 0}$, $\{W_t{}^2\}_{t\geq 0}$ are correlated Brownian motion processes. The six parameters of the model are:

μ: the drift of the process of the stock

κ: the mean reversion speed for the variance, κ>0

θ: the long run variance, θ>0

σ: the volatility of the variance, σ>0

ρ: the correlation between volatility of the asset and log-returns

$V_0$: the initial (time zero) level of the variance (current variance)

The reasons for choosing such a form for a detailed empirical analysis are mathematical, empirical and economic.

Empirical studies have shown that an asset's log-return distribution is non-Gaussian. It is characterized by heavy tails and high peaks (leptokurtic). Moreover, there are economic arguments and empirical evidence that suggest that equity returns and implied volatility are negatively correlated. This departure plagues the Black-Scholes-Merton model with many problems.

In contrast, Heston's model can imply a number of different distributions. ρ, which can be interpreted as the correlation between the log-returns and volatility of the asset, affects the heaviness of the tails. Intuitively, if $\rho>0$, then volatility will increase as the asset price/return increases. This will spread the right tail and squeeze the left tail of the distribution creating a fat right-tailed distribution. Conversely, if $\rho<0$, then volatility will increase when the asset price/return decreases, thus spreading the left tail and squeezing the right tail of the distribution creating a fat left-tailed distribution (emphasizing the fact that equity returns and its related volatility are negatively correlated). ρ, therefore, affects the skewness of the distribution.

In the Black-Scholes-Merton model, a contingent claim is dependent on one or more tradable assets. The randomness in the option value is solely due to the randomness of these assets. Since the assets are tradable, the option can be hedged by continuously trading the underlying. This makes the market complete, every contingent claim can be replicated.

In a stochastic volatility model, a contingent claim is dependent on the randomness of the asset ($\{S_t\}_{t\geq 0}$) and the randomness associates with the volatility of the asset's return($\{u_t\}_{t\geq 0}$). Only one of these is tradable, the asset. Volatility is not a traded asset. This renders the market incomplete and has many implications to the pricing of options.

## 7.2 PDE APPROACH BY HEDGING

Now we will explain how to derive the partial differential equation from the Heston model. This derivation is a special case of a PDE for general stochastic volatility models. Form a portfolio consisting of one option U=U(S,V,t) Δ units of the stock S, and φ units of another option U=U(S,V,t) that is used to hedge the volatility. The portfolio has value

Π=U+ΔS+φu

Where Π=$\Pi_t$. Assuming the portfolio is self-financing, the change in portfolio value is

dΠ=dV+ΔdS+φdU

We use Ito's Lemma to dV. We must differentiate with respect to the variables t,S,V. Hence,

dV=$\frac{\partial V}{\partial t}dt+\frac{\partial V}{\partial S}ds+\frac{\partial V}{\partial u}du+\frac{1}{2}uS^2\frac{\partial^2 V}{\partial S^2}dt+\frac{1}{2}\sigma^2 u\frac{\partial^2 V}{\partial u^2}dt+\sigma u\rho S\frac{\partial^2 V}{\partial u\partial S}dt$ (7.4)

since $(dS)^2 uS^2(dW_t{}^1)^2=uS^2dt, (du)^2=\sigma^2 udt,$ and dSdV $=\mathrm{VS}\sigma dW_t{}^1W_t{}^2=uS\rho dt$.We have used the fact that $(dt)^2$=0 and d$W_t{}^1dt=dW_t{}^2dt=0$. Applying Ito's Lemma to dV produces the identical result, but in V

dU=$\frac{\partial U}{\partial t}dt+\frac{\partial U}{\partial S}ds+\frac{\partial U}{\partial u}du+\frac{1}{2}uS^2\frac{\partial^2 U}{\partial S^2}dt+\frac{1}{2}\sigma^2 u\frac{\partial^2 U}{\partial u^2}dt+\sigma u\rho S\frac{\partial^2 U}{\partial u\partial S}dt$(7.5)

Combining these two expressions, we can write the change in portfolio value, dΠ, as

dΠ=dV+ΔdS+φdU

$$=\left\{\frac{\partial V}{\partial t}+\frac{1}{2}uS^2\frac{\partial^2 V}{\partial S^2}+\frac{1}{2}\sigma^2 u\frac{\partial^2 V}{\partial u^2}+\sigma u\rho S\frac{\partial^2 V}{\partial u\partial S}\right\}dt+\varphi\left\{\begin{matrix}\frac{\partial U}{\partial t}+\frac{1}{2}uS^2\frac{\partial^2 U}{\partial S^2}+\frac{1}{2}\sigma^2 u\frac{\partial^2 U}{\partial u^2}+\\ \sigma u\rho S\frac{\partial^2 U}{\partial u\partial S}\end{matrix}\right\}dt+$$

$$\left\{\frac{\partial V}{\partial S}+\varphi\frac{\partial U}{\partial S}+\Delta\right\}dS+\left\{\frac{\partial V}{\partial u}+\varphi\frac{\partial U}{\partial u}\right\}du \quad (7.6)$$

