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Time value of absolute ruin with debit interest rate

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ΤΜΗΜΑ ΣΤΑΤΙΣΤΙΚΗΣ ΚΑΙ ΑΣΦΑΛΙΣΤΙΚΗΣ ΕΠΙΣΤΗΜΗΣ

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To Christos, Aristea(s) & Nantia

Dedication

I would like to dedicate this thesis to my parents and my partner and express my profound gratitude for their unfailing support and encouragement throughout my studies and the process of researching and writing this thesis. This accomplishment would not have been possible without them.

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Abstract

In this Project, the surplus process is going to be studied under the scope of the classical continuous time risk model. Over the time, the surplus is continuously changing, receiving different values. In our study, two possible situations for the surplus are of great importance and significance. The first one is called "classical ruin", or just "ruin", and it happens when the surplus drops below zero for the first time (ruin time). The second one is called "absolute ruin" and it holds when the surplus falls below a negative critical value for the first time (absolute ruin time). In the last case, during the period in which the surplus is negative, insurance institutions can borrow money, with a debit interest rate, in order to compensate their obligations and the claims happening. On the other hand, their debt is paid back by the premiums earned. If the debt remains at a reasonable level, it is possible for the surplus to become positive again, resulting in the absolute ruin to be avoided.

The objective of this thesis is to study and analyze risk measures related to both, classical ruin (chapter 1) and absolute ruin (chapter 2). Such measures will be the ruin probability and the Laplace transform of the ruin time. For this purpose, the expected discounted penalty function, or just Gerber-Shiu function, is defined and examined thoroughly, because, under specific circumstances, it is reduced to the aforementioned measures. Furthermore, at the end of each chapter, there are examples with explicit results for exponential claims.

Finally, having assumed that there are dividend payments to shareholders according to a barrier strategy, we provide expressions for the moments of the present value of all dividends paid until the absolute ruin time, when the claims are exponentially distributed.

Keywords: *Compound Poisson process, Surplus process, Gerber-Shiu function, time of ruin, ruin probability, Laplace transform for ruin time, absolute ruin, debit interest, absolute ruin time, absolute ruin probability, Laplace transform for absolute ruin time, deficit at ruin/absolute ruin, surplus just before ruin/absolute ruin, Dickson-Hipp operator, renewal argument, defective renewal equation, compound geometric, equilibrium function, dividend barrier, moment-generating function*

Contents

Chapter 1

The Classical Model of Ruin Theory

The objective of this chapter is to present the solutions of the defective renewal equation satisfied by Gerber-Shiu function using a compound geometric distribution. This concept has been introduced and analysed thorougly by Lin and Willmot (1999). To reach at this point, firstly, they are mentioned all the definitions/notions deemed essential to our study of the classical continuous time risk model, which have been analysed extensively by Gerber and Shiu (1997,1998). Moreover, regarding the research of Gerber and Shiu, we define the Gerber-Shiu function and provide some of its properties. Then, there are two sub-sections, the Dickson-Hipp operator and the solutions of a defective renewal equation, which facilitate the gradual construction of our objective presented in section 1.5. Finally, analytical expressions for the ruin probability and the Laplace transform of the ruin time are given when the claims are distributed exponentially.

1.1 Introduction

Depended on the definitions and assumptions of Gerber and Shiu (1997,1998) and on the digital-online academic notes of professor K. Politis (2013-2014, see reference [28]), we set up in section 1.1 all the necessary definitions and properties on which we will base our study.

Definition 1.1.1. *Stochastic process is a collection $\{X(t) : t \in T\}$ of random variables representing numerical values changing over the time. The index set T usually denotes the time. Regarding the values received by the random variables $X(t)$, stochastic processes can be divided into discrete-valued or continuous-valued stochastic processes. A further segmentation is based on the nature of set T. If T consists of countable or uncountable number of elements, processes are said to be in discrete or continuous time, respectively.*

As the size of portfolio is developing over the time, a random variable which describes the number of claims in each time period should be used.

Definition 1.1.2. *Counting process is a stochastic process* $\{N(t) : t \geq 0\}$*, in which the values of N(t) are non-negative, integer and non-decreasing. This means:*

1. $N(t) \geq 0, \ \forall \, t \geq 0$
2. *N(t) is an integer*
3. *if* $t \leq s \Longrightarrow N(t) \leq N(s)$

Let $\{N(t) : t \geq 0\}$ be a counting process in which the random variable N(t) indicates the number of claims arising within [0,t] time space. Two sequences of random variables, which can be defined by a counting process, are:

1. the sequence of arrival times $T_1, T_2, T_3, \cdots$ where T_i is the time needed until i-event occurred
2. the sequence of interarrival times $W_1, W_2, W_3, \cdots$ where W_i is the time difference between the (i - 1) and (i) events.

So,

$$T_i = \inf\{t : N(t) = i\}$$

and,

$$W_i = T_i - T_{i-1}, \ i = 2, 3, 4, ... \qquad \text{with} \quad W_1 = T_1$$

The relation between them is that:

$$T_n = \sum_{i=1}^{n} W_i \qquad \forall \, n = 1, 2, 3, ...$$

It is worth mentioning that T_i and W_i are continuous random variables, although N(t) is a discrete random variable.

Example 1.1. According to Figure 1.1, it can be observed that no claim has happened in [0, t_1]. As a result, the value of N(t) at t_1 is N(t_1) = 0. Similarly, N(t_2) = 2 and N(t_3) = 3.

Consequently, a counting process is a discrete-valued stochastic process in continuous time. Examples of counting processes, which have been studied and used extensively in ruin theory, are Poisson process and Renewal process.

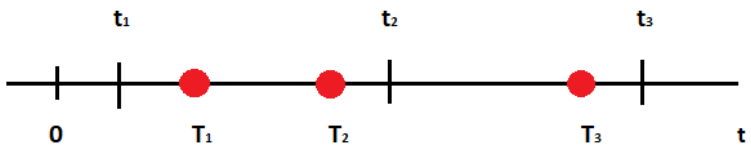


Figure 1.1: Example of sequences T_i and W_i

Definition 1.1.3. *Let $\{N(t) : t \geq 0\}$ be a counting process, in which N(t) represents the number of events in the time interval [0,t]. This process is called Poisson process with rate $\lambda > 0$, if the following three properties are satisfied:*

1. *N(0) = 0*

2. $Pr\left(N(t+h) = n+k|N(t) = n\right) = \begin{cases} \lambda h + o(h) & k = 1 \\ 1 - \lambda h + o(h) & k = 0 \\ o(h) & k \geq 2 \end{cases}$

   where function $o(h)$ satisfies: $\lim_{h \to 0} \frac{o(h)}{h} = 0$

3. *For disjoint time intervals, the random variables, which denote the number of claims in each of them, are independent. For instance, for all $t < s$, the random variable $N(s) - N(t)$ is independent from $N(t)$.*

Two fundamental properties of Poisson process are:

- The random variable N(t) follows Poisson distribution with intensity λt (see book [30], Proposition 5.5.1, pg 316-317)

- The interarrival times W_i are independent and identically distributed with Exponential distribution with parameter λ (therefore, $\lambda = \frac{1}{E(W_i)}$ represents the expected number of claims arising per unit time) (see reference [28], pg 11)

Consequently, the arrival time T_n obeys Erlang distribution with parameters n and λ.

Indeed, let $M_W(r)$ be the moment generating function of the interarrival times W_i. Expanding the moment generating function of T_n yields:

$$\begin{aligned} M_{T_n}(t) &= E\left[e^{tT_n}\right] = E\left[e^{t(W_1+W_2+\cdots W_n)}\right] \\ &= E\left[e^{tW_1}e^{tW_2}\cdots e^{tW_n}\right] = E\left[e^{tW_1}\right]E\left[e^{tW_2}\right]\cdots E\left[e^{tW_n}\right] = \left(\frac{\lambda}{\lambda - t}\right)^n \end{aligned}$$

As W_i are independent variables, the mean can be separated. The above moment generating function belongs to Erlang distribution with parameters n and λ. So, $T_n \sim^d \text{Erl}(n, \lambda)$.

$\diamondsuit$

Definition 1.1.4. *Let $\{N(t) : t \geq 0\}$ be a counting process in which the random variable N(t) represents the claims happened within [0,t]. The random variable X_i indicates the size of the i-claim. The total loss in the time interval [0,t] is described by the random variable $S(t)$:*

$$S(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i, & N \geq 1 \\ 0, & N = 0 \end{cases}$$

The compound stochastic process $\{S(t) : t \geq 0\}$ is known as the aggregate claims process.

Insurance companies are obliged by the legislation to start any operation having an initial capital in their portfolio. This initial surplus is denoted by u. Moreover, in the classical approach of ruin, there is no uncertainty about the total amount of revenues in each interval [0,t]. This is because, it is assumed that the only source of income is the payments of premiums. Thus, only a mathematical function $P(t)$ is about to declare the total size of revenues in [0,t]. Hence, the random variable

$$U(t) = u + P(t) - S(t) \qquad \forall\, t \geq 0$$

denotes the size of surplus in the time period [0,t] (where $U(0) = u$).

Definition 1.1.5. *The collection $\{U(t) : t \geq 0\}$, which contains the random variables of the surplus U(t), is called Surplus process.*

The objective of this project is to study the process of surplus while it is developing over the time and focus on the first time that it becomes negative. This analysis is going to be described under the scope of the classical model.

Definition 1.1.6. *In the classical continuous time risk model, the components of the surplus process $\{U(t) : t \geq 0\}$ satisfy the following:*

1. *P(t) is a linear function with positive slope c. That means,* $P(t) = ct,\ c > 0$

2. *The claim sizes X_i, independent of $N(t)$ $\forall t \geq 0$, are independent and identically distributed nonnegative random variables*

3. *The counting process* $\{N(t) : t \geq 0\}$ *is a Poisson process, with intensity rate* $\lambda > 0$. *Thus, the* $\{S(t) : t \geq 0\}$ *is a compound Poisson aggregate claims process*

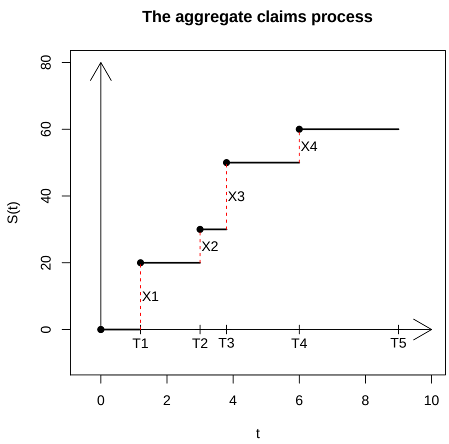


Figure 1.2: The aggregate claims process S(t)

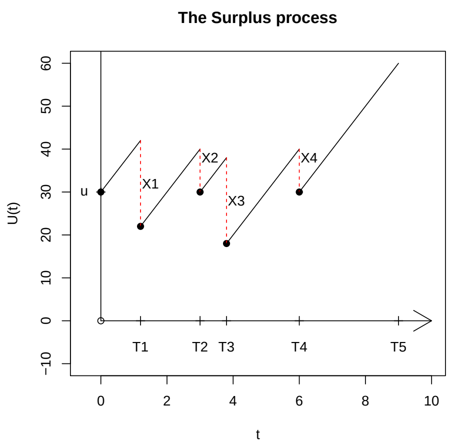


Figure 1.3: The Surplus process U(t)

The Figures 1.2 and 1.3 depict a general picture of how the processes of surplus U(t) and aggregate claims S(t) are developing over the time. It can be observed that the surplus U(t) increases continuously between two successive arrival times of claims T_{i-1} and T_i, due to the earned premiums, whereas the total loss S(t) remains constant in the same intervals. It is to be noted that both of them are right-continuous and consist of jumps whenever claims occur. Thus, the respective jumps of S(t) and U(t), which happen at arrival times T_i, have the same size, whereas the former has an upward trend and the latter a downward trend.

Some further notions under this concept:

- The slope c of premiums' function is referred to as premium rate per unit time ($c = \frac{P(t)}{t}$)
- The cumulative distribution function of X_i is symbolized by F, with $F(0) = 0$ and the moments of X_i by

$$\mu_k = E\left[X^k\right] = \int_0^\infty x^k dF(x)$$

- The intensity λ of Poisson process denotes the number of claims happening per unit time ($\lambda = \frac{E[N(t)]}{t}$, because $N(t) \sim \text{Poisson}(\lambda t)$)
- As far as the expected total loss in [0,t] is concerned:

$$\begin{aligned}
E\left[S(t)\right] &= E\left[\, E\left[S(t)|N(t)\right] \,\right] = \sum_{n=0}^{\infty} E\left[S(t)|N(t)=n\right] Pr[N(t)=n] \\
&= \sum_{n=0}^{\infty} E\left[X_1 + X_2 \cdots + X_{N(t)}|N(t)=n\right] Pr[N(t)=n] \\
&= \sum_{n=0}^{\infty} E\left[X_1 + X_2 \cdots + X_n\right] Pr[N(t)=n] \\
&= \sum_{n=0}^{\infty} n \cdot E\left[X_i\right] Pr[N(t)=n] = E\left[X_i\right] \sum_{n=0}^{\infty} n \cdot Pr[N(t)=n] \\
&= E\left[X_i\right] E\left[N(t)\right] \\
&= \mu_1 \cdot \lambda \cdot t
\end{aligned}$$

Therefore, the expected size of loss per unit time is:

$$\lambda\mu_1 = \frac{E\left[S(t)\right]}{t}$$

Definition 1.1.7. *The security loading factor is defined by* $\theta = \dfrac{c}{\lambda\mu_1} - 1$ *and satisfies the net profit condition in the classical model*

$$\theta > 0 \Longleftrightarrow c > \lambda\mu_1$$

The assumption of θ being positive means that the expected income is greater than the expected loss per unit time. On the other hand, θ can not take any positive value without any limitation, because, in this way, the portfolio will not be competitive enough to survive in the insurance market. As a result, the percentage of profit θ is normally restricted between 0 and 1.

Definition 1.1.8. *In the classical model, the adjustment coefficient R is defined as the smallest positive root of the equation*

$$M_X(r) = 1 + (1+\theta)\mu_1 r$$

in which $M_X(r) = \displaystyle\int_0^{\infty} e^{rx} f(x)dx$ is the moment generating function (m.g.f) of claims X_i.

1.2 Measures of Ruin

It is of great interest not only when the surplus becomes negative for the first time, but also how likely this is to happen. The definitions of ruin probability, ruin time, Gerber-Shiu function and the special cases of Gerber-Shiu function can be found in the work of Gerber and Shiu (1997,1998). We present the respective definitions and properties of them.

Definition 1.2.1. *Let $\psi(u)$ define the probability of ruin, provided that the initial surplus is u. It denotes how likely is for the surplus to drop below zero for the first time. This means,*

$$\psi(u) = Pr\left[U(t) < 0,\ for\, t \geq 0\ |U(0) = u\right], \quad u \geq 0$$

Of course, the notion of ruin should not be linked with the terminology of bankruptcy of an insurance company. Ruin is a technical term which controls and tests the adequacy of any portfolio. As a result, companies do not stop their operation when ruin happens, but they change their risk management by borrowing money, being reinsured etc.

Properties satisfied by $\psi(u)$:

- $\psi(u)$ is a decreasing function of the initial surplus u:
  $(u_1 < u_2 \Longrightarrow \psi(u_1) \geq \psi(u_2)\ )$
- $\lim\limits_{u\to\infty} \psi(u) = 0$
- Regarding the net profit condition,

$$\psi(u) < 1\ \forall\, u \geq 0 \iff c > \lambda\mu_1$$

Another useful notion is the random variable T, which denotes the time of ruin.

$$T = \begin{cases} inf\{t : U(t) < 0\} \\ \infty, \quad if\ U(t) > 0 \quad \forall\, t \geq 0 \end{cases}$$

Note that:

- $\psi(u) = Pr\left[T < \infty \mid U(0) = u\right]$

- the variable T is a *defective random variable*, which means

$$Pr\,(T = \infty) > 0$$

Indeed,
$$\begin{aligned} Pr\,[T=\infty] &= Pr\,(U(t) > 0 \quad \forall\, t \geq 0 \mid U(0) = u) \\ &= 1 - Pr\,[U(t) < 0 \text{ for } t \geq 0 \mid U(0) = u] \\ &= 1 - \psi(u) \\ &= \delta(u) > 0 \end{aligned}$$

The symbol of $\delta(u)$ denotes the non-ruin probability. Moreover, the definitions of $\psi(u)$, $\delta(u)$ and T imply that these quantities are depended on the initial surplus u. Hence, their values are going to change regarding the values of u.

Example 1.2.1. *Consider the following table of elements:*

number of claims (N(t))	*1*	*2*	*3*	*4*	*5*
arrival times (T_i)	*1*	*2*	*2.5*	*5.5*	*7*
size of claims (X_i)	*3*	*10*	*5*	*35*	*5*
u = 8 and c = 5					

The values of the aggregate claims process S(t) are:

$$S(t) = \sum_{i=1}^{N(t)} X_i = \begin{cases} 0, & t < 1 \\ 3, & 1 \leq t < 2 \\ 13, & 2 \leq t < 2.5 \\ 18, & 2.5 \leq t < 5.5 \\ 53, & 5.5 \leq t < 7 \\ 58, & 7 \leq t \end{cases}$$

and the respective values of the surplus process $U(t)$ are:

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i = \begin{cases} 8+5t & & & t<1 \\ 8+5t-3 & = & 5+5t, & 1 \leq t < 2 \\ 8+5t-13 & = & -5+5t, & 2 \leq t < 2.5 \\ 8+5t-18 & = & -10+5t, & 2.5 \leq t < 5.5 \\ 8+5t-53 & = & -45+5t, & 5.5 \leq t < 7 \\ 8+5t-58 & = & -50+5t, & 7 \leq t \end{cases}$$

The development of the surplus over the time is depicted by Figure 1.4. The time in which the surplus becomes negative for the first time is called *time of ruin* and this random variable is symbolised by T. Another two variables associated with the time of ruin are the surplus immediately before the time of ruin and the deficit exactly at the time of ruin. These two are symbolised by U(T-) and U(T) respectively. Note that the random variable of deficit, U(T), takes always negative values, thus the absolute amount of it, $|U(T)|$, is going to be studied.

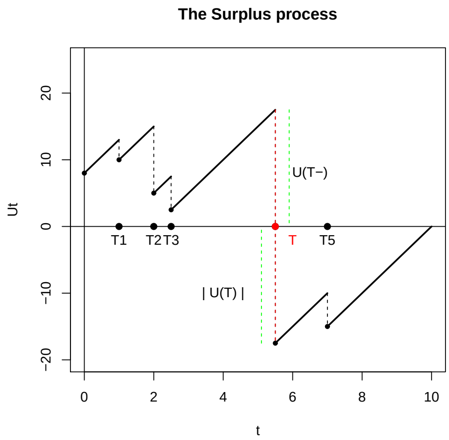


Figure 1.4: Time of Ruin

Definition 1.2.2. *For $w : \mathcal{R}x\mathcal{R} \longrightarrow \mathcal{R}^+$ and $\delta \geq 0$, the Gerber-Shiu function is defined by:*

$$\phi_\delta(u) = E\left[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty) \mid U(0) = u\right], \quad u \geq 0$$

where,

$$I(T < \infty) = \begin{cases} 1, & T < \infty \\ 0, & T = \infty \end{cases} \quad \text{and} \quad U(T-) = \lim_{t \to T^-} U(t)$$

Let $f(x, y, t|u)$ be the joint probability density function of U(T-), $|U(T)|$ and T, provided that the initial syrplus is u. So, the Gerber-Shiu function can be written,

$$\phi_\delta(u) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta t} w(x, y) f(x, y, t|u) dx dy dt, \quad u \geq 0$$

The function w(x,y) is known as *penalty function*. It can be assumed that the insurer has to be fined at the time of ruin T. The amount of this fine depends on U(T-) and $|U(T)|$. Moreover, if δ is interpreted as a force of interest, the random variable

$$e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty)$$

can be explained as the present value of the fine that the insurer is going to pay in case ruin happens. Thus, the Gerber-Shiu function is called the *expected discounted penalty function*, as well. However, δ can be also considered as the argument for the Laplace transform of the ruin time T.

Note. It has been mentioned that $\lim_{u\to+\infty} \psi(u) = 0$. As a result, it is expected for the surplus U(t) not to fall below zero while $u \to +\infty$. Thus,

$$\lim_{u\to+\infty} \phi_\delta(u) = 0$$

The Gerber-Shiu function is a general function which can be reduced to some interesting functions for the risk theory, regarding the values received by the arguments δ and penalty function $w(x,y)$. Namely,

- If $\underline{\delta = 0 \text{ and } w(x,y) = 1}$, $\phi_\delta(u)$ equals to the probability of ruin $\psi(u)$

$$\begin{aligned} \phi_\delta(u) &= E\left[I(T<\infty) \mid U(0)=u\right] \\ &= Pr\left[T<\infty \mid U(0)=u\right] \\ &= \psi(u) \end{aligned}$$

  or,

$$\begin{aligned} \phi_\delta(u) &= \int_0^\infty \int_0^\infty \int_0^\infty f(x,y,t|u)dxdydt = \int_0^\infty f_T(t|u)dt \\ &= Pr\left[T<\infty \mid U(0)=u\right] \\ &= \psi(u) < 1 \end{aligned}$$

  which indicates that $f(x,y,t|u)$ is a *defective probability density function*

- If $\underline{w(x,y) = 1}$, the Laplace transform for the time of ruin T is received

$$e^{-\delta T} I(T<\infty) = \begin{cases} e^{-\delta T}, & T<\infty \\ 0, & T=\infty \end{cases}$$

  So,

$$\begin{aligned} \phi_\delta(u) &= E\left[e^{-\delta T} I(T<\infty) \mid U(0)=u\right] = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta t} f(x,y,t|u)dxdydt \\ &= \int_0^\infty e^{-\delta t} f_T(t|u)dt = \widehat{L}_T(\delta) \end{aligned}$$

- For $\underline{w(x,y) = I(X\le x)I(Y\le y)}$, Gerber-Shiu function is reduced to the discounted joint cumulative distribution function of U(T-) and $|U(T)|$

$$e^{-\delta T} I(U(T-)\le x) I(|U(T)|\le y) I(T<\infty) = \begin{cases} e^{-\delta T},\ U(T-)\le x,\ |U(T)|\le y,\ T<\infty \\ 0,\ \text{otherwise} \end{cases}$$

  So,

$$\begin{aligned}\phi_\delta(u) &= E\left[e^{-\delta T} I(U(T-) \leq x) I(|U(T)| \leq y) I(T < \infty) \mid U(0) = u\right] \\ &= \int_0^x \int_0^y \int_0^\infty e^{-\delta t} f(k,l,t|u) dt dl dk\end{aligned}$$

and if $\delta = 0$, it is received the joint cumulative distribution function of U(T-) and $|U(T)|$

$$\phi_\delta(u) = \int_0^x \int_0^y \int_0^\infty f(k,l,t|u) dt dl dk = \int_0^x \int_0^y f(k,l|u) dl dk = F_{U(T-),|U(T)|}(x,y|u)$$

- Similarly, for $\underline{w(x,y) = I(X = x)I(Y = y)}$, the discounted joint probability density function of U(T-) and $|U(T)|$ is produced

$$e^{-\delta T} I(U(T-) = x) I(|U(T)| = y) I(T < \infty) = \begin{cases} e^{-\delta T}, & U(T-) = x,\ |U(T)| = y,\ T < \infty \\ 0, & \text{otherwise} \end{cases}$$

So,

$$\phi_\delta(u) = E\left[e^{-\delta T} I(U(T-) = x) I(|U(T)| = y) I(T < \infty) \mid U(0) = u\right] = \int_0^\infty e^{-\delta t} f(x,y,t|u) dt$$

and if $\delta = 0$, it is received just the joint probability density function of U(T-) and $|U(T)|$

$$\phi_\delta(u) = \int_0^\infty f(x,y,t|u) dt = f_{U(T-),|U(T)|}(x,y|u)$$

- $\underline{w(x,y) = I(X = x)}$ generates the discounted marginal probability density function of U(T-)

$$e^{-\delta T} I(U(T-) = x) I(T < \infty) = \begin{cases} e^{-\delta T}, & U(T-) = x,\ T < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}\phi_\delta(u) &= E\left[e^{-\delta T} I(U(T-) = x) I(T < \infty) \mid U(0) = u\right] = \int_0^\infty \int_0^\infty e^{-\delta t} f(x,y,t|u) dy dt \\ &= \int_0^\infty e^{-\delta t} f(x,t|u) dt\end{aligned}$$

and $\delta = 0$ yields merely the marginal probability density function of U(T-)

$$\phi_\delta(u) = \int_0^\infty f(x,t|u) dt = f_{U(T-)}(x|u)$$

- Similarly, $\underline{w(x,y) = I(Y = y)}$ produces the discounted marginal probability density function of $|U(T)|$

$$e^{-\delta T} I(|U(T)| = y) I(T < \infty) = \begin{cases} e^{-\delta T}, & |U(T)| = y,\ T < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}\phi_\delta(u) &= E\left[e^{-\delta T}I(|U(T)|=y)I(T<\infty) \mid U(0)=u\right] = \int_0^\infty\int_0^\infty e^{-\delta t}f(x,y,t|u)dxdt \\ &= \int_0^\infty e^{-\delta t}f(y,t|u)dt\end{aligned}$$

where $\delta = 0$ gives the marginal probability density function of $|U(T)|$

$$\phi_\delta(u) = \int_0^\infty f(y,t|u)dt = f_{|U(T)|}(y|u)$$

1.3 The Dickson-Hipp operator

The definition and properties of Dickson-Hipp operator, which are mentioned below, are based on the book [26] (Chapter 2.2, pg 12) of Willmot and Woo (2017).

Definition 1.3.1. *Let f(x) be an integrable function and $r \in \mathcal{R}$. Define*

$$T_r f(x) = e^{rx}\int_x^\infty e^{-ry}f(y)dy = \int_x^\infty e^{-r(y-x)}f(y)dy \tag{1.1}$$

to be the Dickson-Hipp operator. This operator consists of some interesting and useful properties, such as:

1. T_r is a linear operator. Indeed,
$$\begin{aligned}T_r(f(x)+g(x)) &= e^{rx}\int_x^\infty e^{-ry}(f(y)+g(y))dy \\ &= e^{rx}\int_x^\infty e^{-ry}f(y)dy \;+\; e^{rx}\int_x^\infty e^{-ry}g(y)dy \\ &= T_r f(x) \;+\; T_r g(x)\end{aligned}$$

2. The Laplace transform and the tail distribution of f(x) are special cases of (1.1), i.e.
$$\begin{cases} T_r f(0) = \displaystyle\int_0^\infty e^{-ry}f(y)dy = \hat{f}(r) \\ T_0 f(x) = \displaystyle\int_x^\infty f(y)dy = \bar{F}(x)\end{cases}$$

3. Applying integration by parts implies the Laplace transform of T_r

$$\begin{aligned}
\hat{T}_r f(s) &= \int_0^\infty e^{-sx} T_r f(x) dx = \int_0^\infty e^{-sx} e^{rx} \int_x^\infty e^{-ry} f(y) dy dx \\
&= \int_0^\infty e^{-(s-r)x} \int_x^\infty e^{-ry} f(y) dy dx = \int_0^\infty \frac{d}{dx} \left[\frac{-e^{-(s-r)x}}{s-r} \right] \int_x^\infty e^{-ry} f(y) dy dx \\
&= \left[-\frac{e^{-(s-r)x}}{s-r} \int_x^\infty e^{-ry} f(y) dy \right]_{x=0}^\infty - \int_0^\infty \left[\frac{e^{-(s-r)x}}{s-r} \right] e^{-rx} f(x) dx \\
&= \frac{\int_0^\infty e^{-ry} f(y) dy}{s-r} - \frac{\int_0^\infty e^{-sx} f(x) dx}{s-r} \\
&= \frac{\hat{f}(r) - \hat{f}(s)}{s-r}
\end{aligned}$$

4. A double repeated application of the operator is given by

$$\begin{aligned}
T_{r_1,r_2} f(x) &= e^{r_1 x} \int_x^\infty e^{-r_1 y} T_{r_2} f(y) dy \\
&= e^{r_1 x} \int_x^\infty e^{-r_1 y}\ e^{r_2 y} \int_y^\infty e^{-r_2 t} f(t) dt\ dy \\
&= e^{r_1 x} \int_x^\infty e^{-(r_1 - r_2) y} \int_y^\infty e^{-r_2 t} f(t) dt\ dy \\
&= e^{r_1 x} \int_x^\infty \int_y^\infty e^{-(r_1 - r_2) y}\ e^{-r_2 t} f(t) dt\ dy
\end{aligned}$$

Reversing the order of integration, the new boundaries are $x \le t < \infty$ and $x \le y \le t$. Thus, we obtain

$$\begin{aligned}
T_{r_1,r_2} f(x) &= e^{r_1 x} \int_x^\infty \int_x^t e^{-(r_1 - r_2) y}\ e^{-r_2 t} f(t) dy\ dt \\
&= e^{r_1 x} \int_x^\infty e^{-r_2 t} f(t) \int_x^t e^{-(r_1 - r_2) y} dy\ dt \\
&= e^{r_1 x} \int_x^\infty e^{-r_2 t} f(t) \left[-\frac{e^{-(r_1 - r_2) y}}{r_1 - r_2} \right]_{y=x}^t dt \\
&= e^{r_1 x} \int_x^\infty e^{-r_2 t} f(t) \left[\frac{e^{-(r_1 - r_2) x} - e^{-(r_1 - r_2) t}}{r_1 - r_2} \right] dt \quad \Longrightarrow
\end{aligned}$$

$$
\begin{aligned}
T_{r_1,r_2}f(x) &= \frac{1}{r_1 - r_2}\left[e^{r_2 x}\int_x^\infty e^{-r_2 t}f(t)dt \;-\; e^{r_1 x}\int_x^\infty e^{-r_1 t}f(t)dt\right] \\
&= \frac{T_{r_2}f(x) - T_{r_1}f(x)}{r_1 - r_2}
\end{aligned}
$$

1.4 Solutions of a defective renewal equation

The idea of using a compound geometric distribution to solve a defective renewal equation is based on Lin and Willmot (1999) and Willmot and Woo (2017). In this subsection we present their results. Moreover, based on E. Chadjikonstantinidis (2016), we present some useful properties of a compound geometric distribution. In the following section, in which these properties will be used, we will make the appropriate connection with the classical model studied.

Lemma 1.4.1. *Let*

$$
S = \begin{cases} \sum_{i=1}^{N} X_i, & N \geq 1 \\ 0, & N = 0 \end{cases}
$$

be a compound geometric random variable, where G(x) is the distribution function of S, $N \sim Geom(1-\varphi)$, $0 < \varphi < 1$, F(x) is the distribution function of X_i with F(0) = 0 and $\bar{F}^{*n}(x) = Pr\left(X_1 + X_2 + \cdots + X_n > x\right)$ is the tail of the n-fold convolution of F(x). Then,

i. $\bar{G}(x) = \sum_{n=1}^{\infty}(1-\varphi)\varphi^n \bar{F}^{*n}(x), \qquad x \geq 0$

ii. $\hat{g}(s) = \int_{0^+}^{\infty} e^{-sx}dG(x) + G(0), \qquad s \geq 0$

iii. $\hat{g}(s) = \dfrac{1-\varphi}{1-\varphi\hat{f}(s)}, \qquad s \geq 0$

iv. $\hat{\bar{G}}(s) = \varphi\hat{\bar{G}}(s)\hat{f}(s) + \varphi\hat{\bar{F}}(s) \Longleftrightarrow \hat{\bar{G}}(s) = \dfrac{\varphi\hat{\bar{F}}(s)}{1-\varphi\hat{f}(s)}, \qquad s \geq 0$

Proof.

(i.) Using $Pr(N = n) = \varphi^n(1-\varphi), \quad n = 0, 1, 2, \cdots$, and the law of total probability, we have

$$
\begin{aligned}
\bar{G}(x) &= Pr(S > x) = Pr\,(X_1 + X_2 + \cdots + X_N > x) \\
&= \sum_{n=1}^{\infty} Pr\,(X_1 + X_2 + \cdots + X_N > x,\ N = n) \\
&= \sum_{n=1}^{\infty} Pr\,(X_1 + X_2 + \cdots + X_N > x|\ N = n)\ \ Pr(N = n) \\
&= \sum_{n=1}^{\infty} Pr\,(X_1 + X_2 + \cdots + X_n > x)\ \ Pr(N = n) \\
&= \sum_{n=1}^{\infty} (1 - \varphi)\varphi^n \bar{F}^{\,*n}(x)
\end{aligned}
$$

(ii.) S follows a mixed distribution with partial probability mass function at 0 and partial probability density function in $(0, +\infty)$. Indeed,

$$G(0) = Pr(S = 0) = Pr(N = 0) = 1 - \varphi > 0$$

Thus, the Laplace transform of g is given by

$$\hat{g}(s) = E[e^{-sS}] = \int_{0^+}^{\infty} e^{-sx} dG(x) + Pr(S = 0) = \int_{0^+}^{\infty} e^{-sx} dG(x) + G(0)$$

(iii.) Another form of $\hat{g}(s)$ is described by

$$
\begin{aligned}
\hat{g}(s) &= E[^{-sS}] = E\left[E[e^{-sS}|N]\right] = \sum_{n=0}^{\infty} E[e^{-s(X_1+X_2+\cdots+X_N)}|N = n]Pr(N = n) \\
&= \sum_{n=0}^{\infty} E[e^{-s(X_1+X_2+\cdots+X_n)}]Pr(N = n) = \sum_{n=0}^{\infty} [E(e^{-sX})]^n Pr(N = n) \\
&= P_N[E(e^{-sX})] = P_N(\hat{f}(s))
\end{aligned}
$$

Considering the probability generating function of N, which is $P_N(u) = \dfrac{1 - \varphi}{1 - \varphi u}$, we receive

$$\hat{g}(s) = \frac{1 - \varphi}{1 - \varphi \hat{f}(s)}$$

(iv.) For any given distribution F, it is known that $\hat{\bar{F}}(s) = \dfrac{1 - \hat{f}(s)}{s}$ (it is proven later, in section (1.5.4)). So,

$$
\begin{aligned}
\hat{\bar{G}}(s) &= \frac{1 - \hat{g}(s)}{s} = \frac{1 - \dfrac{1 - \varphi}{1 - \varphi \hat{f}(s)}}{s} = \frac{\varphi(1 - \hat{f}(s))}{s(1 - \varphi \hat{f}(s))} = \left[\frac{1 - \hat{f}(s)}{s}\right] \left[\frac{\varphi}{1 - \varphi \hat{f}(s)}\right] \\
&= \hat{\bar{F}}(s) \left[\frac{\varphi}{1 - \varphi \hat{f}(s)}\right]
\end{aligned}
$$

$\diamondsuit$

Proposition 1.4.1. *Let $0 < \varphi < 1$, F(x) be a distribution function defined in $[0, +\infty)$ with F(0) = 0, $r(x)$ be a continuous function in $[0, +\infty)$ and m(x) satisfy the following defective renewal equation*

$$m(x) = \varphi \int_0^x m(x-y)dF(y) \; + \; r(x), \qquad x \geq 0$$

Then, the solution of m(x) is given by

$$m(x) = \frac{1}{1-\varphi} \int_{0^+}^x r(x-y)dG(y) \; + \; r(x), \qquad x \geq 0 \tag{1.2}$$

or

$$m(x) = \frac{1}{1-\varphi} r(x) - \frac{r(0)}{1-\varphi} \bar{G}(x) - \frac{1}{1-\varphi} \int_{0^+}^x r'(x-y)\bar{G}(y)dy \tag{1.3}$$

where

$$S = \begin{cases} \sum_{i=1}^N X_i, & N \geq 1 \\ 0, & N = 0 \end{cases}$$

is a compound geometric random variable, G(x) is the distribution function of S, $N \sim Geom(1-\varphi)$ and $F_X(x) = F(x)$ is the distribution function of X_i.

Proof. Applying Laplace transform in m(x) yields

$$\hat{m}(s) = \varphi \hat{m}(s)\hat{f}(s) + \hat{r}(s) \Longrightarrow \hat{m}(s) = \frac{\hat{r}(s)}{1 - \varphi \hat{f}(s)}$$

From Lemma 1.4.1 (iii), where $\hat{g}(s) = \dfrac{1-\varphi}{1-\varphi\hat{f}(s)}$, we receive

$$\hat{m}(s) = \frac{\hat{r}(s)\hat{g}(s)}{1-\varphi}$$

From Lemma 1.4.1(ii), where $\hat{g}(s) = \int_{0^+}^{\infty} e^{-sx} dG(x) + G(0)$, we obtain

$$\begin{aligned} \hat{m}(s) &= \frac{1}{1-\varphi} \left[\int_{0^+}^{\infty} e^{-sx} dG(x) + G(0) \right] \hat{r}(s) \\ &= \frac{1}{1-\varphi} \left[\int_{0^+}^{\infty} e^{-sx} dG(x) + (1-\varphi) \right] \hat{r}(s) \\ &= \frac{1}{1-\varphi} \left[\int_{0^+}^{\infty} e^{-sx} dG(x) \right] \hat{r}(s) + \hat{r}(s) \end{aligned}$$

Finally, applying the inverse Laplace transform, we obtain the first solution

$$m(x) = \frac{1}{1-\varphi}\int_{0^+}^{x} r(x-y)dG(y) \;+\; r(x)$$

Now, the integral in the above equation equals to

$$\begin{aligned}
\int_{0^+}^{x} r(x-y)dG(y) &= -\int_{0^+}^{x} r(x-y)\bar{G}'(y)dy \\
&= -\left[r(x-y)\bar{G}(y)\right]_{y=0}^{x} - \int_{0^+}^{x}\left[\frac{d}{dy}r(x-y)\right]\bar{G}(y)dy \\
&= r(x)\bar{G}(0) - r(0)\bar{G}(x) - \int_{0^+}^{x} r'(x-y)\bar{G}(y)dy \\
&= \varphi r(x) - r(0)\bar{G}(x) - \int_{0^+}^{x} r'(x-y)\bar{G}(y)dy
\end{aligned}$$

where, it has been used the fact that

$$G(0) = Pr(S=0) = Pr(N=0) = 1-\varphi \Longrightarrow \bar{G}(0) = \varphi$$

Substituting it in (1.2), we receive the second solution of m(x)

$$\begin{aligned}
m(x) &= \frac{1}{1-\varphi}\left[\varphi r(x) - r(0)\bar{G}(x) - \int_{0}^{x} r'(x-y)\bar{G}(y)dy\right] \;+\; r(x) \\
&= \frac{1}{1-\varphi}r(x) - \frac{r(0)}{1-\varphi}\bar{G}(x) - \frac{1}{1-\varphi}\int_{0}^{x} r'(x-y)\bar{G}(y)dy
\end{aligned}$$

$\diamondsuit$

Lemma 1.4.2. *Let*

$$S = \begin{cases} \sum_{i=1}^{N} X_i, & N \geq 1 \\ 0, & N = 0 \end{cases}$$

be a compound geometric random variable, where G(x) is the distribution function of S, $N \sim$ Geom(p), $0 < p < 1$, $q = 1-p$ and F(x) is the distribution function of X_i with F(0) = 0. Then,

$$G(x) = p + qF_Y(x) \Longleftrightarrow \bar{G}(x) = q\bar{F}_Y(x)$$

where $Y = S|S > 0$ and its moment generating function is given by

$$M_Y(t) = \frac{pM_X(t)}{1-qM_X(t)}$$

Proof. As N follows a Geometric distribution with parameter p, it is known that $Pr(N=n) = q^n p,\;\; n = 0,1,2,\cdots$ and the probability generating function is given by

$$P_N(u) = E(u^N) = \frac{p}{1-qu}$$

It has already been mentioned that S follows a mixed distribution, so the probability of receiving the value 0 is positive and equal to

$$G(0) = Pr(S=0) = Pr(N=0) = p > 0$$

Thus, it is expected for its moment generating function to consist of a discrete part equal to p and a continuous part with distribution function F_Y. So,

$$\begin{aligned} M_S(t) &= P_N(M_X(t)) = \frac{p}{1-q\,M_X(t)} = p + \frac{p}{1-q\,M_X(t)} - p \\ &= p + \frac{p - p(1-q\,M_X(t))}{1-q\,M_X(t)} = p + q\,\frac{p\,M_X(t)}{1-q\,M_X(t)} \end{aligned}$$

As a result, setting $Y = S|S>0$ implies

$$M_Y(t) = \frac{p\,M_X(t)}{1-q\,M_X(t)}$$

and

$$G(x) = p + qF_Y(x) \Longleftrightarrow \bar{G}(x) = q\bar{F}_Y(x)$$

$\diamondsuit$

1.5 Gerber-Shiu function

1.5.1 Integro-Differential Equation for Gerber-Shiu function

In the following Theorem 1.5.1.1, we use the idea of the renewal argument by conditioning on the time and size of the first claim. This concept is extracted from Gerber and Shiu (1997,1998) and Cai (2000). Meanwhile, the latter applies it in the absolute ruin, so we modify his work properly in order to be applied in our case of ruin. Furthermore, the academic notes of E. Chadjikonstantinidis (2016) have been used to offer detailed explanations in our analysis.

Theorem 1.5.1.1. *The Gerber-Shiu function $\phi_\delta(u)$ satisfies the following integro-differential equation*

$$c\phi_\delta'(u) = (\lambda+\delta)\phi_\delta(u) - \lambda\int_0^u \phi_\delta(u-x)f(x)dx - \lambda\gamma(u), \quad \forall\, u \geq 0 \quad (1.4)$$

where

$$\gamma(x) = \int_x^\infty w(x, y-x)f(y)dy \quad (1.5)$$

Proof. By conditioning on the time t and size x of the first claim and using the renewal argument, the law of total probability gives

$$\begin{aligned}
\phi_\delta(u) &= E\left[e^{-\delta T} w(U(T-), |U(T)|) I(T<\infty) \mid U(0)=u\right] \\
&= \int_0^\infty \int_0^\infty E\left[e^{-\delta T} w(U(T-), |U(T)|) I(T<\infty) \mid U(0)=u, T_1=t, X_1=x\right] \\
&\quad f_{X_1}(x) f_{T_1}(t) dx\, dt \\
&= \int_0^\infty \int_0^\infty \phi_\delta(u|t,x) f_{X_1}(x) f_{T_1}(t) dx dt \\
&= \int_0^\infty \lambda e^{-\lambda t}\left[\int_0^\infty \phi_\delta(u|t,x) f(x) dx\right] dt
\end{aligned}$$

Considering the time t, when the first claim happens, there are two possible cases for the surplus $U(t)$

$$U(t) = u + ct - x \Longrightarrow \begin{cases} U(t) \geq 0, & 0 \leq x \leq u+ct \\ U(t) < 0, & x > u+ct \end{cases}$$

Observing the Figure 1.5 for the progress of the surplus $U(t)$, in the first case there is not ruin. So the procedure is renewed, starting with initial surplus equal to $u + ct - x$. In the second case, ruin happens and $U(T-) = u + ct$, whereas $|U(T)| = x - u - ct$.

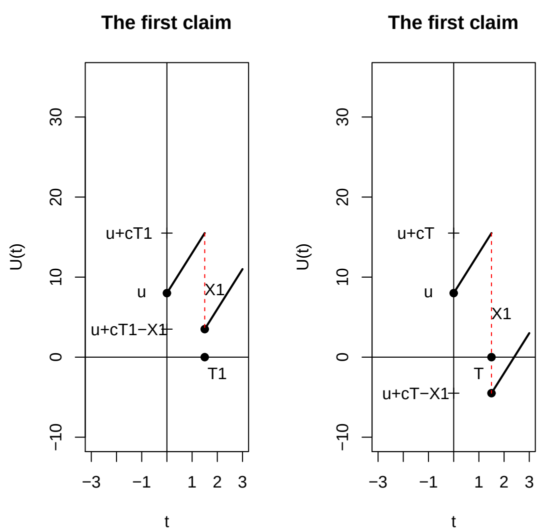


Figure 1.5: Situations of the first claim

Applying the above in $\phi_\delta(u|t,x)$ leads to

$$\begin{aligned}
\phi_\delta(u) &= \int_0^\infty \lambda e^{-\lambda t}\left[\int_0^{u+ct} e^{-\delta t}\phi_\delta(u+ct-x)f(x)dx\right.\\
&+ \left.\int_{u+ct}^\infty e^{-\delta t}w(u+ct,x-u-ct)f(x)dx\right]dt\\
&= \int_0^\infty \lambda e^{-(\lambda+\delta)t}\left[\int_0^{u+ct} \phi_\delta(u+ct-x)f(x)dx\right.\\
&+ \left.\int_{u+ct}^\infty w(u+ct,x-u-ct)f(x)dx\right]dt
\end{aligned}$$

Putting $s = u + ct \Longrightarrow t = \frac{s-u}{c}$, & $dt = \frac{1}{c}ds$, the boundaries of integration are converted into $u \le s \le \infty$ and the last equation is written as

$$\begin{aligned}
\phi_\delta(u) &= \lambda\int_u^\infty e^{-\frac{(\lambda+\delta)(s-u)}{c}}\int_0^s \phi_\delta(s-x)f(x)dx\frac{1}{c}ds\\
&+ \lambda\int_u^\infty e^{-\frac{(\lambda+\delta)(s-u)}{c}}\int_s^\infty w(s,x-s)f(x)dx\frac{1}{c}ds
\end{aligned}$$

Put

$$\gamma(x) = \int_x^\infty w(x,y-x)f(y)dy$$

to obtain

$$c\phi_\delta(u) = \lambda\int_u^\infty e^{-\frac{(\lambda+\delta)(s-u)}{c}}\int_0^s \phi_\delta(s-x)f(x)dxds + \lambda\int_u^\infty e^{-\frac{(\lambda+\delta)(s-u)}{c}}\gamma(s)ds$$

Let

$$g(u,s) = e^{-\frac{(\lambda+\delta)(s-u)}{c}}\int_0^s \phi_\delta(s-x)f(x)dx$$

and

$$h(u,s) = e^{-\frac{(\lambda+\delta)(s-u)}{c}}\gamma(s)$$

Substituting them in the last equation implies,

$$c\phi_\delta(u) = \lambda\int_u^\infty g(u,s)ds + \lambda\int_u^\infty h(u,s)ds \tag{1.6}$$

The derivative of the components in (1.6) with respect to u yields

$$\begin{aligned}\frac{d}{du}\int_u^\infty g(u,s)ds &= -g(u,u)+\int_u^\infty \frac{dg(u,s)}{du}ds \\ &= -\int_0^u \phi_\delta(u-x)f(x)dx+\frac{\lambda+\delta}{c}\int_u^\infty g(u,s)ds\end{aligned}$$

and

$$\begin{aligned}\frac{d}{du}\int_u^\infty h(u,s)ds &= -h(u,u)+\int_u^\infty \frac{dh(u,s)}{du}ds \\ &= -\gamma(u)+\frac{\lambda+\delta}{c}\int_u^\infty h(u,s)ds\end{aligned}$$

As a result, differentiating the equation (1.6) with respect to u yields

$$\begin{aligned}c\phi_\delta'(u) &= \lambda\left[-\int_0^u \phi_\delta(u-x)f(x)dx+\frac{\lambda+\delta}{c}\int_u^\infty e^{-\frac{(\lambda+\delta)(s-u)}{c}}\int_0^s \phi_\delta(s-x)f(x)dxds\right] \\ &+ \lambda\left[-\gamma(u)+\frac{\lambda+\delta}{c}\int_u^\infty e^{-\frac{(\lambda+\delta)(s-u)}{c}}\gamma(s)ds\right] \\ &= -\lambda\left[\int_0^u \phi_\delta(u-x)f(x)dx+\gamma(u)\right] \\ &+ \frac{\lambda+\delta}{c}\cdot\left[\lambda\int_u^\infty e^{-\frac{(\lambda+\delta)(s-u)}{c}}\int_0^s \phi_\delta(s-x)f(x)dxds+\lambda\int_u^\infty e^{-\frac{(\lambda+\delta)(s-u)}{c}}\gamma(s)ds\right] \\ &= -\lambda\left[\int_0^u \phi_\delta(u-x)f(x)dx+\gamma(u)\right]+\frac{\lambda+\delta}{c}\cdot c\phi_\delta(u)\end{aligned}$$

This leads to the desirable integro-differential equation (1.4)

$$c\phi_\delta'(u)=(\lambda+\delta)\phi_\delta(u)-\lambda\int_0^u \phi_\delta(u-x)f(x)dx-\lambda\gamma(u), \quad u\geq 0$$

$\diamondsuit$

1.5.2 Roots of Lundberg's Equation

Gerber and Shiu (1997,1998) show that the Lundberg's fundamental equation has a unique nonnegative root. In this subsection we present this result.

Definition 1.5.2.1. *The equation*

$$l(s)=\lambda\hat{f}(s) \tag{1.7}$$

where

$$l(s) = \lambda + \delta - cs$$

and

$$\hat{f}(s) = \int_0^\infty e^{-sx} f(x) dx$$

is known as the Lundberg's fundamental equation.

Proposition 1.5.2.1. *The Lundberg's fundamental equation has a unique nonnegative root $\rho = \rho(\delta)$, which is an increasing function of δ and* $\lim_{\delta \to 0^+} \rho(\delta) = \rho(0) = 0.$

Proof. Taking into consideration the fact that the Laplace transform $\hat{f}(s)$, of each integrable function f(x), is a decreasing, convex function and the l(s) is a decreasing function (having negative slope -c) with

$$l(0) = \lambda + \delta \geq \lambda = \lambda \hat{f}(0)$$

it can be extracted that the lines of l(s) and $\lambda \hat{f}(s)$ have only one common point, $\rho = \rho(\delta)$, in the nonnegative axis (See Figure 1.6). Thus, $\rho(\delta)$ satisfies

$$\lambda + \delta - c\rho(\delta) = \lambda \hat{f}(\rho(\delta))$$

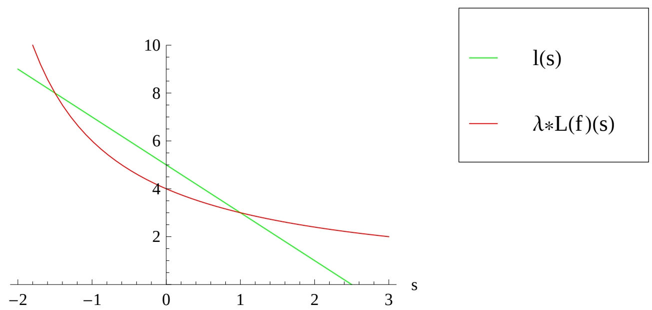


Figure 1.6: Depiction of Lundberg's equation solution

Differentiating with respect to δ yields

$$1 - c\rho^{'}(\delta) = \lambda \rho^{'}(\delta) \hat{f}^{\,'}(\rho(\delta)) \Longrightarrow \rho^{'}(\delta) = \frac{1}{c + \lambda \hat{f}^{\,'}(\rho(\delta))}$$

Using the security loading factor θ, the premium rate c is equal to

$$c = (1+\theta)\lambda \mu_1 = \lambda(1+\theta) \int_0^\infty x f(x) dx$$

The derivative of $\hat{f}\,(\rho(\delta))$ with respect to $\rho(\delta)$ is

$$\hat{f}'(\rho(\delta)) = -\int_0^\infty e^{-\rho(\delta)x} x f(x) dx$$

Hence, the denominator of the above fraction can be written

$$\begin{aligned} c + \lambda \hat{f}'(\rho(\delta)) &= \lambda(1+\theta) \int_0^\infty x f(x) dx \; - \; \lambda \int_0^\infty e^{-\rho(\delta)x} x f(x) dx \\ &= \lambda \left[\int_0^\infty \left[(1+\theta) - e^{-\rho(\delta)x} \right] x f(x) dx \right] \; > \; 0 \end{aligned}$$

Because of $\rho(\delta) \geq 0$, we have

$$e^{-\rho(\delta)x} \leq 1 < 1 + \theta \quad \forall\, x \geq 0 \;\text{ and }\; \theta \in (0,1)$$

As a result, $\rho'(\delta) > 0 \Longrightarrow \rho(\delta)$ is an increasing function of δ.

Now, for $\delta = 0$ the Lundberg's equation is

$$\lambda - c\rho(0) = \lambda \hat{f}(\rho(0))$$

where $\rho(0) = 0$ verifies the Lundberg's equation, because $\hat{f}(0) = \int_0^\infty f(x) dx = 1$. Due to the uniqueness of the root, 0 is the only acceptable solution of $\rho(\delta)$ in case of $\delta = 0$.

$\diamondsuit$

Proposition 1.5.2.2. *For $\delta = 0$, the adjustment coefficient R is the largest negative solution of the equation (1.7) (at absolute value).*

Proof. It has already been demonstrated that the adjustment coefficient R is the smallest positive solution of

$$M_X(r) = 1 + (1+\theta)\mu_1 r$$

Substituting

$$1 + \theta = \frac{c}{\lambda \mu_1}$$

it is obtained

$$\lambda M_X(r) = \lambda + cr$$

For r = -s, in the left-hand side, Laplace tranform appears

$$\lambda \hat{f}(s) = \lambda - cs \tag{1.8}$$

Due to the replacement of r with -s, the largest negative solution of (1.8) (at absolute value) is now the adjustment coefficient R.

$\diamondsuit$

1.5.3 Integral Equation for Gerber-Shiu function through Laplace Transform

In the following calculations, they will be used some properties satisfied by Laplace transform. Namely,

$$
\begin{aligned}
(i) \quad \int_0^\infty e^{-sx} f'(x)dx &= e^{-sx} f(x)|_{x=0}^\infty - \int_0^\infty (e^{-sx})' f(x)dx \\
&= -f(0) + s\int_0^\infty e^{-sx} f(x)dx \\
&= s\hat{f}(s) - f(0)
\end{aligned}
$$

If

$$(f*g)(x) = \int_0^x f(t)g(x-t)dt$$

is the convolution of f and g defined in $(0, +\infty)$, then its Laplace transform equals to

$$
\begin{aligned}
(ii) \quad (f \hat{*} g)(s) &= \int_0^\infty e^{-sx}(f*g)(x)dx \\
&= \left(\int_0^\infty e^{-sx} f(x)dx\right)\left(\int_0^\infty e^{-sx} g(x)dx\right) \\
&= \hat{f}(s) \cdot \hat{g}(s)
\end{aligned}
$$

The Theorem 1.5.3.1, which follows, describes the integral equation satisfied by $\phi_\delta(u)$ and it can be found in the work of Gerber and Shiu (1998). In comparison to their approach and based on E.Chadjikonstantinidis (2016), we use the Dickson-Hipp operator to simplify our final results.

Theorem 1.5.3.1. *The Gerber-Shiu function $\phi_\delta(u)$ satisfies the integral equation*

$$\phi_\delta(u) = \frac{\lambda}{c}\int_0^u \phi_\delta(u-x)T_\rho f(x)dx \; + \; \frac{\lambda}{c}\, T_\rho\gamma(u), \quad u \geq 0 \qquad (1.9)$$

where ρ is the positive root of Lundberg's equation (1.7) and T_ρ is the Dickson-Hipp operator defined in (1.1).

Proof. Applying Laplace transform in (1.4), it is obtained

$$\begin{aligned}
c\phi_\delta'(u) &= (\lambda+\delta)\phi_\delta(u) - \lambda\int_0^u \phi_\delta(u-x)f(x)dx - \lambda\gamma(u) && \Longrightarrow \\
c\int_0^\infty e^{-sx}\phi_\delta'(x)dx &= (\lambda+\delta)\int_0^\infty e^{-sx}\phi_\delta(x)dx - \lambda\int_0^\infty e^{-sx}\int_0^x \phi_\delta(x-y)f(y)dydx \\
& -\lambda\int_0^\infty e^{-sx}\gamma(x)dx && \Longrightarrow \\
c\left[s\hat{\phi}_\delta(s) - \phi_\delta(0)\right] &= (\lambda+\delta)\hat{\phi}_\delta(s) - \lambda\,\hat{\phi}_\delta(s)\hat{f}(s) - \lambda\hat{\gamma}(s) && \Longrightarrow \\
\hat{\phi}_\delta(s) &= \frac{c\phi_\delta(0) - \lambda\hat{\gamma}(s)}{cs - (\lambda+\delta) + \lambda\hat{f}(s)}
\end{aligned}$$

Let

$$\hat{\phi}_\delta(s) = \frac{c\phi_\delta(0) - \lambda\hat{\gamma}(s)}{cs - (\lambda+\delta) + \lambda\hat{f}(s)} = \frac{\lambda\hat{\gamma}(s) - c\phi_\delta(0)}{\lambda+\delta - cs - \lambda\hat{f}(s)} = \frac{A(s)}{B(s)} \tag{1.10}$$

The denominator of the second fraction in (1.10) is the Lundberg's fundamental equation described in section 1.5.2. It has been proven that it has a unique positive root symbolized by ρ. So,

$$B(\rho) = 0$$

Moreover, it is known that

$$\hat{\phi}_\delta(s) = \int_0^\infty e^{-sx}\phi_\delta(x)dx \ < \ \infty \qquad \forall \quad s \geq 0$$

Consequently,

$$A(\rho) = 0$$

Otherwise, if $A(\rho) \neq 0$, then (1.10) would give $\hat{\phi}_\delta(\rho) = \infty$. Now, (1.10) is going to be converted into another form in which inverse Laplace transform can be applied.

$$A(\rho) = 0 \quad \Longrightarrow \quad c\phi_\delta(0) = \lambda\hat{\gamma}(\rho) \quad \Longrightarrow$$

$$c\phi_\delta(0) - \lambda\hat{\gamma}(s) = \lambda\hat{\gamma}(\rho) - \lambda\hat{\gamma}(s) \qquad \Longrightarrow$$

$$A(s) = \lambda(s-\rho)\frac{\hat{\gamma}(\rho) - \hat{\gamma}(s)}{s-\rho}$$

Moreover,

$$\begin{aligned}
B(s) &= B(s) - B(\rho) \\
&= cs - (\lambda+\delta) + \lambda\hat{f}(s) - [c\rho - (\lambda+\delta) + \lambda\hat{f}(\rho)] \\
&= (s-\rho)\left[c - \lambda\frac{\hat{f}(\rho)-\hat{f}(s)}{s-\rho}\right]
\end{aligned}$$

Substituting $A(s)$ and $B(s)$ in (1.10), it is obtained

$$\hat{\phi}_\delta(s) = \frac{\lambda(s-\rho)\dfrac{\hat{\gamma}(\rho)-\hat{\gamma}(s)}{s-\rho}}{(s-\rho)\left[c-\lambda\dfrac{\hat{f}(\rho)-\hat{f}(s)}{s-\rho}\right]} = \frac{\lambda\dfrac{\hat{\gamma}(\rho)-\hat{\gamma}(s)}{s-\rho}}{c-\lambda\dfrac{\hat{f}(\rho)-\hat{f}(s)}{s-\rho}}$$

Applying the third property of Dickson-Hipp operator, $\hat{\phi}_\delta(s)$ can be written

$$\hat{\phi}_\delta(s) = \frac{\lambda\hat{T}_\rho\gamma(s)}{c-\lambda\hat{T}_\rho f(s)} \qquad \Longrightarrow$$

$$\hat{\phi}_\delta(s)\left[c-\lambda\hat{T}_\rho f(s)\right] = \lambda\hat{T}_\rho\gamma(s) \qquad \Longrightarrow$$

$$c\hat{\phi}_\delta(s) = \lambda\hat{\phi}_\delta(s)\hat{T}_\rho f(s) + \lambda\hat{T}_\rho\gamma(s)$$

The inverse Laplace transform converts the last equation into

$$c\phi_\delta(u) = \lambda\int_0^u \phi_\delta(u-x)T_\rho f(x)dx + \lambda T_\rho\gamma(u)$$

So, it can be easily received the final form (1.9)

$$\phi_\delta(u) = \frac{\lambda}{c}\int_0^u \phi_\delta(u-x)T_\rho f(x)dx + \frac{\lambda}{c}T_\rho\gamma(u), \quad u \geq 0$$

$\diamondsuit$

Proposition 1.5.3.1. *The probability of ruin $\psi(u)$ satisfies the integral equation*

$$\psi(u) = \frac{\lambda}{c}\int_0^u \psi(u-x)\bar{F}(x)dx + \frac{\lambda}{c}\int_u^\infty \bar{F}(x)dx$$

where F(x) is the distribution function of the individual claim sizes X_i.

Proof. For $\delta = 0$ and $w(x,y) = 1$, the Gerber-Shiu function is reduced to the probability of ruin, i.e. $\phi_0(u) = \psi(u)$. Regarding the root $\rho = \rho(\delta)$, it has been mentioned that $\rho = \rho(0) = 0$. The second property of Dickson-Hipp operator implies

$$T_0 f(x) = \int_x^\infty f(y)dy = \bar{F}(x)$$

Moreover, using the definition (1.5) of $\gamma(x)$, they can be estimated

$$\gamma(u) = \int_u^\infty w(u, y-u)f(y)dy = \int_u^\infty f(y)dy = \bar{F}(u)$$

and

$$T_0\gamma(u) = \int_u^\infty \gamma(x)dx = \int_u^\infty \bar{F}(x)dx$$

So, from (1.9) it can be derived

$$\psi(u) \;=\; \frac{\lambda}{c}\int_0^u \psi(u-x)\bar{F}(x)dx \;+\; \frac{\lambda}{c}\int_u^\infty \bar{F}(x)dx$$

$\diamondsuit$

1.5.4 The defective renewal equation satisfied by Gerber-Shiu function

Lin and Willmot (1999) use the definition of equilibrium distribution function F_e which simplifies the calculations in our study.

Definition 1.5.4.1. *The equilibrium function of the survival function $\bar{F}(x)$ is defined as*

$$F_e(x) \;=\; \frac{\int_0^x \bar{F}(y)dy}{\int_0^\infty \bar{F}(y)dy} \;=\; \frac{\int_0^x \bar{F}(y)dy}{E[X]} \;=\; \frac{\int_0^x \bar{F}(y)dy}{\mu_1}$$

It is actually a cumulative distribution function where

$$\lim_{x\to\infty} F_e(x) = \frac{\int_0^\infty \bar{F}(y)dy}{\int_0^\infty \bar{F}(y)dy} = 1$$

Consider now the respective probability density function, which is defined as

$$f_e(x) \;=\; \frac{d}{dx}F_e(x) \;=\; \frac{\bar{F}(x)}{\mu_1},\ \textit{where}\ \int_0^\infty f_e(x)dx \;=\; 1$$

Lin and Willmot (1999) convert the integral equation (1.9) satisfied by $\phi_\delta(u)$ into a defective renewal equation, by defining some new quantities which facilitate the procedure. We present their work in the following Theorem 1.5.4.1.

Theorem 1.5.4.1. *The Gerber-Shiu function, $\phi_\delta(u)$, satisfies the defective renewal equation*

$$\phi_\delta(u) \;=\; \frac{1}{1+\xi_\delta}\int_0^u \phi_\delta(u-x)g_\delta(x)dx \;+\; \frac{1}{1+\xi_\delta}H_\delta(u), \qquad u \geq 0 \quad (1.11)$$

where the new quantities are defined as below:

- $\dfrac{1}{1+\xi_\delta} \;=\; 1-\dfrac{\delta}{c\rho} \;=\; \dfrac{\lambda}{c}\hat{\bar{F}}(\rho) \;=\; \dfrac{1}{1+\theta}\,\hat{f}_e(\rho),$

  $\rho = \rho(\delta)$, *the positive solution of Lundberg's equation defined in (1.7)*
- $g_\delta(x) = (1+\xi_\delta)\,\dfrac{\lambda}{c}\,T_\rho f(x) = \dfrac{T_\rho f(x)}{\hat{\bar{F}}(\rho)}$
- $\bar{G}_\delta(x) = \dfrac{T_\rho \bar{F}(x)}{\hat{\bar{F}}(\rho)}$
- $H_\delta(u) = (1+\xi_\delta)\,\dfrac{\lambda}{c}\,T_\rho\gamma(u) = \dfrac{T_\rho\gamma(u)}{\hat{\bar{F}}(\rho)}$

Proof. In this proof we are going to define new functions which will be applied in (1.9) and they will give the (1.11). Firstly, consider

$$\frac{\lambda}{c}T_\rho f(x) = Z(x) \qquad (1.12)$$

Integrating Z(x) implies

$$\begin{aligned}\int_0^\infty Z(x)dx \;&=\; \frac{\lambda}{c}\int_0^\infty T_\rho f(x)dx = \frac{\lambda}{c}\int_0^\infty e^{-0x}T_\rho f(x)dx = \frac{\lambda}{c}\hat{T}_\rho f(0)\\ &=\; \frac{\lambda}{c}\frac{\hat{f}(0)-\hat{f}(\rho)}{\rho-0}\end{aligned}$$

resulting to

$$\int_0^\infty Z(x)dx = \frac{\lambda}{c}\left[\frac{1-\hat{f}(\rho)}{\rho}\right] \qquad (1.13)$$

As ρ satisfies Lundberg's equation, we obtain

$$\lambda+\delta-c\rho = \lambda\hat{f}(\rho) \Longrightarrow \lambda(1-\hat{f}(\rho)) = c\rho-\delta$$

Consequently, (1.13) can be written as

$$\int_0^\infty Z(x)dx = \frac{c\rho-\delta}{c\rho} = 1-\frac{\delta}{c\rho}$$

The above integral will be denoted, from now on, by

$$\frac{1}{1+\xi_\delta} = \int_0^\infty Z(x)dx = 1 - \frac{\delta}{c\rho} \tag{1.14}$$

Using (1.14), as well as the concept of equilibrium function, we define

$$G_\delta(x) \;=\; \frac{\displaystyle\int_0^x Z(y)dy}{\displaystyle\int_0^\infty Z(y)dy} = \left(\frac{1}{\displaystyle\int_0^\infty Z(y)dy}\right)\int_0^x Z(y)dy$$

i.e.

$$G_\delta(x) \;=\; (1+\xi_\delta)\int_0^x Z(y)dy \tag{1.15}$$

This is considered as a distribution function where $\lim_{x\to\infty} G_\delta(x) = 1$ and $G_\delta(0) = 0$. The respective probability density function is

$$g_\delta(x) \;=\; \frac{d}{dx}G_\delta(x) = \frac{Z(x)}{\displaystyle\int_0^\infty Z(y)dy} = \left(\frac{1}{\displaystyle\int_0^\infty Z(y)dy}\right)Z(x)$$

Thus,

$$g_\delta(x) = (1+\xi_\delta)Z(x) \tag{1.16}$$

Finally, let

$$H_\delta(x) = (1+\xi_\delta)\frac{\lambda}{c}T_\rho\gamma(x) \tag{1.17}$$

Applying firstly (1.12) in (1.9) yields

$$\phi_\delta(u) = \int_0^u \phi_\delta(u-x)Z(x)dx + \frac{\lambda}{c}T_\rho\gamma(u)$$

Then, substituting (1.16) and (1.17) leads to the desirable result

$$\phi_\delta(u) = \frac{1}{1+\xi_\delta}\int_0^u \phi_\delta(u-x)g_\delta(x)dx + \frac{1}{1+\xi_\delta}H_\delta(u)$$

Now, they will be proven some further mathematical expressions for $\dfrac{1}{1+\xi_\delta}$, $g_\delta(x)$, the survival function $\bar{G}_\delta(x)$ and $H_\delta(u)$. We have that

$$\begin{aligned}
\hat{\bar{F}}(s) &= \int_0^\infty e^{-sx}\bar{F}(x)dx = \int_0^\infty \left(-\frac{e^{-sx}}{s}\right)'\bar{F}(x)dx \\
&= \left[-\frac{e^{-sx}}{s}\bar{F}(x)\right]_{x=0}^\infty - \int_0^\infty \frac{e^{-sx}}{s}f(x)dx \\
&= \frac{1}{s} - \frac{1}{s}\hat{f}(s) = \frac{1-\hat{f}(s)}{s}
\end{aligned}$$

Using the second and third property of Dickson-Hipp operator gives as well

$$\bar{F}(x) = T_0 f(x) \Longrightarrow \hat{\bar{F}}(s) = \hat{T}_0 f(s) = \frac{\hat{f}(s) - \hat{f}(0)}{0 - s} \Longrightarrow \hat{\bar{F}}(s) = \frac{1 - \hat{f}(s)}{s}$$

The Laplace transform of the equilibrium function is equal to

$$\hat{f}_e(s) = \int_0^\infty e^{-sx} f_e(x) dx = \int_0^\infty e^{-sx} \frac{\bar{F}(x)}{\mu_1} dx = \frac{\hat{\bar{F}}(s)}{\mu_1}$$

From (1.13) and (1.14) we obtain

$$\frac{1}{1 + \xi_\delta} = 1 - \frac{\delta}{c\rho} = \frac{\lambda}{c} \frac{1 - \hat{f}(\rho)}{\rho} = \frac{\lambda}{c} \hat{\bar{F}}(\rho) \tag{1.18}$$

Using the definition of security loading factor θ, we have

$$c = (1 + \theta)\lambda\mu_1 \Longrightarrow \frac{\lambda}{c} = \frac{1}{\mu_1(1 + \theta)}$$

and through it we conclude to

$$\frac{1}{1 + \xi_\delta} = \frac{1}{1 + \theta} \frac{\hat{\bar{F}}(\rho)}{\mu_1} = \frac{1}{1 + \theta} \hat{f}_e(\rho) \tag{1.19}$$

From the definitions in (1.12), (1.16) and (1.18), $g_\delta(x)$ can be written as

$$g_\delta(x) = (1 + \xi_\delta) Z(x) = (1 + \xi_\delta) \frac{\lambda}{c} T_\rho f(x) = \frac{T_\rho f(x)}{\hat{\bar{F}}(\rho)} = \frac{e^{\rho x} \displaystyle\int_x^\infty e^{-\rho y} f(y) dy}{\displaystyle\int_0^\infty e^{-\rho y} \bar{F}(y) dy}$$

In the following proposition, we will use the form of

$$g_\delta(x) = \frac{T_\rho f(x)}{\hat{\bar{F}}(\rho)} \tag{1.20}$$

Meanwhile, two more properties of Dickson-Hipp operator $T_r f(x)$ are going to be used, namely

i. Using the 4th property of Dickson-Hipp operator, it can be estimated

$$\begin{aligned} \int_x^\infty T_r f(t) dt &= e^{0x} \int_x^\infty e^{-0t} T_r f(t) dt = T_{0,r} f(x) = \frac{T_r f(x) - T_0 f(x)}{0 - r} \\ &= \frac{\bar{F}(x) - T_r f(x)}{r} \end{aligned}$$

ii. Applying integration by parts in the operator $T_r f(x)$ leads to

$$\begin{aligned} T_r f(x) &= e^{rx}\int_x^\infty e^{-ry} f(y)dy = -e^{rx}\int_x^\infty e^{-ry}\left[\bar{F}(y)\right]' dy \\ &= -e^{rx}\left[e^{-ry}\bar{F}(y)\right]_{y=x}^\infty + e^{rx}\int_x^\infty -re^{-ry}\bar{F}(y)dy \\ &= -e^{rx}\left[0 - e^{-rx}\bar{F}(x)\right] - re^{rx}\int_x^\infty e^{-ry}\bar{F}(y)dy \\ &= \bar{F}(x) - r\, T_r\bar{F}(x) \end{aligned}$$

Subsequently, (i) yields

$$\begin{aligned} \bar{G}_\delta(x) &= \int_x^\infty g_\delta(t)dt = \int_x^\infty \frac{T_\rho f(t)}{\hat{\bar{F}}(\rho)}\, dt = \frac{\int_x^\infty T_\rho f(t)dt}{\hat{\bar{F}}(\rho)} = \frac{\frac{\bar{F}(x) - T_\rho f(x)}{\rho}}{\hat{\bar{F}}(\rho)} \\ &= \frac{\bar{F}(x) - T_\rho f(x)}{\rho\hat{\bar{F}}(\rho)} \end{aligned}$$

whereas, (ii) yields

$$\bar{G}_\delta(x) = \frac{\bar{F}(x) - T_\rho f(x)}{\rho\hat{\bar{F}}(\rho)} = \frac{\bar{F}(x) - \left[\bar{F}(x) - \rho\, T_\rho\bar{F}(x)\right]}{\rho\hat{\bar{F}}(\rho)} = \frac{T_\rho\bar{F}(x)}{\hat{\bar{F}}(\rho)}$$

Now, it has been proven that

$$\frac{1}{1+\xi_\delta} = \frac{\lambda}{c}\hat{\bar{F}}(\rho)$$

So, $H_\delta(u)$ can be calculated by

$$\begin{aligned} H_\delta(u) &= \frac{\lambda}{c}(1+\xi_\delta)T_\rho\gamma(u) = \frac{\lambda}{c}\,\frac{c}{\lambda}\,\frac{1}{\hat{\bar{F}}(\rho)}\, T_\rho\gamma(u) = \frac{T_\rho\gamma(u)}{\hat{\bar{F}}(\rho)} \\ &= \frac{e^{\rho u}\int_u^\infty e^{-\rho x}\int_x^\infty w(x, y-x)f(y)dy\, dx}{\int_0^\infty e^{-\rho y}\bar{F}(y)dy} \end{aligned}$$

$\diamondsuit$

Proposition 1.5.4.1. *The probability of ruin satisfies the defective renewal equation*

$$\psi(u) = \frac{1}{1+\theta}\int_0^u \psi(u-x)f_e(x)dx + \frac{1}{1+\theta}\,\bar{F}_e(u) \qquad u \geq 0 \tag{1.21}$$

Proof. Substituting $\delta = 0$ and $w(x,y) = 1$ in Gerber-Shiu function gives the probability of ruin, i.e. $\phi_0(u) = \psi(u)$. From (1.19) we have

$$\frac{1}{1+\xi_\delta} = \frac{1}{1+\theta}\,\hat{f}_e(\rho(\delta))$$

Moreover, $\delta = 0$ implies that $\rho = \rho(0) = 0$ and

$$\hat{f}_e(0) = \int_0^\infty f_e(x)dx = \int_0^\infty \frac{\bar{F}(x)}{\mu_1}dx = \frac{\int_0^\infty \bar{F}(x)dx}{\mu_1} = 1$$

Consequently,

$$\frac{1}{1+\xi_0} = \frac{1}{1+\theta} \Longleftrightarrow \theta = \xi_0$$

Taking into account the fact that

$$\hat{\hat{F}}(0) = \int_0^\infty \bar{F}(x)dx = \mu_1$$

and using (1.20), we are led to

$$g_0(x) = \frac{T_0 f(x)}{\hat{\hat{F}}(0)} = \frac{\bar{F}(x)}{\mu_1} = f_e(x) \tag{1.22}$$

For $\delta = 0$, (1.17) equals to

$$H_0(u) = (1+\xi_0)\,\frac{\lambda}{c}\,T_0\gamma(u)$$

Observing that

$$\begin{aligned} T_0\gamma(u) &= \int_u^\infty \gamma(y)dy = \int_u^\infty \int_y^\infty w(y, t-y)f(t)dt\,dy = \int_u^\infty \int_y^\infty f(t)dt\,dy \quad (w(x,y)=1) \\ &= \int_u^\infty \bar{F}(y)dy \end{aligned}$$

and

$$c = (1+\theta)\lambda\mu_1 \Longrightarrow (1+\theta)\frac{\lambda}{c} = \frac{1}{\mu_1} \quad \text{where} \quad \xi_0 = \theta$$

we receive

$$H_0(u) = \frac{\int_u^\infty \bar{F}(y)dy}{\mu_1} = \bar{F}_e(u) \tag{1.23}$$

In conclusion, substituting (1.22) and (1.23) in (1.11) implies (1.21).

$\diamondsuit$

Finally, we reach the central Theorem of this chapter, which is the solution of the defective renewal equation satisfied by $\phi_\delta(u)$, through a compound geometric distribution. The Theorem 1.5.4.2, the Proposition 1.5.4.3 and their proofs are based on Lin and Willmot (1999). Furthermore, our analytical presentation of this work is based on E. Chadjikonstantinidis (2016), as well. This result allows us to estimate in section 1.6 the ruin probability $\psi(u)$ and the Laplace transform for ruin time T, when the claims X_i follow an exponential distribution.

Theorem 1.5.4.2. *The solution of the defective renewal equation, which is satisfied by Gerber-Shiu function $\phi_\delta(u)$ and described in Theorem 1.5.4.1, is given by*

$$\phi_\delta(u) = \frac{1}{\xi_\delta}\int_0^u H_\delta(u-x)dK_\delta(x) + \frac{1}{1+\xi_\delta}H_\delta(u), \quad u \geq 0 \tag{1.24}$$

or

$$\phi_\delta(u) = -\frac{1}{\xi_\delta}\int_0^u \bar{K}_\delta(x)H_\delta^{'}(u-x)dx - \frac{1}{\xi_\delta}H_\delta(0)\bar{K}_\delta(u) + \frac{1}{\xi_\delta}H_\delta(u), \quad u \geq 0 \tag{1.25}$$

where

$$\bar{K}_\delta(u) = \sum_{n=1}^{\infty}\frac{\xi_\delta}{1+\xi_\delta}\left(\frac{1}{1+\xi_\delta}\right)^n \bar{G}_\delta^{*n}(u), \quad u \geq 0$$

Proof. In Theorem 1.5.4.1 we have seen that Gerber-Shiu function satisfies the defective renewal equation (1.11)

$$\phi_\delta(u) \;=\; \frac{1}{1+\xi_\delta}\int_0^u \phi_\delta(u-x)g_\delta(x)dx \;+\; \frac{1}{1+\xi_\delta}H_\delta(u), \qquad u \geq 0$$

In Proposition 1.4.1 we have solved the defective renewal equation through a compound geometric distribution. So, based on it, let

$$S = \begin{cases} \sum_{i=1}^{M} W_i, & M \geq 1 \\ 0, & M = 0 \end{cases} \tag{1.26}$$

be a compound geometric random variable, where $K_\delta(u)$ is the distribution function of S, $G_\delta(u)$ is the distribution function of W_i with $G_\delta(0) = 0$ and $M \sim \text{Geom}(\frac{\xi_\delta}{1+\xi_\delta})$. From Lemma 1.4.1(i), the survival function of S, $\bar{K}_\delta(u)$, equals to

$$\bar{K}_\delta(u) = \sum_{n=1}^{\infty}\frac{\xi_\delta}{1+\xi_\delta}\left(\frac{1}{1+\xi_\delta}\right)^n \bar{G}_\delta^{*n}(u), \quad u \geq 0$$

Hence, regarding the solutions given by Proposition 1.4.1, if we replace $\varphi = \frac{1}{1+\xi_\delta} < 1$, $dF(x) = dG_\delta(x)$, $r(x) = \frac{1}{1+\xi_\delta} H_\delta(x)$, $\bar{G}(x) = \bar{K}_\delta(x)$ in (1.2) and (1.3), the solutions of $\phi_\delta(u)$ will be

$$\begin{aligned}
\phi_\delta(u) &= \frac{1}{1-\phi}\int_0^u r(u-x)dK_\delta(x) + r(u) \\
&= \frac{1+\xi_\delta}{\xi_\delta}\int_0^u \frac{1}{1+\xi_\delta} H_\delta(u-x)dK_\delta(x) + \frac{1}{1+\xi_\delta}H_\delta(u) \\
&= \frac{1}{\xi_\delta}\int_0^u H_\delta(u-x)dK_\delta(x) + \frac{1}{1+\xi_\delta}H_\delta(u)
\end{aligned}$$

or

$$\begin{aligned}
\phi_\delta(u) &= \frac{1}{1-\varphi}r(u) - \frac{r(0)}{1-\varphi}\bar{G}(u) - \frac{1}{1-\varphi}\int_0^u r^{'}(u-x)\bar{G}(x)dx \\
&= \frac{1+\xi_\delta}{\xi_\delta}\frac{1}{1+\xi_\delta}H_\delta(u) - \frac{1+\xi_\delta}{\xi_\delta}\frac{1}{1+\xi_\delta}H_\delta(0)\bar{K}_\delta(u) \\
&\quad -\frac{1+\xi_\delta}{\xi_\delta}\int_0^u \frac{1}{1+\xi_\delta}H_\delta^{'}(u-x)\bar{K}_\delta(x)dx \\
&= \frac{1}{\xi_\delta}H_\delta(u) - \frac{1}{\xi_\delta}H_\delta(0)\bar{K}_\delta(u) - \frac{1}{\xi_\delta}\int_0^u H_\delta^{'}(u-x)\bar{K}_\delta(x)dx
\end{aligned}$$

$\diamondsuit$

Proposition 1.5.4.2. *Putting $\delta = 0$ and $w(x,y) = 1$, from Theorem 1.5.4.2, we obtain the solution of the defective renewal equation satisfied by the ruin probability $\psi(u)$ i.e.*

$$\psi(u) = \frac{1}{\theta}\int_0^u \bar{F}_e(u-x)dK_0(x) + \frac{1}{1+\theta}\bar{F}_e(u), \quad u \geq 0 \tag{1.27}$$

or

$$\psi(u) = -\frac{1}{\theta}\int_0^u \bar{K}_0(x)\bar{F}_e^{'}(u-x)dx - \frac{1}{\theta}\bar{F}_e(0)\bar{K}_0(u) + \frac{1}{\theta}\bar{F}_e(u), \quad u \geq 0 \tag{1.28}$$

where

$$\bar{K}_0(u) = \sum_{n=1}^{\infty}\frac{\theta}{1+\theta}\left(\frac{1}{1+\theta}\right)^n \bar{F}_e^{*n}(u), \quad u \geq 0$$

Proof. It is a direct result from Theorem 1.5.4.2, considering $\xi_0 = \theta$, (1.22) with $g_0(x) = f_e(x) \Longrightarrow \bar{G}_0(x) = \bar{F}_e(x)$ and (1.23) with $H_0(u) = \bar{F}_e(u)$.

◊

Proposition 1.5.4.3. *When $w(x, y) = 1$, the Laplace transform for the time of ruin T is equal to the survival function $\bar{K}_\delta(u)$ of the compound geometric distribution defined in (1.26), i.e.*

$$\bar{K}_\delta(u) = E\left[e^{-\delta T} I(T < \infty)|U(0) = u\right] = \phi_\delta(u) \tag{1.29}$$

Remark. *If $\delta = 0$ as well, the probability of ruin will be*

$$\psi(u) = \bar{K}_0(u)$$

Proof. Consider, once more, the compound geometric distribution S defined in (1.26) and the replacements taken place in the above proof in order for Proposition 1.4.1 to be used. Regarding the property (iv) in Lemma 1.4.1, the Laplace transform of $\bar{K}_\delta(u)$ equals to

$$\hat{\bar{K}}_\delta(s) = \frac{1}{1+\xi_\delta}\hat{\bar{K}}_\delta(s)\hat{g}_\delta(s) + \frac{1}{1+\xi_\delta}\hat{\bar{G}}_\delta(s)$$

The inverse Laplace transform yields

$$\bar{K}_\delta(u) = \frac{1}{1+\xi_\delta}\int_o^u \bar{K}_\delta(u-x)g_\delta(x)dx + \frac{1}{1+\xi_\delta}\bar{G}_\delta(u), \quad u \geq 0 \tag{1.30}$$

Now, we have seen that the Gerber-Shiu function $\phi_\delta(u)$, for $w(x, y) = 1$, is reduced to the Laplace transform of the time of ruin T. So,

$$\gamma(x) = \int_x^\infty w(x, y-x)f(y)dy = \int_x^\infty f(y)dy = \bar{F}(x)$$

and

$$H_\delta(u) = \frac{T_\rho\gamma(u)}{\hat{\bar{F}}(\rho)} = \frac{T_\rho\bar{F}(u)}{\hat{\bar{F}}(\rho)} = \bar{G}_\delta(u)$$

Substituting the above in (1.11) implies

$$\phi_\delta(u) \;=\; \frac{1}{1+\xi_\delta}\int_0^u \phi_\delta(u-x)g_\delta(x)dx \;+\; \frac{1}{1+\xi_\delta}\bar{G}_\delta(u) \tag{1.31}$$

From (1.30) and (1.31) we conclude to the proof, i.e. $\bar{K}_\delta(u) = \phi_\delta(u)$ when $w(x, y) = 1$.

Now, considering $w(x, y) = 1$ and $\delta = 0$, (1.29) is reduced to

$$\bar{K}_0(u) \;=\; E\left[I(T < \infty)|U(0) = u\right] = Pr\left[T < \infty|U(0) = u\right] = \psi(u)$$

We have seen that $\delta = 0 \Longrightarrow \rho = 0$, so $\xi_0 = \theta$ and

$$\bar{G}_0(x) = \frac{T_0 \bar{F}(x)}{\hat{\bar{F}}(0)} = \frac{\displaystyle\int_x^\infty \bar{F}(y)dy}{\displaystyle\int_0^\infty \bar{F}(y)dy} = \bar{F}_e(x)$$

Briefly, the probability of ruin $\psi(u)$ is the survival function $\bar{K}_0(u)$ of a compound geometric random variable,

$$L = \begin{cases} \displaystyle\sum_{i=1}^{K} L_i, & K \geq 1 \\ 0, & K = 0 \end{cases}$$

where $\bar{G}_0(u) = \bar{F}_e(u)$ is the distribution function of L_i and $K \sim \text{Geom}(\dfrac{\xi_0}{1+\xi_0}) = \text{Geom}(\dfrac{\theta}{1+\theta})$.

$\diamondsuit$

1.6 Exponential Claims

At this point, we are going to apply our study in exponential claims providing explicit formulas for the root of Lundberg's equation, the ruin probability, the Laplace transform of ruin time and the Gerber-Shiu function. Moreover, by solving a numerical example we present the respective results for the aforementioned quantities.