In order for the portfolio to be hedged against movements in the stock and against volatility, the last two terms in equation (7.6) involving dS and dV must be zero. This implies that the hedge parameters must be

$$\varphi=-\frac{\frac{\partial V}{\partial u}}{\frac{\partial U}{\partial u}} \quad (7.7)$$

$$\Delta=-\varphi\frac{\partial U}{\partial S}-\frac{\partial V}{\partial S} \quad (7.8)$$

Moreover, the portfolio must earn the risk free rate r. Hence

dΠ=rΠdt=r(V+ΔS+φV)dt. (7.9)

Now with the values of φ and Δ from equation (7.6) the change in value of the riskless portfolio is dΠ $=\left\{\frac{\partial V}{\partial t}+\frac{1}{2}uS^2\frac{\partial^2 V}{\partial S^2}+\frac{1}{2}\sigma^2 u\frac{\partial^2 V}{\partial u^2}+\sigma u\rho S\frac{\partial^2 V}{\partial u\partial S}\right\}dt$

$$+\varphi\left\{\frac{\partial U}{\partial t}+\frac{1}{2}uS^2\frac{\partial^2 U}{\partial S^2}+\frac{1}{2}\sigma^2 u\frac{\partial^2 U}{\partial u^2}+\sigma u\rho S\frac{\partial^2 U}{\partial u\partial S}\right\}dt \quad (7.10)$$

Which we write as dΠ=(A+φB)dt (7.11)

Hence we have A+φB=r(V+ΔS+φV)

Substituting for φ and re-arranging, produces the equality

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$$\frac{A - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial u}} = \frac{B - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial u}} \quad (7.12)$$

The left-hand of equation (7.12) is a function of V only, and the right-hand side is a function of U only. This implies that both sides can be written as a function f(S,V,t) of S,V and t. According to Heston's model, specify this function as f(S,V,t)=$-\kappa(\theta - u) + \lambda(S,u,t)$, where $\lambda(S,u,t)$ is the market price of volatility risk. Heston assumes that the market price of volatility risk is proportional to volatility,

$$\lambda(S,u,t) = k\sqrt{u} \text{ for some constant k}$$

$$\Lambda(S,u,t) = k\sigma u = \lambda(S,V,t) \quad (7.13)$$

say, λ(S,V,t) therefore represents the market price of volatility risk and must be independent from the particular asset. This parameter appears in the pricing formula and hence needs to be approximated. This is no easy task as it well known that the market price of volatility risk is nearly impossible to estimate. This problem is overcome due to the parametric nature of the model and the existence of a closed-form solution.

Write the left-hand side of equation (7.12) as

$$\frac{A - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial u}} = -\kappa(\theta - u) + \lambda(S,u,t) \quad (7.14)$$

Substitute for A and rearrange to produce the Heston PDE expressed in terms of the price S

$$\frac{\partial V}{\partial t} + \frac{1}{2}uS^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma^2 u\frac{\partial^2 V}{\partial u^2} + \sigma u\rho S\frac{\partial^2 V}{\partial u \partial S} - rV + rS\frac{\partial V}{\partial S} + [\kappa(\theta - u) - \lambda(S,u,t)]\frac{\partial V}{\partial u} = 0. \quad (7.15)$$

The partial differential equation in (7.15) can be written

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$\frac{\partial V}{\partial t} + AV - rV = 0$

Where

$$A = rS\frac{\partial}{\partial S} + \frac{1}{2}uS^2\frac{\partial}{\partial S^2}$$

$$+[\kappa(\theta - u) - \lambda(S, u, t)]\frac{\partial}{\partial u} + \frac{1}{2}\sigma^2 u\frac{\partial^2}{\partial u^2} + \sigma u\rho S\frac{\partial}{\partial u\partial S} \text{ (7.16)}$$

The first line in equation (7.15) is the generator of the Black-Scholes model, while the second line adds the corrections for stochastic volatility.

Before we consider the closed-form solution of Heston's model, we will present the risk-neutral approach. Risk-neutral valuation is the pricing of a contingent claim in an equivalent martingale measure (EMM). The price is evaluated as the expected discounted payoff of the contingent claim, under the EMM $\mathbb{Q}$, say. So,

Option Value=$\mathbb{E}_t{}^{\mathbb{Q}}\left[e^{r(T-t)}H(T)\right]$ (7.17)