Proposition 1.6.1. *Let N(t) be the number of claims in $[0, t]$, which follows a Poisson process with mean $\lambda > 0$, c the premium rate per unit time, θ the security loading factor and δ the force of interest. If the size of claims X_i follows an Exponential distribution with mean $\frac{1}{\beta} > 0$, we will have*

i. $\dfrac{1}{1+\xi_\delta} = \dfrac{\beta}{(1+\theta)(\beta+\rho)}$

ii. $g_\delta(x) = \beta e^{-\beta x}$

iii. $\bar{G}_\delta(x) = e^{-\beta x}$

iv. $\bar{K}_\delta(x) = \dfrac{1}{1+\xi_\delta} e^{-\beta \frac{\xi_\delta}{1+\xi_\delta} x} = \dfrac{\beta}{(1+\theta)(\beta+\rho)} e^{-\beta \frac{\rho + (\beta+\rho)\theta}{(1+\theta)(\beta+\rho)} x}$

v. The probability of ruin equals to $\psi(u) = \dfrac{1}{1+\theta} e^{-\frac{\beta\theta}{1+\theta} u}, \quad u \geq 0$

Moreover, let $w(x, y) = e^{-\rho y}$ be the penalty function and ρ the root of Lundberg's equation. Then, *vi. To be calculated an analytical expression for the root ρ*

vii. To be found the Gerber-Shiu function

Proof. As $X \sim \mathrm{Exp}(\beta)$, we have
$$f(x) = \beta e^{-\beta x}, \quad \bar{F}(x) = e^{-\beta x}, \quad \hat{f}(s) = \frac{\beta}{\beta + s}, \quad M_X(t) = \frac{\beta}{\beta - t}, \ t < \beta$$

and
$$\hat{\bar{F}}(s) = \int_0^\infty e^{-sx} e^{-\beta x} dx = \left| -\frac{e^{-(\beta+s)x}}{\beta + s} \right|_{x=0}^{\infty} = \frac{1}{\beta + s}$$

As far as the equilibrium function is concerned, we have
$$f_e(x) = \frac{\bar{F}(x)}{E(X)} = \frac{e^{-\beta x}}{\frac{1}{\beta}} = \beta e^{-\beta x}$$

which is the probability density function of $\mathrm{Exp}(\beta)$. So,
$$\bar{F}_e(x) = e^{-\beta x} \ \text{ and } \ \hat{f}_e(s) = \frac{\beta}{\beta + s}$$

Moreover, the following Dickson-Hipp operators will be needed

$$\begin{aligned}
(1)\ T_\rho f(x) &= e^{\rho x} \int_x^\infty e^{-\rho y} f(y) dy = e^{\rho x} \int_x^\infty e^{-\rho y} \beta e^{-\beta y} dy \\
&= \beta e^{\rho x} \int_x^\infty e^{-(\rho+\beta) y} dy = \beta e^{\rho x} \left[-\frac{e^{-(\rho+\beta) y}}{\rho + \beta} \right]_{y=x}^{\infty} \\
&= \beta e^{\rho x} \frac{e^{-(\rho+\beta) x}}{\rho + \beta} = \frac{\beta e^{-\beta x}}{\beta + \rho}
\end{aligned}$$

$$\begin{aligned}
(2)\ T_\rho \bar{F}(x) &= e^{\rho x} \int_x^\infty e^{-\rho y} \bar{F}(y) dy = e^{\rho x} \int_x^\infty e^{-\rho y} e^{-\beta y} dy \\
&= e^{\rho x} \int_x^\infty e^{-(\rho+\beta) y} dy = e^{\rho x} \left[-\frac{e^{-(\rho+\beta) y}}{\rho + \beta} \right]_{y=x}^{\infty} \\
&= e^{\rho x} \frac{e^{-(\rho+\beta) x}}{\rho + \beta} = \frac{e^{-\beta x}}{\beta + \rho}
\end{aligned}$$

Consequently, if ρ is the positive root of Lundberg's equation, we will have

$$\begin{aligned}
\text{i.} \quad \frac{1}{1 + \xi_\delta} &= \frac{1}{1 + \theta} \hat{f}_e(\rho) = \frac{1}{1 + \theta} \frac{\beta}{\beta + \rho} = \frac{\beta}{(1 + \theta)(\beta + \rho)} \\
\frac{\xi_\delta}{1 + \xi_\delta} &= 1 - \frac{1}{1 + \xi_\delta} = 1 - \frac{\beta}{(1 + \theta)(\beta + \rho)} = \frac{\rho + (\beta + \rho)\theta}{(1 + \theta)(\beta + \rho)}
\end{aligned}$$

ii. $$g_\delta(x) = \frac{T_\rho f(x)}{\hat{\bar{F}}(\rho)} = \frac{\dfrac{\beta e^{-\beta x}}{\beta+\rho}}{\dfrac{1}{\beta+\rho}} = \beta e^{-\beta x} \ ,$$

which is the density function of Exp(β)

iii. $$\bar{G}_\delta(x) = \frac{T_\rho \bar{F}(x)}{\hat{\bar{F}}(\rho)} = \frac{\dfrac{e^{-\beta x}}{\beta+\rho}}{\dfrac{1}{\beta+\rho}} = e^{-\beta x}$$

which is the survival function of Exp(β)

iv. It has been proven that $\bar{K}_\delta(x)$ is the survival function of a compound geometric random variable S,

$$S = \begin{cases} \sum_{i=1}^{M} W_i, & M \geq 1 \\ 0, & M = 0 \end{cases}$$

where $G_\delta(x)$ is the distribution function of W_i with $G_\delta(0) = 0$ and $M \sim \text{Geom}(\frac{\xi_\delta}{1+\xi_\delta})$. From Lemma 1.4.2, if we substitute

$$p = \frac{\xi_\delta}{1+\xi_\delta} \Longrightarrow q = 1 - p = \frac{1}{1+\xi_\delta}$$

and

$$\bar{F}_X(x) = \bar{F}(x) = \bar{G}_\delta(x)$$

it is obtained

$$\bar{K}_\delta(x) = q\ \bar{F}_Y(x) = \frac{1}{1+\xi_\delta}\bar{F}_Y(x) \tag{1.32}$$

where

$$M_Y(t) = \frac{p\ M_W(t)}{1 - q\ M_W(t)}$$

In (iii.) we have seen that $\bar{G}_\delta(x) = e^{-\beta x}$, which implies $W \sim \text{Exp}(\beta) \Longrightarrow M_W(t) = \frac{\beta}{\beta - t}, \quad t < \beta$. Hence,

$$M_Y(t) = \frac{p\ M_W(t)}{1 - q\ M_W(t)} = \frac{p\ \dfrac{\beta}{\beta - t}}{1 - q\ \dfrac{\beta}{\beta - t}} = \frac{p\beta}{(1-q)\beta - t} = \frac{p\beta}{p\beta - t}$$

As a result, $Y \sim \text{Exp}(p\beta)$ which implies

$$\bar{F}_Y(x) = e^{-\beta p x}$$

Substituting them in (1.32), we obtain

$$\bar{K}_\delta(x) = q\,\bar{F}_Y(x) = q\,e^{-\beta p x} = \frac{1}{1+\xi_\delta} e^{-\beta(\frac{\xi_\delta}{1+\xi_\delta})x} \Longrightarrow$$

$$\bar{K}_\delta(x) = \frac{\beta}{(1+\theta)(\beta+\rho)}\; e^{-\beta\frac{\rho+(\beta+\rho)\theta}{(1+\theta)(\beta+\rho)}x} \qquad (1.33)$$

v. From Proposition 1.5.4.3, the probability of ruin is given by $\bar{K}_\delta(u)$ when $\delta = 0$. Then, we have $\rho = 0$, as well. So, from (1.33) we obtain that

$$\psi(u) = \bar{K}_0(u) = \frac{1}{1+\theta} e^{-\frac{\beta\theta}{1+\theta}u} \qquad \forall\, u \geq 0 \qquad (1.34)$$

Comment. We are led to the same result for $\psi(u)$ as above, by using the formula (1.28) of Proposition 1.5.4.2. Indeed, substituting $\bar{F}_e(x) = e^{-\beta x}$ and $\bar{K}_0(u) = \dfrac{1}{1+\theta} e^{-\frac{\beta\theta}{1+\theta}u}$ in (1.28) implies

$$\begin{aligned}
\psi(u) &= -\frac{1}{\theta}\int_0^u \bar{K}_0(x)\bar{F}_e^{'}(u-x)dx - \frac{1}{\theta}\bar{F}_e(0)\bar{K}_0(u) + \frac{1}{\theta}\bar{F}_e(u) \\
&= \frac{1}{\theta}\int_0^u \beta e^{-\beta(u-x)}\frac{1}{1+\theta}\; e^{-\frac{\beta\theta}{1+\theta}x}dx - \frac{1}{\theta}\bar{K}_0(u) + \frac{1}{\theta}e^{-\beta u} \\
&= \frac{\beta}{\theta(1+\theta)}e^{-\beta u}\int_0^u e^{\left(\beta-\frac{\beta\theta}{1+\theta}\right)x}dx - \frac{1}{\theta}\bar{K}_0(u) + \frac{1}{\theta}e^{-\beta u} \\
&= \frac{\beta}{\theta(1+\theta)}e^{-\beta u}\int_0^u e^{\left(\frac{\beta}{1+\theta}\right)x}dx - \frac{1}{\theta}\bar{K}_0(u) + \frac{1}{\theta}e^{-\beta u} \\
&= \frac{\beta}{\theta(1+\theta)}e^{-\beta u}\left[\tfrac{1+\theta}{\beta}\; e^{\left(\frac{\beta}{1+\theta}\right)x}\right]_{x=0}^{u} - \frac{1}{\theta}\bar{K}_0(u) + \frac{1}{\theta}e^{-\beta u} \\
&= \frac{1}{\theta}e^{-\beta u}\left[e^{\left(\frac{\beta}{1+\theta}\right)u} \;-\; 1\right] - \frac{1}{\theta}\bar{K}_0(u) + \frac{1}{\theta}e^{-\beta u} \\
&= \frac{1}{\theta}\; e^{-\left(\frac{\beta\theta}{1+\theta}\right)u} \;-\; \frac{1}{\theta}\bar{K}_0(u) \\
&= \frac{1}{\theta}\;(1+\theta)\;\bar{K}_0(u) \;-\; \frac{1}{\theta}\bar{K}_0(u) \\
&= \bar{K}_0(u)
\end{aligned}$$

vi. Solving the Lundberg's equation with respect to s yields

$$l(s) = \lambda \hat{f}(s) \Longrightarrow \lambda + \delta - cs = \lambda \frac{\beta}{\beta + s} \Longrightarrow$$

$$\lambda(\beta + s) + \delta(\beta + s) - cs(\beta + s) = \lambda\beta \Longrightarrow$$

$$c\, s^2 + (c\,\beta - \delta - \lambda)\, s - \delta\beta = 0$$

This is a quadratic equation of s, so the two algebraic solutions are given by

$$s_{1,2} = \frac{-(c\,\beta - \delta - \lambda) \pm \sqrt{(c\,\beta - \delta - \lambda)^2 + 4c\delta\beta}}{2c} \tag{1.35}$$

The positive one is the root ρ whereas, by putting $\delta = 0$ the negative one is the adjustment coefficient R (at absolute value).

vii. Firstly, we should calculate the $H_\delta(u)$ provided that $w(x, y) = e^{-\rho y}$. So,

$$\begin{aligned}
T_\rho\gamma(u) &= e^{\rho u} \int_u^\infty e^{-\rho x}\gamma(x)dx = e^{\rho u} \int_u^\infty e^{-\rho x} \int_x^\infty w(x, y - x) f(y) dy\, dx \\
&= e^{\rho u} \int_u^\infty e^{-\rho x} \int_x^\infty e^{-\rho(y-x)} \beta e^{-\beta y} dy\, dx \\
&= \beta e^{\rho u} \int_u^\infty \int_x^\infty e^{-(\rho+\beta)y} dy\, dx = \beta e^{\rho u} \int_u^\infty \left[-\frac{e^{-(\rho+\beta)y}}{\rho + \beta} \right]_{y=x}^\infty dx \\
&= \beta e^{\rho u} \int_u^\infty \frac{e^{-(\rho+\beta)x}}{\rho + \beta} dx = \frac{\beta}{\beta + \rho}\, e^{\rho u} \left[-\frac{e^{-(\rho+\beta)x}}{\rho + \beta} \right]_{x=u}^\infty \\
&= \frac{\beta}{\beta + \rho}\, e^{\rho u}\, \frac{e^{-(\rho+\beta)u}}{\rho + \beta} = \frac{\beta e^{-\beta u}}{(\rho + \beta)^2}
\end{aligned}$$

Consequently,

$$H_\delta(u) = \frac{T_\rho\gamma(u)}{\hat{\bar{F}}(\rho)} = \frac{\dfrac{\beta e^{-\beta u}}{(\rho + \beta)^2}}{\dfrac{1}{\beta + \rho}} = \frac{\beta e^{-\beta u}}{\beta + \rho}$$

In order to find the Gerber-Shiu function, we will use the solution (1.25) in Theorem 1.5.4.2, i.e.

$$\phi_\delta(u) = -\frac{1}{\xi_\delta} \int_0^u \bar{K}_\delta(u - x) H_\delta'(x) dx - \frac{1}{\xi_\delta} H_\delta(0) \bar{K}_\delta(u) + \frac{1}{\xi_\delta} H_\delta(u) \tag{1.36}$$

To facilitate the procedure, we denote

$$\varphi = \frac{1}{1+\xi_\delta} \quad \text{and} \quad 1-\varphi = \frac{\xi_\delta}{1+\xi_\delta}$$

So,

$$\bar{K}_\delta(x) = \frac{1}{1+\xi_\delta} e^{-\beta(\frac{\xi_\delta}{1+\xi_\delta})x} = \varphi e^{-\beta(1-\varphi)x}$$

We are going to estimate separately the components of (1.36). Thus,

$$H_\delta^{'}(x) = -\frac{\beta^2 e^{-\beta x}}{\beta+\rho} \qquad \text{and} \quad H_\delta(0) = \frac{\beta}{\beta+\rho}$$

and

$$\begin{aligned}
\int_0^u \bar{K}_\delta(u-x) H_\delta^{'}(x)dx &= \int_0^u \varphi e^{-\beta(1-\varphi)(u-x)}(-\frac{\beta^2 e^{-\beta x}}{\beta+\rho})dx \\
&= -\frac{\beta^2}{\beta+\rho}\varphi e^{-\beta(1-\varphi)u} \int_0^u e^{\beta(1-\varphi)x} e^{-\beta x} dx \\
&= -\frac{\beta^2}{\beta+\rho}\varphi e^{-\beta(1-\varphi)u} \int_0^u e^{-\beta\varphi x} dx \\
&= -\frac{\beta^2}{\beta+\rho}\varphi e^{-\beta(1-\varphi)u} \frac{1}{\beta\varphi} \int_0^u \beta\varphi e^{-\beta\varphi x} dx \\
&= -\frac{\beta^2}{\beta+\rho}\varphi e^{-\beta(1-\varphi)u} \frac{1}{\beta\varphi}(1-e^{-\beta\varphi u}) \\
&= -\frac{\beta}{\beta+\rho} e^{-\beta(1-\varphi)u}(1-e^{-\beta\varphi u}) \\
&= -\frac{\beta}{\beta+\rho} e^{-\beta(1-\varphi)u} + \frac{\beta}{\beta+\rho} e^{-\beta u}
\end{aligned}$$

As a result, (1.36) is written as

$$\begin{aligned}
\phi_\delta(u) &= -\frac{1}{\xi_\delta}\left[-\frac{\beta}{\beta+\rho}e^{-\beta(1-\varphi)u} + \frac{\beta}{\beta+\rho}e^{-\beta u}\right] - \frac{1}{\xi_\delta}\frac{\beta}{\beta+\rho}\varphi e^{-\beta(1-\varphi)x} \\
&\quad + \frac{1}{\xi_\delta}\frac{\beta e^{-\beta u}}{\beta+\rho} \\
&= \frac{1}{\xi_\delta}\frac{\beta}{\beta+\rho}e^{-\beta(1-\varphi)u} - \frac{1}{\xi_\delta}\frac{\beta}{\beta+\rho}e^{-\beta u} - \frac{1}{\xi_\delta}\frac{\beta}{\beta+\rho}\varphi e^{-\beta(1-\varphi)x} \\
&\quad + \frac{1}{\xi_\delta}\frac{\beta e^{-\beta u}}{\beta+\rho} \Longrightarrow
\end{aligned}$$

$$\phi_\delta(u) \;=\; (1-\varphi)\frac{1}{\xi_\delta}\frac{\beta}{\beta+\rho}e^{-\beta(1-\varphi)u} = \frac{\xi_\delta}{1+\xi_\delta}\frac{1}{\xi_\delta}\frac{\beta}{\beta+\rho}e^{-\beta(\frac{\xi_\delta}{1+\xi_\delta})u}$$

Substituting $\dfrac{\xi_\delta}{1+\xi_\delta}$ and $\dfrac{1}{1+\xi_\delta}$, we finally obtain

$$\phi_\delta(u) = \frac{\beta}{\beta+\rho}\,\frac{\beta}{(1+\theta)(\beta+\rho)}\; e^{-\beta\left(\frac{\rho+(\beta+\rho)\theta}{(1+\theta)(\beta+\rho)}\right)u} \tag{1.37}$$

$\diamondsuit$

Example 1.6.1. *We assume that the mean of Poisson process is $\lambda = 4$, the premium rate $c = 2$, the force of interest $\delta = 1$, $X \sim Exp(3)$ and the penalty function $w(x, y) = e^{-\rho y}$, where ρ is the root of Lundberg's equation. They are going to be calculated*

i. the root ρ of Lundberg's equation and we will give a graph depiction of it. Furthermore, it will be presented by a graph the fact that $\rho = \rho(\delta)$ is an increasing function of δ

ii. the Laplace transform for the time of ruin T

iii. the probability of ruin

iv. the Gerber-Shiu function

Solution. According to the data we have

$$f(x) = 3e^{-3x}, \quad \mu_1 = \frac{1}{3}, \quad \hat{f}(s) = \frac{3}{3+s} \text{ and } M_X(t) = \frac{3}{3-t},\ t < 3$$

Firstly, it is observed that the net profit condition is satisfied

$$c = 2 > \frac{4}{3} = \lambda\,\mu_1$$

The security loading factor is equal to

$$\theta = \frac{c}{\lambda\mu_1} - 1 = \frac{1}{2}$$

Using the formula (1.35) for the root ρ yields

$$s_1 = \frac{-(c\,\beta-\delta-\lambda)+\sqrt{(c\,\beta-\delta-\lambda)^2+4c\delta\beta}}{2c} = 1$$

$$s_2 = \frac{-(c\,\beta-\delta-\lambda)-\sqrt{(c\,\beta-\delta-\lambda)^2+4c\delta\beta}}{2c} = -1.5$$

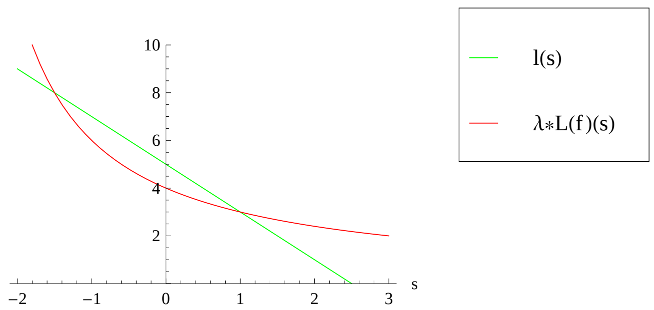


Figure 1.7: The root ρ

The positive one is the root of Lundberg's equation, so $\rho = 1$. The Figure 1.7 of $l(s) = \lambda + \delta - cs$ and $\lambda \hat{f}(s)$ depicts this solution. Remark that if $\delta = 0$, we will obtain the adjustment coefficient R, which is the negative solution (at absolute value) between the above formulas. Thus, for $\delta = 0$ we have

$$s_1 = \frac{-(c\,\beta - \lambda) + \sqrt{(c\,\beta - \lambda)^2}}{2c} = 0, \ \ s_2 = \frac{-(c\,\beta - \lambda) - \sqrt{(c\,\beta - \lambda)^2}}{2c} = -1$$

So, R = 1. The same result for R is given by solving the equation

$$M_X(t) = 1 + (1+\theta)\mu_1\, t \Longrightarrow \frac{3}{3-t} = 1 + \frac{3}{2}\,\frac{1}{3}\, t \Longrightarrow t = 0 \quad or \quad R = t = 1$$

Furthermore, considering the root ρ as function of δ, we have

$$\rho(\delta) = \frac{-(c\,\beta - \delta - \lambda) + \sqrt{(c\,\beta - \delta - \lambda)^2 + 4c\delta\beta}}{2c} = \frac{-(2-\delta) + \sqrt{(2-\delta)^2 + 24\delta}}{4}$$

and its developing over δ can be observed in Figure 1.8, verifying that it is an increasing function of δ.

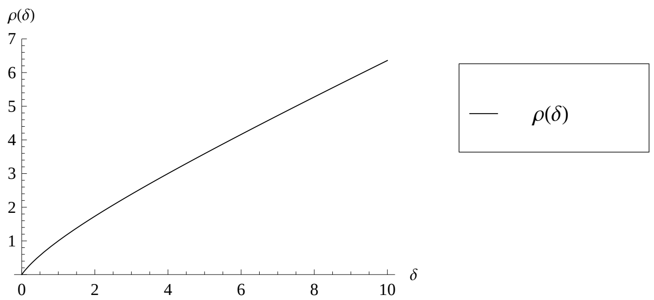


Figure 1.8: The increase of $\rho(\delta)$

In Proposition 1.5.4.3, we have proven that the Laplace transform for the time of ruin T is equal to $\bar{K}_\delta(u)$, which is given by the formula (1.33). Thus, in our exercise we have

$$\bar{K}_\delta(x) = \frac{\beta}{(1+\theta)(\beta+\rho)} e^{-\beta \frac{\rho+(\beta+\rho)\theta}{(1+\theta)(\beta+\rho)} x} \;=\; 0.5\; e^{-1.5x}$$

In Figure 1.9 we can observe that $\bar{K}_\delta(u)$ has a decreasing drift to 0, while the initial surplus u increases.

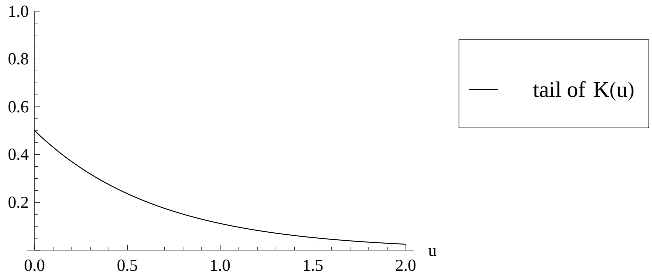


Figure 1.9: The Laplace transform of T

Regarding the same Proposition 1.5.4.3 and the formula (1.34), the probability of ruin is given by

$$\psi(u) = \bar{K}_0(u) = \frac{1}{1+\theta} e^{-\frac{\beta\theta}{1+\theta} u} = \frac{2}{3}\; e^{-u} \qquad \forall\; u \geq 0$$

Its developing, while initial surplus u increases, is depicted by Figure 1.10. Remark that it is

$$\psi(u) = \psi(0)\; e^{-Ru} \qquad \forall\; u \geq 0$$

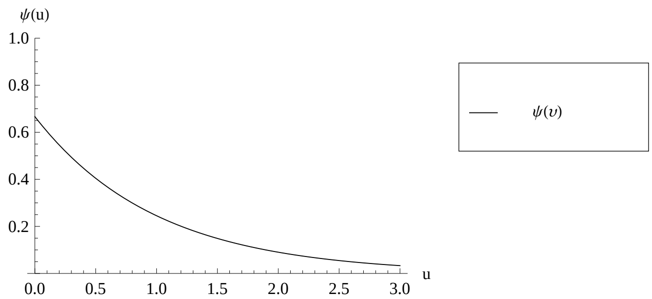


Figure 1.10: The probability of ruin $\psi(u)$

Comment. We are led to the same result for $\psi(u)$ using the formula (1.28) of Proposition 1.5.4.2, as we have already proven.

Finally, the Gerber-Shiu function is given by the formula (1.37), i.e.

$$\phi_\delta(u) = \frac{\beta}{\beta+\rho}\,\frac{\beta}{(1+\theta)(\beta+\rho)}\; e^{-\beta\left(\frac{\rho+(\beta+\rho)\theta}{(1+\theta)(\beta+\rho)}\right)u} \;=\; 0.375\, e^{-1.5u}$$

and Figure 1.11 depicts its movement according to the values of the initial surplus u.

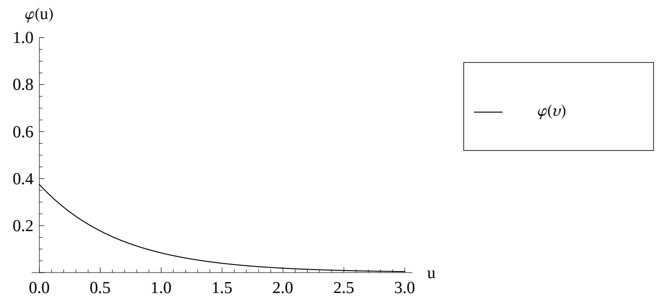


Figure 1.11: The Gerber-Shiu function $\phi_\delta(u)$

Remark 1.6.1. *We are going to present the Laplace transform, $\bar{K}_\delta(x)$, of ruin time T and the Gerber-Shiu function, $\phi_\delta(u)$, for four different values of the discounting interest force δ, namely for $\delta_1 = 0.2,\ \delta_2 = 0.6,\ \delta_3 = 1,\ \delta_4 =$ 1.5.*

Solution. All the necessary calculations have been conducted in Mathematica program, so we will give only the final results by explaining briefly the following steps. Firstly, solving the Lundberg's fundamental equation (1.7),

$$\lambda + \delta_i - cs \;=\; \lambda \hat{f}(s)$$

yields the following roots ρ for each $\delta_i,\ i = 1,\ 2,\ 3,\ 4,$

$$\rho(\delta) = \begin{cases} 0.258872, & \delta_1 = 0.2 \\ 0.661187, & \delta_2 = 0.6 \\ 1, & \delta_3 = 1 \\ 1.3802, & \delta_4 = 1.5 \end{cases}$$

Substituting them in (1.33), we obtain the corresponding forms of the Laplace transform, $\bar{K}_\delta(u)$, of ruin time T, namely

$$\bar{K}_\delta(x) = \frac{\beta}{(1+\theta)(\beta+\rho)}\, e^{-\beta\frac{\rho+(\beta+\rho)\theta}{(1+\theta)(\beta+\rho)}x} = \begin{cases} 0.613709e^{-1.15887x}, & \delta_1 = 0.2 \\ 0.546271e^{-1.36119x}, & \delta_2 = 0.6 \\ 0.5e^{-1.5x}, & \delta_3 = 1 \\ 0.4566e^{-1.6302x}, & \delta_4 = 1.5 \end{cases}$$

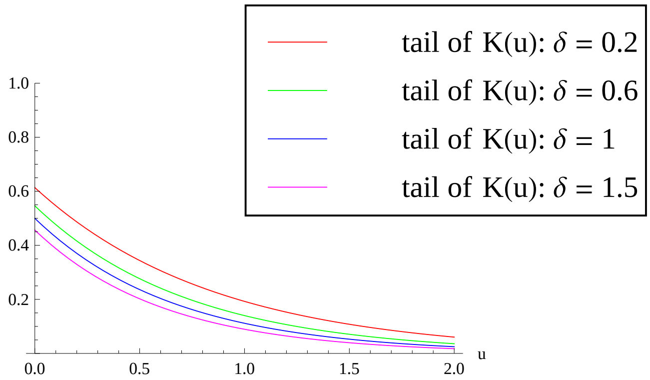


Figure 1.12: The Laplace transform of T

It can be observed in Figure 1.12 that the greater values received by the discounting interest force δ, the less the Laplace transform $\bar{K}_\delta(u)$ is, which is a completely reasonable result considering the role of the discounting factor. Finally, regarding the Gerber-Shiu function $\phi_\delta(u)$, from (1.37) we obtain

$$\phi_\delta(u) = \frac{\beta}{\beta+\rho}\,\frac{\beta}{(1+\theta)(\beta+\rho)}\, e^{-\beta\left(\frac{\rho+(\beta+\rho)\theta}{(1+\theta)(\beta+\rho)}\right)u} = \begin{cases} 0.564959e^{-1.15887u}, & \delta_1 = 0.2 \\ 0.447618e^{-1.36119u}, & \delta_2 = 0.6 \\ 0.375e^{-1.5u}, & \delta_3 = 1 \\ 0.312726e^{-1.6302u}, & \delta_4 = 1.5 \end{cases}$$

where, as far as the discounting interest force δ is concerned, $\phi_\delta(u)$ appears the same tendency with $\bar{K}_\delta(x)$, which is depicted by Figure 1.13.

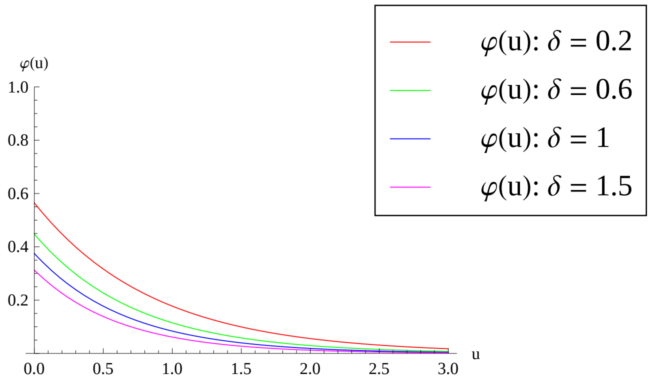


Figure 1.13: The Gerber-Shiu function $\phi_\delta(u)$

1.6.1 Code of Mathematica

In Example 1.6.1 the calculations and the graphs have been developed in Mathematica. In purpose of offering a better monitoring of this work, we include the respective code.

```
c = 2
d = 1
l = 4
f[x_] = 3 * Exp[-3 * x]
m1 = Integrate[x * f[x], {x, 0, Infinity}]
LaplOff[s_] = LaplaceTransform[f[x], x, s]
MomentOff[t_] = Integrate[Exp[x * t] * f[x], {x, 0, Infinity}, Assumptions → t < 3]
b = 1 / m1

c > l * m1

True

theta = (c / (l * m1)) - 1

1
—
2

NSolve[l + d - c * x - l * LaplOff[x] == 0, x]

{{x → -1.5}, {x → 1.}}

root1 = (-(c * b - d - l) + Sqrt[((c * b - d - l)^2) + 4 * c * d * b]) / (2 * c) // N
root2 = (-(c * b - d - l) - Sqrt[((c * b - d - l)^2) + 4 * c * d * b]) / (2 * c) // N

1.

-1.5

NSolve[MomentOff[t] == (1 + (1 + theta) * m1 * t), t]
Solve[l - c * x - l * LaplOff[x] == 0, x]
root1d0 = (-(c * b - l) + Sqrt[((c * b - l)^2)]) / (2 * c) // N
root2d0 = (-(c * b - l) - Sqrt[((c * b - l)^2)]) / (2 * c) // N

{{t → 1.}, {t → 0.}}

{{x → -1}, {x → 0}}

0.

-1.
```

Figure 1.14: Example 1.6.1, (i), Code 1/6

```
root[y_] = (-(c*b - y - l) + Sqrt[((c*b - y - l)^2) + 4*c*y*b]) / (2*c)
```

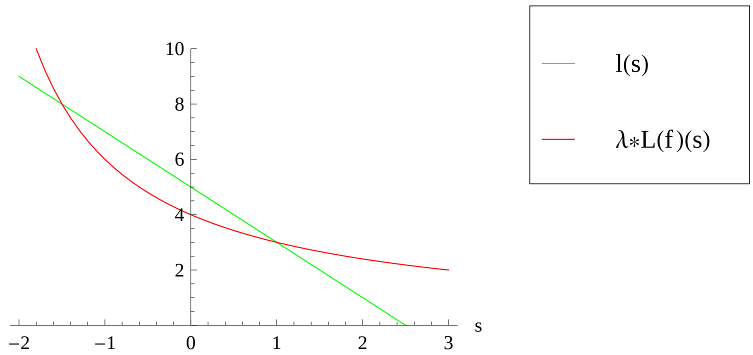
$$\frac{1}{4}\left(-2+y+\sqrt{(2-y)^2+24\,y}\right)$$

```
lun[x_] = l + d - c*x
```

$5 - 2x$

```
<< PlotLegends`;
```

```
Plot[{lun[x], l*LaplOff[x]}, {x, -2, 3}, PlotRange → {0, 10},
 AxesLabel → {"s", ""}, PlotStyle → {RGBColor[0, 1, 0], RGBColor[1, 0, 0]},
 PlotLegend → {Style["l(s)", 12], Style["λ*L(f)(s)", 12]},
 LegendPosition → {.9, 0}, LegendTextSpace → 2, LegendLabel → "",
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```




```
Plot[root[x], {x, 0, 10}, PlotRange → {0, 7}, AxesLabel → {"δ", "ρ(δ)"},
 PlotStyle → RGBColor[0, 0, 0], PlotLegend → Style["ρ(δ)", 12],
 LegendPosition → {.9, 0}, LegendTextSpace → 2, LegendLabel → "",
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```

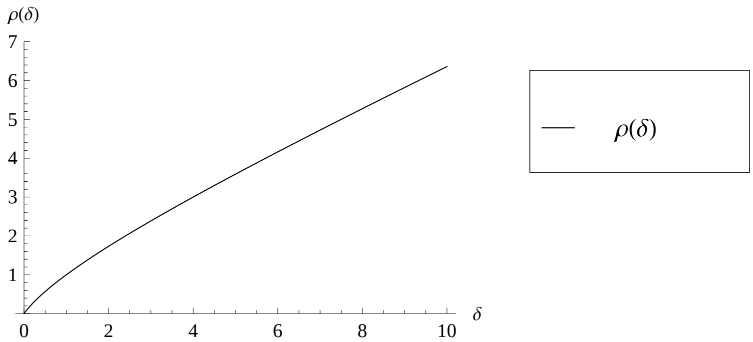


Figure 1.15: Example 1.6.1, (i), Code 2/6

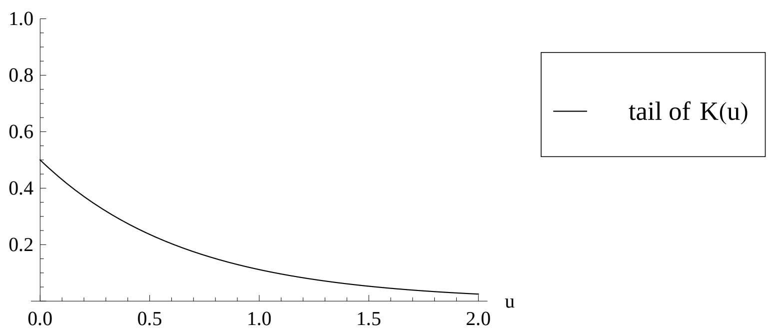
```
K[x_] = (b / ((1 + theta) * (b + root1))) *
  Exp[-b * ((root1 + (b + root1) * theta) / ((1 + theta) * (b + root1))) * x]
```

$0.5\, e^{-1.5\, x}$

```
psi[u_] = (1 / (1 + theta)) * Exp[((-b * theta) / (1 + theta)) * u]
```

$\frac{2\, e^{-u}}{3}$

```
Plot[K[x], {x, 0, 2}, PlotRange → {0, 1}, AxesLabel → {"u", ""},
 PlotStyle → RGBColor[0, 0, 0], PlotLegend → Style["tail of K(u)", 12],
 LegendPosition → {.9, 0}, LegendTextSpace → 2, LegendLabel → "",
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```




```
Plot[psi[x], {x, 0, 3}, PlotRange → {0, 1}, AxesLabel → {"u", "ψ(u)"},
 PlotStyle → RGBColor[0, 0, 0], PlotLegend → Style["ψ(υ)", 12],
 LegendPosition → {.9, 0}, LegendTextSpace → 2, LegendLabel → "",
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```

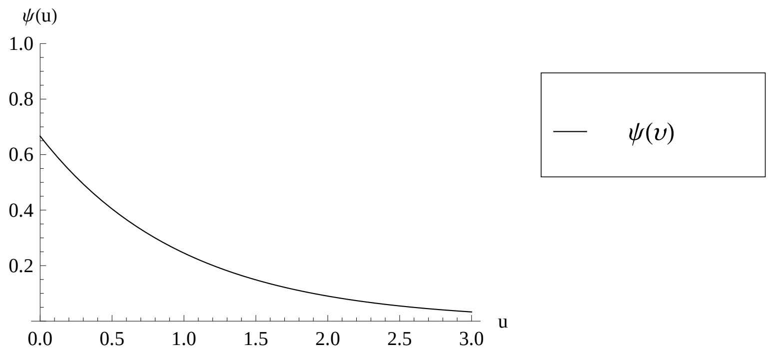
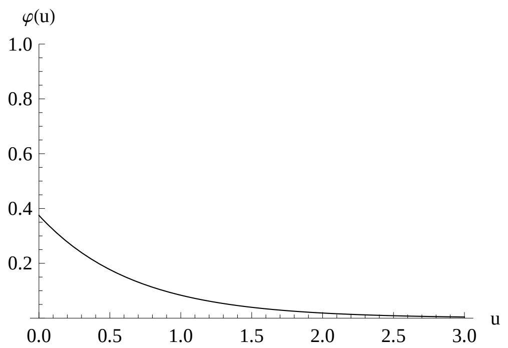


Figure 1.16: Example 1.6.1, (ii) & (iii), Code 3/6

```
phi[u_] = (b / (b + root1)) * (b / ((1 + theta) * (b + root1))) *
  Exp[-b * ((root1 + (b + root1) * theta) / ((1 + theta) * (b + root1))) * u]
```

$0.375\, e^{-1.5\, u}$

```
Plot[phi[x], {x, 0, 3}, PlotRange → {0, 1}, AxesLabel → {"u", "φ(u)"},
 PlotStyle → RGBColor[0, 0, 0], PlotLegend → Style["φ(υ)", 12],
 LegendPosition → {.9, 0}, LegendTextSpace → 2, LegendLabel → "",
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```




```
SurvivalOfFe[x_] = Exp[-b * x]
DerivativOfSurvivalOfFe[x_] = Derivative[1][SurvivalOfFe][x]
```

$e^{-3\, x}$

$-3\, e^{-3\, x}$

```
VerificationOfProbabilityOfRuin[u_] =
 -(1 / theta) * Integrate[DerivativOfSurvivalOfFe[u - x] * psi[x], {x, 0, u}] -
  (1 / theta) * SurvivalOfFe[0] * psi[u] + (1 / theta) * SurvivalOfFe[u]
FullSimplify[VerificationOfProbabilityOfRuin[u]]
```

$2\, e^{-3\, u} - \frac{4\, e^{-u}}{3} - 2\left(e^{-3\, u} - e^{-u}\right)$

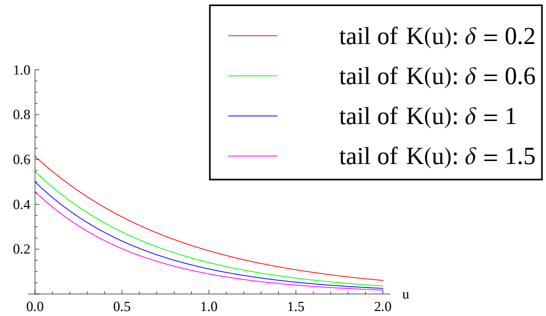
$\frac{2\, e^{-u}}{3}$

Figure 1.17: Example 1.6.1, (iv), Code 4/6

```
d1 = 0.2
d2 = 0.6
d3 = 1
d4 = 1.5

NSolve[lambda + d1 - c * s - lambda * LaplaceTransform[f[x], x, s] == 0, s]

{{s → -1.15887}, {s → 0.258872}}

root1 = 0.2588723439378912

NSolve[lambda + d2 - c * s - lambda * LaplaceTransform[f[x], x, s] == 0, s]

{{s → -1.36119}, {s → 0.661187}}

root2 = 0.6611874208078342

NSolve[lambda + d3 - c * s - lambda * LaplaceTransform[f[x], x, s] == 0, s]

{{s → -1.5}, {s → 1.}}

root3 = 1.

NSolve[lambda + d4 - c * s - lambda * LaplaceTransform[f[x], x, s] == 0, s]

{{s → -1.6302}, {s → 1.3802}}

root4 = 1.380199322349037

K1[x_] = (b / ((1 + theta) * (b + root1))) *
  Exp[-b * ((root1 + ((b + root1) * theta)) / ((1 + theta) * (b + root1))) * x]

0.613709 e^(-1.15887 x)

K2[x_] = (b / ((1 + theta) * (b + root2))) *
  Exp[-b * ((root2 + ((b + root2) * theta)) / ((1 + theta) * (b + root2))) * x]

0.546271 e^(-1.36119 x)

K3[x_] = (b / ((1 + theta) * (b + root3))) *
  Exp[-b * ((root3 + ((b + root3) * theta)) / ((1 + theta) * (b + root3))) * x]

0.5 e^(-1.5 x)

K4[x_] = (b / ((1 + theta) * (b + root4))) *
  Exp[-b * ((root4 + ((b + root4) * theta)) / ((1 + theta) * (b + root4))) * x]

0.4566 e^(-1.6302 x)
```

Figure 1.18: Remark 1.6.1, Code 5/6

```
Plot[{K1[x], K2[x], K3[x], K4[x]}, {x, 0, 2}, PlotRange → {0, 1}, AxesLabel → {"u", ""},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1], RGBColor[1, 0, 1]},
 PlotLegend → {Style["tail of K(u): δ = 0.2", 16], Style["tail of K(u): δ = 0.6", 16],
   Style["tail of K(u): δ = 1", 16], Style["tail of K(u): δ = 1.5", 16]},
 LegendPosition → {.9, 0}, LegendTextSpace → 2, LegendLabel → "",
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```




```
phi1[u_] = (b / (b + root1)) * (b / ((1 + theta) * (b + root1))) *
  Exp[-b * ((root1 + ((b + root1) * theta)) / ((1 + theta) * (b + root1))) * u]
```

$0.564959\, e^{-1.15887\, u}$

```
phi2[u_] = (b / (b + root2)) * (b / ((1 + theta) * (b + root2))) *
  Exp[-b * ((root2 + ((b + root2) * theta)) / ((1 + theta) * (b + root2))) * u]
```

$0.447618\, e^{-1.36119\, u}$

```
phi3[u_] = (b / (b + root3)) * (b / ((1 + theta) * (b + root3))) *
  Exp[-b * ((root3 + ((b + root3) * theta)) / ((1 + theta) * (b + root3))) * u]
```

$0.375\, e^{-1.5\, u}$

```
phi4[u_] = (b / (b + root4)) * (b / ((1 + theta) * (b + root4))) *
  Exp[-b * ((root4 + ((b + root4) * theta)) / ((1 + theta) * (b + root4))) * u]
```