Where $H(T)$ is the payoff of the option at time T and r is the risk-free rate of interest over $[t, T]$ (we are assuming, of course, that interest rates are deterministic and pre-visible, and that the numeraire is the money market instrument). Moving from a real world measure to an EMM is achieved by Girsanov's Theorem. In particular, we have

$$d\widetilde{W_t}^1 = dW_t{}^1 + \partial_t dt \text{ (7.18)}$$

$$d\widetilde{W_t}^2 = dW_t{}^2 + \Lambda(S, V, t)dt \text{ (7.19)}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = exp\left\{-\frac{1}{2}\int_0^t\left(\partial_s{}^2 + \Lambda(S, V, s)^2\right)ds - \int_0^t \partial_s dW_s{}^1 - \int_0^t \Lambda(S, V, t)dW_s{}^2\right\} \text{ (7.20)}$$

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$$\partial_t = \frac{\mu - r}{\sqrt{V_t}} \ (7.21)$$

Where $\mathbb{P}$ is the real world measure and $\left\{\widetilde{W_t}^1\right\}_{t\geq 0}$ and $\left\{\widetilde{W_t}^2\right\}_{t\geq 0}$ are $\mathbb{Q}$-Brownian Motions. Under measure $\mathbb{Q}$, (7.1), (7.2), (7.3), become,

$$dS_t = rS_t dt + \sqrt{V_t} S_t d\widetilde{W}_t^{\,1} (7.22)$$

$$dV_t = k^*(\theta^* - V_t)dt + \sigma\sqrt{V_t} d\widetilde{W}_t^{\,2} (7.23)$$

$$d\widetilde{W}_t^{\,1} d\widetilde{W}_t^{\,2} = \rho dt (7.24)$$

Where $k^* = k + \lambda$

$$\theta^* = \frac{\kappa\theta}{\kappa + \lambda} \ (7.25)$$

This is an important result. Under the risk-neutral measure, λ has effectively been eliminated.

So, the closed-form solution of a European call option on a non-dividend paying asset for the Heston model is:

$$C(S_t, V_t, t, T) = S_t P_1 - K e^{-r(T-t)} P_2 \ \ (7.26)$$

where,

$$P_j(x, V_t, T, K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\varphi \ln(K)} f_j(x, V_t, T, \varphi)}{i\varphi}\right) d\varphi \ \ (7.27)$$

$$\chi = \ln(S_t)$$

$$f_j(x, V_t, T, \varphi) = \exp\{C(T-t,\varphi) + D(T-t,\varphi)V_t + i\varphi x\} \quad (7.28)$$

$$C(T-t,\varphi) = r\varphi i\rho + \frac{\alpha}{\sigma^2}\left[\left(b_j - \rho\sigma\varphi i + d\right)r - 2\ln\left(\frac{1-ge^{dT}}{1-g}\right)\right] \quad (7.29)$$

$$D(T-t,\varphi)V_t = \frac{b_j-\rho\sigma\varphi i+d}{\sigma^2}\left(\frac{1-e^{dT}}{1-ge^{dT}}\right) \quad (7.30)$$

$$g = \frac{b_j-\rho\sigma\varphi i+d}{b_j-\rho\sigma\varphi i-d} \quad (7.31)$$

$$d = \sqrt{\left(\rho\sigma\varphi i - b_j\right)^2 - \sigma^2\left(2u_j\varphi i - \varphi^2\right)} \quad (7.32)$$

For j=1,2 where,

$u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, α=kθ, $b_1 = \kappa + \lambda - \rho\sigma$, $b_2 = \kappa + \lambda$

The only part that poses a slight problem is the limits of the integral in (7.27). This integral cannot be evaluated exactly, but can be approximated with reasonable accuracy by using some numerical integration, but can be approximated with reasonable accuracy by using some numerical technique. Under equivalent martingale measure $\mathbb{Q}$ some parameter simplification takes place viz,

α=$\kappa^*\theta^*$, $b_1 = \kappa^* - \rho\sigma$, $b_2 = \kappa^*$

The parameter λ has been eliminated. A method to evaluate formulas in the form of (7.26) is using a numerical method for the said integrals.

The basic Simpson's Rule for numerical integration of a function, f(x), over two equal subintervals, with partition points α, α+h, and α+2h (we choose this form of partition for ease of calculation and notation. A general formula will be given later), is

$$\int_{a}^{a+2h} f(x)dx \approx \frac{h}{3}[f(\alpha) + 4f(a+h) + f(\alpha+2h)] \ (7.33)$$

The error of the above approximate can be established with a basic Taylor series. Consider,

$$f(\alpha+h) = f(a) + hf'(a) + \frac{1}{2!}h^2 f''(a) + \frac{1}{3!}h^3 f'''(a) + \cdots$$

$$f(a+2h) = f(a) + 2hf'(a) + 2h^2 f''(a) + \frac{4}{3}h^3 f'''(a) + \cdots$$

So,

(3.13) will be equal with:2hf+$2h^2 f' + \frac{4}{3}h^3 f'' + \frac{2}{3}h^4 f''' + \frac{20}{3\cdot 4!}h^5 f^{(4)}(a) + \cdots$ (7.34)