$0.312726\, e^{-1.6302\, u}$

```
Plot[{phi1[u], phi2[u], phi3[u], phi4[u]},
 {u, 0, 3}, PlotRange → {0, 1}, AxesLabel → {"u", "φ(u)"},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1], RGBColor[1, 0, 1]},
 PlotLegend → {Style["φ(u): δ = 0.2", 16], Style["φ(u): δ = 0.6", 16],
   Style["φ(u): δ = 1", 16], Style["φ(u): δ = 1.5", 16]},
 LegendPosition → {.9, 0}, LegendTextSpace → 2, LegendLabel → "",
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```

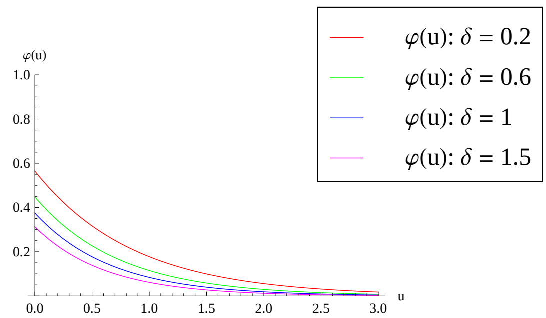


Figure 1.19: Remark 1.6.1, Code 6/6

Chapter 2

The Absolute Ruin

In this chapter, Cai (2000) adds one more feature to the suplus process. More specifically, it is assumed that the insurer can borrow money, equal to the deficit, at a debit interest force, whenever the surplus falls below zero. On the other hand, the debt will be repaid by the premium revenues. However, when the surplus attains or becomes less than a critical value, then absolute ruin is said to happen. Our study for the absolute ruin is conducted through the Gerber-Shiu function properly adapted for this case. The Gerber-Shiu function at absolute ruin, as it is called, embraces many interesting quantities, such as the absolute ruin probability, the Laplace transform of the absolute ruin time, the surplus just before absolute ruin, the deficit at absolute ruin, etc. Firstly, using the renewal argument and the law of total probability, we express the integro-differential equations satisfied by Gerber-Shiu function. Then, we derive the defective renewal equation for the Gerber-Shiu function throughout the Laplace transform and Dickson-Hipp operator. The solution to the defective renewal equation is given by a compound geometric distribution. Finally, we present explicit results for exponential claims, by estimating the absolute ruin probability and the Laplace transform of the absolute ruin time.

Based on the work of Cai (2000), we set up all the respective initial definitions and notions, which are presented in the Introduction 2.1.

2.1 Introduction

In this chapter we will continue working in the classical continuous time risk model. So, let N(t) denote the number of claims occur in [0, t] and $\{N(t) : t \geq 0\}$ be a Poisson process with mean $\lambda > 0$. Moreover, $\{X_i\}_{i=1}^{\infty}$ is the sequence of the claim sizes, independent of N(t) $\forall\, t > 0$ and it consists of independent and identical distributed nonnegative random variables with distribution function F(x), $F(0) = 0$, moments

$$\mu_k = E\left[X^k\right] = \int_0^\infty x^k f(x)dx$$

and mean

$$\mu_1 = E\left[X\right] = \int_0^\infty x f(x)dx = \int_0^\infty \bar{F}(x)dx$$

where $\bar{F}(x) = 1 - F(x)$ is the survival function of F. The Surplus process is denoted by $U(t) = u + ct - S(t),\ t \geq 0$, where $u \geq 0$ is the initial surplus, c is the premium rate per unit time and $S(t),\ t \geq 0$, is the aggregate claims process with

$$S(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i, & N(t) \geq 1 \\ 0, & N(t) = 0 \end{cases}$$

By contrast to our previous study, the process does not stop when the surplus becomes negative for the first time. However, whether the surplus falls below 0, an amount of money, equal to the deficit, could be borrowed according to a debit interest force $\delta > 0$. Then, the debt will be repaid constantly by the premiums. In this case, the surplus, in association with δ, is symbolised by $U_\delta(t)$ and satisfies

$$\begin{aligned} dU_\delta(t) &= [c + \delta\ U_\delta(t) I\left(U_\delta(t) < 0\right)]dt - dS(t) \\ &= \begin{cases} c\ dt + \delta\ U_\delta(t)\ dt - dS(t), & U_\delta(t) < 0 \\ c\ dt - dS(t), & \text{otherwise} \end{cases} \end{aligned}$$

Thus, when $U_\delta(t) < 0$, the revenues gained per unit time are

$$c + \delta U_\delta(t)$$

There is no chance for the surplus to become positive again if

$$c + \delta U_\delta(t) \leq 0 \Longleftrightarrow U_\delta(t) \leq -\frac{c}{\delta}$$

Consequently, $-\frac{c}{\delta}$ is a critical value for the surplus. If the surplus reaches $-\frac{c}{\delta}$ or drops below $-\frac{c}{\delta}$, then it never becomes positive again and it is said that absolute ruin happens.

Apart from Cai (2000), many other results and discussion related to absolute ruin can be found in the actuarial literature, see for example Dassios and Embrechts (1989), Embrechts and Schmidli (1994), Dickson and Egidio dos Reis (1997). Furthermore, there are studies for the impact of the interest or credit interest on positive surpluses, for instance Asmussen (2000), Cai (2004), Cai and Dickson (2002), Sundt and Teugels (1995) and many others.

Definition 2.1.1. *The absolute ruin time is denoted by T_δ where*

$$T_\delta = \begin{cases} inf\Big\{t \geq 0 : \; U_\delta(t) \leq -\dfrac{c}{\delta}\Big\} \\ \\ \infty, \qquad if \quad U_\delta(t) > -\dfrac{c}{\delta} \quad \forall\, t \geq 0 \end{cases}$$

In this chapter, Cai (2000) approaches the absolute ruin by using the Gerber-Shiu function at absolute ruin, as it is defined below. His definition is based on the classical Gerber-Shiu function (the expected discounted penalty function at ruin), which was introduced firstly by Gerber and Shiu (1997, 1998). Applying similar arguments to the Definition 1.2.2, we can see that Gerber-Shiu function at absolute ruin can be reduced to the absolute ruin probability, the Laplace transform of the absolute ruin time, the surplus just before absolute ruin, the deficit at absolute ruin, etc.

Definition 2.1.2. *The Gerber-Shiu function at absolute ruin (or the expected discounted penalty function at absolute ruin) is defined by*

$$\phi(u) = E\left[e^{-aT_\delta} w\left(U_\delta(T_\delta^-), |U_\delta(T_\delta)|\right) I\left(T_\delta < \infty\right) \;|\; U_\delta(0) = u\right]$$

where

- $u > -\dfrac{c}{\delta}$
- $w(x, y)$, with $x > -\frac{c}{\delta}$ and $y \geq \frac{c}{\delta}$, is a bivariate nonnegative penalty function
- the argument $a \geq 0$ can be seen as both, a discounting interest force calculating the present value of the penalty function $w(x, y)$ or as the argument for the Laplace transform of the absolute ruin time T_δ
- we can borrow money with debit interest force $\delta > 0$
- $U_\delta(T_\delta^-) = \lim\limits_{t \to T_\delta^-} U_\delta(t)$ denotes the surplus just before the absolute ruin and receives values in $(-\dfrac{c}{\delta}, \; +\infty)$
- $|U_\delta(T_\delta)|$ denotes the deficit exactly at absolute ruin and always satisfies $|U_\delta(T_\delta)| \geq \dfrac{c}{\delta}$
- $I\left(T_\delta < \infty\right)$ is the indicator function of the event $\{T_\delta < \infty\}$, i.e.

$$I\left(T_\delta < \infty\right) = \begin{cases} 1, & T_\delta < \infty \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.1.3. *The absolute ruin probability, which is symbolised as $\psi_\delta(u)$, is defined by*

$$\psi_\delta(u) = Pr\left[T_\delta < \infty \mid U_\delta(0) = u\right]$$

Remark 2.1.1. Cai (2000) makes a segmentation in Gerber-Shiu function at absolute ruin into two different parts, regarding whether or not the values received by the initial surplus u are positive. Hence,

$$\phi(u) = \begin{cases} \phi_+(u), & u \geq 0 \\ \phi_-(u), & -\frac{c}{\delta} < u < 0 \end{cases}$$

The respective absolute ruin probability obeys the same rule

$$\psi_\delta(u) = \begin{cases} \psi_+(u), & u \geq 0 \\ \psi_-(u), & -\frac{c}{\delta} < u < 0 \end{cases}$$

Furthermore, it is asssumed that

$$\lim_{u\to\infty} \phi(u) = \lim_{u\to\infty} \phi_+(u) = 0 \tag{2.1}$$

Remarks 2.1.2.

- Note that in the current notions and definitions the argument δ is the debit interest force and not the discounting interest force, as it was in the previous chapter. The discounting role belongs to a now.

- For $a = 0$ and $w(x, y) = 1$, the Gerber-Shiu function at absolute ruin $\phi(u)$ is reduced to the absolute ruin probability $\psi_\delta(u)$. Indeed,

$$\begin{aligned} \phi(u) &= E\left[I\left(T_\delta < \infty\right) \mid U_\delta(0) = u\right] \\ &= Pr\left[T_\delta < \infty \mid U_\delta(0) = u\right] \\ &= \psi_\delta(u) \end{aligned}$$

- For $w(x, y) = 1$, the Gerber-Shiu function at absolute ruin $\phi(u)$ is reduced to the Laplace transform of T_δ with argument a. Indeed,

$$\phi(u) = E\left[e^{-aT_\delta} I\left(T_\delta < \infty\right) \mid U_\delta(0) = u\right]$$

- We recall that

$$T = \begin{cases} \inf\{t \geq 0 : \ U(t) < 0\} \\ \\ \infty, \qquad if \quad U(t) \geq 0 \quad \forall\, t \geq 0 \end{cases}$$

is the classical ruin time and $\psi(u) = Pr[T < \infty \mid U(0) = u]$ is the classical ruin probability. It can be observed that

  * $T \leq T_\delta$
  * $\psi_+(u) \leq \psi(u) < 1 \qquad \forall\, u \geq 0.$

Moreover, the security loading factor is defined by $\theta = \left(\frac{c}{\lambda\mu_1}\right) - 1$ and it is assumed that $\theta > 0 \iff c > \lambda\mu_1$ (net profit condition). Under this assumption, we have

  * $0 < \psi_+(u) \leq \psi(u) < 1 \qquad \forall\, u \geq 0$
  * $\lim_{u\to\infty} \psi_+(u) = 0$ (because $\lim_{u\to\infty} \psi(u) = 0$ and $\psi_+(u) \leq \psi(u)$)

2.2 Integro-differential equations for $\phi(u)$

The integral and integro-differential equations satisfied by $\phi_+(u)$ and $\phi_-(u)$ have been proven by Cai (2000). He uses the renewal argument by conditioning on the time and size of the first claim. In the following two Theorems, we present this concept offering detailed explanations and calculations of his methodology.

Theorem 2.2.1. *For $u \geq 0$, $\phi_+(u)$ satisfies the following integro-differential equation*

$$\phi_+^{'}(u) = \frac{\lambda + a}{c}\phi_+(u) \; - \; \frac{\lambda}{c}\left[\int_0^u \phi_+(u-x)f(x)dx + B(u)\right] \tag{2.2}$$

where

$$B(u) = \int_u^{u+\frac{c}{\delta}} \phi_-(u-x)f(x)dx \; + \; \gamma_\delta(u) \tag{2.3}$$

and

$$\gamma_\delta(u) = \int_{u+\frac{c}{\delta}}^{\infty} w(u, x-u)f(x)dx \tag{2.4}$$

Proof. Conditioning on the time t and size x of the first claim and using the renewal argument, the law of total probability yields

$$\begin{aligned} \phi_+(u) &= \int_0^\infty \int_0^\infty \phi_+(u|t,x)f(x)f_T(t)dxdt \\ &= \int_0^\infty \lambda e^{-\lambda t} \int_0^\infty \phi_+(u|t,x)f(x)dx \; dt \end{aligned}$$

Regarding the size x of the first claim, there are three potential situations for Gerber-Shiu function $\phi_+(u|t,x)$. Namely,

- for $x \leq u+ct$, the procedure is renewed and the initial surplus receives the value $u + ct - x \geq 0$
- for $u+ct < x < u+ct+\frac{c}{\delta}$, the procedure is renewed, with the difference that the initial surplus is negative, equal to $u + ct - x < 0$
- for $x \geq u + ct + \frac{c}{\delta}$, absolute ruin happens and the values of the surplus just before the absolute ruin and the deficit exactly at the absolute ruin are $U_\delta(T_\delta^-) = u + ct$ and $U_\delta(T_\delta) = u + ct - x$, respectively

Overall, we have

$$\phi_+(u|t,x) = \begin{cases} e^{-at}\,\phi_+(u+ct-x), & x \leq u+ct \\ \\ e^{-at}\,\phi_-(u+ct-x), & u+ct < x < u+ct+\frac{c}{\delta} \\ \\ e^{-at}\,w(u+ct, x-u-ct), & x \geq u+ct+\frac{c}{\delta} \end{cases}$$

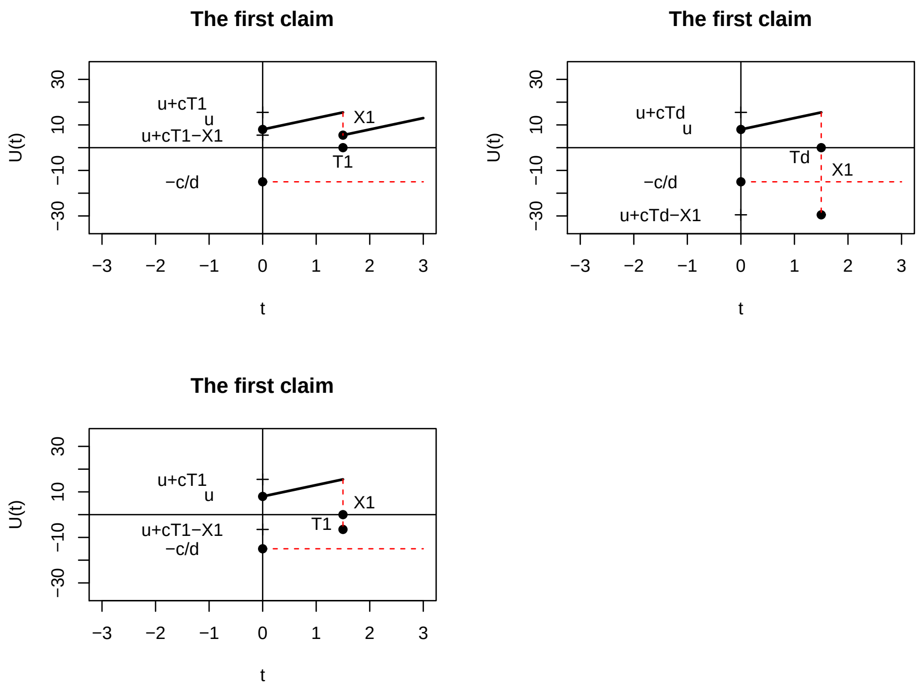


Figure 2.1: The size x_1 of the first claim

The above situations can be depicted by the Figure 2.1. Thus, we obtain

$$\begin{aligned}
\phi_+(u) &= \int_0^\infty \lambda e^{-\lambda t} \left[\int_0^{u+ct} e^{-at}\, \phi_+(u+ct-x)f(x)dx + \int_{u+ct}^{u+ct+\frac{c}{\delta}} e^{-at}\, \phi_-(u+ct-x)f(x)dx \right.\\
&\left. + \int_{u+ct+\frac{c}{\delta}}^\infty e^{-at}\, w(u+ct, x-u-ct)f(x)dx \right] dt\\
&= \int_0^\infty \lambda e^{-(\lambda+a)t} \left[\int_0^{u+ct} \phi_+(u+ct-x)f(x)dx + \int_{u+ct}^{u+ct+\frac{c}{\delta}} \phi_-(u+ct-x)f(x)dx \right.\\
&\left. + \int_{u+ct+\frac{c}{\delta}}^\infty w(u+ct, x-u-ct)f(x)dx \right] dt
\end{aligned}$$

Replacing $s = u + ct \Longrightarrow t = \frac{s-u}{c}$ & $dt = \frac{1}{c}ds$, the boundaries of the external integral are converted into $u \le s < +\infty$. Hence, we are led to

$$
\begin{aligned}
\phi_+(u) &= \int_u^\infty \lambda e^{-\frac{(\lambda+a)(s-u)}{c}} \left[\int_0^s \phi_+(s-x)f(x)dx + \int_s^{s+\frac{c}{\delta}} \phi_-(s-x)f(x)dx \right. \\
&+ \left. \int_{s+\frac{c}{\delta}}^\infty w(s,x-s)f(x)dx\right] \frac{1}{c}ds \\
&= \frac{\lambda}{c}\left[\int_u^\infty e^{-\frac{(\lambda+a)(s-u)}{c}} \left(\int_0^s \phi_+(s-x)f(x)dx\right) ds \right. \\
&+ \int_u^\infty e^{-\frac{(\lambda+a)(s-u)}{c}} \left(\int_s^{s+\frac{c}{\delta}} \phi_-(s-x)f(x)dx\right) ds \\
&+ \left. \int_u^\infty e^{-\frac{(\lambda+a)(s-u)}{c}} \left(\int_{s+\frac{c}{\delta}}^\infty w(s,x-s)f(x)dx\right) ds\right]
\end{aligned}
$$

Defining

$$\gamma_\delta(x) = \int_{x+\frac{c}{\delta}}^\infty w(x,y-x)f(y)dy$$

yields

$$
\begin{aligned}
\phi_+(u) &= \frac{\lambda}{c}\left[\int_u^\infty e^{-\frac{(\lambda+a)(s-u)}{c}} \left(\int_0^s \phi_+(s-x)f(x)dx\right) ds \right. \\
&+ \int_u^\infty e^{-\frac{(\lambda+a)(s-u)}{c}} \left(\int_s^{s+\frac{c}{\delta}} \phi_-(s-x)f(x)dx\right) ds \\
&+ \left. \int_u^\infty e^{-\frac{(\lambda+a)(s-u)}{c}} \gamma_\delta(s)ds\right]
\end{aligned}
\tag{2.5}
$$

Putting $u = 0$ and applying the definition (2.3) in (2.5), we obtain

$$
\begin{aligned}
\phi_+(0) &= \frac{\lambda}{c}\left[\int_0^\infty e^{-\frac{s(\lambda+a)}{c}} \left(\int_0^s \phi_+(s-x)f(x)dx + \int_s^{s+\frac{c}{\delta}} \phi_-(s-x)f(x)dx + \gamma_\delta(s)\right) ds\right] \\
&= \frac{\lambda}{c}\left[\int_0^\infty e^{-\frac{s(\lambda+a)}{c}} \left(\int_0^s \phi_+(s-x)f(x)dx + B(s)\right) ds\right]
\end{aligned}
\tag{2.6}
$$

Let

- $g_1(u,s) = e^{-\frac{(\lambda+a)(s-u)}{c}} \left(\int_0^s \phi_+(s-x)f(x)dx\right)$

- $g_2(u,s) = e^{-\frac{(\lambda+a)(s-u)}{c}} \left(\int_s^{s+\frac{c}{\delta}} \phi_-(s-x)f(x)dx \right)$
- $g_3(u,s) = e^{-\frac{(\lambda+a)(s-u)}{c}} \gamma_\delta(s)ds$

Now, (2.5) can be written as

$$\phi_+(u) \quad = \frac{\lambda}{c}\left[\int_u^\infty g_1(u,s)ds + \int_u^\infty g_2(u,s)ds + \int_u^\infty g_3(u,s)\right] \tag{2.7}$$

Differentiating each term of (2.7) separately, with respect to u, we obtain

$$\begin{aligned}
\frac{d}{du}\left(\int_u^\infty g_1(u,s)ds\right) &= -g_1(u,u) + \int_u^\infty \frac{d}{du}\left(g_1(u,s)\right)ds \\
&= -\int_0^u \phi_+(u-x)f(x)dx \\
&+ \int_u^\infty \left(\frac{\lambda+a}{c}\right) e^{-\frac{(\lambda+a)(s-u)}{c}} \left(\int_0^s \phi_+(s-x)f(x)dx\right) ds \\
&= -\int_0^u \phi_+(u-x)f(x)dx \; + \; \frac{\lambda+a}{c} \; \int_u^\infty g_1(u,s)ds \\
\frac{d}{du}\left(\int_u^\infty g_2(u,s)ds\right) &= -g_2(u,u) + \int_u^\infty \frac{d}{du}\left(g_2(u,s)\right)ds \\
&= -\int_u^{u+\frac{c}{\delta}} \phi_-(u-x)f(x)dx \\
&+ \int_u^\infty \left(\frac{\lambda+a}{c}\right) e^{-\frac{(\lambda+a)(s-u)}{c}} \left(\int_s^{s+\frac{c}{\delta}} \phi_-(s-x)f(x)dx\right) ds \\
&= -\int_u^{u+\frac{c}{\delta}} \phi_-(u-x)f(x)dx \; + \; \frac{\lambda+a}{c} \; \int_u^\infty g_2(u,s)ds \\
\frac{d}{du}\left(\int_u^\infty g_3(u,s)ds\right) &= -g_3(u,u) + \int_u^\infty \frac{d}{du}\left(g_3(u,s)\right)ds \\
&= -\gamma_\delta(u) + \int_u^\infty \left(\frac{\lambda+a}{c}\right) e^{-\frac{(\lambda+a)(s-u)}{c}} \gamma_\delta(s)ds \\
&= -\gamma_\delta(u) \; + \; \frac{\lambda+a}{c} \; \int_u^\infty g_3(u,s)ds
\end{aligned}$$

Differentiating (2.7) in total, with respect to u, and substituting the above results, we obtain

$$\begin{aligned}
\phi_+^{'}(u) &= \frac{\lambda}{c}\Bigg[-\int_0^u \phi_+(u-x)f(x)dx \;+\; \frac{\lambda+a}{c}\int_u^\infty g_1(u,s)ds - \int_u^{u+\frac{c}{\delta}} \phi_-(u-x)f(x)dx \\
&+ \frac{\lambda+a}{c}\int_u^\infty g_2(u,s)ds - \gamma_\delta(u) \;+\; \frac{\lambda+a}{c}\int_u^\infty g_3(u,s)ds\Bigg] \\
&= \frac{\lambda}{c}\Bigg[-\int_0^u \phi_+(u-x)f(x)dx - \int_u^{u+\frac{c}{\delta}} \phi_-(u-x)f(x)dx - \gamma_\delta(u)\Bigg] \\
&+ \frac{\lambda+a}{c}\Bigg[\frac{\lambda}{c}\Bigg[\int_u^\infty g_1(u,s)ds + \int_u^\infty g_2(u,s)ds + \int_u^\infty g_3(u,s)ds\Bigg]\Bigg] \\
&= \frac{\lambda+a}{c}\,\phi_+(u) \;-\; \frac{\lambda}{c}\Bigg[\int_0^u \phi_+(u-x)f(x)dx + \int_u^{u+\frac{c}{\delta}} \phi_-(u-x)f(x)dx + \gamma_\delta(u)\Bigg]
\end{aligned}$$

Having defined in (2.3)

$$B(x) = \int_x^{x+\frac{c}{\delta}} \phi_-(x-y)f(y)dy \;+\; \gamma_\delta(x)$$

we are led exactly to the desirable result

$$\phi_+^{'}(u) = \frac{\lambda+a}{c}\,\phi_+(u) \;-\; \frac{\lambda}{c}\Bigg[\int_0^u \phi_+(u-x)f(x)dx + B(u)\Bigg]$$

$\diamondsuit$

Theorem 2.2.2. *For $-\frac{c}{\delta} < u < 0$, $\phi_-(u)$ satisfies the following integro-differential equation*

$$\phi_-^{'}(u) = \frac{\lambda+a}{\delta u+c}\,\phi_-(u) \;-\; \frac{\lambda}{\delta u+c}\Bigg[\int_0^{u+\frac{c}{\delta}} \phi_-(u-x)f(x)dx + \gamma_\delta(u)\Bigg] \quad (2.8)$$

Proof. Let $-\frac{c}{\delta} < u < 0$. Then, the surplus $U_\delta(t)$ is depended on the debit interest force δ. Let t_0 be the first time that the negative surplus becomes zero, provided that there is no claim in $[0,\, t_0]$. Furthermore, we assume that $h(t,u),\ t \le t_0$ depicts the values of the surplus, provided that there is no claim in [0, t]. As a result, $h(t_0,u) = 0,\ h(t,u) < 0\ \forall\, t < t_0$ and $h(0,u) = u$ (the initial surplus). It always holds that

$$h(t,u) = ue^{\delta t} + c\left[\frac{e^{\delta t}-1}{\delta}\right]$$

Solving the equation $h(t,u) = 0$ with respect to t, we receive the value of t_0, i.e.

$$t_0 = \ln\left(\frac{c}{\delta u + c}\right)^{\frac{1}{\delta}}$$

Conditioning on the time t and size x of the first claim and using the renewal argument, the law of total probability yields

$$\begin{aligned} \phi_-(u) &= \int_0^\infty \int_0^\infty \phi_-(u|t,x) f(x) f_T(t) dx dt \\ &= \int_0^\infty \lambda e^{-\lambda t} \int_0^\infty \phi_-(u|t,x) f(x) dx \, dt \end{aligned}$$

According to the arrival time t of the first claim, two possible cases can be detected; the first claim happening either before or after the time t_0. Taking into consideration the size x of the first claim as well, we obtain five situations in total. Namely,

(I.) For $t \leq t_0$

* if $x < h(t,u) + \frac{c}{\delta}$, the procedure will be renewed, with initial surplus equal to $h(t,u) - x < 0$
* if $x \geq h(t,u) + \frac{c}{\delta}$, absolute ruin will happen and the values of the surplus just before the absolute ruin and the deficit exactly at the absolute ruin will be $U_\delta(T_\delta^-) = h(t,u)$ and $U_\delta(T_\delta) = h(t,u) - x$, respectively

So, we can write for $\phi_-(u|t,x)$ that

$$\phi_-(u|t,x) = \begin{cases} e^{-at}\phi_-(h(t,u) - x), & x < h(t,u) + \frac{c}{\delta} \\ \\ e^{-at} w(h(t,u),\ x - h(t,u)), & x \geq h(t,u) + \frac{c}{\delta} \end{cases}$$

(II.) For $t > t_0$

* if $x \leq c(t - t_0)$, the procedure will be renewed, with positive initial surplus equal to $c(t - t_0) - x$
* if $c(t - t_0) < x < c(t - t_0) + \frac{c}{\delta}$, the procedure will be renewed, with negative initial surplus equal to $c(t - t_0) - x$
* if $x \geq c(t - t_0) + \frac{c}{\delta}$, absolute ruin will happen and the values of the surplus just before the absolute ruin and the deficit exactly at the absolute ruin will be $U_\delta(T_\delta^-) = c(t - t_0)$ and $U_\delta(T_\delta) = c(t - t_0) - x$, respectively

So, we can write for $\phi_-(u|t,x)$ that

$$\phi_-(u|t,x) = \begin{cases} e^{-at}\phi_+(c(t-t_0)-x), & x \le c(t-t_0) \\ e^{-at}\phi_-(c(t-t_0)-x), & c(t-t_0) < x < c(t-t_0)+\frac{c}{\delta} \\ e^{-at}w(c(t-t_0),\ x-c(t-t_0)), & x \ge c(t-t_0)+\frac{c}{\delta} \end{cases}$$

The above results could be illustrated by the Figures 2.2 & 2.3.

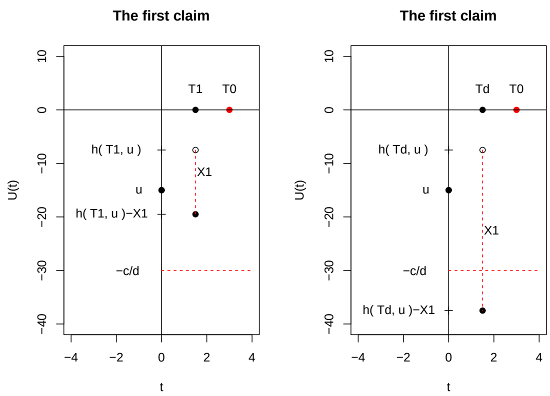


Figure 2.2: The first claim (1/2)

Applying the above possible situations in $\phi_-(u)$ leads to

$$\begin{aligned} \phi_-(u) \;&= \int_0^{t_0} \lambda e^{-\lambda t} \left[\int_0^{h(t,u)+\frac{c}{\delta}} e^{-at}\phi_-(h(t,u)-x)f(x)dx \right. \\ &+ \left. \int_{h(t,u)+\frac{c}{\delta}}^{\infty} e^{-at}w(h(t,u),\ x-h(t,u))f(x)dx \right] dt \\ &+ \int_{t_0}^{\infty} \lambda e^{-\lambda t} \left[\int_0^{c(t-t_0)} e^{-at}\phi_+(c(t-t_0)-x)f(x)dx \right. \\ &+ \int_{c(t-t_0)}^{c(t-t_0)+\frac{c}{\delta}} e^{-at}\phi_-(c(t-t_0)-x)f(x)dx \\ &+ \left. \int_{c(t-t_0)+\frac{c}{\delta}}^{\infty} e^{-at}w(c(t-t_0),\ x-c(t-t_0))f(x)dx \right] dt \end{aligned}$$

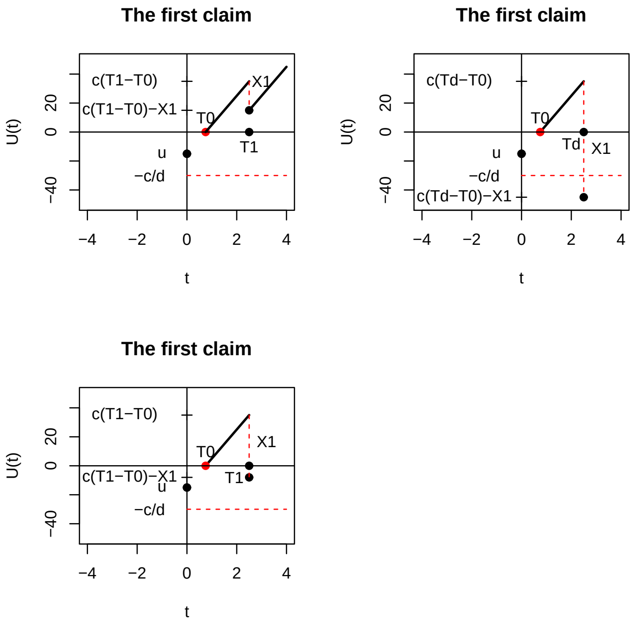


Figure 2.3: The first claim (2/2)

$$
\begin{aligned}
&= \int_0^{t_0} \lambda e^{-(\lambda+a)t} \left[\int_0^{h(t,u)+\frac{c}{\delta}} \phi_-(h(t,u)-x) f(x)dx \right. \\
&+ \left. \int_{h(t,u)+\frac{c}{\delta}}^{\infty} w(h(t,u),\ x-h(t,u)) f(x)dx \right] dt \\
&+ \int_{t_0}^{\infty} \lambda e^{-(\lambda+a)t} \left[\int_0^{c(t-t_0)} \phi_+(c(t-t_0)-x) f(x)dx \right. \\
&+ \int_{c(t-t_0)}^{c(t-t_0)+\frac{c}{\delta}} \phi_-(c(t-t_0)-x) f(x)dx \\
&+ \left. \int_{c(t-t_0)+\frac{c}{\delta}}^{\infty} w(c(t-t_0),\ x-c(t-t_0)) f(x)dx \right] dt \\
&= \int_0^{t_0} \lambda e^{-(\lambda+a)t} \left[\int_0^{h(t,u)+\frac{c}{\delta}} \phi_-(h(t,u)-x) f(x)dx \ + \ \gamma_\delta(h(t,u)) \right] dt \\
&+ \int_{t_0}^{\infty} \lambda e^{-(\lambda+a)t} \left[\int_0^{c(t-t_0)} \phi_+(c(t-t_0)-x) f(x)dx \right. \\
&+ \left. \int_{c(t-t_0)}^{c(t-t_0)+\frac{c}{\delta}} \phi_-(c(t-t_0)-x) f(x)dx + \gamma_\delta(c(t-t_0)) \right] dt
\end{aligned}
\tag{2.9}
$$

(I) Putting $y = h(t,u)$ in the first definite integral, with boundaries $0 \le t \le t_0$, we have the following transformations:

$$y = h(t,u) \implies y = ue^{\delta t} + c\left[\frac{e^{\delta t}-1}{\delta}\right] \implies \delta y = \delta u e^{\delta t} + ce^{\delta t} - c \implies e^{\delta t} = \frac{\delta y + c}{\delta u + c}$$

$$\implies t = \ln\left(\frac{\delta y + c}{\delta u + c}\right)^{\frac{1}{\delta}}$$

$$y = h(t,u) \implies dy = \frac{d\,h(t,u)}{dt}\,dt = (\delta u + c)e^{\delta t}\,dt = (\delta u + c)e^{\delta\,\ln\left(\frac{\delta y + c}{\delta u + c}\right)^{\frac{1}{\delta}}}\,dt$$

$$= (\delta u + c)\left(\frac{\delta y + c}{\delta u + c}\right)\,dt = (\delta y + c)\,dt$$

Consequently,

$$dt = (\delta y + c)^{-1} dy$$

Having defined $h(0,u) = u$ and $h(t_0,u) = 0$, the boundaries of integration are converted from $0 \le t \le t_0$ to $u \le y \le 0$. Finally, in order to facilitate the calculations we count separately the following term:

$$e^{-(\lambda+a)t} = e^{-(\lambda+a)\,\ln\left(\frac{\delta y + c}{\delta u + c}\right)^{\frac{1}{\delta}}} = \left(\frac{\delta y + c}{\delta u + c}\right)^{-\frac{\lambda+a}{\delta}} = (\delta y + c)^{-\frac{\lambda+a}{\delta}}\,(\delta u + c)^{\frac{\lambda+a}{\delta}}$$

(II) Putting $z = c(t - t_0)$ in the second definite integral, with boundaries $t_0 \le t < \infty$, we have the following transformations:

$$z = c(t - t_0) \implies dz = c\,dt \implies dt = c^{-1}\,dz$$

$$z = c(t - t_0) \implies t = \frac{z + c\,t_0}{c} = \frac{z}{c} + t_0 \implies t = \frac{z}{c} + \ln\left(\frac{c}{\delta u + c}\right)^{\frac{1}{\delta}}$$

The new boundaries are $0 \le z < \infty$ and the term $e^{-(\lambda+a)t}$ equals:

$$e^{-(\lambda+a)t} = e^{-(\lambda+a)\left[\frac{z}{c} + \ln\left(\frac{c}{\delta u + c}\right)^{\frac{1}{\delta}}\right]} = e^{-(\lambda+a)\frac{z}{c}}\,e^{-(\lambda+a)\,\ln\left(\frac{c}{\delta u + c}\right)^{\frac{1}{\delta}}}$$

$$= e^{-(\lambda+a)\frac{z}{c}}\left(\frac{c}{\delta u + c}\right)^{-\frac{\lambda+a}{\delta}} = e^{-(\lambda+a)\frac{z}{c}}\,c^{-\frac{\lambda+a}{\delta}}\,(\delta u + c)^{\frac{\lambda+a}{\delta}}$$

Substituting the above findings in (2.9), we obtain

$$
\begin{aligned}
\phi_-(u) &= \int_u^0 \lambda(\delta y+c)^{-\frac{\lambda+a}{\delta}}(\delta u+c)^{\frac{\lambda+a}{\delta}}(\delta y+c)^{-1}\left[\int_0^{y+\frac{c}{\delta}} \phi_-(y-x)f(x)dx+\gamma_\delta(y)\right]dy \\
&+ \int_0^\infty \lambda e^{-(\lambda+a)\frac{z}{c}}\, c^{-\frac{\lambda+a}{\delta}}\, (\delta u+c)^{\frac{\lambda+a}{\delta}}\, c^{-1}\left[\int_0^z \phi_+(z-x)f(x)dx\right. \\
&+ \left.\int_z^{z+\frac{c}{\delta}} \phi_-(z-x)f(x)dx+\gamma_\delta(z)\right]dz \\
&= \int_u^0 \lambda\,(\delta u+c)^{\frac{\lambda+a}{\delta}}\,(\delta y+c)^{-1-\frac{\lambda+a}{\delta}}\left[\int_0^{y+\frac{c}{\delta}} \phi_-(y-x)f(x)dx+\gamma_\delta(y)\right]dy \\
&+ \int_0^\infty \lambda(\delta u+c)^{\frac{\lambda+a}{\delta}}\, c^{-1-\frac{\lambda+a}{\delta}}\, e^{-(\lambda+a)\frac{z}{c}}\left[\int_0^z \phi_+(z-x)f(x)dx+B(z)\right]dz
\end{aligned}
\tag{2.10}
$$

For $u \to 0^-$, we receive

$$
\phi_-(0-) = \frac{\lambda}{c}\int_0^\infty e^{-(\lambda+a)\frac{z}{c}}\left[\int_0^z \phi_+(z-x)f(x)dx+B(z)\right]dz \tag{2.11}
$$

where $\phi_-(0-) = \lim_{u\to 0^-} \phi_-(u)$.

From (2.6) and (2.11), we point out the boundary condition of

$$
\phi_+(0) = \phi_-(0-) \tag{2.12}
$$

Let

$$
g(y) = (\delta y+c)^{-1-\frac{\lambda+a}{\delta}}\left[\int_0^{y+\frac{c}{\delta}} \phi_-(y-x)f(x)dx+\gamma_\delta(y)\right]
$$

and

$$
I = \int_0^\infty c^{-1-\frac{\lambda+a}{\delta}}\, e^{-(\lambda+a)\frac{z}{c}}\left[\int_0^z \phi_+(z-x)f(x)dx+B(z)\right]dz \quad \in \mathcal{R}
$$

Hence, (2.10) can be written as

$$
\phi_-(u) = \lambda\,(\delta u+c)^{\frac{\lambda+a}{\delta}}\int_u^0 g(y)dy + \lambda\,(\delta u+c)^{\frac{\lambda+a}{\delta}}\, I \qquad (*)
$$

Differentiating the $(*)$, with respect to u, leads to

$$\begin{aligned}
\phi_-^{'}(u) &= \lambda\frac{\lambda+a}{\delta}\left(\delta u+c\right)^{\frac{\lambda+a}{\delta}}\left(\delta u+c\right)^{-1}\ \delta\int_u^0 g(y)dy+\lambda\left(\delta u+c\right)^{\frac{\lambda+a}{\delta}}\left(-g(u)\right)\\
&+\lambda\frac{\lambda+a}{\delta}\left(\delta u+c\right)^{\frac{\lambda+a}{\delta}}\left(\delta u+c\right)^{-1}\ \delta\ I\\
&= \frac{\lambda+a}{\delta u+c}\left[\lambda\left(\delta u+c\right)^{\frac{\lambda+a}{\delta}}\int_u^0 g(y)dy+\lambda\left(\delta u+c\right)^{\frac{\lambda+a}{\delta}}\ I\right]\\
&-\lambda\left(\delta u+c\right)^{\frac{\lambda+a}{\delta}}\left(\delta u+c\right)^{-1-\frac{\lambda+a}{\delta}}\left[\int_0^{u+\frac{c}{\delta}}\phi_-(u-x)f(x)dx+\gamma_\delta(u)\right]
\end{aligned}$$

Taking into consideration the expression for $\phi_-(u)$ in $(*)$, we conclude to what we want to prove,

$$\phi_-^{'}(u)=\frac{\lambda+a}{\delta u+c}\,\phi_-(u)\ -\ \frac{\lambda}{\delta u+c}\left[\int_0^{u+\frac{c}{\delta}}\phi_-(u-x)f(x)dx+\gamma_\delta(u)\right]$$

$\diamondsuit$

Remark 2.2.1. *It always holds that*

$$\phi_+^{'}(0)=\phi_-^{'}(0-) \tag{2.13}$$

Indeed, substituting $u=0$ in (2.2) yields

$$\begin{aligned}
\phi_+^{'}(o) &= \frac{\lambda+a}{c}\phi_+(0)-\frac{\lambda}{c}B(0)\\
&= \frac{\lambda+a}{c}\phi_+(0)-\frac{\lambda}{c}\left[\int_0^{\frac{c}{\delta}}\phi_-(-x)f(x)dx+\gamma_\delta(0)\right]
\end{aligned}$$

whereas, for $u\to 0^-$, (2.8) leads to the same result as above, i.e.

$$\phi_-^{'}(0^-)=\frac{\lambda+a}{c}\phi_-(0^-)-\frac{\lambda}{c}\left[\int_0^{\frac{c}{\delta}}\phi_-(-x)f(x)dx+\gamma_\delta(0)\right]$$

which are identical, because of (2.12), where $\phi_+(0)=\phi_-(0^-)$.

$\diamondsuit$

2.3 Defective Renewal Equation for $\phi_+(u)$

2.3.1 Integral Equation for $\phi_+(u)$

The integral equation for $\phi_+(u)$ in Proposition 2.3.1.1 is based on Gerber and Shiu (1998), whereas its transformation into a defective renewal equation in Theorem 2.3.1.1 is based on Lin and Willmot (1999). This equation is going to be solved in the subsection 2.3.2. Moreover, the following steps and methodology appear similarities to those in subsections 1.5.3 and 1.5.4, with the difference that now the calculations consist of the term $\phi_-(u)$, as well.

Proposition 2.3.1.1. *For $u \geq 0$ the Gerber-Shiu function $\phi(u) = \phi_+(u)$ satisfies the following integral equation*

$$\phi_+(u) = \frac{\lambda}{c} \int_0^u \phi_+(u-x)\, T_\rho f(x)\, dx \; + \; \frac{\lambda}{c}\, T_\rho B(u) \tag{2.14}$$

where ρ is the positive root of Lundberg's equation and $T_r f(x) = \int_x^\infty e^{-r(y-x)} f(y) dy$ is the Dickson-Hipp operator defined in Chapter 1.

Proof. For $u \geq 0$, multiplying (2.2) by c, it is received

$$c\, \phi_+^{'}(u) = (\lambda + a)\phi_+(u) \; - \; \lambda \left[\int_0^u \phi_+(u-x) f(x) dx + B(u) \right]$$

Applying the Laplace transform and following the same methodology used in Therorem 1.5.3.1, it is obtained

$$\begin{aligned} c \int_0^\infty e^{-sx} \phi_+^{'}(x) dx \;\; &= \;\; (\lambda + a) \int_0^\infty e^{-sx} \phi_+(x) dx \\ &\quad - \lambda \left[\int_0^\infty e^{-sx} \left(\int_0^x \phi_+(x-y) f(y) dy \right) dx + \int_0^\infty e^{-sx} B(x) dx \right] \Longrightarrow \end{aligned}$$

$$c \left[s\, \hat{\phi}_+(s) - \phi_+(0) \right] = (\lambda + a)\hat{\phi}_+(s) - \lambda\, \hat{\phi}_+(s) \hat{f}(s) - \lambda\, \hat{B}(s) \Longrightarrow$$

$$\left[cs - (\lambda + a) + \lambda \hat{f}(s) \right] \hat{\phi}_+(s) = c\, \phi_+(0) - \lambda \hat{B}(s) \Longrightarrow$$

$$\hat{\phi}_+(s) = \frac{c\, \phi_+(0) - \lambda \hat{B}(s)}{cs - (\lambda + a) + \lambda \hat{f}(s)} = \frac{\lambda \hat{B}(s) - c\, \phi_+(0)}{(\lambda + a) - cs - \lambda \hat{f}(s)} = \frac{P(s)}{Q(s)} \tag{2.15}$$

It can be observed that the denominator of (2.15) is the Lundberg's fundamental equation which has a unique positive root defined as $\rho = \rho(a) > 0$. Thus, it holds $Q(\rho) = 0 \Longrightarrow P(\rho) = 0$, because of $\hat{\phi}_+(s) < \infty$. Now, we rewrite the numerator and denominator using the 3rd property of Dickson-Hipp operator in Definition 1.3.1:

$$
\begin{aligned}
* \quad P(\rho) = 0 \quad & \Longrightarrow c\, \phi_+(0) = \lambda \hat{B}(\rho) \\
& \Longrightarrow c\, \phi_+(0) - \lambda \hat{B}(s) = \lambda \hat{B}(\rho) - \lambda \hat{B}(s) \\
& \Longrightarrow P(s) = \lambda \left(\hat{B}(\rho) - \hat{B}(s) \right) \\
& \Longrightarrow P(s) = \lambda (s-\rho) \left(\frac{\hat{B}(\rho) - \hat{B}(s)}{s-\rho} \right) \\
& \Longrightarrow P(s) = \lambda (s-\rho)\, \hat{T}_\rho B(s)
\end{aligned}
$$

$$
\begin{aligned}
* \quad Q(s) \quad & = Q(s) - Q(\rho) \\
& = cs - (\lambda + a) + \lambda \hat{f}(s) - \left[c\rho - (\lambda + a) + \lambda \hat{f}(\rho) \right] \\
& = (s-\rho) \left[c - \lambda\, \frac{\hat{f}(\rho) - \hat{f}(s)}{s-\rho} \right] \\
& = (s-\rho) \left[c - \lambda\, \hat{T}_\rho f(s) \right]
\end{aligned}
$$

Substituting them in (2.15) we have that

$$
\hat{\phi}_+(s) = \frac{\lambda (s-\rho)\, \hat{T}_\rho B(s)}{(s-\rho) \left[c - \lambda\, \hat{T}_\rho f(s) \right]} \;=\; \frac{\lambda\, \hat{T}_\rho B(s)}{c - \lambda\, \hat{T}_\rho f(s)} \;\Longrightarrow
$$

$$
c\, \hat{\phi}_+(s) - \lambda\, \hat{\phi}_+(s)\, \hat{T}_\rho f(s) = \lambda\, \hat{T}_\rho B(s) \;\Longrightarrow
$$

$$
\hat{\phi}_+(s) = \frac{\lambda}{c}\, \hat{\phi}_+(s)\, \hat{T}_\rho f(s) \;+\; \frac{\lambda}{c}\, \hat{T}_\rho B(s)
$$

Applying the inverse Laplace transform leads to

$$
\phi_+(u) = \frac{\lambda}{c} \int_0^u \phi_+(u-x)\, T_\rho f(x)\, dx \;+\; \frac{\lambda}{c}\, T_\rho B(u)
$$

$\diamondsuit$

Corollary 2.3.1.1. *For $u \geq 0$ the absolute ruin probability $\psi_\delta(u) = \psi_+(u)$ satisfies the integral equation*

$$
\psi_+(u) = \frac{\lambda}{c} \int_0^u \psi_+(u-x)\, \bar{F}(x)\, dx \;+\; \frac{\lambda}{c} \int_u^\infty B(y) dy \tag{2.16}
$$

Proof. It has been mentioned that by substituting $a = 0$ and $w(x, y) = 1$, the Gerber-Shiu function $\phi_+(u)$ is reduced to the absolute ruin probability $\psi_+(u)$. Furthermore, we have proven in Chapter 1 that for $a = 0$, it holds $\rho = \rho(a) = \rho(0) = 0$. Hence,

$$T_\rho f(x) = T_0 f(x) = \int_x^\infty f(y)dy = \bar{F}(x)$$

$$T_\rho B(u) = T_0 B(u) = \int_u^\infty B(y)dy$$

Consequently, under this scope, (2.14) leads to

$$\psi_+(u) = \frac{\lambda}{c} \int_0^u \psi_+(u-x)\, \bar{F}(x)\, dx \;+\; \frac{\lambda}{c} \int_u^\infty B(y)dy$$

$\diamondsuit$

Theorem 2.3.1.1. *For $u \geq 0$ the Gerber-Shiu function $\phi(u) = \phi_+(u)$ satisfies the following defective renewal equation*

$$\phi_+(u) = \frac{1}{1+\xi_a} \int_0^u \phi_+(u-x) g_a(x)dx \;+\; \frac{1}{1+\xi_a} H_a(u) \tag{2.17}$$

where the new quantities are defined as below:

- $\dfrac{1}{1+\xi_a} = \dfrac{\lambda}{c}\, \hat{\bar{F}}(\rho) = \dfrac{1}{1+\theta}\, \hat{f}_e(\rho)$
- $g_a(x) = \dfrac{T_\rho f(x)}{\hat{\bar{F}}(\rho)}$
- $\bar{G}_a(x) = \dfrac{T_\rho \bar{F}(x)}{\hat{\bar{F}}(\rho)}$
- $H_a(u) = \dfrac{T_\rho B(u)}{\hat{\bar{F}}(\rho)}$

in which, $f_e(x) = \frac{d}{dx} F_e(x)$ and $F_e(x)$ is the equilibrium function of the survival function $\bar{F}(x)$ defined in Definition 1.5.4.1.

Proof. In the Theorem 1.5.4.1 we have already defined the above functions using as indicator the discounting interest force δ. Now, this indicator will be replaced by a, because δ represents the debit interest force in this chapter, whereas a plays the role of the discounting interest force. So, avoiding further details, which can be found in the Theorem 1.5.4.1, we recall some results:

- $Z(x) = \dfrac{\lambda}{c}\, T_\rho f(x)$ and $\dfrac{1}{1+\xi_a} = \displaystyle\int_0^\infty Z(x)dx = \dfrac{\lambda}{c}\, \hat{\bar{F}}(\rho) = \dfrac{1}{1+\theta}\, \hat{f}_e(\rho)$
- $G_a(x) = \dfrac{\displaystyle\int_0^x Z(y)dy}{\displaystyle\int_0^\infty Z(y)dy} = (1+\xi_a) \displaystyle\int_0^x Z(y)dy$

- $\bar{G}_a(x) = \dfrac{T_\rho \bar{F}(x)}{\hat{\bar{F}}(\rho)}$
- $g_a(x) = \dfrac{d}{dx} G_a(x) = (1+\xi_a) Z(x) = \dfrac{\lambda}{c}\,(1+\xi_a)\, T_\rho f(x) = \dfrac{T_\rho f(x)}{\hat{\bar{F}}(\rho)}$

By contrast, the only difference is in the definition of $H_a(u)$, where

$$H_a(u) = \frac{\lambda}{c}\,(1+\xi_a)\, T_\rho B(u) = \frac{\lambda}{c}\left(\frac{c}{\lambda}\,\frac{1}{\hat{\bar{F}}(\rho)}\right) T_\rho B(u) = \frac{T_\rho B(u)}{\hat{\bar{F}}(\rho)}$$

Applying the corresponding substitutions in (2.14), we obtain

$$\begin{aligned} \phi_+(u) &= \int_0^u \phi_+(u-x)\frac{\lambda}{c}\, T_\rho f(x)dx \;+\; \frac{\lambda}{c}\, T_\rho B(u) \\ &= \frac{1}{1+\xi_a}\int_0^u \phi_+(u-x) g_a(x)dx + \frac{1}{1+\xi_a} H_a(u) \end{aligned}$$

$\diamondsuit$

Remark 2.3.1.1. *Putting $a = 0$ and $u = 0$ in (2.17), we obtain*

$$\phi_+(0) = \frac{\lambda}{c}\int_0^\infty B(y)dy \tag{2.18}$$

Indeed,

$$\phi_+(0) = \frac{1}{1+\xi_0} H_0(0) = \frac{1}{1+\xi_0}\,(1+\xi_0)\,\frac{\lambda}{c}\, T_0 B(0) = \frac{\lambda}{c}\int_0^\infty B(y)dy$$

Corollary 2.3.1.2. *For $u \geq 0$, the absolute ruin probability $\psi_\delta(u) = \psi_+(u)$ satisfies the defective renewal equation*

$$\psi_+(u) = \frac{1}{1+\theta}\int_0^u \psi_+(u-x) f_e(x)dx \;+\; \frac{\lambda}{c}\int_u^\infty B(y)dy \tag{2.19}$$

and

$$\psi_+(0) \;=\; \frac{\lambda}{c}\int_0^\infty B(y)dy$$

Proof. Substituting $a = 0$ and $w(x, y) = 1$, the Gerber-Shiu function $\phi_+(u)$ is reduced to the absolute ruin probability $\psi_+(u)$. For $a = 0$, we have $\rho = \rho(0) = 0$ and $\xi_0 = \theta$, as we have already seen in Proposition 1.5.4.1. Moreover, using the formula $c = (1+\theta)\lambda\mu_1$ of premium c, we obtain

$$\begin{aligned} (i)\; g_a(x) &= g_0(x) = \frac{\lambda}{c}\,(1+\xi_0)\, T_0 f(x) = \frac{\lambda}{c}\,(1+\theta)\, \bar{F}(x) \\ &= \frac{\lambda(1+\theta)}{(1+\theta)\lambda\mu_1}\, \bar{F}(x) = \frac{\bar{F}(x)}{\mu_1} = f_e(x) \end{aligned}$$

$$(ii)\ H_a(u) \quad = H_0(u) = (1+\xi_0)\ \frac{\lambda}{c}\ T_0 B(u) = (1+\theta)\ \frac{\lambda}{c}\ \int_u^{\infty} B(y)dy$$

Thus, we derive from (2.17)

$$\begin{aligned} \psi_+(u) \quad &= \frac{1}{1+\theta} \int_0^u \psi_+(u-x) f_e(x)dx \ + \ \frac{1}{1+\theta}\ (1+\theta)\ \frac{\lambda}{c}\ \int_u^{\infty} B(y)dy \\ &= \frac{1}{1+\theta} \int_0^u \psi_+(u-x) f_e(x)dx \ + \ \frac{\lambda}{c}\ \int_u^{\infty} B(y)dy \end{aligned}$$

$\diamondsuit$

2.3.2 Solution of the Defective Renewal Equation

The idea of using a compound geometric distribution to solve the defective renewal equation for $\phi_+(u)$ belongs to Lin and Willmot (1999), Willmot and Woo (2017). The following Theorem and its proof stem from those papers and the academic notes of professor E. Chadjikonstantinidis (2016) about the properties satisfied by a compound geometric distribution.

Theorem 2.3.2.1. *For $u \geq 0$, the solution of the defective renewal equation (2.17), which is satisfied by the Gerber-Shiu function and described in Theorem 2.3.1.1, is given by*

$$\phi_+(u) = \frac{1}{\xi_a}\ \int_0^u H_a(u-x)dK_a(x) \ + \ \frac{1}{1+\xi_a} H_a(u) \tag{2.20}$$

or

$$\phi_+(u) = \frac{1}{\xi_a} H_a(u) - \frac{1}{\xi_a} H_a(0)\bar{K}_a(u) - \frac{1}{\xi_a}\ \int_0^u H_a^{'}(u-x)\bar{K}_a(x)dx \tag{2.21}$$

where

$$\bar{K}_a(x) = \sum_{n=1}^{\infty} \left(\frac{\xi_a}{1+\xi_a}\right)\left(\frac{1}{1+\xi_a}\right)^n \bar{G}_a^{*n}(x)$$

Proof. For $u \geq 0$, we have seen that the Gerber-Shiu function $\phi_+(u)$ obeys the defective renewal equation (2.17)

$$\phi_+(u) = \frac{1}{1+\xi_a}\ \int_0^u \phi_+(u-x) g_a(x)dx \ + \ \frac{1}{1+\xi_a}\ H_a(u)$$

Putting

$$\varphi = \frac{1}{1+\xi_a} \quad \text{and} \quad r(x) = \frac{1}{1+\xi_a} H_a(x)$$

(2.17) is converted into

$$\phi_+(u) = \varphi \int_0^u \phi_+(u-x) g_a(x) dx \; + \; r(x)$$

Applying Laplace transform leads to

$$\hat{\phi}_+(s) = \varphi \cdot \hat{\phi}_+(s) \cdot \hat{g}_a(s) \; + \; \hat{r}(s) \Longrightarrow [1 - \varphi \cdot \hat{g}_a(s)] \cdot \hat{\phi}_+(s) = \hat{r}(s)$$

So,

$$\hat{\phi}_+(s) = \frac{\hat{r}(s)}{1 - \varphi \cdot \hat{g}_a(s)} \tag{2.22}$$

Let

$$S = \begin{cases} \sum_{i=1}^{M} W_i, & M \geq 1 \\ 0, & M = 0 \end{cases}$$

be a compound geometric random variable, where $K_a(u)$ is the distribution function of S, $g_a(u) = \frac{d}{du} G_a(u)$ is the probability density function of W_i with $G_a(0) = 0$ and $M \sim \text{Geom}(1-\varphi)$. From Lemma 1.4.1, we have

i. $K_a(0) = Pr\,(S \leq 0) = Pr\,(S = 0) = Pr\,(M = 0) = 1 - \varphi$

ii. $\hat{k}_a(s) = \int_0^\infty e^{-sx} dK_a(x) \; + \; K_a(0) = \int_0^\infty e^{-sx} dK_a(x) \; + \; (1-\varphi)$

iii. $\hat{k}_a(s) = \dfrac{1-\varphi}{1 - \varphi \cdot \hat{g}_a(s)} \Longrightarrow \dfrac{1}{1 - \varphi \cdot \hat{g}_a(s)} = \dfrac{\hat{k}_a(s)}{1-\varphi}$

iv. $\bar{K}_a(x) = \sum_{n=1}^{\infty} (1-\varphi)\varphi^n \bar{G}_a^{*n}(x) = \sum_{n=1}^{\infty} \left(\dfrac{\xi_a}{1+\xi_a}\right) \left(\dfrac{1}{1+\xi_a}\right)^n \bar{G}_a^{*n}(x)$

Applying (iii) and (ii) successively in (2.22), we obtain

$$\begin{aligned} \hat{\phi}_+(s) &= \frac{1}{1-\varphi}\, \hat{k}_a(s)\, \hat{r}(s) = \frac{1}{1-\varphi} \left\{ \left[\int_0^\infty e^{-sx} dK_a(x) \; + \; (1-\varphi) \right] \hat{r}(s) \right\} \\ &= \frac{1}{1-\varphi} \left(\int_0^\infty e^{-sx} dK_a(x) \right) \cdot \hat{r}(s) \; + \; \hat{r}(s) \end{aligned}$$

Now, the inverse Laplace transform yields

$$\phi_+(u) = \frac{1}{1-\varphi} \int_0^u r(u-x) dK_a(x) \; + \; r(u)$$

Substituting

- $\varphi = \dfrac{1}{1+\xi_a} \Longrightarrow 1 - \varphi = \dfrac{\xi_a}{1+\xi_a} \Longrightarrow \dfrac{1}{1-\varphi} = \dfrac{1+\xi_a}{\xi_a}$

- $r(u) = \frac{1}{1+\xi_a} H_a(u)$

leads to the first form (2.20) of the solution

$$\begin{aligned}
\phi_+(u) &= \frac{1+\xi_a}{\xi_a} \int_0^u \frac{1}{1+\xi_a} H_a(u-x) dK_a(x) \; + \; \frac{1}{1+\xi_a} H_a(u) \\
&= \frac{1}{\xi_a} \int_0^u H_a(u-x) dK_a(x) \; + \; \frac{1}{1+\xi_a} H_a(u) \qquad (*)
\end{aligned}$$

In order to receive the second form (2.21) of the solution, the integral, in the last equation $(*)$, can be written as

$$\begin{aligned}
\int_0^u H_a(u-x) dK_a(x) &= -\int_0^u H_a(u-x) \bar{K}_a^{'}(x) dx \\
&= -\left[H_a(u-x) \bar{K}_a(x) \right]_{x=0}^u - \int_0^u H_a^{'}(u-x) \bar{K}_a(x) dx \\
&= -H_a(0) \bar{K}_a(u) + H_a(u) \bar{K}_a(0) - \int_0^u H_a^{'}(u-x) \bar{K}_a(x) dx
\end{aligned}$$

where

$$K_a(0) = 1 - \varphi \Longrightarrow \bar{K}_a(0) = \varphi = \frac{1}{1+\xi_a}$$

Finally, substituting them in $(*)$, we are led to the desirable result

$$\begin{aligned}
\phi_+(u) &= \frac{1}{\xi_a} \left[-H_a(0) \bar{K}_a(u) + \frac{1}{1+\xi_a} H_a(u) - \int_0^u H_a^{'}(u-x) \bar{K}_a(x) dx \right] \; + \; \frac{1}{1+\xi_a} H_a(u) \\
&= \left[1 + \frac{1}{\xi_a} \right] \left[\frac{1}{1+\xi_a} H_a(u) \right] - \frac{1}{\xi_a} H_a(0) \bar{K}_a(u) - \frac{1}{\xi_a} \int_0^u H_a^{'}(u-x) \bar{K}_a(x) dx \\
&= \frac{1}{\xi_a} H_a(u) - \frac{1}{\xi_a} H_a(0) \bar{K}_a(u) - \frac{1}{\xi_a} \int_0^u H_a^{'}(u-x) \bar{K}_a(x) dx
\end{aligned}$$

$\diamondsuit$

Remark 2.3.2.1. *The expressions for ξ_a, $g_a(x)$, $\bar{G}_a(x)$ and $\bar{K}_a(x)$ remain unchanged in comparison to Chapter 1. The only thing that needs further study is the $H_a(u)$, which includes the term $\phi_-(u)$.*

Proposition 2.3.2.1. *For $u \geq 0$, the solution to the defective renewal equation (2.19), which is satisfied by the absolute ruin probability and described in Corollary 2.3.1.2, is given by*

$$\psi_+(u) = \frac{1}{\theta} \int_0^u H_0(u-x) dK_0(x) \; + \; \frac{1}{1+\theta} H_0(u) \qquad (2.23)$$

where

$$H_0(u) = \frac{\lambda}{c}(1+\theta)\int_u^\infty B(y)dy$$

and

$$\bar{K}_0(x) = \sum_{n=1}^{\infty}\left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)^n \bar{F}_e^{*n}(x)$$

Proof. As it is known, the absolute ruin probability arises from the Gerber-Shiu function for $a = 0$ and $w(x, y) = 1$. Furthermore, in this case, we have seen that $\rho = 0$, $\xi_0 = \theta$ and

- $\hat{\bar{F}}(0) = \mu_1$
- $\bar{G}_0(x) = \dfrac{T_0\bar{F}(x)}{\hat{\bar{F}}(0)} = \dfrac{\int_x^\infty \bar{F}(y)dy}{\int_0^\infty \bar{F}(y)dy} = \bar{F}_e(x)$
- $H_0(u) = (1+\xi_0)\,\dfrac{\lambda}{c}\,T_0B(u) = (1+\theta)\,\dfrac{\lambda}{c}\,\int_u^\infty B(y)dy$

So, from (2.20) we obtain

$$\begin{aligned}\psi_+(u) &= \frac{1}{\xi_0}\int_0^u H_0(u-x)dK_0(x) + \frac{1}{1+\xi_0}H_0(u)\\ &= \frac{1}{\theta}\int_0^u H_0(u-x)dK_0(x) + \frac{1}{1+\theta}H_0(u)\end{aligned}$$

where

$$\begin{aligned}\bar{K}_0(x) &= \sum_{n=1}^{\infty}\left(\frac{\xi_0}{1+\xi_0}\right)\left(\frac{1}{1+\xi_0}\right)^n \bar{G}_0^{*n}(x)\\ &= \sum_{n=1}^{\infty}\left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)^n \bar{F}_e^{*n}(x)\end{aligned}$$

$\diamondsuit$

2.4 Boundary Conditions for $\phi(u)$

The boundary conditions, which will be presented in this paragraph, are indispensable when we want to find explicit expressions for $\phi_-(u)$. Cai (2000) shows these boundary conditions and we express them, by giving the detailed calculations which are described briefly by him. Up to now, some boundary conditions have been already mentioned. More specifically, from (2.1) we have

$$\lim_{u\to\infty} \phi(u) = \lim_{u\to\infty} \phi_+(u) = 0$$

and from (2.12)

$$\phi_+(0) = \phi_-(0-)$$

Regarding (2.10), $\phi_-(u)$ can be written as

$$\begin{aligned}
\phi_-(u) \quad = \quad & \lambda\, (\delta u + c)^{\frac{\lambda+a}{\delta}} \int_u^0 (\delta y + c)^{-1-\frac{\lambda+a}{\delta}} \left[\int_0^{y+\frac{c}{\delta}} \phi_-(y-x) f(x) dx \right] dy \\
& + \lambda\, (\delta u + c)^{\frac{\lambda+a}{\delta}} \int_u^0 (\delta y + c)^{-1-\frac{\lambda+a}{\delta}} \gamma_\delta(y) dy \\
& + \lambda (\delta u + c)^{\frac{\lambda+a}{\delta}} \left[c^{-1-\frac{\lambda+a}{\delta}} \int_0^\infty e^{-(\lambda+a)\frac{z}{c}} \left[\int_0^z \phi_+(z-x) f(x) dx + B(z) \right] dz \right]
\end{aligned}$$

Let

$$g(y) = (\delta y + c)^{-1-\frac{\lambda+a}{\delta}} \int_0^{y+\frac{c}{\delta}} \phi_-(y-x) f(x) dx$$

and

$$I = c^{-1-\frac{\lambda+a}{\delta}} \int_0^\infty e^{-(\lambda+a)\frac{z}{c}} \left[\int_0^z \phi_+(z-x) f(x) dx + B(z) \right] dz \qquad \in \mathcal{R}$$

Hence, $\phi_-(u)$ can be written as

$$\begin{aligned}
\phi_-(u) \quad = \quad & \lambda\, (\delta u + c)^{\frac{\lambda+a}{\delta}} \int_u^0 g(y) dy \\
& + \lambda\, (\delta u + c)^{\frac{\lambda+a}{\delta}} \int_u^0 (\delta y + c)^{-1-\frac{\lambda+a}{\delta}} \gamma_\delta(y) dy \\
& + \lambda\, (\delta u + c)^{\frac{\lambda+a}{\delta}}\; I
\end{aligned}$$

As $u \to -\left(\frac{c}{\delta}\right)^+$, the last term, on the right-hand side of the equation, is equal to 0. Hence,

$$\begin{aligned}
\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \phi_-(u) \quad = \quad & \lim_{u\to-\left(\frac{c}{\delta}\right)^+} \lambda\, (\delta u + c)^{\frac{\lambda+a}{\delta}} \int_u^0 g(y) dy \\
& + \lim_{u\to-\left(\frac{c}{\delta}\right)^+} \lambda\, (\delta u + c)^{\frac{\lambda+a}{\delta}} \int_u^0 (\delta y + c)^{-1-\frac{\lambda+a}{\delta}} \gamma_\delta(y) dy \qquad (**)
\end{aligned}$$

According to the first limit in $(**)$, it can be observed that

- if $\displaystyle\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \int_u^0 g(y) dy < \infty$, then

$$\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \lambda\,(\delta u+c)^{\frac{\lambda+a}{\delta}} \int_u^0 g(y)dy = 0$$

- if $\displaystyle\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \int_u^0 g(y)dy = \infty$, applying L'Hospital's rule, we will obtain

$$\begin{aligned}
\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \lambda\,(\delta u+c)^{\frac{\lambda+a}{\delta}} \int_u^0 g(y)dy &= \lim_{u\to-\left(\frac{c}{\delta}\right)^+} \frac{-\lambda g(u)}{-\frac{\lambda+a}{\delta}\,\delta\,(\delta u+c)^{-1-\frac{\lambda+a}{\delta}}} \\
&= \lim_{u\to-\left(\frac{c}{\delta}\right)^+} \frac{\lambda}{\lambda+a}\int_0^{u+\frac{c}{\delta}} \phi_-(u-x)f(x)dx \\
&= 0
\end{aligned}$$

As a result, in all circumstances, while $u \to -\left(\frac{c}{\delta}\right)^+$, the first limit in $(**)$ equals 0, which leads to

$$\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \phi_-(u) = \lim_{u\to-\left(\frac{c}{\delta}\right)^+} \lambda\,(\delta u+c)^{\frac{\lambda+a}{\delta}} \int_u^0 (\delta y+c)^{-1-\frac{\lambda+a}{\delta}}\gamma_\delta(y)dy$$

Now, there are two possible options for this limit:

i. if $\displaystyle\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \int_u^0 (\delta y+c)^{-1-\frac{\lambda+a}{\delta}}\gamma_\delta(y)dy < \infty$, then obviously

$$\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \phi_-(u) = 0 \tag{2.24}$$

ii. if $\displaystyle\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \int_u^0 (\delta y+c)^{-1-\frac{\lambda+a}{\delta}}\gamma_\delta(y)dy = \infty$, using once more L'Hospital's rule, we are led to

$$\begin{aligned}
\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \phi_-(u) &= \lim_{u\to-\left(\frac{c}{\delta}\right)^+} \frac{\left(\lambda \displaystyle\int_u^0 (\delta y+c)^{-1-\frac{\lambda+a}{\delta}}\gamma_\delta(y)dy\right)'}{\left((\delta u+c)^{-\frac{\lambda+a}{\delta}}\right)'} \\
&= \lim_{u\to-\left(\frac{c}{\delta}\right)^+} \frac{-\lambda(\delta u+c)^{-1-\frac{\lambda+a}{\delta}}\gamma_\delta(u)}{-\dfrac{\lambda+a}{\delta}\,\delta\,(\delta u+c)^{-1-\frac{\lambda+a}{\delta}}}
\end{aligned}$$

So,

$$\lim_{u\to-\left(\frac{c}{\delta}\right)^+} \phi_-(u) = \frac{\lambda}{\lambda+a}\,\gamma_\delta(-\frac{c}{\delta}) \tag{2.25}$$

2.5 Exponential Claims

We are going to study the case whose claim sizes X_i obey an exponential distribution with parameter β, i.e. $X_i \sim \text{Exp}(\beta)$ and $f(x) = f_X(x) = \beta e^{-\beta x}$, $\beta > 0$, $x \geq 0$. Based on the explicit results for exponential claims of Cai (2000) and using the general solution of a second order linear differential equation described by Alikakos and Kalogeropoulos (2003), we present, step by step, the final form of $\phi_-(u)$, which will be applied in the following subsection in order for the absolute ruin probability $\psi_+(u)$ to be estimated.

Proposition 2.5.1. *For $-\frac{c}{\delta} < u < 0$, the Gerber-Shiu function $\phi(u) = \phi_-(u)$ satisfies the following second order linear differential equation*

$$(\delta u+c)\phi_-^{''}(u)+[\beta\delta u + \beta c + \delta - \lambda - a]\,\phi_-^{'}(u)-\beta a\phi_-(u) = -\lambda\left(\gamma_\delta^{'}(u) + \beta\gamma_\delta(u)\right) \tag{2.26}$$

Proof. Referring to (2.8), for $-\frac{c}{\delta} < u < 0$, the Gerber-Shiu function $\phi_-(u)$ satisfies

$$\phi_-^{'}(u) = \frac{\lambda + a}{\delta u + c}\,\phi_-(u) \; - \; \frac{\lambda}{\delta u + c}\left[\int_0^{u+\frac{c}{\delta}} \phi_-(u-y)f(y)dy + \gamma_\delta(u)\right] \Longrightarrow$$

$$(\delta u + c)\,\phi_-^{'}(u) = (\lambda + a)\,\phi_-(u) \; - \; \lambda\int_0^{u+\frac{c}{\delta}} \phi_-(u-y)\beta e^{-\beta y}dy - \lambda\,\gamma_\delta(u)$$

In the above integral, we set $x = u - y \Longrightarrow y = u - x$ & $dy = -dx$. The new boundaries of integration are $x \to u$ while $y \to 0$ and $x \to -\frac{c}{\delta}$ while $y \to u + \frac{c}{\delta}$. Thus,

$$\begin{aligned}(\delta u + c)\,\phi_-^{'}(u) &= (\lambda + a)\,\phi_-(u) \; - \; \lambda\int_{-\frac{c}{\delta}}^{u} \phi_-(x)\beta e^{-\beta(u-x)}dx - \lambda\,\gamma_\delta(u) \\ &= (\lambda + a)\,\phi_-(u) \; - \; \lambda\beta e^{-\beta u}\int_{-\frac{c}{\delta}}^{u} \phi_-(x)e^{\beta x}dx - \lambda\,\gamma_\delta(u)\end{aligned}$$

Differentiating with respect to u yields

$$\begin{aligned}(\delta u + c)\,\phi_-^{''}(u) + \delta\,\phi_-^{'}(u) &= (\lambda + a)\,\phi_-^{'}(u) + \beta\lambda\beta e^{-\beta u}\int_{-\frac{c}{\delta}}^{u} \phi_-(x)e^{\beta x}dx \\ &\quad -\lambda\beta e^{-\beta u}\phi_-(u)e^{\beta u} - \lambda\gamma_\delta^{'}(u) \qquad \Longrightarrow\end{aligned}$$

$$\begin{aligned}(\delta u + c)\,\phi_-^{''}(u) + \delta\,\phi_-^{'}(u) - (\lambda + a)\,\phi_-^{'}(u) + \lambda\beta\phi_-(u) &= \lambda\beta\int_{-\frac{c}{\delta}}^{u} \phi_-(x)\beta e^{-\beta(u-x)}dx \\ &\quad -\lambda\gamma_\delta^{'}(u)\end{aligned}$$

Bringing back the initial variable of integration, we obtain

$$(\delta u + c)\,\phi_-^{''}(u) + \delta\,\phi_-^{'}(u) - (\lambda + a)\,\phi_-^{'}(u) + \lambda\beta\phi_-(u) \;=\; \lambda\beta\int_0^{u+\frac{c}{\delta}} \phi_-(u-y)\beta e^{-\beta y}dy$$
$$-\lambda\gamma_\delta^{'}(u)$$

Multiplying (2.8) by $\beta(\delta u + c)$ and adding it down to the above equation, we obtain

$$(\delta u + c)\,\phi_-^{''}(u) + \delta\,\phi_-^{'}(u) - (\lambda + a)\,\phi_-^{'}(u) + \lambda\beta\phi_-(u) + \beta(\delta u + c)\phi_-^{'}(u) \;=$$

$$\lambda\beta\int_0^{u+\frac{c}{\delta}} \phi_-(u-y)\beta e^{-\beta y}dy - \lambda\gamma_\delta^{'}(u) + \beta(\lambda + a)\,\phi_-(u)$$
$$-\,\lambda\beta\int_0^{u+\frac{c}{\delta}} \phi_-(u-y)\beta e^{-\beta y}dy - \lambda\beta\gamma_\delta(u) \qquad \Longrightarrow$$
$$(\delta u+c)\phi_-^{''}(u) + [\beta\delta u + \beta c + \delta - \lambda - a]\,\phi_-^{'}(u) - \beta a\phi_-(u) = -\lambda\left(\gamma_\delta^{'}(u) + \beta\gamma_\delta(u)\right)$$

which is exactly what we want to prove.

$\diamondsuit$

Since our final objective is to find the absolute ruin probability $\psi_+(u)$, substituting $a = 0$ in (2.26) yields

$$(\delta u + c)\phi_-^{''}(u) + [\beta\delta u + \beta c + \delta - \lambda]\,\phi_-^{'}(u) = -\lambda\left(\gamma_\delta^{'}(u) + \beta\gamma_\delta(u)\right)$$

which leads to the next corollary.

Corollary 2.5.1. *For $-\frac{c}{\delta} < u < 0$ and $a = 0$, the Gerber-Shiu function $\phi_-(u)$ satisfies the following second order linear differential equation*

$$\phi_-^{''}(u) + f(u)\,\phi_-^{'}(u) = g(u) \tag{2.27}$$

where

$$f(u) \;=\; \frac{\beta\delta u + \beta c + \delta - \lambda}{\delta u + c} \;=\; \frac{\delta u + c + \frac{1}{\beta}(\delta - \lambda)}{\frac{1}{\beta}(\delta u + c)}$$

and

$$g(u) \;=\; -\frac{\lambda\left(\gamma_\delta^{'}(u) + \beta\gamma_\delta(u)\right)}{\delta u + c} \;=\; -\frac{\lambda\left(\frac{1}{\beta}\gamma_\delta^{'}(u) + \gamma_\delta(u)\right)}{\frac{1}{\beta}(\delta u + c)}$$

$\diamondsuit$

By solving the previous differential equation (see book [27] of Alikakos and Kalogeropoulos (2003), chapter 1.3, pg 9, or Appendix A.1), the general solution of (2.27) is given by

$$
\begin{aligned}
\phi_-(u) &= c_1 + \int_0^u e^{-\int f(x)dx} \left(c_2 + \int_0^x e^{\int f(y)dy} g(y)dy \right) dx \\
&= c_1 + c_2 \int_0^u e^{-\int f(x)dx} dx \ + \ \int_0^u e^{-\int f(x)dx} \left(\int_0^x e^{\int f(y)dy} g(y)dy \right) dx
\end{aligned}
$$

So, by replacing properly the integrals in the above expression, we obtain its short form

$$
\phi_-(u) \ = \ c_1 - c_2 \ P(u) - Q(u) \tag{2.28}
$$

Now, in order to find the $\phi_-(u)$, we should estimate the terms $P(u)$, $Q(u)$, c_1 and c_2. We observe that $f(u)$ can be written as

$$
f(u) \ = \ \frac{\delta u + c + \frac{1}{\beta}(\delta - \lambda)}{\frac{1}{\beta}(\delta u + c)} \ = \ \beta \ + \ \frac{\delta - \lambda}{\delta u + c}
$$

So, integrating $f(u)$ implies

$$
\begin{aligned}
\int f(u)du &= \int \left(\beta \ + \ \frac{\delta - \lambda}{\delta u + c} \right) du \ = \ \beta u \ + \ \frac{\delta - \lambda}{\delta} \ \ln(\delta u + c) \\
&= \beta u \ + \ \ln(\delta u + c)^{\frac{\delta - \lambda}{\delta}}
\end{aligned}
$$

Consequently,

$$
e^{\int f(u)du} \ = \ e^{\beta u} \ e^{\ln(\delta u + c)^{\frac{\delta - \lambda}{\delta}}} \ = \ (\delta u + c)^{\frac{\delta - \lambda}{\delta}} \ e^{\beta u}
$$

and

$$
e^{-\int f(u)du} \ = \ (\delta u + c)^{-\frac{\delta - \lambda}{\delta}} \ e^{-\beta u}
$$

Now, $P(u)$ and $Q(u)$ are given by

$$
P(u) \ = \ -\int_0^u e^{-\int f(x)dx} \ dx \ = \ \int_u^0 (\delta x + c)^{-\frac{\delta - \lambda}{\delta}} \ e^{-\beta x} \ dx
$$

and

$$
\begin{aligned}
Q(u) &= -\int_0^u e^{-\int f(x)dx} \left(\int_0^x e^{\int f(y)dy} g(y)dy \right) dx \\
&= \int_u^0 (\delta x + c)^{-\frac{\delta - \lambda}{\delta}} \ e^{-\beta x} \left(\int_0^x (\delta y + c)^{\frac{\delta - \lambda}{\delta}} \ e^{\beta y} \ g(y) \ dy \right) \ dx
\end{aligned}
$$

Remark 2.5.1. *From the above formulas, it is obvious that*

$$
P(0) = Q(0) = 0
$$

Depended on (2.28), in order for $\phi_-(u)$ to be computed, the only thing that has been left is the estimation of the constants c_1 and c_2. For this purpose we should use initial conditions. Firstly, from (2.28) and Remark 2.5.1, it can be extracted that

$$\phi_-(0^-) \;=\; \lim_{u\to 0^-} \phi_-(u) \;=\; \lim_{u\to 0^-} (c_1 - c_2\, P(u) - Q(u)) \quad\Longrightarrow$$

$$\phi_-(0^-) \;=\; c_1$$

From (2.12) we have

$$\phi_-(0^-) = \phi_+(0)$$

whereas, from (2.18)

$$\phi_+(0) = \frac{\lambda}{c}\int_0^\infty B(t)dt$$

Thus, we conclude that

$$c_1 \;=\; \frac{\lambda}{c}\int_0^\infty B(t)dt \tag{2.29}$$

Now, we are going to estimate the integral $\displaystyle\int_0^\infty B(t)dt$. Using the definition (2.3) of $B(u)$ yields

$$\int_0^\infty B(t)dt \;=\; \int_0^\infty \left(\int_t^{t+\frac{c}{\delta}} \phi_-(t-y)f(y)dy + \gamma_\delta(t)\right) dt$$

Substituting $x = t - y \Longrightarrow y = t - x$ & $dy = -dx$, the new boundaries of integration are $x \to 0$ while $y \to t$ and $x \to -\frac{c}{\delta}$ while $y \to t + \frac{c}{\delta}$. Hence,

$$\begin{aligned}
\int_0^\infty B(t)dt \;&=\; \int_0^\infty \left(\int_{-\frac{c}{\delta}}^0 \phi_-(x)\beta e^{-\beta(t-x)}dx + \gamma_\delta(t)\right) dt \\
&=\; \int_0^\infty \beta e^{-\beta t}\left(\int_{-\frac{c}{\delta}}^0 \phi_-(x)e^{\beta x}dx\right) dt \;+\; \int_0^\infty \gamma_\delta(t)dt \\
&=\; \left(\int_{-\frac{c}{\delta}}^0 \phi_-(x)e^{\beta x}dx\right)\left(\int_0^\infty \beta e^{-\beta t}dt\right) \;+\; \int_0^\infty \gamma_\delta(t)dt
\end{aligned}$$

Notice that

$$\int_0^\infty \beta e^{-\beta t}dt \;=\; \int_0^\infty f(t)dt \;=\; 1$$

So,

$$\begin{aligned}
\int_0^\infty B(t)dt &= \int_{-\frac{c}{\delta}}^0 \phi_-(x)e^{\beta x}dx \;+\; \int_0^\infty \gamma_\delta(t)dt \\
&= \int_{-\frac{c}{\delta}}^0 (c_1 - c_2\, P(x) - Q(x))\, e^{\beta x}dx \;+\; \int_0^\infty \gamma_\delta(t)dt \\
&= c_1 \int_{-\frac{c}{\delta}}^0 e^{\beta x}dx - c_2 \int_{-\frac{c}{\delta}}^0 P(x)e^{\beta x}dx - \left[\int_{-\frac{c}{\delta}}^0 Q(x)e^{\beta x}dx - \int_0^\infty \gamma_\delta(t)dt\right]
\end{aligned}$$

Making the corresponding substitutions in the above expression, we obtain a more convenient form, which will be used below, i.e.

$$\int_0^\infty B(t)dt \;=\; c_1\beta_1 - c_2\beta_2 - \beta_3 \tag{2.30}$$

where

$$\beta_1 \;=\; \int_{-\frac{c}{\delta}}^0 e^{\beta x}dx \;=\; \left[\frac{1}{\beta}\, e^{\beta x}\right]_{x=-\frac{c}{\delta}}^0 \;=\; \frac{1}{\beta}\left(1 - e^{-\beta\frac{c}{\delta}}\right)$$

$$\begin{aligned}
\beta_2 &= \int_{-\frac{c}{\delta}}^0 P(x)e^{\beta x}dx \;=\; \int_{-\frac{c}{\delta}}^0 e^{\beta x}\left(\int_x^0 e^{-\beta y}(\delta y + c)^{-\frac{\delta-\lambda}{\delta}}dy\right)dx \\
&= \left[\frac{1}{\beta}\, e^{\beta x}\left(\int_x^0 e^{-\beta y}(\delta y + c)^{-\frac{\delta-\lambda}{\delta}}dy\right)\right]_{x=-\frac{c}{\delta}}^0 \;+\; \int_{-\frac{c}{\delta}}^0 \frac{1}{\beta}\, e^{\beta x}e^{-\beta x}(\delta x + c)^{-\frac{\delta-\lambda}{\delta}}dx \\
&= -\frac{1}{\beta}e^{-\beta\frac{c}{\delta}} \int_{-\frac{c}{\delta}}^0 e^{-\beta y}(\delta y + c)^{-\frac{\delta-\lambda}{\delta}}dy \;+\; \frac{1}{\beta}\int_{-\frac{c}{\delta}}^0 (\delta x + c)^{\frac{\lambda}{\delta}-1}dx \\
&= -\frac{1}{\beta}\, e^{-\beta\frac{c}{\delta}}\, P\left(-\tfrac{c}{\delta}\right) \;+\; \left[\frac{1}{\beta}\,\frac{\delta}{\lambda}\,\frac{(\delta x + c)^{\frac{\lambda}{\delta}}}{\delta}\right]_{x=-\frac{c}{\delta}}^0 \\
&= -\frac{1}{\beta}\, e^{-\beta\frac{c}{\delta}}\, P\left(-\tfrac{c}{\delta}\right) \;+\; \frac{1}{\beta}\,\frac{c^{\frac{\lambda}{\delta}}}{\lambda} \\
&= \frac{1}{\beta}\left[-P\left(-\tfrac{c}{\delta}\right)\, e^{-\beta\frac{c}{\delta}} \;+\; \frac{c^{\frac{\lambda}{\delta}}}{\lambda}\right]
\end{aligned}$$

and

$$\beta_3 \;=\; \int_{-\frac{c}{\delta}}^0 Q(x)e^{\beta x}dx \;-\; \int_0^\infty \gamma_\delta(t)dt$$

To sum up, we have found

- $\beta_1 \ = \ \dfrac{1}{\beta}\left(1 - e^{-\beta\frac{c}{\delta}}\right)$
- $\beta_2 \ = \ \dfrac{1}{\beta}\left[-P\left(-\frac{c}{\delta}\right)\ e^{-\beta\frac{c}{\delta}} \ + \ \dfrac{c^{\frac{\lambda}{\delta}}}{\lambda}\right]$
- $\beta_3 \ = \ \displaystyle\int_{-\frac{c}{\delta}}^{0} Q(x)e^{\beta x}dx \ - \ \int_0^{\infty} \gamma_\delta(t)dt$

From (2.29) and (2.30), we obtain

$$\phi_-(0^-) \ = \ \frac{\lambda}{c}\int_0^{\infty} B(t)dt \quad \Longrightarrow \quad c_1 \ = \ \frac{\lambda}{c}\ [c_1\beta_1 - c_2\beta_2 - \beta_3] \Longrightarrow$$

$$c_1 \ = \ \frac{-\lambda\beta_2 c_2 - \lambda\ \beta_3}{c - \lambda\beta_1} \tag{2.31}$$

Another initial condition that we will use is the boundary condition studied in section 2.4. We have seen that there are two possible options for the $\lim\limits_{u\to -\frac{c}{\delta}^+} \phi_-(u)$, which are mentioned in (2.25) and (2.24). We will examine both of them, however, in this project, we will use the results from the (2.25). Namely,

1st Case

Let $\displaystyle\lim_{u\to -\left(\frac{c}{\delta}\right)^+} \int_u^0 (\delta x + c)^{-1-\frac{\lambda+a}{\delta}}\gamma_\delta(x)dx \ = \ \infty$. Then, from (2.25), it holds that

$$\lim_{u\to -\left(\frac{c}{\delta}\right)^+} \phi_-(u) = \frac{\lambda}{\lambda + a}\ \gamma_\delta(-\frac{c}{\delta})$$

and applying $a = 0$ we are led to

$$\lim_{u\to -\left(\frac{c}{\delta}\right)^+} \phi_-(u) \ = \ \gamma_\delta(-\frac{c}{\delta})$$

Nevertheless, considering (2.28) we point out

$$\lim_{u\to -\left(\frac{c}{\delta}\right)^+} \phi_-(u) \ = \ c_1 - c_2\ P(-\frac{c}{\delta}) - Q(-\frac{c}{\delta})$$

i.e.

$$c_1 - c_2\ P(-\tfrac{c}{\delta}) - Q(-\tfrac{c}{\delta}) \ = \ \gamma_\delta(-\tfrac{c}{\delta})$$

Substituting c_1, from (2.31), implies

$$\frac{-\lambda\beta_2 c_2 - \lambda\,\beta_3}{c-\lambda\beta_1} - c_2\,P(-\tfrac{c}{\delta}) - Q(-\tfrac{c}{\delta}) \;=\; \gamma_\delta(-\tfrac{c}{\delta}) \qquad \Longrightarrow$$

$$-\lambda\beta_2 c_2 - \lambda\,\beta_3 - c_2\,(c-\lambda\beta_1)\,P(-\tfrac{c}{\delta}) - (c-\lambda\beta_1)\,Q(-\tfrac{c}{\delta}) \;=\; (c-\lambda\beta_1)\,\gamma_\delta(-\tfrac{c}{\delta}) \qquad \Longrightarrow$$

$$\left[\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\tfrac{c}{\delta})\right] c_2 \;=\; -\lambda\beta_3 - (c-\lambda\beta_1)\,Q(-\tfrac{c}{\delta}) - (c-\lambda\beta_1)\,\gamma_\delta(-\tfrac{c}{\delta})$$

So,

$$c_2 = \frac{-\lambda\beta_3 - (c-\lambda\beta_1)\left[Q(-\tfrac{c}{\delta}) + \gamma_\delta(-\tfrac{c}{\delta})\right]}{\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\tfrac{c}{\delta})} \tag{2.32}$$

Applying (2.32) in (2.31) implies the value of c_1 as well

$$\begin{aligned}
(c-\lambda\beta_1)c_1 &= -\lambda\beta_2 c_2 - \lambda\,\beta_3 \\
&= -\lambda\beta_2\,\frac{-\lambda\beta_3 - (c-\lambda\beta_1)\left[Q(-\tfrac{c}{\delta}) + \gamma_\delta(-\tfrac{c}{\delta})\right]}{\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\tfrac{c}{\delta})} - \lambda\,\beta_3 \\
&= \frac{\lambda\beta_2\beta_3 + \lambda\beta_2(c-\lambda\beta_1)\left[Q(-\tfrac{c}{\delta}) + \gamma_\delta(-\tfrac{c}{\delta})\right] - \lambda\beta_2\beta_3 - \lambda\beta_3(c-\lambda\beta_1)P(-\tfrac{c}{\delta})}{\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\tfrac{c}{\delta})} \\
&= \frac{\lambda(c-\lambda\beta_1)\left[\beta_2\left[Q(-\tfrac{c}{\delta}) + \gamma_\delta(-\tfrac{c}{\delta})\right] - \beta_3 P(-\tfrac{c}{\delta})\right]}{\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\tfrac{c}{\delta})}
\end{aligned}$$

which yields

$$c_1 = \frac{-\lambda\beta_3 P(-\tfrac{c}{\delta}) \;+\; \lambda\beta_2\left[Q(-\tfrac{c}{\delta}) + \gamma_\delta(-\tfrac{c}{\delta})\right]}{\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\tfrac{c}{\delta})} \tag{2.33}$$

2nd Case

Let $\displaystyle\lim_{u\to-\left(\frac{c}{\delta}\right)^+}\int_u^0 (\delta x + c)^{-1-\frac{\lambda+a}{\delta}}\gamma_\delta(x)dx \;<\; \infty$. Then, from (2.24), it holds that

$$\lim_{u\to-\left(\frac{c}{\delta}\right)^+}\phi_-(u) = 0$$

Considering (2.28), we point out

$$c_1 - c_2\,P(-\tfrac{c}{\delta}) - Q(-\tfrac{c}{\delta}) \;=\; 0$$

Substituting c_1, from (2.31), implies

$$\frac{-\lambda\beta_2 c_2 - \lambda\,\beta_3}{c-\lambda\beta_1} - c_2\,P(-\tfrac{c}{\delta}) - Q(-\tfrac{c}{\delta}) \;=\; 0 \Longrightarrow$$

$$-\lambda\beta_2 c_2 - \lambda\,\beta_3 - c_2(c-\lambda\beta_1)\,P(-\tfrac{c}{\delta}) - (c-\lambda\beta_1)\,Q(-\tfrac{c}{\delta}) \;=\; 0 \Longrightarrow$$

$$\left[\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\tfrac{c}{\delta})\right] c_2 \;=\; -\lambda\,\beta_3 - (c-\lambda\beta_1)\,Q(-\tfrac{c}{\delta}) \Longrightarrow$$

So,

$$c_2 \;=\; \frac{-\lambda\,\beta_3 - (c-\lambda\beta_1)\,Q(-\frac{c}{\delta})}{\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\frac{c}{\delta})} \tag{2.34}$$

Substituting it in (2.31), we obtain the expression of c_1, i.e.

$$\begin{aligned}
(c-\lambda\beta_1)c_1 &= -\lambda\beta_2 c_2 - \lambda\,\beta_3\\
&= -\lambda\beta_2\,\frac{-\lambda\,\beta_3 - (c-\lambda\beta_1)\,Q(-\frac{c}{\delta})}{\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\frac{c}{\delta})} - \lambda\beta_3\\
&= \frac{\lambda\beta_2\beta_3 + \lambda\beta_2(c-\lambda\beta_1)Q(-\frac{c}{\delta}) - \lambda\beta_2\beta_3 - \lambda\beta_3(c-\lambda\beta_1)P(-\frac{c}{\delta})}{\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\frac{c}{\delta})}\\
&= \frac{\lambda(c-\lambda\beta_1)\left[\beta_2 Q(-\frac{c}{\delta}) - \beta_3 P(-\frac{c}{\delta})\right]}{\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\frac{c}{\delta})}
\end{aligned}$$

which implies

$$c_1 = \frac{-\lambda\beta_3 P(-\frac{c}{\delta}) \;+\; \lambda\beta_2 Q(-\frac{c}{\delta})}{\lambda\beta_2 + (c-\lambda\beta_1)\,P(-\frac{c}{\delta})} \tag{2.35}$$

2.5.1 Absolute Ruin Probability

Based on the respective paragraph of Cai (2000), we are going to apply our findings for the $\phi(u)$ in exponential claims, when $a = 0$ and $w(x, y) = 1$. We recall that the Gerber-Shiu function is reduced to the absolute ruin probability when $a = 0$ and $w(x, y) = 1$. i.e.

$$\phi(u) = \psi_\delta(u) \Longleftrightarrow \begin{cases} \phi_+(u) = \psi_+(u), & u \geq 0\\ \phi_-(u) = \psi_-(u), & -\frac{c}{\delta} < u < 0 \end{cases}$$

Our final objective is to estimate the $\psi_+(u)$, through the Theorem 2.3.2.1, after having estimated firstly the $\psi_-(u)$, via the results of Section 2.5.

Step 1. *Firstly, we will calculate the absolute ruin probability $\psi_-(u)$ when $-\frac{c}{\delta} < u < 0$.*

Solution. As the claims are exponentially distributed, the $\gamma_\delta(u)$ is given by

$$\gamma_\delta(u) = \int_{u+\frac{c}{\delta}}^{\infty} w(u, x-u) f(x)dx = \int_{u+\frac{c}{\delta}}^{\infty} f(x)dx = \bar{F}\left(u + \frac{c}{\delta}\right) \Longrightarrow$$

$$\gamma_\delta(u) \;=\; e^{-\beta u} e^{-\beta \frac{c}{\delta}} \tag{2.36}$$

So, it is obvious for $\gamma_\delta(u)$ that

$$\gamma_\delta'(u) = -\beta\, \gamma_\delta(u) \qquad \text{and} \qquad \gamma_\delta(-\tfrac{c}{\delta}) = 1$$

Regarding $g(u)$ defined in Corollary 2.5.1, we obtain

$$g(u) \;=\; -\frac{\lambda\left(\frac{1}{\beta}\gamma_\delta'(u) + \gamma_\delta(u)\right)}{\frac{1}{\beta}(\delta u + c)} = -\frac{\lambda\left(-\frac{1}{\beta}\;\beta\; \gamma_\delta(u) + \gamma_\delta(u)\right)}{\frac{1}{\beta}(\delta u + c)} \Longrightarrow g(u) = 0$$

Two direct results are

$$Q(u) = 0$$

and

$$\beta_3 = -\int_0^\infty \gamma_\delta(t)dt = -\int_0^\infty e^{-\beta t} e^{-\beta\frac{c}{\delta}} dt = -\frac{1}{\beta} e^{-\beta\frac{c}{\delta}} \int_0^\infty \beta e^{-\beta t} dt = -\frac{1}{\beta} e^{-\beta\frac{c}{\delta}}$$

In order to estimate the $\psi_-(u) = \phi_-(u)$ given in (2.28), we should first calculate the constants c_1 and c_2. As $\gamma_\delta(u)$ satisfies the (2.25) (see Appendix A.3), using (2.33) we obtain

$$c_1 \;=\; \frac{-\lambda\beta_3 P(-\frac{c}{\delta}) \;+\; \lambda\beta_2\left[Q(-\frac{c}{\delta}) + \gamma_\delta(-\frac{c}{\delta})\right]}{\lambda\beta_2 + (c - \lambda\beta_1)\; P(-\frac{c}{\delta})} = \frac{-\lambda\beta_3 P(-\frac{c}{\delta}) + \lambda\beta_2}{\lambda\beta_2 + cP(-\frac{c}{\delta}) - \lambda\beta_1 P(-\frac{c}{\delta})}$$

Replacing the values of $\beta_i,\; i = 1, 2, 3$

- $\beta_1 \;=\; \dfrac{1}{\beta}\left(1 - e^{-\beta\frac{c}{\delta}}\right)$
- $\beta_2 \;=\; \dfrac{1}{\beta}\left[-P\left(-\frac{c}{\delta}\right)\; e^{-\beta\frac{c}{\delta}} \;+\; \dfrac{c^{\frac{\lambda}{\delta}}}{\lambda}\right]$
- $\beta_3 \;=\; -\dfrac{1}{\beta} e^{-\beta\frac{c}{\delta}}$

we are led to

$$c_1 \;=\; \frac{-\lambda(-\frac{1}{\beta}e^{-\beta\frac{c}{\delta}})P(-\frac{c}{\delta}) + \lambda\frac{1}{\beta}\left[-P\left(-\frac{c}{\delta}\right)\; e^{-\beta\frac{c}{\delta}} \;+\; \frac{c^{\frac{\lambda}{\delta}}}{\lambda}\right]}{\lambda\frac{1}{\beta}\left[-P\left(-\frac{c}{\delta}\right)\; e^{-\beta\frac{c}{\delta}} \;+\; \frac{c^{\frac{\lambda}{\delta}}}{\lambda}\right] + cP(-\frac{c}{\delta}) - \lambda\frac{1}{\beta}\left(1 - e^{-\beta\frac{c}{\delta}}\right) P(-\frac{c}{\delta})}$$

$$=\; \frac{c^{\frac{\lambda}{\delta}}}{c^{\frac{\lambda}{\delta}} \;+\; \beta\left(c - \frac{\lambda}{\beta}\right) P(-\frac{c}{\delta})}$$

Considering the security loading factor θ which satisfies

$$c \;=\; (1+\theta)\lambda\mu_1 \;=\; (1+\theta)\lambda\frac{1}{\beta} \quad\Longrightarrow\quad \beta\left(c-\frac{\lambda}{\beta}\right)=\lambda\theta$$

implies

$$c_1 \;=\; \frac{c^{\frac{\lambda}{\delta}}}{c^{\frac{\lambda}{\delta}} \;+\; \lambda\theta P(-\frac{c}{\delta})}$$

Using (2.32), we can find for c_2 that

$$\begin{aligned}
c_2 \;&=\; \frac{-\lambda\beta_3-(c-\lambda\beta_1)\left[Q(-\frac{c}{\delta})+\gamma_\delta(-\frac{c}{\delta})\right]}{\lambda\beta_2+(c-\lambda\beta_1)\;P(-\frac{c}{\delta})} = \frac{-\lambda\beta_3-c+\lambda\beta_1}{\lambda\beta_2+cP(-\frac{c}{\delta})-\lambda\beta_1P(-\frac{c}{\delta})}\\
&=\; \frac{-\lambda(-\frac{1}{\beta}e^{-\beta\frac{c}{\delta}})-c+\lambda\frac{1}{\beta}\left(1-e^{-\beta\frac{c}{\delta}}\right)}{\lambda\frac{1}{\beta}\left[-P\left(-\frac{c}{\delta}\right)\;e^{-\beta\frac{c}{\delta}}\;+\;\frac{c^{\frac{\lambda}{\delta}}}{\lambda}\right]+cP(-\frac{c}{\delta})-\lambda\frac{1}{\beta}\left(1-e^{-\beta\frac{c}{\delta}}\right)P(-\frac{c}{\delta})}\\
&=\; \frac{-\beta\left(c-\frac{\lambda}{\beta}\right)}{c^{\frac{\lambda}{\delta}}+\beta\left(c-\frac{\lambda}{\beta}\right)P(-\frac{c}{\delta})} \;=\; \frac{-\lambda\theta}{c^{\frac{\lambda}{\delta}}+\lambda\theta P(-\frac{c}{\delta})}
\end{aligned}$$

Substituting the constants c_1 and c_2 in (2.28) implies

$$\begin{aligned}
\psi_-(u) \;&=\; c_1-c_2\;P(u)-Q(u)\\
&=\; \frac{c^{\frac{\lambda}{\delta}}}{c^{\frac{\lambda}{\delta}}\;+\;\lambda\theta P(-\frac{c}{\delta})} \;-\; \frac{-\lambda\theta}{c^{\frac{\lambda}{\delta}}+\lambda\theta P(-\frac{c}{\delta})}\;P(u)
\end{aligned}$$

Thus,

$$\psi_-(u)=\frac{c^{\frac{\lambda}{\delta}}\;+\;\lambda\theta\;P(u)}{c^{\frac{\lambda}{\delta}}+\lambda\theta P(-\frac{c}{\delta})} \quad \text{for } -\frac{c}{\delta}<u<0 \tag{2.37}$$