Now, define

$$F(x) := \int_{a}^{x} f(t)dt$$

Again, using a Taylor series,

$$F(a+2h) = F(a) + 2hF'(a) + 2h^2 F''(a) + \frac{4}{3}h^3 F'''(a) + \cdots$$

Noting, by the Fundamental Theorem of Calculus, that $F^{(n+1)}(a) = f^{n}(a)$and F(α) $= 0$ we have

$$\int_{a}^{a+2h} f(x)dx = F(a+2h)$$

$=2hf(\alpha)+2h^2f'(\alpha)+\frac{4}{3}h^3f''(\alpha)+\frac{2}{3}h^4f'''(\alpha)+\cdots$ (7.35)

Subtracting (7.34) from (7.35), we get

Error in approximation=$-\frac{h^5}{90}f^{(4)}(\xi)=O(h^5),\quad \xi\epsilon(\alpha,\alpha+h)$ We know state a general result for the basic Simpson's Rule,

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$Error=-\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi),\qquad \xi\epsilon(a,b)$$

An Adaptive Simpson's Rule divides the interval of integration into more than 2 subintervals. The number of subintervals is dependent on the required accuracy of the estimate. This is quantified by the error term given above. The function will, in general, behave differently over different parts of the interval and hence the intervals chosen will not be of uniform length. Adaptive Simpson's Rule produces a result that has an error less than $10^{-6}$.

**7.3 HESTON'S INTEGRAL**

In order to evaluate (7.27) we need to compute the integral in the below equation

$$P_j(x,V_t,T,K)=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\varphi\ln(K)f_j(x,V_t,T,\varphi)}}{i\varphi}\right)d\varphi \quad (7.36)$$

We know that the integrand converges very quickly to zero. So, sufficiently large b, the integral can be evaluated with the required accuracy. I have put b=100 according to bibliography.

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### 7.4 EFFECT OF PARAMETERS

Heston's model has five parameters which affect differently on option prices. Below we consider the effects of each parameter through implied volatility graphs.


**Figure 6:** implied volatilities of call options on S&P 500 as a function of their strikes for different values of the correlation in Heston's model. Other parameters: κ=4,9, θ=0,4, σ=4,9, $V_0$=0,11, the underlying price 1460,26 and option maturity=1 month

- The correlation determines the direction of skew. This feature is extremely interesting as it is the only parameter to play such a role. The negative correlation corresponding negative skewness, in contrast with positive correlation which provides positive skewness. In special case, where correlation is zero, the implied volatility figure is close to volatility smile.
- In the money options, are more expensive when the correlation is negative while out of the money options are cheaper.
- When ρ=0, the smile is symmetric about the at the money volatility.
- Non-zero correlation control's the smile's asymmetry which is important in equity markets.

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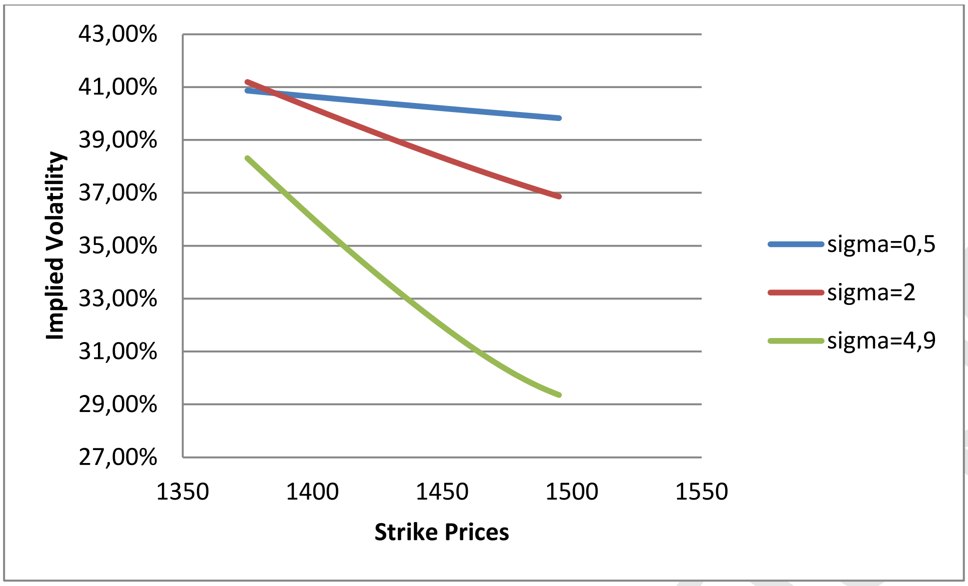


**Figure 7:** implied volatilities of call options on S&P 500 as a function of their strikes for different values of the volatility of the volatility in Heston's model. Other parameters: κ=4,9, θ=0,4, ρ=-0,5, $V_o$=0,11, the underlying price 1460,26 and option maturity=1 month

- The main effect of the volatility of the volatility is that creates imbalance on the in the money options relatively with the out of the money options. This is considered especially from the first two graphs in figure 7. The third graph, is for a more extreme value of sigma, so affects all the type of options.

**Figure 8:** implied volatilities of call options on S&P 500 as a function of their strikes for different values of the long run variance in Heston's model. Other parameters: volatility=11%, κ=2, ρ=-0,5, σ=0,1, $V_o$=0,11, the underlying price 1460,26 and option maturity=1 month

- The long-run variance does not change the skewness or the straight pitch's line. As it is constant for a long time horizon, the effect which we can observe is that as it increases, implied volatility gets bigger for all the types of options.

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## SECTION 8 COMPARISON BETWEEN MODELS

### 8.1 MERTON AND KOU MODEL

Let’s consider the relation between double exponential jump-diffusion model and Normal jump-diffusion model. In Merton’s paper (1976) Y are normally distributed. Both the double exponential and normal jump-diffusion models can lead to the leptokurtic feature (although the kurtosis from the double exponential jump-diffusion model is significantly more pronounced), implied volatility smile, and analytical solutions for call and put options, and interest rate derivatives. The main difference between the double exponential jump-diffusion model and the normal jump-diffusion model is the analytical tractability for the path-dependent options. Path-dependent options give the right, but not the obligation, to buy or sell an underlying asset at a predetermined price during a specified time period, where the price is based on the fluctuations in the underlying's value during all or part of the contract term. A path dependent option's payoff is determined by the path of the underlying asset's price.

Now, we are going to explain why the double-exponential jump-diffusion model can lead to closed-form solutions for path-dependent options, while the normal jump-diffusion model cannot. To price for example American options for general jump-diffusion processes, it is crucial to study the first passage time of a jump-diffusion process to a flat boundary. When a jump-diffusion process crosses a boundary sometimes it hits the boundary and sometimes it incurs an ‘’overshoot’’ over the boundary.

The overshoot presents several problems for option pricing. First, one needs to get the exact distribution of the overshoot. It is well known from stochastic renewal theory that this is only possible if the jump size Y has an exponential-type distribution, thanks to the special memoryless property of the exponential distribution. Moreover, we have to know the dependent structure between the overshoot and the first passage time. The two random variables are conditionally independent, given that the overshoot is bigger than 0, if the jump size Y has an exponential-type distribution, thanks to the memoryless property. This conditionally independent structure seems to be very special to the exponential-type distribution and does not hold for other distributions, such as the normal distribution.

## 8.2 JUMP DIFFUSION MODELS vs STOCHASTIC VOLATILITY MODELS

The main problem with jump diffusion models is that they cannot capture the volatility clustering effects, which can be captured by other models such as stochastic volatility models. In finance, volatility clustering refers to the observation, that "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes."Jump diffusion models and stochastic volatility models implement each other: the stochastic volatility model can incorporate dependent structures better, while the double exponential jump-diffusion model has better analytical tractability, especially for path-dependent options and complex interest rate derivatives. Some alternative models can compute only prices for standard call and put options, and analytical solutions for other equity derivatives and some most liquid interest rate derivatives are unlikely. On the other hand, the Black-Scholes model can compute prices for all the derivatives. This makes it more difficult to persuade practitioners to switch from the Black-Scholes to more realistic alternative models. The double exponential jump-diffusion model attempts to improve the empirical implications of the Black-Scholes model while still retaining its analytical tractability. For example, one empirical phenomenon worth mentioning is that the daily return distribution tends to have more kurtosis than the distribution of monthly returns. As Das and Foreci (1996) point out, this is consistent with models with jumps, but inconsistent with stochastic volatility models. More precisely, in stochastic volatility models the kurtosis decreases as the sampling frequency increases while in jump models the instantaneous jumps are independent of the sampling frequency. This, in particular suggests that jump-diffusion models may capture short-term behavior better, while stochastic volatility may be more useful to model long term behavior.

In summary, there are many alternative models may give some analytical formula for standard European call and put options, but analytical solutions for interest rate derivatives and path-dependent options, such as perpetual American options, barrier and lookback options are difficult, if not impossible. In the double exponential jump diffusion model analytical solution for path-dependent options are possible. However, the jump-diffusion models cannot capture the volatility clustering effect, Therefore, jump-diffusion models are more suitable for pricing short maturity options in which the impact of the volatility clustering effect is less pronounced. In addition, jump-diffusion models can provide a useful benchmark for more complicated models.

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## SECTION 9 RESULTS AND DISCUSSION

This dissertation has reviewed two jump diffusion models and a stochastic volatility model. The first jump-diffusion model, Merton's model includes jumps which follow a log-normal distribution and a Brownian motion such as in the Black and Scholes model. However, due to the fact that mentions parameters in the model make it more difficult in pricing options and tend to have less tractability. This is a part of the reason why practitioner's still like the simplicity of Black and Scholes model. Although reproduces volatility smile, such as Kou's model, this is the second one. This model contains jumps which follow double exponential distribution. The empirical part was the option pricing with both models which have closed-form solutions. Upon implementing the two models we found that some of the parameters which describe the same part of each model have similar effects on options. Moreover, the double exponential jump-diffusion model fits stock data better than the normal jump-diffusion model, and both of them fit the data better than the classical geometric Brownian motion model.