$\diamondsuit$

Step 2. *Now, we are going to estimate the probability of absolute ruin $\psi_+(u)$ when $u \geq 0$.*

Solution. As $a=0$, we have that $\xi_a=\xi_0=\theta$ and $\rho=\rho(a)=\rho(0)=0$, where ρ is the root of Lundberg's equation. Referring to Theorem 2.3.2.1 and formula (2.21), in order to obtain the $\psi_+(u)$, we should find the $H_a(u)=H_0(u)$. We remind that

$$H_a(u) = \frac{\lambda}{c}\,(1+\xi_a)\,T_\rho B(u) \;=\; \frac{T_\rho B(u)}{\hat{\bar{F}}(\rho)}$$

Because of $\rho = 0$, we obtain

$$H_0(u) = \frac{\lambda}{c}\,(1+\theta)\,T_0 B(u) \;=\; \frac{T_0 B(u)}{\hat{\bar{F}}(0)}$$

where

$$\hat{\bar{F}}(0) = \int_0^\infty \bar{F}(x)dx = E\,(X) = \frac{1}{\beta}$$

and substituting the definition (2.3) of $B(u)$, yields

$$\begin{aligned}
T_0 B(u) &= \int_u^\infty B(x)dx \;=\; \int_u^\infty \left(\int_x^{x+\frac{c}{\delta}} \phi_-(x-y)f(y)dy \;+\; \gamma_\delta(x) \right) dx \\
&= \int_u^\infty \left(\int_x^{x+\frac{c}{\delta}} \phi_-(x-y)\beta e^{-\beta y}dy \right) dx \;+\; \int_u^\infty \gamma_\delta(x)dx
\end{aligned}$$

Applying (2.36) and replacing $z = x - y$ imply

$$\begin{aligned}
T_0 B(u) &= \int_u^\infty \left(\int_{-\frac{c}{\delta}}^0 \phi_-(z)\beta e^{-\beta(x-z)}dz \right) dx \;+\; \int_u^\infty e^{-\beta x} e^{-\beta\frac{c}{\delta}}dx \\
&= \left(\int_u^\infty \beta e^{-\beta x}dx \right) \left(\int_{-\frac{c}{\delta}}^0 \phi_-(z)e^{\beta z}dz \right) \;+\; \frac{1}{\beta}e^{-\beta\frac{c}{\delta}} \left(\int_u^\infty \beta e^{-\beta x}dx \right) \\
&= \bar{F}(u) \left[\int_{-\frac{c}{\delta}}^0 \phi_-(z)e^{\beta z}dz \;+\; \frac{1}{\beta}e^{-\beta\frac{c}{\delta}} \right]
\end{aligned}$$

Bringing back the initial variables of integration, by replacing $y = x - z$, we obtain

$$\begin{aligned}
T_0 B(u) &= \bar{F}(u) \left[\int_x^{x+\frac{c}{\delta}} \phi_-(x-y)e^{\beta(x-y)}dy \;+\; \frac{1}{\beta}e^{-\beta\frac{c}{\delta}} \right] \\
&= \bar{F}(u) \left[e^{\beta x} \int_x^{x+\frac{c}{\delta}} \phi_-(x-y)e^{-\beta y}dy \;+\; \frac{1}{\beta}e^{-\beta\frac{c}{\delta}} \right] \\
&= \bar{F}(u) \left[\left(\int_0^\infty \beta e^{-\beta x}dx \right) e^{\beta x} \left(\int_x^{x+\frac{c}{\delta}} \phi_-(x-y)e^{-\beta y}dy \right) \right. \\
&\quad \left. + \frac{e^{-\beta\frac{c}{\delta}}}{\beta} \left(\int_0^\infty \beta e^{-\beta x}dx \right) \right] \quad \Longrightarrow
\end{aligned}$$

$$\begin{aligned} T_0 B(u) &= \bar{F}(u)\left[\int_0^\infty \left(\int_x^{x+\frac{c}{\delta}} \phi_-(x-y)\beta e^{-\beta y}dy\right)dx \ + \int_0^\infty e^{-\beta x}e^{-\beta\frac{c}{\delta}}dx\right] \\ &= \bar{F}(u)\left[\int_0^\infty \left[\int_x^{x+\frac{c}{\delta}} \phi_-(x-y)f(y)dy \ + \ \gamma_\delta(x)\right]dx\right] \end{aligned}$$

In the internal brackets we can recognise the definiton of $B(u)$. So,

$$T_0 B(u) \ = \ \bar{F}(u)\left(\int_0^\infty B(x)dx\right)$$

In (2.12) we have concluded that

$$\phi_-(0^-) = \phi_+(0)$$

where $a = 0$ and $w(x,y) = 1$ implies

$$\psi_-(0^-) = \psi_+(0)$$

whereas in (2.19) we have found that

$$\psi_+(0) \ = \ \frac{\lambda}{c}\int_0^\infty B(y)dy$$

Consequently, all the above yield to

$$T_0 B(u) \ = \ \frac{c}{\lambda}\ \psi_-(0^-)\ \bar{F}(u) \tag{2.38}$$

where $\psi_-(0^-)$ can be estimated by the formula (2.37).

For $u \geq 0$, in order to calculate the $\psi_+(u) = \phi_+(u)$, we can use the formula (2.21), in which

$$\phi_+(u) = \frac{1}{\xi_a}H_a(u) - \frac{1}{\xi_a}H_a(0)\bar{K}_a(u) - \frac{1}{\xi_a}\int_0^u H_a^{'}(u-x)\bar{K}_a(x)dx \quad (*)$$

For $a = 0$ and $w(x,y) = 1$, they have already been proven that

- $\rho = 0$
- $\xi_0 = \theta$
- In Chapter 1, in (1.32), it holds that

$$\bar{K}_0(u) = \psi(u) = \frac{1}{1+\theta}\ e^{-\frac{\beta\theta}{1+\theta}u}$$

  where $\psi(u)$ is the classical ruin probability.

- $H_0(u) = \dfrac{T_0 B(u)}{\hat{\bar{F}}(0)} = \dfrac{\frac{c}{\lambda}\,\psi_-(0^-)\,\bar{F}(u)}{\frac{1}{\beta}} \;=\; (1+\theta)\,\psi_-(0^-)\,\bar{F}(u)$

  because of

$$c = (1+\theta)\lambda\mu_1 = (1+\theta)\,\lambda\,\tfrac{1}{\beta} \implies \frac{\beta c}{\lambda} = (1+\theta)$$

Moreover, the first derivative of $H_0(u)$ is

$$H_0'(u) = -(1+\theta)\,\psi_-(0^-)\,f(u)$$

Substituting all the above in $(*)$, we obtain

$$\begin{aligned}
\psi_+(u) \;&=\; \frac{1}{\theta}(1+\theta)\,\psi_-(0^-)\,\bar{F}(u) - \frac{1}{\theta}(1+\theta)\psi_-(0^-)\bar{F}(0)\psi(u) \\
&\quad -\frac{1}{\theta}\int_0^u -(1+\theta)\psi_-(0^-)f(u-x)\psi(x)dx \\
&= \frac{1+\theta}{\theta}\,\psi_-(0^-)\,e^{-\beta u} - \frac{1+\theta}{\theta}\psi_-(0^-)\frac{1}{1+\theta}\,e^{-\frac{\beta\theta}{1+\theta}u} \\
&\quad + \frac{1}{\theta}\int_0^u (1+\theta)\psi_-(0^-)\beta e^{-\beta(u-x)}\frac{1}{1+\theta}\,e^{-\frac{\beta\theta}{1+\theta}x}dx \\
&= \frac{1+\theta}{\theta}\,\psi_-(0^-)\,e^{-\beta u} - \frac{1}{\theta}\,\psi_-(0^-)e^{-\beta\frac{\theta}{1+\theta}u} \\
&\quad + \frac{1}{\theta}\beta e^{-\beta u}\psi_-(0^-)\int_0^u e^{-\beta\left[\left(\frac{\theta}{1+\theta}\right)-1\right]x}\,dx \\
&= \frac{1+\theta}{\theta}\,\psi_-(0^-)\,e^{-\beta u} - \frac{1}{\theta}\,\psi_-(0^-)e^{-\beta\frac{\theta}{1+\theta}u} \\
&\quad + \frac{1}{\theta}\beta e^{-\beta u}\psi_-(0^-)\int_0^u e^{\beta\left(\frac{1}{1+\theta}\right)x}\,dx
\end{aligned}$$

Let

$$\begin{aligned}
I \;&=\; \int_0^u e^{\beta\left(\frac{1}{1+\theta}\right)x}\,dx \;=\; \left[\frac{e^{\beta\left(\frac{1}{1+\theta}\right)x}}{\beta\left(\frac{1}{1+\theta}\right)}\right]_{x=0}^{x=u} \\
&= \frac{1}{\beta}(1+\theta)e^{\beta\left(\frac{1}{1+\theta}\right)u} \;-\; \frac{1}{\beta}(1+\theta)
\end{aligned}$$

So,

$$\begin{aligned}\psi_+(u) &= \frac{1+\theta}{\theta}\,\psi_-(0^-)\,e^{-\beta u} - \frac{1}{\theta}\,\psi_-(0^-)e^{-\beta(\frac{\theta}{1+\theta})u}\\ &+ \frac{1+\theta}{\theta}e^{-\beta u}\psi_-(0^-)e^{\beta(\frac{1}{1+\theta})u} \;-\; \frac{1+\theta}{\theta}\psi_-(0^-)e^{-\beta u}\\ &= -\frac{1}{\theta}\,\psi_-(0^-)e^{-\beta(\frac{\theta}{1+\theta})u} \;+\; \frac{1+\theta}{\theta}\,\psi_-(0^-)e^{-\beta(\frac{\theta}{1+\theta})u}\end{aligned}$$

Consequently, we are led to the following expresssion for $\psi_+(u),\ u \geq 0$

$$\psi_+(u) \;=\; \psi_-(0^-)\,e^{-\beta(\frac{\theta}{1+\theta})u} \tag{2.39}$$

Taking into consideration (2.37) and the fact that $P(0) = 0$ (according to Remark 2.5.1), we obtain

$$\psi_-(0^-) = \lim_{u\to 0^-}\psi_-(u) = \frac{c^{\frac{\lambda}{\delta}}}{c^{\frac{\lambda}{\delta}} + \lambda\theta P(-\frac{c}{\delta})} \tag{2.40}$$

Finally, (2.39) and (2.40) yield the following form for $\psi_+(u)$

$$\psi_+(u) \;=\; \frac{c^{\frac{\lambda}{\delta}}}{c^{\frac{\lambda}{\delta}} + \lambda\theta P(-\frac{c}{\delta})}\;e^{-\beta(\frac{\theta}{1+\theta})u} \qquad \forall\, u \geq 0 \tag{2.41}$$

$\diamondsuit$

Example 2.5.1. *Let use the same data frame as in numerical Example 1.6.1, i.e. the mean of Poisson process is $\lambda = 4$, the premium rate is $c = 2$ and the claim sizes X follow an Exponential distribution with mean $\frac{1}{3}$. Furthermore, as we want to estimate the absolute ruin probability $\psi_+(u)$, we assume that the discounting interest force is $a = 0$ and the penalty function $w(x,y) = 1$. Finally, we will present the results for three different values of the debit interest force δ, namely for $\delta_1 = 0.4,\ \delta_2 = 0.6$ and $\delta_3 = 0.8$.*

Solution. Having done all the above study for exponential claims, the desirable results are obtained throughout the following mere procedure:

Firstly, using the following formula for each $\delta_i,\ i = 1,\ 2,\ 3$

$$P_i(u) \;=\; \int_u^0 (\delta_i x + c)^{-\frac{\delta_i - \lambda}{\delta_i}}\;e^{-\beta x}\,dx, \quad i = 1,\ 2,\ 3$$

we obtain the respective functions of $P(u)$. Then, we substitute them in (2.41) and we receive the corresponding results for $\psi_+(u)$. We recall as well, that the security loading factor θ is given by $\theta = \frac{c}{\lambda\mu_1} - 1 \;=\; \frac{\beta c}{\lambda} - 1$, where $\mu_1 = E(X) = \frac{1}{\beta}$. All the calculations are conducted in Mathematica program (the respective code can be found at the end of this Chapter) and we present directly the results. So, we have that

$$\psi_+(u) = \begin{cases} 0.0946316\, e^{-u}, & \text{for } \delta_1 = 0.4 \\ 0.153823\, e^{-u}, & \text{for } \delta_2 = 0.6 \\ 0.201299\, e^{-u}, & \text{for } \delta_2 = 0.8 \end{cases} \qquad \forall\, u \geq 0$$

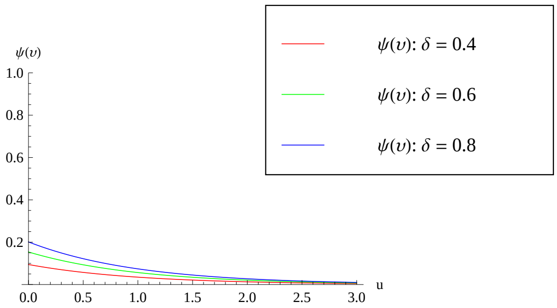


Figure 2.4: The absolute ruin probability $\psi_+(u)$ for $\delta = 0.4,\ 0.6,\ 0.8$

We can observe exactly what we expected to obtain, i.e. the greater values the debit interest force δ receives, the higher the absolute ruin probability $\psi_+(u)$ is. This conclusion is depicted by the Figure 2.4, as well. Moreover, we have found the (classical) ruin probability $\psi(u)$ through (1.34)

$$\psi(u) \;=\; \frac{1}{1+\theta} e^{-\frac{\beta\theta}{1+\theta}u} \;=\; \frac{2}{3}\, e^{-u} \qquad \forall\, u \geq 0$$

and we have mentioned in Introduction 2.1 that it always holds

$$\psi_+(u) \leq \psi(u) \quad \forall\, u \geq 0$$

as we can see in the Figure 2.5.

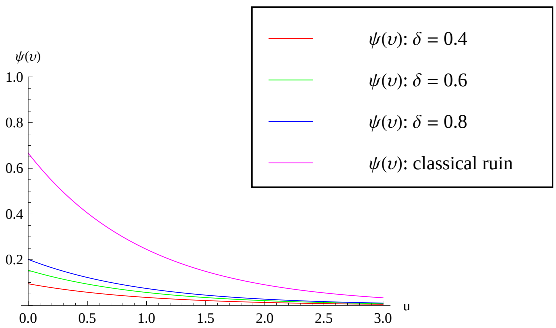


Figure 2.5: Comparison between the absolute ruin probability $\psi_+(u)$ and the ruin probability $\psi(u)$

2.5.2 The Laplace Transform of the Absolute Ruin Time T_δ

We have mentioned that the Gerber-Shiu function, $\phi(u)$, is reduced to the Laplace transform of the absolute ruin time T_δ, when the penalty function is constant and equal to one, i.e. $w(x, y) = 1$. In this case, Cai (2000) derives and solves the differential equations for the Gerber-Shiu function $\phi(u)$. More specifically, he proves that $\phi_-(u)$ satisfies a confluent hypergeometric equation, whereas $\phi_+(u)$ satisfies merely a second order linear homogeneous differential equation with constant coefficients. His solution for the former differential equation is based on Abramowitz and Stegun (1972). Finally, using initial conditions, which have been mentioned in this chapter, he extracts explicit expressions for the arbitrary constants appeared in the general forms of the solutions.

Lemma 2.5.2.1. *When $w(x, y) = 1$ and $X \sim Exp(\beta)$, we have that*

i. $\gamma_\delta(u) = e^{-\beta(u+\frac{c}{\delta})}$

ii. $\gamma_\delta'(u) = -\beta\gamma_\delta(u)$

iii. $\gamma_\delta(-\frac{c}{\delta}) = 1$

iv. $B'(u) = -\beta B(u)$

v. $\lim_{u\to-\frac{c}{\delta}^+} \phi_-(u) = \dfrac{\lambda}{\lambda + a}$

Proof.
i. Using the definition of $\gamma_\delta(u)$ in (2.4), we have

$$\gamma_\delta(u) = \int_{u+\frac{c}{\delta}}^{\infty} w(u, x-u)f(x)dx = \int_{u+\frac{c}{\delta}}^{\infty} f(x)dx = \bar{F}\left(u + \frac{c}{\delta}\right) = e^{-\beta(u+\frac{c}{\delta})}$$

ii. & iii. are derived directly from i.

iv. Using the definition of $B(u)$ in (2.3), we have

$$B(u) = \int_{u}^{u+\frac{c}{\delta}} \phi_-(u-x)f(x)dx + \gamma_\delta(u) = \int_{u}^{u+\frac{c}{\delta}} \phi_-(u-x)\beta e^{-\beta x}dx + \gamma_\delta(u)$$

Changing the variable of integration into $y = u - x \implies dy = -dx$, the boundaries of integration are converted into $y \to 0$ when $x \to u$ and $y \to -\frac{c}{\delta}$ when $x \to u + \frac{c}{\delta}$. Thus,

$$B(u) = -\int_{0}^{-\frac{c}{\delta}} \phi_-(y)\beta e^{-\beta(u-y)}dy + \gamma_\delta(u) = \beta e^{-\beta u}\int_{-\frac{c}{\delta}}^{0} \phi_-(y)e^{\beta y}dy + \gamma_\delta(u)$$

Differentiating with respect to u and using the result in (ii) yield

$$\begin{aligned}
B^{'}(u) &= -\beta^2 e^{-\beta u}\int_{-\frac{c}{\delta}}^{0}\phi_-(y)e^{\beta y}dy + \gamma_\delta^{'}(u) \\
&= -\beta\left[\beta e^{-\beta u}\int_{-\frac{c}{\delta}}^{0}\phi_-(y)e^{\beta y}dy + \gamma_\delta(u)\right] \\
&= -\beta B(u)
\end{aligned}$$

v. In Appendix A.3, for $w(x,y) = 1$ and $X \sim \text{Exp}(\beta)$, we prove that $\gamma_\delta(u)$ satisfies

$$\lim_{u\to-\frac{c}{\delta}^+}\int_u^0 (\delta y + c)^{-1-\frac{\lambda+a}{\delta}}\gamma_\delta(y)dy = \infty$$

So, from (2.25) and the result in (iii), we obtain

$$\lim_{u\to-\frac{c}{\delta}^+}\phi_-(u) = \frac{\lambda}{\lambda+a}\cdot\gamma_\delta\left(-\frac{c}{\delta}\right) = \frac{\lambda}{\lambda+a}$$

$\diamondsuit$

Corollary 2.5.2.1. *For $-\frac{c}{\delta} < u < 0$, the equation (2.26) satisfied by $\phi_-(u)$ is reduced to the second order linear homogeneous differential equation*

$$(\delta u + c)\phi_-^{''}(u) + [\beta(\delta u + c) + \delta - \lambda - a]\,\phi_-^{'}(u) - \beta a\phi_-(u) = 0 \qquad (2.42)$$

Proof. According to Lemma 2.5.2.1 (ii), we have

$$\gamma_\delta^{'}(u) = -\beta\gamma_\delta(u) \Longrightarrow \gamma_\delta^{'}(u) + \beta\gamma_\delta(u) = 0$$

Substituting it in (2.26), we are led to (2.42).

$\diamondsuit$

Proposition 2.5.2.1. *The equation (2.42) can be transformed into a confluent hypergeometric equation*

$$xy^{''} + \left(1 - \frac{\lambda+a}{\delta} - x\right)y^{'} + \frac{a}{\delta}y = 0, \qquad -\frac{\beta c}{\delta} < x < 0 \qquad (2.43)$$

Proof. Cai (2000) applies in (2.42) the transforms

$$\phi_-(u) = y(x(u)) = y(x) \text{ where } x = x(u) = -\frac{\beta(\delta u + c)}{\delta}$$

in order to derive the equation (2.43). Firstly, we evaluate the derivatives of the first and second order, by using the chain rule, i.e.

- $\dfrac{dx}{du} = -\beta$ and $\dfrac{d^2x}{du^2} = 0$

- $\dfrac{d\phi_-(u)}{du} = \dfrac{dy(x)}{du} = \dfrac{dy}{dx} \cdot \dfrac{dx}{du} = -\beta y^{'}$
- $\dfrac{d^2\phi_-(u)}{du^2} = \dfrac{d^2y(x)}{du^2} = \left(\dfrac{d^2y}{dx^2} \cdot \dfrac{dx}{du}\right)\dfrac{dx}{du} + \dfrac{dy}{dx}\left(\dfrac{d^2x}{du^2}\right) = \beta^2 y^{''}$

Substituting them in (2.42), we obtain

$$(\delta u + c)\beta^2 y^{''} + [\beta(\delta u + c) + \delta - (\lambda + a)]\,(-\beta y^{'}) - \beta a y = 0$$

Multiplying by $-\dfrac{1}{\beta\delta}$ yields

$$-\frac{\beta(\delta u + c)}{\delta} y^{''} + \left[\frac{\beta(\delta u + c)}{\delta} + \frac{\delta}{\delta} - \frac{(\lambda + a)}{\delta}\right] y^{'} + \frac{a}{\delta} y = 0$$

Replacing $x = -\dfrac{\beta(\delta u + c)}{\delta}$, we are led to

$$xy^{''} + \left(1 - \frac{\lambda + a}{\delta} - x\right) y^{'} + \frac{a}{\delta} y = 0$$

Finally,
$-\dfrac{c}{\delta} < u < 0 \Longrightarrow 0 > -\dfrac{\beta}{\delta}(\delta u + c) > -\dfrac{\beta c}{\delta} \Longrightarrow -\dfrac{\beta c}{\delta} < x < 0$

♢

Theorem 2.5.2.1. *The Gerber-Shiu function $\phi_-(u)$ is given by*

$$\phi_-(u) = c_1 h_1(u) + c_2 h_2(u), \quad -\frac{c}{\delta} < u < 0 \tag{2.44}$$

with

$$h_1(u) = e^{-\frac{\beta(\delta u + c)}{\delta}} U\left(1 - \frac{\lambda}{\delta}, 1 - \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right) \tag{2.45}$$

$$h_2(u) = \left[\frac{\beta(\delta u + c)}{\delta}\right]^{\frac{\lambda + a}{\delta}} e^{-\frac{\beta(\delta u + c)}{\delta}} M\left(1 + \frac{a}{\delta}, 1 + \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right) \tag{2.46}$$

where $M(x, y, z)$, $U(x, y, z)$ are the confluent hypergeometric functions of the first and second kinds, respectively, and c_1, c_2 are arbitrary constants.

Proof. Regarding (13.1.15) and (13.1.18) of Abramowitz and Stegun (1972), the general solution of (2.43) is given by

$$y(x) = c_1 e^x U\left(1 - \tfrac{\lambda}{\delta}, 1 - \tfrac{\lambda + a}{\delta}, -x\right) + c_2(-x)^{\frac{\lambda + a}{\delta}} e^x M\left(1 + \tfrac{a}{\delta}, 1 + \tfrac{\lambda + a}{\delta}, -x\right)$$

where $-\frac{\beta c}{\delta} < x < 0$, $M(x, y, z)$, $U(x, y, z)$ are the confluent hypergeometric functions of the first and second kinds, respectively, and c_1, c_2 are arbitrary constants. Hence,

$$\phi_-(u) = y(x) = y\left(-\frac{\beta(\delta u + c)}{\delta}\right) = c_1 h_1(u) + c_2 h_2(u), \quad -\frac{c}{\delta} < u < 0$$

where

$$h_1(u) = e^{-\frac{\beta(\delta u+c)}{\delta}} U\left(1 - \frac{\lambda}{\delta}, 1 - \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)$$

$$h_2(u) = \left[\frac{\beta(\delta u + c)}{\delta}\right]^{\frac{\lambda+a}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}} M\left(1 + \frac{a}{\delta}, 1 + \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)$$

$\diamondsuit$

Lemma 2.5.2.2. *Using the properties of*

$$\begin{aligned} \frac{d}{dz} M(x, y, z) &= \frac{x}{y} M(x + 1, y + 1, z) \\ \frac{d}{dz} U(x, y, z) &= -x U(x + 1, y + 1, z) \end{aligned}$$

which can be found in Abramowitz and Stegun (1972) (see also Appendix A.4, Proposition A.4.1), and differentiating (2.45) and (2.46) with respect to u, we obtain

$$\begin{aligned} h_1'(u) &= -\beta e^{-\frac{\beta(\delta u+c)}{\delta}} \left[U\left(1 - \frac{\lambda}{\delta}, 1 - \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)\right. \\ &\quad \left. + \left(1 - \frac{\lambda}{\delta}\right) U\left(2 - \frac{\lambda}{\delta}, 2 - \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)\right] \end{aligned} \tag{2.47}$$

$$\begin{aligned} h_2'(u) &= \beta \left[\frac{\beta(\delta u + c)}{\delta}\right]^{\frac{\lambda+a}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}} \left[\frac{\lambda + a - \beta(\delta u + c)}{\beta(\delta u + c)} M\left(1 + \frac{a}{\delta}, 1 + \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)\right. \\ &\quad \left. + \frac{\delta + a}{\lambda + \delta + a} M\left(2 + \frac{a}{\delta}, 2 + \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)\right] \end{aligned} \tag{2.48}$$

Proof. Differentiating (2.45) and (2.46) with respect to u, we obtain

$$\begin{aligned} h_1'(u) &= -\beta e^{-\frac{\beta(\delta u+c)}{\delta}} U\left(1 - \frac{\lambda}{\delta}, 1 - \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right) \\ &\quad -\beta e^{-\frac{\beta(\delta u+c)}{\delta}} \left(1 - \frac{\lambda}{\delta}\right) U\left(2 - \frac{\lambda}{\delta}, 2 - \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right) \\ &= -\beta e^{-\frac{\beta(\delta u+c)}{\delta}} \left[U\left(1 - \frac{\lambda}{\delta}, 1 - \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)\right. \\ &\quad \left. + \left(1 - \frac{\lambda}{\delta}\right) U\left(2 - \frac{\lambda}{\delta}, 2 - \frac{\lambda + a}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)\right] \end{aligned}$$

$$\begin{aligned}
h_2'(u) &= \beta\left[\frac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+a}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}}\left[\frac{\frac{\lambda+a}{\delta}}{\frac{\beta(\delta u+c)}{\delta}}-1\right] M\left(1+\frac{a}{\delta}, 1+\frac{\lambda+a}{\delta}, \frac{\beta(\delta u+c)}{\delta}\right)\\
&+\beta\left(\frac{\delta+a}{\lambda+\delta+a}\right)\left[\frac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+a}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}} M\left(2+\frac{a}{\delta}, 2+\frac{\lambda+a}{\delta}, \frac{\beta(\delta u+c)}{\delta}\right)\\
&= \beta\left[\frac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+a}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}}\left[\frac{\lambda+a-\beta(\delta u+c)}{\beta(\delta u+c)} M\left(1+\frac{a}{\delta}, 1+\frac{\lambda+a}{\delta}, \frac{\beta(\delta u+c)}{\delta}\right)\right.\\
&\left.+\frac{\delta+a}{\lambda+\delta+a} M\left(2+\frac{a}{\delta}, 2+\frac{\lambda+a}{\delta}, \frac{\beta(\delta u+c)}{\delta}\right)\right]
\end{aligned}$$

$\diamondsuit$

Proposition 2.5.2.2. *The Geber-Shiu function $\phi_+(u)$ satisfies the second order linear homogeneous differential equation with constant coefficients*

$$\phi_+''(u)+p\phi_+'(u)+q\phi_+(u)=0, \qquad u\geq 0 \tag{2.49}$$

where

$$p=\beta-\frac{\lambda+a}{c} \quad \textit{and} \quad q=-\frac{\beta a}{c} \tag{2.50}$$

Proof. From (2.2), we have

$$\begin{aligned}
c\phi_+'(u) &= (\lambda+a)\phi_+(u)-\lambda\left[\int_o^u \phi_+(u-x)f(x)dx+B(u)\right] \Longrightarrow\\
c\phi_+'(u) &= (\lambda+a)\phi_+(u)-\lambda\int_o^u \phi_+(u-x)\beta e^{-\beta x}dx-\lambda B(u) \qquad (2.51)
\end{aligned}$$

Changing the variable of integration into $y=u-x \Longrightarrow dy=-dx$, the boundaries of integration are converted into $y\to u$ when $x\to 0$ whereas $y\to 0$ when $x\to u$. So,

$$\begin{aligned}
c\phi_+'(u) &= (\lambda+a)\phi_+(u)+\lambda\int_u^0 \phi_+(y)\beta e^{-\beta(u-y)}dy-\lambda B(u) \Longrightarrow\\
c\phi_+'(u) &= (\lambda+a)\phi_+(u)-\lambda\beta e^{-\beta u}\int_0^u \phi_+(y)e^{\beta y}dy-\lambda B(u)
\end{aligned}$$

Differentiating with respect to u yields

$$c\phi_+''(u)=(\lambda+a)\phi_+'(u)+\lambda\beta^2 e^{-\beta u}\int_0^u \phi_+(y)e^{\beta y}dy-\lambda\beta e^{-\beta u}\phi_+(u)e^{\beta u}-\lambda B'(u)$$

Bringing back the initial variable of integration, we have

$$c\phi_+''(u)=(\lambda+a)\phi_+'(u)+\lambda\beta\int_0^u \phi_+(u-x)\beta e^{-\beta x}dx-\lambda\beta\phi_+(u)-\lambda B'(u) \tag{2.52}$$

Multiplying (2.51) by β implies

$$\beta c\phi_+^{'}(u) = \beta(\lambda + a)\phi_+(u) - \lambda\beta \int_o^u \phi_+(u-x)\beta e^{-\beta x}dx - \lambda\beta B(u) \quad (2.53)$$

Adding down (2.52) and (2.53) and making the cancellations needed, we obtain

$$c\phi_+^{''}(u) + \beta c\phi_+^{'}(u) = (\lambda + a)\phi_+^{'}(u) - \lambda\beta\phi_+(u) - \lambda B^{'}(u) + \beta(\lambda + a)\phi_+(u) - \lambda\beta B(u) \Longrightarrow$$

$$c\phi_+^{''}(u) + [\beta c - (\lambda + a)]\,\phi_+^{'}(u) - \beta a\phi_+(u) = -\lambda\left[B^{'}(u) + \beta B(u)\right]$$

From Lemma 2.5.2.1 (iv), we have that

$$B^{'}(u) = -\beta B(u) \Longrightarrow B^{'}(u) + \beta B(u) = 0$$

Thus, dividing by c, we obtain

$$\phi_+^{''}(u) + \left[\beta - \frac{\lambda + a}{c}\right]\phi_+^{'}(u) - \frac{\beta a}{c}\phi_+(u) = 0$$

Setting

$$p = \beta - \frac{\lambda + a}{c} \quad \text{and} \quad q = -\frac{\beta a}{c}$$

we are led to the desirable result

$$\phi_+^{''}(u) + p\phi_+^{'}(u) + q\phi_+(u) = 0$$

$\diamondsuit$

Theorem 2.5.2.2. *The Gerber-Shiu function $\phi_+(u)$ is given by*

$$\phi_+(u) = c_4 e^{r_2 u}, \qquad u \geq 0 \quad (2.54)$$

where

$$r_2 = \frac{-p - \sqrt{p^2 - 4q}}{2} \quad (2.55)$$

p, q are defined by (2.50) and c_4 is an arbitrary constant.

Proof. The equation (2.49) is a second order linear homogeneous differential equation with constant coefficients

$$p = \beta - \frac{\lambda + a}{c} \quad \text{and} \quad q = -\frac{\beta a}{c}$$

We observe that the corresponding characteristic equation

$$r^2 + pr + q = 0$$

always has a positive Discriminant, indeed

$$D = p^2 - 4q = (\beta - \frac{\lambda + a}{c})^2 + 4\frac{\beta a}{c} > 0 \quad \text{(as all the terms are positive).}$$

Thus, there are two distinct real roots,

$$r_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} \qquad \text{and} \qquad r_2 = \frac{-p - \sqrt{p^2 - 4q}}{2} \tag{2.56}$$

According to chapter 4.2.2, page 159, of Alikakos and Kalogeropoulos (2003) (see also Appendix A.2), the general solution of (2.49) is given by

$$\phi_+(u) = c_3 e^{r_1 u} + c_4 e^{r_2 u}, \qquad u \geq 0 \tag{2.57}$$

where $c_3,\ c_4$ are arbitrary constants. From (2.1), we have that

$$\lim_{u \to \infty} \phi_+(u) = 0$$

Letting $u \to \infty$ in (2.57) and considering the fact that $\lim\limits_{u \to \infty} e^{r_2 u} = 0$, we obtain $c_3 = 0$. Thus, (2.57) is reduced to

$$\phi_+(u) = c_4 e^{r_2 u} = c_4 e^{\frac{-p - \sqrt{p^2 - 4q}}{2} u}, \qquad u \geq 0$$

$\diamondsuit$

Specifying the arbitrary constants

In order to obtain explicit solutions of (2.44) and (2.54), we should determine the constants $c_1,\ c_2$ and c_4. For this reason, we will use the initial conditions (2.12), (2.13) and Lemma 2.5.2.1 (v). Now, we are going to describe step by step the procedure.

Step 1. Differentiating (2.44) and (2.54) with respect to u, we obtain

$$\phi_-^{'}(u) = c_1 h_1^{'}(u) + c_2 h_2^{'}(u), \quad -\tfrac{c}{\delta} < u < 0$$

$$\phi_+^{'}(u) = c_4 r_2 e^{r_2 u}, \qquad u \geq 0$$

Step 2. From (2.12), by setting $u = 0$ in (2.44) and (2.54), we obtain

$$\begin{aligned} \phi_-(0^-) &= \phi_+(0) \Longrightarrow \\ c_1 h_1(0) + c_2 h_2(0) &= c_4 \end{aligned} \tag{2.58}$$

Step 3. From (2.13), by setting $u = 0$ in the derivatives in Step 1, we obtain

$$\begin{aligned} \phi_-^{'}(0^-) &= \phi_+^{'}(0) \Longrightarrow \\ c_1 h_1^{'}(0) + c_2 h_2^{'}(0) &= c_4 r_2 \end{aligned} \tag{2.59}$$

According to Cai (2000) and (13.5.10), (13.5.12) of Abramowitz and Stegun (1972), if $\frac{\lambda+a}{\delta} \neq 1$[1] then

$$\lim_{u \to -\frac{c}{\delta}^+} h_1(u) = \frac{\Gamma\left(\dfrac{\lambda+a}{\delta}\right)}{\Gamma\left(\dfrac{\delta+a}{\delta}\right)} \qquad \text{and} \qquad \lim_{u \to -\frac{c}{\delta}^+} h_2(u) = 0$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

is the Gamma function (see Appendix A.4, Corollary A.4.1, for $n = 1$).

Step 4. From Lemma 2.5.2.1 (v), by letting $u \to -\frac{c}{\delta}^+$ in (2.44), we obtain

$$\lim_{u \to -\frac{c}{\delta}^+} \phi_-(u) = \frac{\lambda}{\lambda+a} \quad \Longrightarrow$$

$$c_1 \lim_{u \to -\frac{c}{\delta}^+} h_1(u) + c_2 \lim_{u \to -\frac{c}{\delta}^+} h_2(u) = \frac{\lambda}{\lambda+a} \quad \Longrightarrow$$

$$c_1 \frac{\Gamma\left(\dfrac{\lambda+a}{\delta}\right)}{\Gamma\left(\dfrac{\delta+a}{\delta}\right)} = \frac{\lambda}{\lambda+a} \quad \Longrightarrow$$

$$c_1 = \frac{\lambda\Gamma\left(\dfrac{\delta+a}{\delta}\right)}{(\lambda+a)\Gamma\left(\dfrac{\lambda+a}{\delta}\right)} \tag{2.60}$$

Step 5. As a result, (2.58)-(2.59)-(2.60) compose a system of three equations with three unknowns, namely

$$(\Sigma) = \begin{cases} c_1 h_1(0) + c_2 h_2(0) = c_4 \\[2ex] c_1 h_1'(0) + c_2 h_2'(0) = c_4 r_2 \\[2ex] c_1 = \dfrac{\lambda\Gamma\left(\dfrac{\delta+a}{\delta}\right)}{(\lambda+a)\Gamma\left(\dfrac{\lambda+a}{\delta}\right)} \end{cases}$$

[1]Cai (2000) mentions that the case $\delta = \lambda + a$ does not appear in reality because the debit interest force δ is usually less than one, while the intensity of Poisson process λ is greater than one.

where r_2 is given by (2.55), $h_1(0),\ h_2(0),\ h_1^{'}(0)$ and $h_2^{'}(0)$ are given by (2.45), (2.46), (2.47) and (2.48), respectively. Solving the system (Σ), we obtain

$$
\begin{aligned}
c_1 &= \frac{\lambda\Gamma\left(\dfrac{\delta+a}{\delta}\right)}{(\lambda+a)\Gamma\left(\dfrac{\lambda+a}{\delta}\right)} \\
c_2 &= \frac{-\lambda\Gamma\left(\dfrac{\delta+a}{\delta}\right)\left[h_1^{'}(0)-r_2h_1(0)\right]}{(\lambda+a)\Gamma\left(\dfrac{\lambda+a}{\delta}\right)\left[h_2^{'}(0)-r_2h_2(0)\right]} \\
c_4 &= \frac{\lambda\Gamma\left(\dfrac{\delta+a}{\delta}\right)\left[h_1(0)h_2^{'}(0)-h_1^{'}(0)h_2(0)\right]}{(\lambda+a)\Gamma\left(\dfrac{\lambda+a}{\delta}\right)\left[h_2^{'}(0)-r_2h_2(0)\right]}
\end{aligned}
\tag{2.61}
$$

$\diamondsuit$

Example 2.5.2 *We use, once more, the same data frame as in previous examples, i.e. the intensity of Poisson process is $\lambda = 4$, the premium rate is $c = 2$ and the claim sizes X_i are exponentially distributed with parameter $\beta = 3$. As we want to estimate the Laplace transform for the absolute ruin time T_δ, we assume that the penalty function is constant and equal to 1, $w(x, y) = 1$. We aim to depict the curves of the Laplace transform for the absolute ruin time in the following two cases:*

i. Maintaining the debit interest force unchanged, $\delta = 0.8$, we will derive the aforementioned curves for three different values of the discounting interest force a, namely, $a_1 = 0.1$, $a_2 = 0.3$ and $a_3 = 0.5$.

ii. Maintaining the discounting interest force unchanged, $a = 0.1$, we will derive the aforementioned curves for three different values of the debit interest force δ, namely, $\delta_1 = 0.4$, $\delta_2 = 0.6$ and $\delta_3 = 0.8$. Moreover, we will compare these curves with the curve of the Laplace transform for the ruin time T estimated in chapter 1.

Solution. All the calculations required for this exercise have been conducted in Mathematica. We are going to present thoroughly the respective methodology.

(i.) Firstly, in order to use the formulas we have found, we should examine whether the quantity $\frac{\lambda+a_i}{\delta}$ is different from one and not integer $\forall\ i = 1, 2, 3$. In our case, this is true. Next, we have to estimate $h_1(u),\ h_2(u),\ h_1^{'}(u)$ and $h_2^{'}(u)$ at $u = 0$ for each value of a. This can be achieved through the formulas (2.45), (2.46), (2.47) and (2.48), respectively. Hence, for $i = 1, 2, 3$ we obtain

$$h_{1i}(0) = e^{-\frac{\beta c}{\delta}} U\left(1-\frac{\lambda}{\delta}, 1-\frac{\lambda+a_i}{\delta}, \frac{\beta c}{\delta}\right)$$

$$h_{2i}(0) = \left[\frac{\beta c}{\delta}\right]^{\frac{\lambda+a_i}{\delta}} e^{-\frac{\beta c}{\delta}} M\left(1+\frac{a_i}{\delta}, 1+\frac{\lambda+a_i}{\delta}, \frac{\beta c}{\delta}\right)$$

and

$$\begin{aligned} h'_{1i}(0) &= -\beta e^{-\frac{\beta c}{\delta}} \left[U\left(1-\frac{\lambda}{\delta}, 1-\frac{\lambda+a_i}{\delta}, \frac{\beta c}{\delta}\right)\right. \\ &\quad \left. + \left(1-\frac{\lambda}{\delta}\right) U\left(2-\frac{\lambda}{\delta}, 2-\frac{\lambda+a_i}{\delta}, \frac{\beta c}{\delta}\right)\right] \end{aligned}$$

$$\begin{aligned} h'_{2i}(0) &= \beta \left[\frac{\beta c}{\delta}\right]^{\frac{\lambda+a_i}{\delta}} e^{-\frac{\beta c}{\delta}} \left[\frac{\lambda+a_i-\beta c}{\beta c} M\left(1+\frac{a_i}{\delta}, 1+\frac{\lambda+a_i}{\delta}, \frac{\beta c}{\delta}\right)\right. \\ &\quad \left. + \frac{\delta+a_i}{\lambda+\delta+a_i} M\left(2+\frac{a_i}{\delta}, 2+\frac{\lambda+a_i}{\delta}, \frac{\beta c}{\delta}\right)\right] \end{aligned}$$

Now, using (2.50), we will find the constant coefficients, p and q, of equation (2.49), for $i = 1, 2, 3$. So,

$$p_i = \beta - \frac{\lambda+a_i}{c} \quad \text{and} \quad q_i = -\frac{\beta a_i}{c}$$

The respective value of the root r_2 is given by (2.55), i.e.

$$r_{2i} = \frac{-p_i - \sqrt{p_i^2 - 4q_i}}{2}, \ \forall\, i = 1, 2, 3$$

From formulas (2.61) we obtain the values of the arbitrary constants c_1, c_2 and c_4, for all $i = 1, 2, 3$. So,

$$c_{1i} = \frac{\lambda \Gamma\left(\frac{\delta+a_i}{\delta}\right)}{(\lambda+a_i)\Gamma\left(\frac{\lambda+a_i}{\delta}\right)}$$

$$c_{2i} = \frac{-\lambda \Gamma\left(\frac{\delta+a_i}{\delta}\right)\left[h'_{1i}(0) - r_{2i} h_{1i}(0)\right]}{(\lambda+a_i)\Gamma\left(\frac{\lambda+a_i}{\delta}\right)\left[h'_{2i}(0) - r_{2i} h_{2i}(0)\right]}$$

$$c_{4i} = \frac{\lambda \Gamma\left(\frac{\delta+a_i}{\delta}\right)\left[h_{1i}(0) h'_{2i}(0) - h'_{1i}(0) h_{2i}(0)\right]}{(\lambda+a_i)\Gamma\left(\frac{\lambda+a_i}{\delta}\right)\left[h'_{2i}(0) - r_{2i} h_{2i}(0)\right]}$$

Finally, substituting them in formulas (2.44) and (2.54), we are led to the Laplace transform for the absolute ruin time T_δ, i.e.

$$\phi_{-,i}(u) = c_{1i}h_{1i}(u) + c_{2i}h_{2i}(u), \quad -\frac{c}{\delta} < u < 0, \qquad i = 1,2,3$$

and

$$\phi_{+,i}(u) = c_{4i}e^{r_{2i}u}, \qquad u \geq 0, \qquad i = 1,2,3$$

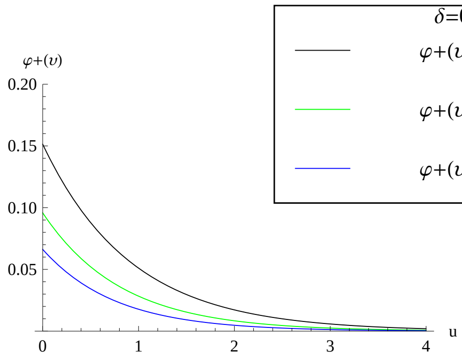


Figure 2.6: Laplace transform for absolute ruin time

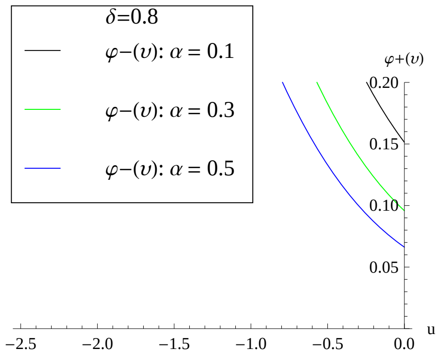


Figure 2.7: Laplace transform for absolute ruin time

In Figures 2.6 & 2.7, we observe that the greater values received by the discounting interest force a, the lower the curves of Laplace transform are. This is a reasonable result considering the role of the discounting factor.

(ii.) In this case, we follow exactly the same steps, as in question (i), in order to estimate the same quantities. The main difference is that the discounting interest force a is now constant, whereas the debit interest force δ

is our variable. Having estimated the respective $\phi_{-,i}(u),\ -\dfrac{c}{\delta} < u < 0$, and $\phi_{+,i}(u),\ u \geq 0$, we are led to Figure 2.8.

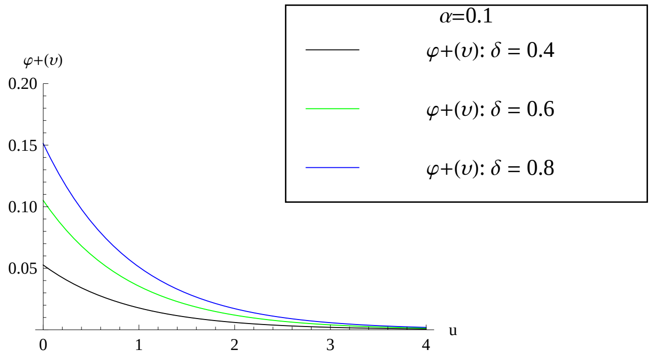


Figure 2.8: Laplace transform for absolute ruin time

In Figure 2.8 we observe an opposite situation than in Figure 2.6, regarding the debit interest force δ. The greater values of the debit interest force δ, the less the absolute ruin time T_δ is. This leads to higher curves for the corresponding Laplace transforms.

In the second part of this question, we want to compare the Laplace transform for the absolute ruin time with the Laplace transform for the ruin time, which is studied in chapter 1. Having estimated the former, we remind how we estimate the Laplace transform for the ruin time.
Firstly, we observe that the net profit condition, $c > \lambda\frac{1}{\beta}$, is valid. Then, the security loading factor θ is given by

$$\theta = \frac{c}{\lambda\dfrac{1}{\beta}} - 1$$

Solving the Lundberg's equation,

$$l(s) = \lambda\hat{f}(s) \Longrightarrow \lambda + a - cs = \lambda\frac{\beta}{\beta + s}$$

we are led to a quadratic equation of s (where $f_X(x) = f(x) = \beta e^{-\beta x}$ is the probability density function of X)

$$cs^2 + (c\beta - a - \lambda)s - a\beta = 0$$

From (1.35), the two real roots of this equation are

$$s_{1,2} = \frac{-(c\beta - a - \lambda) \pm \sqrt{(c\beta - a - \lambda)^2 + 4ca\beta}}{2c}$$

The positive one is the root ρ we need. Finally, the Laplace transform for the ruin time T is given by the formula (1.33), which is

$$\bar{K}_a(u) = \frac{\beta}{(1+\theta)(\beta+\rho)} e^{-\beta \frac{\rho + (\beta+\rho)\theta}{(1+\theta)(\beta+\rho)} u}, \quad u \geq 0$$

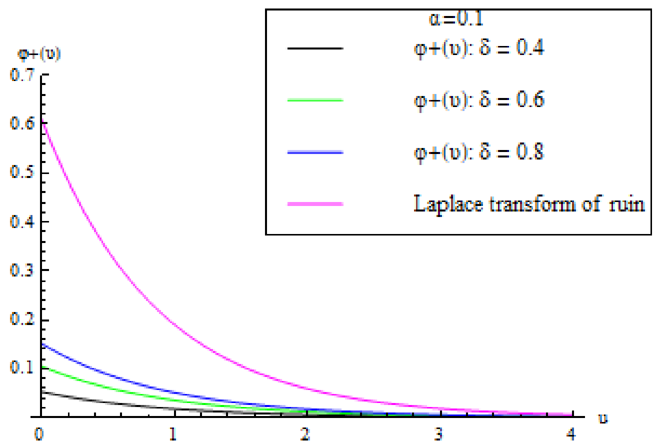


Figure 2.9: Laplace transform for absolute ruin time

Putting all the curves together in Figure 2.9, we observe that the curve of the Laplace transform for the ruin time T is above from all the curves of the Laplace transform for the absolute ruin time T_δ. This is because, it always holds $T \leq T_\delta$, which means the corresponding discounting of T is greater than T_δ.

$\diamondsuit$

Note. In order to calculate the confluent hypergeometric functions of the first and second kinds in Mathematica, we apply the following commands

First Kind	$M(a,b,z)$	$\longleftrightarrow$	Hypergeometric1F1[a,b,z]
Second Kind	$U(a,b,z)$	$\longleftrightarrow$	HypergeometricU[a,b,z]

2.5.3 Code of Mathematica

In the Examples 2.5.1 and 2.5.2 the calculations and the graphs have been developed in Mathematica. In purpose of offering a better monitoring of this work, we include the respective code.