Furthermore, we have presented a stochastic volatility model without jumps. Due to the fact that it has much more parameters than Merton's and Kou's model, it is difficult on the option pricing, too. As we have no jumps, the parameters are not the same with these of jump-diffusion models. The most important is the correlation between volatility and the spot price is necessary to generate skewness. Without this correlation, stochastic volatility only changes the kurtosis. With proper choice of parameters, the stochastic volatility model appears to be a very flexible and promising description of option prices.

Each model has its advantages and limitations. However, one of the most important facts which jump diffusion models and stochastic volatility models differ is the volatility clustering effect. Due to this fact, jump diffusion models are more suitable for short maturity options in which the volatility clustering effect is less pronounced. In contrast with stochastic volatility models which are more suitable for long maturity options.

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**APPENDIX A**

**A$_1$.** From Girsanov's theorem about jump diffusion processes, we know that under$P^*$, $W'_1(t) \coloneqq W_1(t) - \sigma_1(\alpha-1)t$ is a new Brownian motion and under $P^*$ the jump rate of N(t) is $\lambda^* = \lambda E(\breve{V}^{a-1}{}_i)$=λ($\zeta_1{}^{(\alpha-1)}+1$, and $\breve{V}_\iota$ has a new density

$$f_{\breve{V}}{}^*(x) = \frac{1}{\zeta_1{}^{(\alpha-1)}+1} x^{a-1} f_{\breve{V}}(x).$$

Therefore, the dynamics S(t) is given by

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma\left\{\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right\} + \varDelta\left[\sum_{\iota=1}^{N(t)}\left(\breve{V}_i{}^\beta - 1\right)\right]$$

$$= \{\mu + \sigma_1\sigma\rho(\alpha-1)\}dt + \sigma\left\{\rho dW'_1(t) + \sqrt{1-\rho^2}dW_2(t)\right\} + \varDelta\left[\sum_{\iota=1}^{N(t)}\left(\breve{V}_i{}^\beta - 1\right)\right]$$

Because

$E^*\left(\breve{V}_i{}^\beta\right) = \int_0^\infty x^\beta \frac{1}{\zeta_1{}^{(\alpha-1)}+1}\chi^{\alpha-1} f_{\breve{V}}(x)dx = \frac{1}{\zeta_1{}^{(\alpha-1)}+1} E\left(\breve{V}^{a+\beta-1}\right) = \frac{\zeta_1{}^{(\alpha+\beta-1)}+1}{\zeta_1{}^{(\alpha-1)}+1}$, we have $\lambda^*\left\{E^*\left(\breve{V}_i{}^\beta\right) - 1\right\} = \lambda\left(\zeta_1{}^{(\alpha+\beta-1)} - \zeta_1{}^{(\alpha-1)} + 1\right)$. Therefore,

$$\frac{dS(t)}{S(t-)} = \left\{\mu + \sigma_1\sigma\rho(\alpha-1) + \lambda\left(\zeta_1{}^{(\alpha+\beta-1)} - \zeta_1{}^{(\alpha-1)} + 1\right)\right\}dt$$

$$- \lambda^*\left\{E^*\left(\breve{V}_i{}^\beta\right) - 1\right\}dt + \sigma\left\{\rho dW'_1(t) + \sqrt{1-\rho^2}dW_2(t)\right\} + \varDelta\left[\sum_{\iota=1}^{N(t)}\left(\breve{V}_i{}^\beta - 1\right)\right]$$

Hence, to satisfy the rational equilibrium requirement S(t)= $e^{-r(T-t)}E^*(S(T)/F_t)$, we must have $\mu+\sigma_1\sigma\rho(\alpha-1)+\lambda\left(\zeta_1{}^{(\alpha+\beta-1)}-\zeta_1{}^{(\alpha-1)}+1\right)=r$, from which (14) follows. If (14) is satisfied, under the measure $P^*$, the dynamics of S(t) is given by

$$\frac{dS(t)}{S(t-)}=rdt-\lambda^*E^*(\check{V}_i{}^{\beta}-1)dt+\sigma\left\{\rho dW'_1(t)+\sqrt{1-\rho^2}dW_2(t)\right\}$$
$$+\varDelta\left[\sum_{\iota=1}^{N(t)}\left(\check{V}_i{}^{\beta}-1\right)\right]$$

For which (15) follows.