```
c = 2
lambda = 4
f[x_] = 3 * Exp[-3 * x]
m1 = Integrate[x * f[x], {x, 0, Infinity}]
b = 1 / m1

c > lambda * m1

True

theta = (c / (lambda * m1)) - 1

1
—
2

d1 = 0.4
d2 = 0.6
d3 = 0.8


P1[u_] = Integrate[ ((d1 * x + c) ^ (- (d1 - lambda) / d1)) * Exp[-b * x], {x, u, 0}]

P2[u_] = Integrate[ ((d2 * x + c) ^ (- (d2 - lambda) / d2)) * Exp[-b * x], {x, u, 0}]

P3[u_] = Integrate[ ((d3 * x + c) ^ (- (d3 - lambda) / d3)) * Exp[-b * x], {x, u, 0}]

psiNegative1[u_] = ((c ^ (lambda / d1)) + (lambda * theta * P1[u])) /
  ((c ^ (lambda / d1)) + (lambda * theta * P1[-c / d1]))

0.0000924137 (1024. + 2 (-367.872 - e^(-3. u) (-367.872 +
          u (-591.616 + u (-426.624 + u (-180.864 + u (-49.6322 + u (-9.13549 + u (-1.1271 +
                    u (-0.089828 + (-0.0041943 - 0.0000873813 u) u))))))))))

psiNegative2[u_] = ((c ^ (lambda / d2)) + (lambda * theta * P2[u])) /
  ((c ^ (lambda / d2)) + (lambda * theta * P2[-c / d2]))

ConditionalExpression[
 0.0015141 (101.594 + 2 (-33.1622 + e^(-3. u) ((2. + 0.6 u)^(2/3) (20.8879 + u (26.4953 +
               u (13.7187 + u (3.60896 + (0.48096 + 0.02592 u) u)))) +
                                                                    10
          230.848 e^(3. u) Gamma[0.666667, 10. + 3. u]))), Re[u] ≥ - —— || u ∉ Reals]
                                                                    3

psiNegative3[u_] = ((c ^ (lambda / d3)) + (lambda * theta * P3[u])) /
  ((c ^ (lambda / d3)) + (lambda * theta * P3[-c / d3]))

0.0062906 (32. +
   2 (-9.65942 - e^(-3. u) (-9.65942 + u (-12.9783 + u (-6.66738 + (-1.54738 - 0.136533 u) u)))))
```

Figure 2.10: Example 2.5.1, Code 1/2

```
psiPositive1[u_] = psiNegative1[0] * Exp[-b * (theta / (1 + theta)) * u]
psiPositive2[u_] = psiNegative2[0] * Exp[-b * (theta / (1 + theta)) * u]
psiPositive3[u_] = psiNegative3[0] * Exp[-b * (theta / (1 + theta)) * u]
```

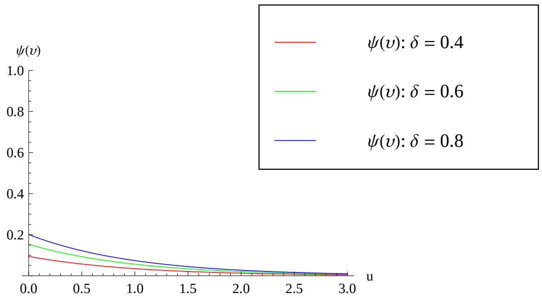
$0.0946316\, e^{-u}$

$0.153823\, e^{-u}$

$0.201299\, e^{-u}$

```
<< PlotLegends`;
```

```
Plot[{psiPositive1[u], psiPositive2[u], psiPositive3[u]},
 {u, 0, 3}, PlotRange → {0, 1}, AxesLabel → {"u", "ψ(υ)"},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]}, PlotLegend →
  {Style["ψ(υ): δ = 0.4", 12], Style["ψ(υ): δ = 0.6", 12], Style["ψ(υ): δ = 0.8", 12]},
 LegendPosition → {.9, 0}, LegendTextSpace → 2, LegendLabel → "",
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```




```
psiClassical[u_] = (1 / (1 + theta)) * Exp[-b * (theta / (1 + theta)) * u]
```

$\frac{2\, e^{-u}}{3}$

```
Plot[{psiPositive1[u], psiPositive2[u], psiPositive3[u], psiClassical[u]},
 {u, 0, 3}, PlotRange → {0, 1}, AxesLabel → {"u", "ψ(υ)"}, PlotStyle →
  {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1], RGBColor[1, 0, 1]},
 PlotLegend → {Style["ψ(υ): δ = 0.4", 12], Style["ψ(υ): δ = 0.6", 12],
   Style["ψ(υ): δ = 0.8", 12], Style["ψ(υ): classical ruin", 12]},
 LegendPosition → {.9, 0}, LegendTextSpace → 2, LegendLabel → "",
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```

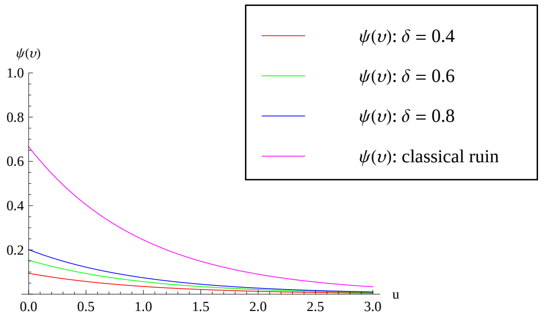


Figure 2.11: Example 2.5.1, Code 2/2

```
l = 4
c = 2
b = 3
a = 0.1
a1 = 0.3
a2 = 0.5
d = 0.8

c > l * (1 / b)

True

(l + a) / d
(l + a1) / d
(l + a2) / d

5.125

5.375

5.625

h1a[u_] =
 Exp[- (b * (d * u + c)) / d] * HypergeometricU[1 - (l / d), 1 - ((l + a) / d), (b * (d * u + c)) / d]
h1a1[u_] = Exp[- (b * (d * u + c)) / d] *
  HypergeometricU[1 - (l / d), 1 - ((l + a1) / d), (b * (d * u + c)) / d]
h1a2[u_] = Exp[- (b * (d * u + c)) / d] *
  HypergeometricU[1 - (l / d), 1 - ((l + a2) / d), (b * (d * u + c)) / d]
h2a[u_] = ((b * (d * u + c)) / d) ^ ((l + a) / d) * Exp[- (b * (d * u + c)) / d] *
  Hypergeometric1F1[1 + (a / d), 1 + ((l + a) / d), (b * (d * u + c)) / d]
h2a1[u_] = ((b * (d * u + c)) / d) ^ ((l + a1) / d) * Exp[- (b * (d * u + c)) / d] *
  Hypergeometric1F1[1 + (a1 / d), 1 + ((l + a1) / d), (b * (d * u + c)) / d]
h2a2[u_] = ((b * (d * u + c)) / d) ^ ((l + a2) / d) * Exp[- (b * (d * u + c)) / d] *
  Hypergeometric1F1[1 + (a2 / d), 1 + ((l + a2) / d), (b * (d * u + c)) / d]

d1h1a[u_] = Derivative[1][h1a][u]
d1h1a1[u_] = Derivative[1][h1a1][u]
d1h1a2[u_] = Derivative[1][h1a2][u]
d1h2a[u_] = Derivative[1][h2a][u]
d1h2a1[u_] = Derivative[1][h2a1][u]
d1h2a2[u_] = Derivative[1][h2a2][u]

pa = b - ((l + a) / c)
pa1 = b - ((l + a1) / c)
pa2 = b - ((l + a2) / c)
qa = - (b * a) / c
qa1 = - (b * a1) / c
qa2 = - (b * a2) / c
```

Figure 2.12: Example 2.5.2, (i), Code 1/6

```
r2a = (-pa - Sqrt[pa^2 - 4 * qa]) / 2
r2a1 = (-pa1 - Sqrt[pa1^2 - 4 * qa1]) / 2
r2a2 = (-pa2 - Sqrt[pa2^2 - 4 * qa2]) / 2

-1.08788

-1.21912

-1.31873

c1a = (l * Gamma[(d + a) / d]) / ((l + a) * Gamma[(l + a) / d])
c1a1 = (l * Gamma[(d + a1) / d]) / ((l + a1) * Gamma[(l + a1) / d])
c1a2 = (l * Gamma[(d + a2) / d]) / ((l + a2) * Gamma[(l + a2) / d])

0.0316585

0.0192918

0.0124293

c2a = (-l * Gamma[(d + a) / d] * (d1h1a[0] - r2a * h1a[0])) /
  ((l + a) * Gamma[(l + a) / d] * (d1h2a[0] - r2a * h2a[0]))
c2a1 = (-l * Gamma[(d + a1) / d] * (d1h1a1[0] - r2a1 * h1a1[0])) /
  ((l + a1) * Gamma[(l + a1) / d] * (d1h2a1[0] - r2a1 * h2a1[0]))
c2a2 = (-l * Gamma[(d + a2) / d] * (d1h1a2[0] - r2a2 * h1a2[0])) /
  ((l + a2) * Gamma[(l + a2) / d] * (d1h2a2[0] - r2a2 * h2a2[0]))

0.000285298

0.0000648126

0.0000175217

c4a = (l * Gamma[(d + a) / d] * (h1a[0] * d1h2a[0] - d1h1a[0] * h2a[0])) /
  ((l + a) * Gamma[(l + a) / d] * (d1h2a[0] - r2a * h2a[0]))
c4a1 = (l * Gamma[(d + a1) / d] * (h1a1[0] * d1h2a1[0] - d1h1a1[0] * h2a1[0])) /
  ((l + a1) * Gamma[(l + a1) / d] * (d1h2a1[0] - r2a1 * h2a1[0]))
c4a2 = (l * Gamma[(d + a2) / d] * (h1a2[0] * d1h2a2[0] - d1h1a2[0] * h2a2[0])) /
  ((l + a2) * Gamma[(l + a2) / d] * (d1h2a2[0] - r2a2 * h2a2[0]))

0.151381

0.0957892

0.0662073

PhiPositivea[u_] = c4a * Exp[r2a * u]
PhiPositivea1[u_] = c4a1 * Exp[r2a1 * u]
PhiPositivea2[u_] = c4a2 * Exp[r2a2 * u]
```

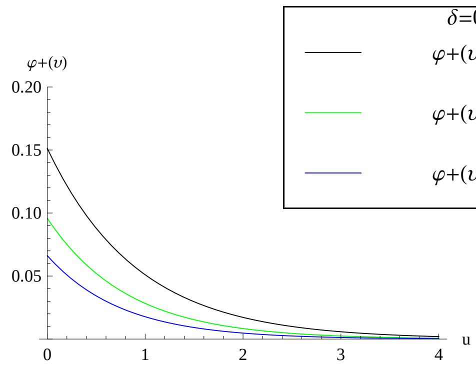
$0.151381\, e^{-1.08788\, u}$

$0.0957892\, e^{-1.21912\, u}$

$0.0662073\, e^{-1.31873\, u}$

Figure 2.13: Example 2.5.2, (i), Code 2/6

```
PhiNegativea[u_] = c1a * h1a[u] + c2a * h2a[u]
PhiNegativea1[u_] = c1a1 * h1a1[u] + c2a1 * h2a1[u]
PhiNegativea2[u_] = c1a2 * h1a2[u] + c2a2 * h2a2[u]

<< PlotLegends';

Plot[{PhiPositivea[u], PhiPositivea1[u], PhiPositivea2[u]},
 {u, 0, 4}, PlotRange → {0, 0.2}, AxesLabel → {"u", "φ+(υ)"},
 PlotStyle → {RGBColor[0, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]},
 PlotLegend → {Style["φ+(υ): α = 0.1", 12], Style["φ+(υ): α = 0.3", 12],
   Style["φ+(υ): α = 0.5", 12]}, LegendPosition → {.9, 0},
 LegendTextSpace → 2, LegendLabel → Style["δ=0.8", 12],
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```




```
Plot[{PhiNegativea[u], PhiNegativea1[u], PhiNegativea2[u]},
 {u, -c / d + 0.001, -0.001}, PlotRange → {0, 0.2}, AxesLabel → {"u", "φ+(υ)"},
 PlotStyle → {RGBColor[0, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]},
 PlotLegend → {Style["φ-(υ): α = 0.1", 12], Style["φ-(υ): α = 0.3", 12],
   Style["φ-(υ): α = 0.5", 12]}, LegendPosition → {.9, 0},
 LegendTextSpace → 2, LegendLabel → Style["δ=0.8", 12],
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```

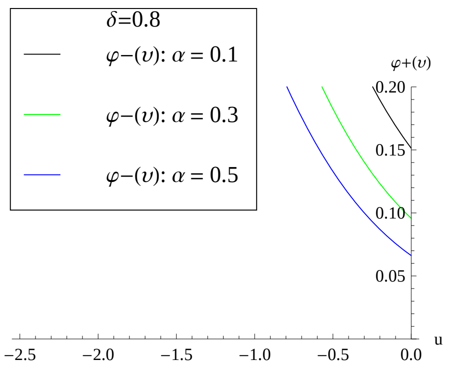


Figure 2.14: Example 2.5.2, (i), Code 3/6

```
l = 4
c = 2
b = 3
a = 0.1
d = 0.4
d1 = 0.6
d2 = 0.8
c > l * (1 / b)
True
(l + a) / d
(l + a) / d1
(l + a) / d2
10.25
6.83333
5.125
h1d[u_] =
 Exp[- (b * (d * u + c)) / d] * HypergeometricU[1 - (l / d), 1 - ((l + a) / d), (b * (d * u + c)) / d]
h1d1[u_] = Exp[- (b * (d1 * u + c)) / d1] *
  HypergeometricU[1 - (l / d1), 1 - ((l + a) / d1), (b * (d1 * u + c)) / d1]
h1d2[u_] = Exp[- (b * (d2 * u + c)) / d2] *
  HypergeometricU[1 - (l / d2), 1 - ((l + a) / d2), (b * (d2 * u + c)) / d2]
h2d[u_] = ((b * (d * u + c)) / d) ^ ((l + a) / d) * Exp[- (b * (d * u + c)) / d] *
  Hypergeometric1F1[1 + (a / d), 1 + ((l + a) / d), (b * (d * u + c)) / d]
h2d1[u_] = ((b * (d1 * u + c)) / d1) ^ ((l + a) / d1) * Exp[- (b * (d1 * u + c)) / d1] *
  Hypergeometric1F1[1 + (a / d1), 1 + ((l + a) / d1), (b * (d1 * u + c)) / d1]
h2d2[u_] = ((b * (d2 * u + c)) / d2) ^ ((l + a) / d2) * Exp[- (b * (d2 * u + c)) / d2] *
  Hypergeometric1F1[1 + (a / d2), 1 + ((l + a) / d2), (b * (d2 * u + c)) / d2]

d1h1d[u_] = Derivative[1][h1d][u]
d1h1d1[u_] = Derivative[1][h1d1][u]
d1h1d2[u_] = Derivative[1][h1d2][u]
d1h2d[u_] = Derivative[1][h2d][u]
d1h2d1[u_] = Derivative[1][h2d1][u]
d1h2d2[u_] = Derivative[1][h2d2][u]

p = b - ((l + a) / c)
q = - (b * a) / c
r2 = (- p - Sqrt[p^2 - 4 * q]) / 2
-1.08788
```

Figure 2.15: Example 2.5.2, (ii), Code 4/6

```
c1d = (l * Gamma[(d + a) / d]) / ((l + a) * Gamma[(l + a) / d])
c1d1 = (l * Gamma[(d1 + a) / d1]) / ((l + a) * Gamma[(l + a) / d1])
c1d2 = (l * Gamma[(d2 + a) / d2]) / ((l + a) * Gamma[(l + a) / d2])
```

1.38337×10^{-6}

0.00171389

0.0316585

```
c2d = (-l * Gamma[(d + a) / d] * (d1h1d[0] - r2 * h1d[0])) /
  ((l + a) * Gamma[(l + a) / d] * (d1h2d[0] - r2 * h2d[0]))
c2d1 = (-l * Gamma[(d1 + a) / d1] * (d1h1d1[0] - r2 * h1d1[0])) /
  ((l + a) * Gamma[(l + a) / d1] * (d1h2d1[0] - r2 * h2d1[0]))
c2d2 = (-l * Gamma[(d2 + a) / d2] * (d1h1d2[0] - r2 * h1d2[0])) /
  ((l + a) * Gamma[(l + a) / d2] * (d1h2d2[0] - r2 * h2d2[0]))
```

1.08646×10^{-9}

6.3125×10^{-6}

0.000285298

```
c4d = (l * Gamma[(d + a) / d] * (h1d[0] * d1h2d[0] - d1h1d[0] * h2d[0])) /
  ((l + a) * Gamma[(l + a) / d] * (d1h2d[0] - r2 * h2d[0]))
c4d1 = (l * Gamma[(d1 + a) / d1] * (h1d1[0] * d1h2d1[0] - d1h1d1[0] * h2d1[0])) /
  ((l + a) * Gamma[(l + a) / d1] * (d1h2d1[0] - r2 * h2d1[0]))
c4d2 = (l * Gamma[(d2 + a) / d2] * (h1d2[0] * d1h2d2[0] - d1h1d2[0] * h2d2[0])) /
  ((l + a) * Gamma[(l + a) / d2] * (d1h2d2[0] - r2 * h2d2[0]))
```

0.0526753

0.105163

0.151381

```
PhiNegatived[u_] = c1d * h1d[u] + c2d * h2d[u]
PhiNegatived1[u_] = c1d1 * h1d1[u] + c2d1 * h2d1[u]
PhiNegatived2[u_] = c1d2 * h1d2[u] + c2d2 * h2d2[u]


PhiPositived[u_] = c4d * Exp[r2 * u]
PhiPositived1[u_] = c4d1 * Exp[r2 * u]
PhiPositived2[u_] = c4d2 * Exp[r2 * u]
```

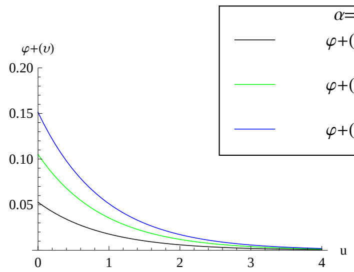
$0.0526753\, e^{-1.08788\, u}$

$0.105163\, e^{-1.08788\, u}$

$0.151381\, e^{-1.08788\, u}$

Figure 2.16: Example 2.5.2, (ii), Code 5/6

```
<< PlotLegends`;