**A$_2$.** Let the price $S_t$ of a risky asset jump at the random times $T_1$, $T_2$,…,$T_n$,… and suppose that the relative/proportional change in its value at a jump time is given by $Y_1$, $Y_2$,…,$Y_n$,… respectively. We may then assume that, between two jump times, the price $S_t$ follows a Black and Scholes model for a Wiener process $w_t$ that $T_n$ are the jump times of a Poisson process $N_t$ with intensity $\lambda_t$ and that $Y_n$ is a sequence of random variables with values in (-1,∞). This description can be formalized by letting, on the intervals $[T_n, T_{n+1})$ ,

$dS_t=S_t(\mu_t dt+\sigma_t dw_t)$ (1)

While, at t= $T_n$, the jump is given by $\Delta S_n=S_{T_n}-S_{T_{n-}}=S_{T_n}-\mathrm{Y_n}$so that

$S_{T_n}=S_{T_{n-}}(1+\mathrm{Y_n})$ (2)

Which, y the assumption that $\mathrm{Y_n}>-1$, leads, always to positive values of the prices. Using the standard Ito formula to obtain the solution to (1) as well as a recursive argument based on (2) , it is easily seen that, at the generic time t, $S_t$ can be given the following equivalent representations.

$$S_t = S_o exp\left[\int_0^t (\mu_s - \frac{{\sigma_s}^2}{2})ds + \int_0^t \sigma_s dw_s\right]\left[\prod_{n=1}^{N_t}(1+Y_n)\right]$$

$$= S_o exp\left[\int_0^t \left(\mu_s - \frac{{\sigma_s}^2}{2}\right)ds + \int_0^t \sigma_s dw_s + \sum_{n=1}^{N_t}\log(1+Y_n)\right]$$

$$= S_o exp\left[\int_0^t \left(\mu_s - \frac{{\sigma_s}^2}{2}\right)ds + \int_0^t \sigma_s dw_s + \int_0^t \log(1+Y_s)\, dN_s\right] (3)$$

Where, as before, $Y_t$ is obtained from $Y_n$ by a piecewise constant and left continuous time interpolation. By the generalized Ito formula

$$dF(t,X_t) = F_t(\cdot)dt + F_x(\cdot)X_t\alpha_t dt + \frac{1}{2}F_{XX}(\cdot){X_t}^2{\beta_t}^2 dt + F_X(\cdot)X_t\beta_t dw_t + \left[F\left(t, X_{t-}\left(1+\gamma(t,Y_t)\right)\right) - F(t,X_{t-})\right]dN_t \ (4)$$

The process $S_t$ in (3) is easily seen to be a solution of

$$dS_t = S_{t-}[\mu_t dt + \sigma_t dw_t + Y_t dN_t] \ (5)$$

This equation corresponds to (1) with the addition of a jump term and is a particular case of the general jump-diffusion model (6) and (7) when $\gamma(t,y) = y$

$$dX_t = X_{t-}(\alpha_t dt + \beta_t dw_t + \int \gamma(t,y)p(dt,dy)) \ (6)$$

Where we write $X_{t-}$ with t- because of the predictability requirement in the last coefficient and where $\gamma(t,y)$>-1. Notice that the last term in (6) can also be written as

$$\int \gamma(t,y)p(dt,dy)) = \gamma(t,Y_t)dN_t \ (7)$$

In what follows we shall consider the more general version of (5) given by

$dS_t = S_{t-}[\mu_t dt + \sigma_t dw_t + \gamma(t, Y_t) dN_t]$ (8)

That corresponds to (6) in the version of (7)and can thus equivalently be represented as $dS_t = S_{t-}[\mu_t dt + \sigma_t dw_t + \int \gamma(t, y) p(dt, dy)]$ (9)

If the marked point process is in particular a multivariate (or univariate) point process $(N_t(1), \dots\ N_t(K))$, then (8) and (9) takes the form

$dS_t = S_{t-}[\mu_t dt + \sigma_t dw_t + \sum_{k=1}^{K} \gamma_t(k) dN_t(k)]$ (10)

Occasionally, in the financial literature one finds model (8) and (9) written in the form

$$dS_t = S_{t-}[\mu_t dt + \sigma_t dw_t + dJ_t]$$

Where, in the specific case when (8) reduces to (5) $J_t = \sum_{n=1}^{N_t} Y_n$, while in the general case $J_t = \sum_{n=1}^{N_t} \gamma(T_n, Y_n)$. Furthermore, in models of the form (5) one may find the last term $Y_t dN_t$ written as $(Y_t - 1) dN_t$ .

**REFERENCES**

Andersen L. & Andreasen J. (2000), *Jump diffusion processes: Volatility smile fitting and numerical methods for pricing*, Review of Derivative Research 4, 231-26

Archambeau Cedric, (2007) A *short introduction to diffusion processes and Ito Calculus, Center* for computational statistics and machine learning

Bates D. (1996), *Testing Option Pricing Models*, In G.S Maddala and C.R. Rao (eds): handbook of Statistics, vol 15: Statistical Methods in Finance, North Holland Amsterdam, 567-611.

Black F. and M. Scholes (1973), *The pricing of options and Corporate Liabilities*, in Journal of Political Economy 81, 637-654

Black F. Jensen M.and M. Scholes (1972), *The valuation of Option Contracts and a Test of Market Efficiency*, Journal of Finance, 27, 399-417

Black, F,& Scholes, M. (1973), *Valuation of options and corporate liabilities,* Journal of Political Economy 81, 637-654

Campbell R.Harvey and Akhtar Siddique, (2000), *Conditional Skewness in Asset Pricing Tests,* The journal of finance

Carr Peter, Helyette Geman, Dilip D. Madan, (2000), *Pricing and hedging in incomplete markets*, Journal of Financial of Economics 62, 131-167

Clifford Ball and Walter Torous, (1985*),On jumps in common stock prices and their impact on call option pricing,* , Journal of Finance,vol.40, 155-173

Cont, R. and Tankov, P. (2004), *Financial modeling with jump processes*, Chapman & Hall/CRC Financial Mathematics Series

S. Das R. and Sundaram, (1999), *Of Smiles and Smirks :A Term-Structure Perspective*, Journal of finance and Quantitive Analysis, 34, 211-239

D. Bates, (1996) *Jumps and Stochastic volatility: Exchange rate processes implicit in Deutche Mark options*, Rev. Financial Studies 9, 69-107

D. Bates , (1988*) Pricing Options under Jump-Diffusion Processes*, Rodney L. White Center

Gerardi Anna and Tardelli Paula, (2010), *Risk-neutral measures and pricing for a pure jump price process,* pp. 47-76

Girsanov I.V. (1960), *On transforming a certain class of stochastic processes by absolutely continuous substitution of measures*, Theory of probability and its application

Henderson V. (2002), *Analytical Comparisons of Option Prices in Stochastic Volatility Models*, Oxford Financial Research Centre Preprint

Heston, S.( 1993) *A closed-form solution of options with stochastic volatility with applications to bond and currency options.* Review of Financial Studies 6, 327-343

Hull C. John, (2000), *Options, futures and other derivatives*, Prentice Hall Finance Series, Fifth edition.

Hull J.C and A. White, (1987), *The Pricing of Options on Assets with Stochastic Volatilities*, Journal of Finance, 42, 281-300

Joost Driessen and Pascal Maenhout, (2006), *The world Price of Jump and Volatility Risk*,

Πανεπιστήμιο Πειραιώς

Kou S.G., *Jump diffusion models for Asset Pricing in Financial Engineering*, Chapter 2, (2008) J.R Birge and V.Linetsky, Handbooks in Operations Research and management Science, vol.15

Kou S. G. ,(2002), *A jump-diffusion model for option pricing,* Management Science 48,1086-1101

Kou, S. & Wang H. (2004), *Option pricing under a double exponential jump diffusion model,* Management Science 50(9),1178-1192

Merton R.C., (1976) ,*Option pricing when underlying stock returns are discontinuous,* Journal Financial Economics 125-144

Merton R.C. (1976b), *The impact on option pricing of specification error in the underlying stock price returns*, The journal of finance 31(2), 333-350

Naik V., (1993) *Option Valuation and hedging strategies with Jumps in the Volatility of Asset Returns*, The journal of Finance, 48, 1969-1984

Nandi S. (1988), *How important is the Correlation between Returns and Volatility in a Stochastic Volatility Model? Empirical Evidence from Pricing and Hedging in the S&P 500 Index Options Market,* Journal of Banking and Finance,22, 589-610

Neftci Salih N., *An introduction to the Mathematics of Financial Derivatives*

Nonthiya Makate and Pairote Sattayatham, (2011), *Stochastic volatility jump-diffusion model for option pricing,* Journal of Mathematical Finance

Pierre Gauthier and Pierre-Yves H. Rivaille, (2009), *Fitting the smile, Smart Parameters for SABR and Heston,*

Πανεπιστήμιο Πειραιώς

Skiadopoulos George, Psychoyios Dimitris, Dotsis George, 2007, *An empirical comparison of continuous-time models of implied volatility indices,* Journal of Financial Banking 31, 3584-3603

Skiadopoulos George, (2000), *Volatility Smile Consistent Option Models: Survey,* International Journal of Theoretical and Applied Finance, vol.4, No.3, 403-437

Spanos Aris, (1999), *Probability Theory and Statistical Inference*, Cambridge University Press

Stan Beckers, (1981) *A Note on Estimating the Parameters of the Diffusion-Jump Model of Stock Returns,* Journal of Financial and Quantitive Analysis 16, 127-40

Tauchen George and Todorov Victor (2008), *Volatility jumps*, ERID Working Paper Number 3.

Vicky Henderson, David Hobson, Sam Howison, Tino Kluge , (2003) *A comparison of option prices under different measures in a stochastic volatility model with correlation,* Nomura Centre for Quantitive Finance, Mathematical Institute

Wilmott, P. (1998), *Derivatives*, John Wiley & Sons

Πανεπιστήμιο Πειραιώς