Plot[{PhiPositived[u], PhiPositived1[u], PhiPositived2[u]},
 {u, 0, 4}, PlotRange → {0, 0.2}, AxesLabel → {"u", "φ+(υ)"},
 PlotStyle → {RGBColor[0, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]},
 PlotLegend → {Style["φ+(υ): δ = 0.4", 12], Style["φ+(υ): δ = 0.6", 12],
   Style["φ+(υ): δ = 0.8", 12]}, LegendPosition → {.9, 0},
 LegendTextSpace → 2, LegendLabel → Style["α=0.1", 12],
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```




```
theta = (c / (l * (1 / b))) - 1

s1 = (-(c * b - a - l) + Sqrt[(c * b - a - l)^2 + 4 * c * a * b]) / 2
s2 = (-(c * b - a - l) - Sqrt[(c * b - a - l)^2 + 4 * c * a * b]) / 2

0.275765

-2.17577

rho = 0.2757650672131262`

Ka[u_] = (b / ((1 + theta) * (b + rho))) *
  Exp[-b * ((rho + (b + rho) * theta) / ((1 + theta) * (b + rho))) * u]

0.610544 e^(-1.16837 u)

Plot[{PhiPositived[u], PhiPositived1[u], PhiPositived2[u], Ka[u]},
 {u, 0, 4}, PlotRange → {0, 0.7}, AxesLabel → {"u", "φ+(υ)"},
 PlotStyle → {RGBColor[0, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1], RGBColor[1, 0, 1]},
 PlotLegend → {Style["φ+(υ): δ = 0.4", 12], Style["φ+(υ): δ = 0.6", 12],
   Style["φ+(υ): δ = 0.8", 12], Style["Laplace transform of ruin time", 12]},
 LegendPosition → {.9, 0}, LegendTextSpace → 2, LegendLabel → Style["α=0.1", 12],
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```

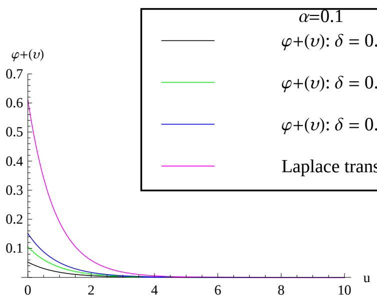


Figure 2.17: Example 2.5.2, (ii), Code 6/6

Chapter 3

Dividend Payments

In this chapter, Wang and Yin (2009), Yuen, Zhou and Guo (2008) consider two more features for the classical surplus process, namely, a debit interest force and dividend payments according to a barrier strategy. So, when the surplus exceeds a threshold, the insurance company will pay dividends to shareholders. Furthermore, when the surplus is negative, but above a critical value, the company can borrow money equal to the deficit, with a debit interest force, as we have already seen in Chapter 2. Our objective in this chapter is to find expressions for the moment-generating function and moments of the present value of the dividend payments and to give explicit results for exponential claims. Specifically, we will focus on the first moment of the present value of the dividend payments which denotes the expected discounted dividends the company has to pay to its shareholders.
Firstly, we present the integro-differential equations satisfied by the moment-generating function of the discounted dividend payments. Through them, we derive the integro-differential equations for the moments of the discounted dividend payments. In case of exponential claims, the latter integro-differential equations are converted into differential equations which are solved offering explicit expressions for the moments of the present value of the dividend payments, focusing mainly on the first moment. Meanwhile, we present the definition of Gerber-Shiu function and the integro-differential equations satisfied by it, under the aforementioned surplus process.

3.1 Introduction

In Chapter 1, where we study the (classical) ruin under the classical continuous time risk model, the surplus process is given by

$$U(t) = u + ct - S(t), \qquad t \geq 0, \tag{3.1}$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the premium rate per unit time and $S(t)$ is the aggregate claims process. We recall that $S(t)$ denotes the total loss for the insurer in the time interval [0,t] and it is defined by

$$S(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i, & N(t) \geq 1 \\ 0, & N(t) = 0 \end{cases}$$

where $N(t)$ is a Poisson process, with intensity $\lambda > 0$, which indicates the number of claims occuring in the time interval [0,t] and $\{X_i\}_{i=1}^{\infty}$ is the sequence of the individual claim sizes, independent of $N(t)$, which consists of independent and identical nonnegative random variables with a common distribution function $F(x)$ that satisfies $F(0) = 0$ and has a positive mean $\mu_1 = \int_0^{\infty} x f(x)dx = \int_0^{\infty} \bar{F}(x)dx$, where $\bar{F}(x) = 1 - F(x)$ is the survival function of $F(x)$ and $f(x) = F^{'}(x)$ is the respective probability density function.

In Chapter 2, in order to study the absolute ruin, we expand the above model by assuming that the insurer can borrow money equal to the deficit, at a debit interest force $\delta > 0$, when the surplus falls below zero. Thus, the surplus process (3.1) is converted into $U_\delta(t)$ and satisfies

$$\begin{aligned} dU_\delta(t) &= [c + \delta U_\delta(t) I\left(U_\delta(t) < 0\right)] dt - dS(t) \\ &= \begin{cases} cdt + \delta U_\delta(t)dt - dS(t), & U_\delta(t) < 0 \\ cdt - dS(t), & \text{otherwise} \end{cases} \end{aligned}$$

We have also mentioned that if the negative surplus attains the critical value $-\frac{c}{\delta}$ or drops below $-\frac{c}{\delta}$, there is no chance for the surplus to be positive again. When this happens, we say that absolute ruin occurs. There are many studies and results for the absolute ruin in the scientific literature, for example Dassios and Embrechts (1989), Embrects and Schmidli (1994), Dickson and Egidio dos Reis (1997), Zhang and Wu (1999), Cai (2007), Gerber and Yang (2007).

In this Chapter, Wang and Yin (2009) add another feature to the surplus process, namely the dividend payments according to a barrier strategy. There are also many papers for the approach through a constant dividend barrier, for example see [2], [8], [14] - [16], [18] - [19] and [22] - [24]. We are going to use the model of Wang and Yin (2009), who were motivated by Cai, Gerber and Yang (2006) and Yuen, Zhou and Guo (2008). According to the model of Wang and Yin (2009), the modified surplus process, under the debit interest force δ and the barrier strategy, is given by

$$dU_b(t) = \begin{cases} cdt + \delta U_b(t)dt - dS(t), & U_b(t) < 0 \\ cdt - dS(t), & 0 \leq U_b(t) < b \\ -dS(t), & U_b(t) = b \end{cases}$$

where $t \geq 0$, $U_b(0) = u$, $b \geq \max\{0, u\}$ is the finite dividend barrier and $D(t)$ is the aggregate dividends paid in the time interval [0,t]. In their work there is the assumption that the company pays dividends at a constant rate equal to the premium rate c, whenever the surplus $U_b(t)$ attains the barrier b. Hence, when the surplus reaches the barrier b, all the premium revenues are given to shareholders as dividends and the surplus remains at the level b until the next claim happens. By contrast, when the surplus is below b, there are not dividend payments. This can be illustrated by the Figure 3.1.

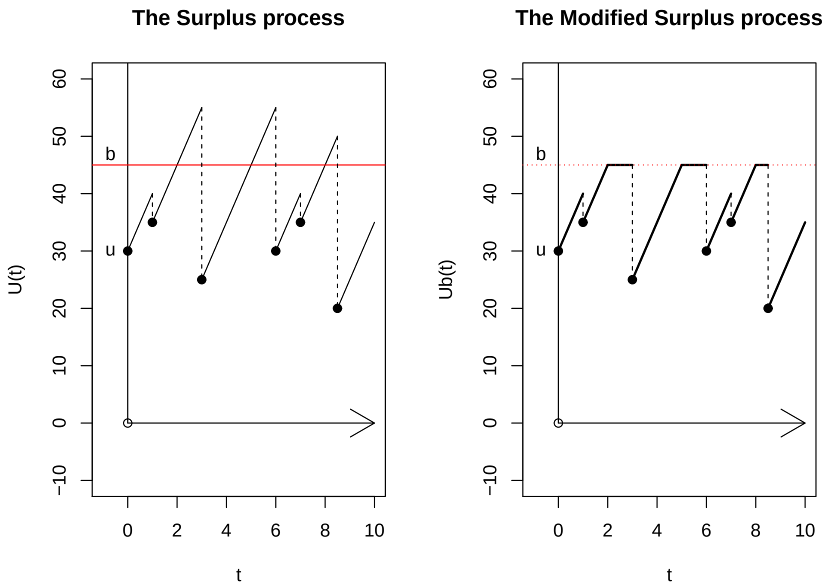


Figure 3.1: The modified surplus process

Definition 3.1.1. *The absolute ruin time of the modified surplus process* $\{U_b(t) : t \geq 0\}$ *is defined by*

$$T_b = \begin{cases} inf\Big\{t \geq 0 : \; U_b(t) \leq -\dfrac{c}{\delta}\Big\} \\ \\ \infty, \qquad if \quad U_b(t) > -\dfrac{c}{\delta} \quad \forall\, t \geq 0 \end{cases}$$

Definition 3.1.2. *If* $a > 0$ *is a discounting interest force, the present value of all dividends paid up to the absolute ruin time* T_b *will be denoted by*

$$D_{u,b} = \int_0^{T_b} e^{-at} dD(t) \; = \; c \int_0^{T_b} e^{-at} I(U_b(t) \geq b) dt$$

Remark 3.1.1. *We observe that for all* t *where* $U_b(t) \geq b$, *we obtain* $T_b = \infty$. *Thus, we are led to the following property*

$$D_{u,b} = c\int_0^{T_b} e^{-at} I(U_b(t) \geq b)dt \leq c\int_0^{\infty} e^{-at}dt = \frac{c}{a}$$

Definition 3.1.3. *The moment-generating function of $D_{u,b}$ is denoted by*

$$M(u,y;b) = E\left[e^{yD_{u,b}}\right], \qquad -\tfrac{c}{\delta} < u \leq b \text{ and } y \text{ is finite}$$

and the n-th moment of $D_{u,b}$ is symbolised by

$$V_n(u,b) = E\left[D_{u,b}^n\right], \qquad -\tfrac{c}{\delta} < u \leq b,\ n \in \mathcal{N}$$

Remarks 3.1.2.

i. We can easily derive that $V_0(u,b) = 1$ and for $u > b$ it holds $V_1(u,b) = u - b + V_1(b,b)$

ii. As $M(u,y;b)$ and $V_n(u,b)$ behave differently for $-\frac{c}{\delta} < u < 0$ and $0 \leq u \leq b$, the following discrimination is made by Wang and Yin (2009)

$$M(u,y;b) = \begin{cases} M_1(u,y;b), & 0 \leq u \leq b \\ M_2(u,y;b), & -\frac{c}{\delta} < u < 0 \end{cases}$$

and

$$V_n(u,b) = \begin{cases} V_{n1}(u,b), & 0 \leq u \leq b \\ V_{n2}(u,b), & -\frac{c}{\delta} < u < 0 \end{cases}$$

3.2 Moment-Generating Function

In order to find expressions for the n^{th} moment of $D_{u,b}$, $V_n(u,b)$, Wang and Yin (2009) describe firstly the integro-differential equations for $M(u,y;b)$ in the following Theorem.

Theorem 3.2.1. *For $0 < u < b$, $M_1(u,y;b)$ satisfies*

$$\begin{aligned} c\frac{\partial M_1(u,y;b)}{\partial u} &= ay\frac{\partial M_1(u,y;b)}{\partial y} + \lambda M_1(u,y;b) - \lambda\left[\int_0^u M_1(u-x,y;b)dF(x)\right. \\ &\qquad \left. + \int_u^{u+\frac{c}{\delta}} M_2(u-x,y;b)dF(x) + \bar{F}(u+\frac{c}{\delta})\right] \end{aligned} \tag{3.2}$$

whereas, for $-\frac{c}{\delta} < u < 0$, $M_2(u, y; b)$ *satisfies*

$$\begin{aligned}(\delta u + c)\frac{\partial M_2(u, y; b)}{\partial u} &= ay\frac{\partial M_2(u, y; b)}{\partial y} + \lambda M_2(u, y; b) \\ &\quad -\lambda\left[\int_0^{u+\frac{c}{\delta}} M_2(u - x, y; b)dF(x) + \bar{F}(u + \frac{c}{\delta})\right]\end{aligned} \tag{3.3}$$

Proof.

i. When $0 < u < b$, we consider a small $t > 0$, so that $U_b(s) < b \ \forall s \in (0, t]$ (i.e. there are not dividend payments in $(0, t]$). Due to the strong Markov property of the surplus process $\{U_b(t) : t \geq 0\}$, it holds

$$M(u, y; b) = E\left[M\left(U_b(t), ye^{-at}; b\right)\right] \tag{3.4}$$

A reasonable explanation for the right side of (3.4) can be depicted by the Figure 3.2. It is known that the surplus at a specific time $t > 0$ is equal to $U_b(t)$. The present value, at this time t, of all dividends paid until the absolute ruin time T_b, is $D_{U_b(t),b}$. However, we want to find the present value for $t = 0$. Thus, by discounting the variable $D_{U_b(t),b}$, we obtain $e^{-at}D_{U_b(t),b}$. The moment-generating function for this variable is: $E\left[e^{ye^{-at}D_{U_b(t),b}}\right] = M\left(U_b(t), ye^{-at}; b\right)$.

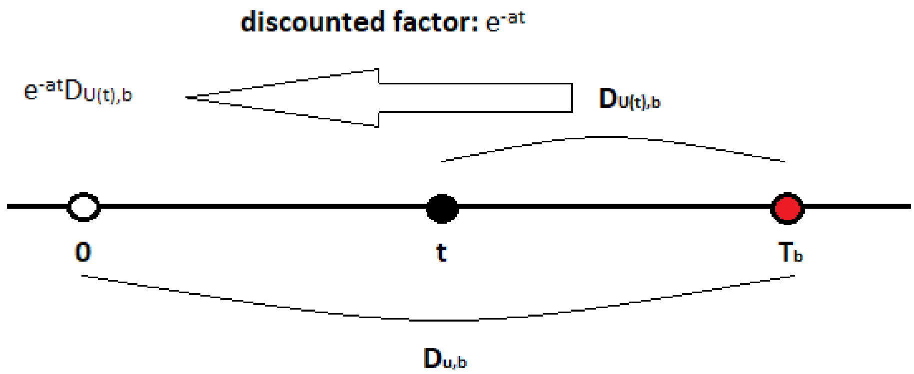


Figure 3.2: The Markov property

In the time interval $(0, t]$, we can have one claim $(N(t) = 1)$, or no claim $(N(t) = 0)$. As we have seen in Definition 1.1.3, the probabilities are $Pr(N(t) = 1) = \lambda t$ and $Pr(N(t) = 0) = 1 - \lambda t$, respectively. Conditioning on the time and size of the first claim and using the strong Markov property (3.4) and the renewal argument, the law of total probability yields

$$\begin{aligned} M_1(u,y;b) &= E\left[M_1\left(U_b(t), ye^{-at}; b\right)\right] \\ &= \sum_{k=0}^{1} E\left[M_1\left(U_b(t), ye^{-at}; b\right) | N(t)=k\right] Pr\left(N(t)=k\right) + o(t) \end{aligned}$$

where,

$$\lim_{t\to 0} \frac{o(t)}{t} = 0$$

When $N(t) = 0$, we have $U_b(t) = u + ct$. Thus,

$$E\left[M_1\left(U_b(t), ye^{-at}; b\right) | N(t) = 0\right] = M_1(u + ct, ye^{-at}; b)$$

When $N(t) = 1$, regarding the size x of the first claim, there are three potential situations for $M_1\left(U_b(t), ye^{-at}; b\right)$, which can be depicted by the Figure 3.3. Namely,

- for $x \leq u + ct$, the procedure is renewed and $U_b(t) = u + ct - x > 0$
- for $u + ct < x < u + ct + \frac{c}{\delta}$, the procedure is renewed and $U_b(t) = u + ct - x < 0$
- for $x \geq u + ct + \frac{c}{\delta}$, absolute ruin happens and the moment-generating function is 1, because there are not dividend payments ($D = 0$)

Overall, we have

$$M_1\left(U_b(t), ye^{-at}; b | N(t) = 1\right) = \begin{cases} M_1\left(u + ct - x, ye^{-at}; b\right), & 0 < x \leq u + ct \\ M_2\left(u + ct - x, ye^{-at}; b\right), & u + ct < x < u + ct + \frac{c}{\delta} \\ 1, & x \geq u + ct + \frac{c}{\delta} \end{cases}$$

Finally, we obtain

$$\begin{aligned} M_1(u,y;b) = \; & (1-\lambda t) M_1(u+ct, ye^{-at}; b) + \lambda t \left[\int_0^{u+ct} M_1(u+ct-x, ye^{-at}; b) dF(x) \right. \\ & \left. + \int_{u+ct}^{u+ct+\frac{c}{\delta}} M_2(u+ct-x, ye^{-at}; b) dF(x) + \int_{u+ct+\frac{c}{\delta}}^{\infty} dF(x)\right] + o(t) \end{aligned} \tag{3.5}$$

Let $M_1(u + ct, ye^{-at}; b) = M_1(u(t), y(t); b)$. Then, the Taylor's expansion evaluated at the point $t = 0$ leads to

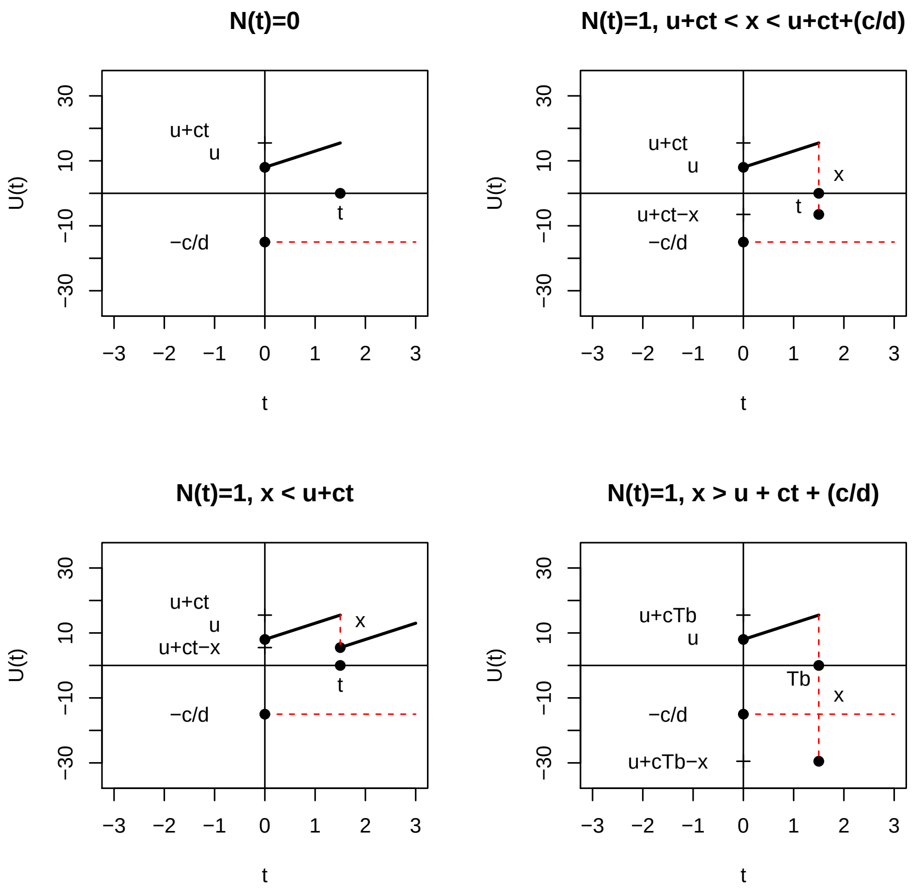


Figure 3.3: Cases of the first claim

$$
\begin{aligned}
M_1(u(t), y(t); b) &= M_1(u(0), y(0); b) + t\frac{d}{dt}[M_1(u(t), y(t); b)]_{t=0} + o(t) \\
&= M_1(u, y; b) + t\left[\frac{\partial M_1(u(0), y(0); b)}{\partial u}\left[\frac{\partial u(t)}{\partial t}\right]_{t=0} + \frac{\partial M_1(u(0), y(0); b)}{\partial y}\left[\frac{\partial y(t)}{\partial t}\right]_{t=0}\right] \\
&\quad + o(t) \\
&= M_1(u, y; b) + ct\frac{\partial M_1(u, y; b)}{\partial u} - ayt\frac{\partial M_1(u, y; b)}{\partial y} + o(t)
\end{aligned}
$$

Substituting the above result in (3.5) yields

$$\begin{aligned}
M_1(u,y;b) &= (1-\lambda t)\left[M_1(u,y;b)+ct\frac{\partial M_1(u,y;b)}{\partial u}-ayt\frac{\partial M_1(u,y;b)}{\partial y}+o(t)\right]\\
&+\lambda t\left[\int_0^{u+ct} M_1(u+ct-x,ye^{-at};b)dF(x)\right.\\
&\left.+\int_{u+ct}^{u+ct+\frac{c}{\delta}} M_2(u+ct-x,ye^{-at};b)dF(x)+\int_{u+ct+\frac{c}{\delta}}^{\infty} dF(x)\right]+o(t)
\end{aligned}$$

Dividing both sides by t, we obtain

$$\begin{aligned}
\frac{M_1(u,y;b)}{t} &= \frac{M_1(u,y;b)}{t}+c\frac{\partial M_1(u,y;b)}{\partial u}-ay\frac{\partial M_1(u,y;b)}{\partial y}+\frac{o(t)}{t}-\lambda M_1(u,y;b)\\
-\lambda tc\frac{\partial M_1(u,y;b)}{\partial u}&+\lambda tay\frac{\partial M_1(u,y;b)}{\partial y}-\lambda t\frac{o(t)}{t}+\lambda\left[\int_0^{u+ct} M_1(u+ct-x,ye^{-at};b)dF(x)\right.\\
&\left.+\int_{u+ct}^{u+ct+\frac{c}{\delta}} M_2(u+ct-x,ye^{-at};b)dF(x)+\bar{F}(u+ct+\frac{c}{\delta})\right]+\frac{o(t)}{t}
\end{aligned}$$

Letting $t\to 0$ yields

$$\begin{aligned}
c\frac{\partial M_1(u,y;b)}{\partial u} &= ay\frac{\partial M_1(u,y;b)}{\partial y}+\lambda M_1(u,y;b)-\lambda\left[\int_0^{u} M_1(u-x,y;b)dF(x)\right.\\
&\left.+\int_{u}^{u+\frac{c}{\delta}} M_2(u-x,y;b)dF(x)+\bar{F}(u+\frac{c}{\delta})\right]
\end{aligned}$$

ii. When $-\frac{c}{\delta}<u<0$, we assume that the surplus, under the effect of the debit interest force δ, will not attain 0 in the time interval $(0,t]$, for a small $t>0$. Let t_0 be the first time the negative surplus becomes zero, provided there is no claim in $[0,t_0]$. Furthermore, let $h(t,u)$, $t\le t_0$, denote the values of the surplus, provided that there is no claim in $[0,t]$. Consequently, $h(t_0,u)=0$ and $h(0,u)=u$. It always holds that

$$h(t,u)=ue^{\delta t}+c\left[\frac{e^{\delta t}-1}{\delta}\right]$$

and t_0 is the solution of $h(t,u)=0$. We assume that $t<t_0$. In the time interval $(0,t]$, we can have one claim ($N(t)=1$), or no claim ($N(t)=0$). As we have seen in Definition 1.1.3, the probabilities are $Pr(N(t)=1)=\lambda t$ and $Pr(N(t)=0)=1-\lambda t$, respectively. Conditioning on the time and size of the first claim and using the strong Markov property (3.4) and the renewal argument, the law of total probability yields

$$\begin{aligned} M_2(u, y; b) &= E\left[M_2\left(U_b(t), ye^{-at}; b\right)\right] \\ &= \sum_{k=0}^{1} E\left[M_2\left(U_b(t), ye^{-at}; b\right) | N(t) = k\right] Pr\left(N(t) = k\right) + o(t) \end{aligned}$$

where, $\lim\limits_{t \to 0} \dfrac{o(t)}{t} = 0$.

When $N(t) = 0$, we have $U_b(t) = h(t, u)$. Thus,

$$E\left[M_2\left(U_b(t), ye^{-at}; b\right) | N(t) = 0\right] = M_2(h(t, u), ye^{-at}; b)$$

When $N(t) = 1$, regarding the size x of the first claim, there are two potential situations for $M_2\left(U_b(t), ye^{-at}; b\right)$, which can be depicted by the Figure 3.4. Namely,

- for $x < h(t, u) + \frac{c}{\delta}$, the procedure is renewed and $U_b(t) = h(t, u) - x < 0$
- for $x \geq h(t, u) + \frac{c}{\delta}$, absolute ruin happens and the moment-generating function is 1, because there are not dividend payments $(D = 0)$

i.e. $$M_2\left(U_b(t), ye^{-at}; b\right) = \begin{cases} M_2\left(h(t, u) - x, ye^{-at}; b\right), & 0 < x < h(t, u) + \frac{c}{\delta} \\ 1, & x \geq h(t, u) + \frac{c}{\delta} \end{cases}$$

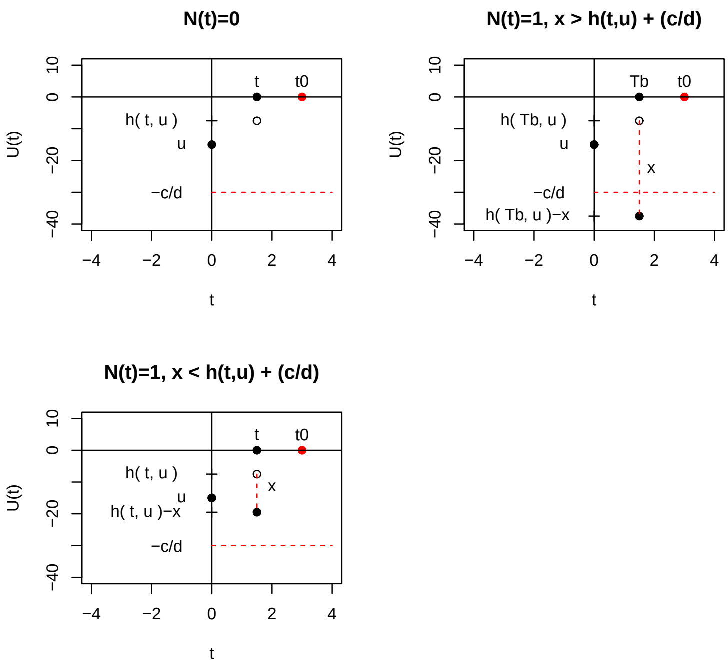


Figure 3.4: Cases of the first claim

So, we obtain

$$
\begin{aligned}
M_2(u,y;b) = \quad & (1-\lambda t)M_2\left(h(t,u), ye^{-at}; b\right) + \lambda t \left[\int_0^{h(t,u)+\frac{c}{\delta}} M_2\left(h(t,u)-x, ye^{-at}; b\right) dF(x)\right. \\
& \left. + \int_{h(t,u)+\frac{c}{\delta}}^{\infty} dF(x)\right] + o(t)
\end{aligned} \tag{3.6}
$$

Let $M_2(h(t,u), ye^{-at}; b) = M_2(ue^{\delta t} + c\left[\frac{e^{\delta t}-1}{\delta}\right], ye^{-at}; b) = M_2(u(t), y(t); b)$. Then, the Taylor's expansion evaluated at the point $t = 0$ leads to

$$
\begin{aligned}
M_2(u(t), y(t); b) &= M_2(u(0), y(0); b) + t\frac{d}{dt}[M_2(u(t), y(t); b)]_{t=0} + o(t) \\
&= M_2(u,y;b) + t\left[\frac{\partial M_2(u(0), y(0); b)}{\partial u}\left[\frac{\partial u(t)}{\partial t}\right]_{t=0}\right. \\
&\quad \left. + \frac{\partial M_2(u(0), y(0); b)}{\partial y}\left[\frac{\partial y(t)}{\partial t}\right]_{t=0}\right] + o(t) \\
&= M_2(u,y;b) + (\delta u + c)t\frac{\partial M_2(u,y;b)}{\partial u} - ayt\frac{\partial M_2(u,y;b)}{\partial y} + o(t)
\end{aligned}
$$

Substituting the above in (3.6) leads to

$$
\begin{aligned}
M_2(u,y;b) = \quad & (1-\lambda t)\left[M_2(u,y;b) + (\delta u + c)t\frac{\partial M_2(u,y;b)}{\partial u} - ayt\frac{\partial M_2(u,y;b)}{\partial y} + o(t)\right] \\
& + \lambda t\left[\int_0^{h(t,u)+\frac{c}{\delta}} M_2\left(h(t,u)-x, ye^{-at}; b\right) dF(x) + \int_{h(t,u)+\frac{c}{\delta}}^{\infty} dF(x)\right] + o(t)
\end{aligned}
$$

Dividing both sides by t, we are led to

$$
\begin{aligned}
\frac{M_2(u,y;b)}{t} = \quad & \frac{M_2(u,y;b)}{t} + (\delta u + c)\frac{\partial M_2(u,y;b)}{\partial u} - ay\frac{\partial M_2(u,y;b)}{\partial y} + \frac{o(t)}{t} - \lambda M_2(u,y;b) \\
& - \lambda t(\delta u + c)\frac{\partial M_2(u,y;b)}{\partial u} + \lambda tay\frac{\partial M_2(u,y;b)}{\partial y} - \lambda t\frac{o(t)}{t} \\
& + \lambda\left[\int_0^{h(t,u)+\frac{c}{\delta}} M_2\left(h(t,u)-x, ye^{-at}; b\right) dF(x) + \bar{F}\left(h(t,u) + \frac{c}{\delta}\right)\right] + \frac{o(t)}{t}
\end{aligned}
$$

If $t \to 0$, then $h(0,u) = u$ and the final integro-differential equation for $M_2(u,y;b)$ is

$$
\begin{aligned}
(\delta u + c)\frac{\partial M_2(u,y;b)}{\partial u} &= ay\frac{\partial M_2(u,y;b)}{\partial y} + \lambda M_2(u,y;b) \\
&\quad -\lambda\left[\int_0^{u+\frac{c}{\delta}} M_2(u-x,y;b)dF(x) + \bar{F}(u+\frac{c}{\delta})\right]
\end{aligned}
$$

$\diamondsuit$

In the following Theorem, we present the boundary conditions and the continuous property of $M(u,y;b)$ at $u=0$, according to the paper of Wang and Yin (2009), without providing any proof.

Theorem 3.2.2. *$M_1(u,y;b)$ and $M_2(u,y;b)$ satisfy*

$$
\left.\frac{\partial M_1(u,y;b)}{\partial u}\right|_{u=b} = yM_1(b,y;b) \tag{3.7}
$$

$$
M_2(-\frac{c}{\delta},y;b) = 1 \tag{3.8}
$$

$$
M_1(0,y;b) = M_2(0^-,y;b) \tag{3.9}
$$

3.3 Moments of the Dividend Payments

Applying the definitions of the moment-generating function $M(u,y;b)$ and the moments $V_n(u,b)$, we can derive the next expression of them.

Lemma 3.3.1. *For any random variable Y, it always holds*

$$
M_Y(t) = E\left[e^{tY}\right] = E\left[\sum_{n=0}^{\infty}\frac{(tY)^n}{n!}\right] = E\left[1+\sum_{n=1}^{\infty}\frac{t^n}{n!}Y^n\right] = 1+\sum_{n=1}^{\infty}\frac{t^n}{n!}E\left[Y^n\right]
$$

As a result, $M(u,y;b)$ and $V_n(u,b)$ are connected through the corresponding expression, which is used by Wang and Yin (2009),

$$
M(u,y;b) = 1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_n(u,b) \tag{3.10}
$$

Indeed,

$$
\begin{aligned}
M(u,y;b) &= E\left[e^{yD_{u,b}}\right] = E\left[\sum_{n=0}^{\infty}\frac{(yD_{u,b})^n}{n!}\right] = E\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}D_{u,b}^n\right] = 1+\sum_{n=1}^{\infty}\frac{y^n}{n!}E\left[D_{u,b}^n\right] \\
&= 1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_n(u,b)
\end{aligned}
$$

$\diamondsuit$

Using the expression (3.10) and the results of Theorem 3.2.1 and Theorem 3.2.2, Wang and Yin (2009) extract the integro-differential equations satisfied by $V_n(u,b)$.

Theorem 3.3.1. *The moments $V_n(u,b)$ of $D_{u,b}$ satisfy the following integro-differential equations:*
i. For $0<u<b$, it holds

$$cV_{n1}^{'}(u,b) = (\lambda+na)V_{n1}(u,b) - \lambda\left[\int_0^u V_{n1}(u-x,b)dF(x) + \int_u^{u+\frac{c}{\delta}} V_{n2}(u-x,b)dF(x)\right] \tag{3.11}$$

ii. For $-\frac{c}{\delta}<u<0$, we have

$$(\delta u+c)V_{n2}^{'}(u,b) = (\lambda+na)V_{n2}(u,b) - \lambda\int_0^{u+\frac{c}{\delta}} V_{n2}(u-x,b)dF(x) \tag{3.12}$$

Proof.
i. When $0<u<b$, by substituting (3.10) in (3.2), we obtain

$$\begin{aligned}
c\frac{\partial}{\partial u}\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n1}(u,b)\right] &= ay\frac{\partial}{\partial y}\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n1}(u,b)\right] + \lambda\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n1}(u,b)\right] \\
&-\lambda\left[\int_0^u\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n1}(u-x,b)\right]dF(x)\right. \\
&\left.+\int_u^{u+\frac{c}{\delta}}\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n2}(u-x,b)\right]dF(x) + \bar{F}(u+\frac{c}{\delta})\right] \Longrightarrow
\end{aligned}$$

$$\begin{aligned}
c\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n1}^{'}(u,b) = \; & ay\left[\sum_{n=1}^{\infty}n\frac{y^{n-1}}{n!}V_{n1}(u,b)\right] + \lambda\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n1}(u,b)\right] \\
&-\lambda\left[\int_0^u dF(x) + \sum_{n=1}^{\infty}\frac{y^n}{n!}\int_0^u V_{n1}(u-x,b)dF(x) + \int_u^{u+\frac{c}{\delta}} dF(x)\right. \\
&\left.+\sum_{n=1}^{\infty}\frac{y^n}{n!}\int_u^{u+\frac{c}{\delta}} V_{n2}(u-x,b)dF(x) + \int_{u+\frac{c}{\delta}}^{\infty} dF(x)\right] \Longrightarrow
\end{aligned}$$

$$\sum_{n=1}^{\infty}\frac{y^n}{n!}cV_{n1}'(u,b) = \sum_{n=1}^{\infty}\frac{y^n}{n!}naV_{n1}(u,b)+\lambda+\sum_{n=1}^{\infty}\frac{y^n}{n!}\lambda V_{n1}(u,b)-\lambda$$
$$+\sum_{n=1}^{\infty}\frac{y^n}{n!}(-\lambda)\int_0^u V_{n1}(u-x,b)dF(x)$$
$$+\sum_{n=1}^{\infty}\frac{y^n}{n!}(-\lambda)\int_u^{u+\frac{c}{\delta}} V_{n2}(u-x,b)dF(x)$$

Comparing the coefficients of y^n, $n\in\mathcal{N}^+$ yields

$$cV_{n1}'(u,b)=(\lambda+na)V_{n1}(u,b)-\lambda\left[\int_0^u V_{n1}(u-x,b)dF(x)+\int_u^{u+\frac{c}{\delta}} V_{n2}(u-x,b)dF(x)\right]$$

ii. When $-\frac{c}{\delta}<u<0$, applying (3.10) in (3.3) yields

$$(\delta u+c)\frac{\partial}{\partial u}\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n2}(u,b)\right] = ay\frac{\partial}{\partial y}\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n2}(u,b)\right]+\lambda\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n2}(u,b)\right]$$
$$-\lambda\left[\int_0^{u+\frac{c}{\delta}}\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n2}(u-x,b)\right]dF(x)+\bar{F}(u+\frac{c}{\delta})\right]$$
$$(\delta u+c)\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n2}'(u,b) = ay\sum_{n=1}^{\infty}n\frac{y^{n-1}}{n!}V_{n2}(u,b)+\lambda\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n2}(u,b)\right]$$
$$-\lambda\left[\int_0^{u+\frac{c}{\delta}}dF(x)+\sum_{n=1}^{\infty}\frac{y^n}{n!}\int_0^{u+\frac{c}{\delta}}V_{n2}(u-x,b)dF(x)+\int_{u+\frac{c}{\delta}}^{\infty}dF(x)\right]$$

$$\sum_{n=1}^{\infty}\frac{y^n}{n!}(\delta u+c)V_{n2}'(u,b) = \sum_{n=1}^{\infty}\frac{y^n}{n!}naV_{n2}(u,b)+\lambda+\sum_{n=1}^{\infty}\frac{y^n}{n!}\lambda V_{n2}(u,b)$$
$$-\lambda+\sum_{n=1}^{\infty}\frac{y^n}{n!}(-\lambda)\int_0^{u+\frac{c}{\delta}}V_{n2}(u-x,b)dF(x)$$

Comparing the coefficients of y^n, $n\in\mathcal{N}^+$ yields

$$(\delta u+c)V_{n2}'(u,b)=(\lambda+na)V_{n2}(u,b)-\lambda\int_0^{u+\frac{c}{\delta}}V_{n2}(u-x,b)dF(x)$$

$\diamond$

In the following Proposition, Wang and Yin (2009) derive the boundary conditions and the continuous property of $V_n(u,b)$ at $u=0$, by applying similar arguments to Theorem 3.3.1.

Proposition 3.3.1. *For the boundary values b and $-\frac{c}{\delta}$, $V_n(u,b)$ satisfies*

$$\begin{aligned} V_{n1}^{'}(u,b)\Big|_{u=b} &= nV_{n-1,1}(b,b), \qquad n \in \mathcal{N}^+ &(3.13)\\ V_{n2}(-\frac{c}{\delta},b) &= 0, \qquad n \in \mathcal{N}^+ &(3.14)\end{aligned}$$

Moreover, $V_n(u,b)$ and $V_n^{'}(u,b)$ are continuous at $u=0$, i.e.

$$\begin{aligned} V_{n1}(0,b) &= V_{n2}(0^-,b), \qquad n \in \mathcal{N}^+ &(3.15)\\ V_{n1}^{'}(0^+,b) &= V_{n2}^{'}(0^-,b), \qquad n \in \mathcal{N}^+ &(3.16)\end{aligned}$$

Proof. Substituting (3.10) in (3.7) leads to

$$\begin{aligned} \frac{\partial}{\partial u}\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n1}(u,b)\right]_{u=b} &= y\left[1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n1}(b,b)\right] \Longrightarrow\\ \sum_{n=1}^{\infty}\frac{y^n}{n!}\,V_{n1}^{'}(u,b)\Big|_{u=b} &= y+\sum_{n=1}^{\infty}\frac{y^{n+1}}{n!}V_{n1}(b,b)\end{aligned}$$

As it holds $V_{01}(b,b)=1$, by setting $k=n+1$ in the serie at the right side, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{y^n}{n!}\,V_{n1}^{'}(u,b)\Big|_{u=b} &= y+\sum_{k=2}^{\infty}k\frac{y^k}{k!}V_{k-1,1}(b,b) \Longrightarrow\\ \sum_{n=1}^{\infty}\frac{y^n}{n!}\,V_{n1}^{'}(u,b)\Big|_{u=b} &= \sum_{k=1}^{\infty}k\frac{y^k}{k!}V_{k-1,1}(b,b)\end{aligned}$$

Taking into consideration the corresponding coefficients of y^n, $n \in \mathcal{N}^+$, we are led to

$$V_{n1}^{'}(u,b)\Big|_{u=b} = nV_{n-1,1}(b,b), \qquad n \in \mathcal{N}^+$$

Now, from (3.10) and (3.8), we obtain

$$1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n2}(-\frac{c}{\delta},b) = 1 \Longrightarrow \sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n2}(-\frac{c}{\delta},b) = 0 \Longrightarrow$$

$$V_{n2}(-\tfrac{c}{\delta},b) = 0, \quad n \in \mathcal{N}^+$$

Regarding the continuous property of $M(u,y;b)$ at $u=0$ in (3.9), substituting (3.10) yields

$$\begin{aligned}
M_1(0, y; b) &= M_2(0^-, y; b) &&\Longrightarrow \\
1 + \sum_{n=1}^{\infty} \frac{y^n}{n!} V_{n1}(0, b) &= 1 + \sum_{n=1}^{\infty} \frac{y^n}{n!} V_{n2}(0^-, b) &&\Longrightarrow \\
V_{n1}(0, b) &= V_{n2}(0^-, b) \quad n \in \mathcal{N}^+
\end{aligned}$$

Finally, letting $u \to 0^+$ in (3.11), we obtain

$$cV_{n1}^{'}(0, b) = (\lambda + na)V_{n1}(0, b) - \lambda \int_0^{\frac{c}{\delta}} V_{n2}(-x, b)dF(x) \tag{3.17}$$

and, letting $u \to 0^-$ in (3.12), we obtain

$$cV_{n2}^{'}(0^-, b) = (\lambda + na)V_{n2}(0^-, b) - \lambda \int_0^{\frac{c}{\delta}} V_{n2}(-x, b)dF(x) \tag{3.18}$$

From (3.15), (3.17) and (3.18), we are led to

$$V_{n1}^{'}(0^+, b) = V_{n2}^{'}(0^-, b), \qquad n \in \mathcal{N}^+$$

$\diamondsuit$

3.4 Gerber-Shiu function

In the paper of Yuen, Zhou and Guo (2008) we can find the definition of Gerber-Shiu function for the surplus process studied at this Chapter, i.e. for the surplus process $\{U_b(t) : t \geq 0\}$ with debit interest force $\delta > 0$ and dividend payments according to a barrier strategy. Yuen, Zhou and Guo based on Gerber and Shiu (1998) give the following definition.

Definition 3.4.1. *The Gerber-Shiu function, or the expected discounted penalty function, for the surplus process $\{U_b(t) : t \geq 0\}$ with debit interest force $\delta > 0$ and dividend payments according to a barrier strategy, is given by*

$$\phi_b(u) = E\left[e^{-aT_b} w\left(U_b(T_b^-), |U_b(T_b)|\right) I(T_b < \infty) | U_b(0) = u\right] \tag{3.19}$$

where u is the initial surplus, T_b is the absolute ruin time, b is the threshold for the dividend payments and $a > 0$ can be considered either as a discounting interest force for the penalty function $w(x, y)$ or as the argument for the Laplace transform of T_b. The function $w(x, y)$ is a bivariate nonnegative function with domain $(-\frac{c}{\delta}, \infty)x[\frac{c}{\delta}, \infty)$, because the surplus prior to absolute ruin, $U_b(T_b^-)$, is greater than $-\frac{c}{\delta}$ and the deficit at absolute ruin, $|U_b(T_b)|$, is at least $\frac{c}{\delta}$.

Furthermore, we point out two different paths for $\phi_b(u)$, regarding the values of u. Namely,

$$\phi_b(u) = \begin{cases} \phi_{b+}(u), & 0 \leq u < b \\ \\ \phi_{b-}(u), & -\dfrac{c}{\delta} < u < 0 \end{cases}$$

Finally, following similar arguments to Definition 1.2.2, we remind that Gerber-Shiu function is a general function and according to the values of its terms, many significant actuarial functions can be derived. For instance, the absolute ruin probability ($a = 0$ and $w(x, y) = 1$), the Laplace transform of T_b ($w(x, y) = 1$), the distribution of $U_b(T_b^-)$ ($a = 0$ and $w(x, y) = I(X \leq x)$), the distribution of $|U_b(T_b)|$ ($a = 0$ and $w(x, y) = I(Y \leq y)$) and the joint distribution of $U_b(T_b^-)$ and $|U_b(T_b)|$ ($a = 0$ and $w(x, y) = I(X \leq x, Y \leq y)$.

In the next Theorem, Yuen, Zhou and Guo (2008) prove the integro-differential equations satisfied by $\phi_b(u)$, by using the same methodology to Theorem 3.2.1.

Theorem 3.4.1. *The Gerber-Shiu function, $\phi_b(u)$, satisfies the following integro-differential equations:*
i. For $0 \leq u < b$,

$$\begin{aligned} c\phi'_{b+}(u) = \quad & (\lambda + a)\phi_{b+}(u) - \lambda \left[\int_0^u \phi_{b+}(u-x)dF(x) + \int_u^{u+\frac{c}{\delta}} \phi_{b-}(u-x)dF(x) \right. \\ & \left. + \int_{u+\frac{c}{\delta}}^{\infty} w(u, x-u)dF(x) \right] \end{aligned}$$

ii. For $-\dfrac{c}{\delta} < u < 0$,

$$(\delta u + c)\phi'_{b-}(u) = \quad (\lambda + a)\phi_{b-}(u) - \lambda \left[\int_0^{u+\frac{c}{\delta}} \phi_{b-}(u-x)dF(x) + \int_{u+\frac{c}{\delta}}^{\infty} w(u, x-u)dF(x) \right]$$

Proof.
i. When $0 \leq u < b$, we assume a small time interval $(0, t]$, in which the surplus does not attain the value b. In this time interval, $(0, t]$, we can have one claim ($N(t) = 1$), or no claim ($N(t) = 0$), with probabilities $Pr(N(t) = 1) = \lambda t$ and $Pr(N(t) = 0) = 1 - \lambda t$, respectively. Conditioning on the time and size of the first claim and using the renewal argument, the law of total probability yields

$$\phi_{b+}(u) = \sum_{k=0}^{1} \phi_{b+}\left(u | N(t) = k\right) Pr\left(N(t) = k\right) + o(t) \tag{3.20}$$

where $\lim_{t\to 0} \frac{o(t)}{t} = 0$. Based on the proof of Theorem 3.2.1 and Figure 3.3, we distinguish the following cases.

When $N(t) = 0$, the surplus process is renewed with initial surplus $u+ct > 0$. Discounting at time $t = 0$, we obtain

$$\phi_{b+}(u|N(t) = 0) = e^{-at}\phi_{b+}(u + ct)$$

When $N(t) = 1$, regarding the size x of the first claim, there are three potential situations for $\phi_{b+}(u|N(t) = 1)$, which can be depicted by the Figure 3.3. Namely,

- for $0 < x \le u + ct$, the surplus process is renewed with initial surplus $u + ct - x > 0$
- for $u + ct < x < u + ct + \frac{c}{\delta}$, the surplus process is renewed with initial surplus $u + ct - x < 0$
- for $x \ge u + ct + \frac{c}{\delta}$, absolute ruin happens. The surplus prior to absolute ruin is $U_b(T_b^-) = u + ct$ and the deficit at absolute ruin is $|U_b(T_b)| = |u + ct - x| = x - (u + ct) > 0$

Discounting all the above at $t = 0$, we obtain

$$\phi_{b+}(u|N(t) = 1) = \begin{cases} e^{-at}\phi_{b+}(u + ct - x), & 0 < x \le u + ct \\ e^{-at}\phi_{b-}(u + ct - x), & u + ct < x < u + ct + \frac{c}{\delta} \\ e^{-at}w(u + ct, x - (u + ct)), & x \ge u + ct + \frac{c}{\delta} \end{cases}$$

As a result, (3.20) can be written as

$$\begin{aligned} \phi_{b+}(u) = \ & (1 - \lambda t)e^{-at}\phi_{b+}(u + ct) + \lambda t \left[\int_0^{u+ct} e^{-at}\phi_{b+}(u + ct - x)dF(x) \right. \\ & + \int_{u+ct}^{u+ct+\frac{c}{\delta}} e^{-at}\phi_{b-}(u + ct - x)dF(x) + \left. \int_{u+ct+\frac{c}{\delta}}^{\infty} e^{-at}w(u + ct, x - (u + ct))dF(x) \right] \\ & + o(t) \end{aligned} \tag{3.21}$$

If we set $g(t) = e^{-at}\phi_{b+}(u + ct) = e^{-at}\phi_{b+}(u(t))$, the Taylor's expansion evaluated at $t = 0$ leads to

$$\begin{aligned}
g(t) &= g(0) + t\left[\frac{d}{dt}g(t)\bigg|_{t=0}\right] + o(t) \\
&= \phi_{b+}(u(0)) + t\left[\frac{d}{dt}e^{-at}\bigg|_{t=0} \cdot \phi_{b+}(u(0)) + e^{-at}\big|_{t=0} \cdot \frac{\partial \phi_{b+}(u(0))}{\partial u} \cdot \frac{\partial u(t)}{\partial t}\bigg|_{t=0}\right] + o(t) \\
&= \phi_{b+}(u) + t\left[-a\phi_{b+}(u) + c\phi_{b+}^{'}(u)\right] + o(t)
\end{aligned}$$

So, we have

$$e^{-at}\phi_{b+}(u+ct) = \phi_{b+}(u) - at\phi_{b+}(u) + ct\phi_{b+}^{'}(u) + o(t) \tag{3.22}$$

Substituting (3.22) in (3.21) yields

$$\begin{aligned}
\phi_{b+}(u) = \quad & (1-\lambda t)\left[\phi_{b+}(u) - at\phi_{b+}(u) + ct\phi_{b+}^{'}(u) + o(t)\right] \\
& +\lambda t\left[\int_0^{u+ct} e^{-at}\phi_{b+}(u+ct-x)dF(x) + \int_{u+ct}^{u+ct+\frac{c}{\delta}} e^{-at}\phi_{b-}(u+ct-x)dF(x)\right. \\
& \left. + \int_{u+ct+\frac{c}{\delta}}^{\infty} e^{-at}w(u+ct, x-(u+ct))dF(x)\right] + o(t)
\end{aligned}$$

Dividing both sides by t, we obtain

$$\begin{aligned}
\frac{\phi_{b+}(u)}{t} = \quad & (1-\lambda t)\left[\frac{\phi_{b+}(u)}{t} - a\phi_{b+}(u) + c\phi_{b+}^{'}(u) + \frac{o(t)}{t}\right] \\
& +\lambda\left[\int_0^{u+ct} e^{-at}\phi_{b+}(u+ct-x)dF(x) + \int_{u+ct}^{u+ct+\frac{c}{\delta}} e^{-at}\phi_{b-}(u+ct-x)dF(x)\right. \\
& \left. + \int_{u+ct+\frac{c}{\delta}}^{\infty} e^{-at}w(u+ct, x-(u+ct))dF(x)\right] + \frac{o(t)}{t} \\
= \quad & \frac{\phi_{b+}(u)}{t} - a\phi_{b+}(u) + c\phi_{b+}^{'}(u) + \frac{o(t)}{t} \\
& -\lambda\phi_{b+}(u) + \lambda at\phi_{b+}(u) - \lambda ct\phi_{b+}^{'}(u) - \lambda t\frac{o(t)}{t} \\
& +\lambda\left[\int_0^{u+ct} e^{-at}\phi_{b+}(u+ct-x)dF(x) + \int_{u+ct}^{u+ct+\frac{c}{\delta}} e^{-at}\phi_{b-}(u+ct-x)dF(x)\right. \\
& \left. + \int_{u+ct+\frac{c}{\delta}}^{\infty} e^{-at}w(u+ct, x-(u+ct))dF(x)\right] + \frac{o(t)}{t}
\end{aligned}$$

Letting $t \to 0$, we are led to

$$c\phi_{b+}^{'}(u) = (\lambda + a)\phi_{b+}(u) - \lambda \left[\int_0^u \phi_{b+}(u-x)dF(x) + \int_u^{u+\frac{c}{\delta}} \phi_{b-}(u-x)dF(x) \right.$$
$$\left. + \int_{u+\frac{c}{\delta}}^{\infty} w(u, x-u)dF(x) \right]$$

ii. When $-\frac{c}{\delta} < u < 0$, we assume that the surplus, under the effect of the debit interest force δ, will not attain 0 in the time interval $(0, t]$, for a small $t > 0$. Let t_0 be the first time the negative surplus becomes zero, provided there is no claim in $[0, t_0]$. Furthermore, let $h(t, u)$, $t \leq t_0$, denote the values of the surplus, provided there is no claim in $[0, t]$. Consequently, $h(t_0, u) = 0$ and $h(0, u) = u$. It always holds that

$$h(t, u) = ue^{\delta t} + c \left[\frac{e^{\delta t} - 1}{\delta} \right]$$

and t_0 is the solution of $h(t, u) = 0$. We assume that $t < t_0$. In the time interval $(0, t]$, we can have one claim ($N(t) = 1$), or no claim ($N(t) = 0$). As we have seen in Definition 1.1.3, the probabilities are $Pr(N(t) = 1) = \lambda t$ and $Pr(N(t) = 0) = 1 - \lambda t$, respectively. Conditioning on the time and size of the first claim and using the renewal argument, the law of total probability yields

$$\phi_{b-}(u) = \sum_{k=0}^{1} \phi_{b-}\left(u | N(t) = k\right) Pr\left(N(t) = k\right) + o(t) \tag{3.23}$$

where $\lim_{t \to 0} \frac{o(t)}{t} = 0$. Based on the proof of Theorem 3.2.1 and Figure 3.4, we distinguish the following cases.

When $N(t) = 0$, the surplus process is renewed with initial surplus $h(t, u) < 0$. Discounting at time $t = 0$, we obtain

$$\phi_{b-}\left(u | N(t) = 0\right) = e^{-at}\phi_{b-}(h(t, u))$$

When $N(t) = 1$, regarding the size x of the first claim, there are two potential situations for $\phi_{b-}\left(u | N(t) = 1\right)$, which can be depicted by the Figure 3.4. Namely,

- for $0 < x < h(t, u) + \frac{c}{\delta}$, the surplus process is renewed with initial surplus $h(t, u) - x < 0$

- for $x \geq h(t,u) + \frac{c}{\delta}$, absolute ruin happens. The surplus prior to absolute ruin is $U_b(T_b^-) = h(t,u) < 0$ and the deficit at absolute ruin is $|U_b(T_b)| = |h(t,u) - x| = x - h(t,u) > 0$

Discounting all the above at $t = 0$, we obtain

$$\phi_{b-}(u|N(t)=1) = \begin{cases} e^{-at}\phi_{b-}(h(t,u)-x), & 0 < x < h(t,u) + \frac{c}{\delta} \\ e^{-at}w(h(t,u), x - h(t,u)), & x \geq h(t,u) + \frac{c}{\delta} \end{cases}$$

Overall, (3.23) is written

$$\begin{aligned} \phi_{b-}(u) = \; & (1-\lambda t)e^{-at}\phi_{b-}(h(t,u)) + \lambda t \left[\int_0^{h(t,u)+\frac{c}{\delta}} e^{-at}\phi_{b-}(h(t,u)-x)dF(x) \right. \\ & \left. + \int_{h(t,u)+\frac{c}{\delta}}^{\infty} e^{-at} w(h(t,u), x-h(t,u))dF(x) \right] + o(t) \end{aligned} \tag{3.24}$$

If we set $y(t) = e^{-at}\phi_{b-}(h(t,u)) = e^{-at}\phi_{b-}\left(ue^{\delta t} + c\left[\frac{e^{\delta t}-1}{\delta}\right]\right)$, the Taylor's expansion evaluated at $t = 0$ yields

$$\begin{aligned} y(t) = \; & y(0) + t\left[\frac{d}{dt}y(t)\bigg|_{t=0}\right] + o(t) \\ = \; & \phi_{b-}(h(0,u)) + t\left[\frac{d}{dt}e^{-at}\bigg|_{t=0} \cdot \phi_{b-}(h(0,u)) + e^{-at}\big|_{t=0} \cdot \frac{\partial\phi_{b-}(h(0,u))}{\partial h} \cdot \frac{\partial h(t,u)}{\partial t}\bigg|_{t=0}\right] \\ & + o(t) \\ = \; & \phi_{b-}(u) + t\left[-a\phi_{b-}(u) + (\delta u + c)\phi_{b-}^{'}(u)\right] + o(t) \end{aligned}$$

So, we have

$$e^{-at}\phi_{b-}(h(t,u)) = \phi_{b-}(u) - at\phi_{b-}(u) + (\delta u + c)t\phi_{b-}^{'}(u) + o(t) \tag{3.25}$$

Substituting (3.25) in (3.24), we obtain

$$\begin{aligned} \phi_{b-}(u) = \; & (1-\lambda t)\left[\phi_{b-}(u) - at\phi_{b-}(u) + (\delta u + c)t\phi_{b-}^{'}(u) + o(t)\right] \\ & + \lambda t \left[\int_0^{h(t,u)+\frac{c}{\delta}} e^{-at}\phi_{b-}(h(t,u)-x)dF(x) \right. \\ & \left. + \int_{h(t,u)+\frac{c}{\delta}}^{\infty} e^{-at} w(h(t,u), x-h(t,u))dF(x) \right] + o(t) \end{aligned}$$

Dividing both sides by t, we have

$$\begin{aligned}
\frac{\phi_{b-}(u)}{t} = & \frac{\phi_{b-}(u)}{t} - a\phi_{b-}(u) + (\delta u + c)\phi_{b-}^{'}(u) + \frac{o(t)}{t} - \lambda\phi_{b-}(u) \\
& +\lambda at\phi_{b-}(u) - \lambda(\delta u + c)t\phi_{b-}^{'}(u) - \lambda t\frac{o(t)}{t} \\
& +\lambda\left[\int_0^{h(t,u)+\frac{c}{\delta}} e^{-at}\phi_{b-}(h(t,u) - x)dF(x)\right. \\
& \left. + \int_{h(t,u)+\frac{c}{\delta}}^{\infty} e^{-at}w(h(t,u), x - h(t,u))dF(x)\right] + \frac{o(t)}{t}
\end{aligned}$$

Letting $t \to 0$, we finally obtain

$$(\delta u + c)\phi_{b-}^{'}(u) = \quad (\lambda + a)\phi_{b-}(u) - \lambda\left[\int_0^{u+\frac{c}{\delta}} \phi_{b-}(u - x)dF(x) + \int_{u+\frac{c}{\delta}}^{\infty} w(u, x - u)dF(x)\right]$$

$\diamondsuit$

3.5 Exponential Claims

3.5.1 Differential Equations for the Moments of the Dividend Payments

In this section we assume that claim sizes $\{X_i\}_{i=1}^{\infty}$ obey exponential distribution with parameter $\beta > 0$, i.e. $f_X(x) = f(x) = \beta e^{-\beta x}$, $x \geq 0$. Based on the study of Wang and Yin (2009), we present explicit expressions for the moments of $D_{u,b}$, $V_n(u,b)$, and more specifically, we focus on the case of $n = 1$. The first moment $V_1(u,b)$ denotes the expected present value of all dividends paid until the absolute ruin time T_b.

Theorem 3.5.1.1. *The moments of $D_{u,b}$, $V_n(u,b)$, satisfy the following second order differential equations:*
i. For $0 < u < b$

$$V_{n1}^{''}(u,b) + \left(\beta - \frac{\lambda + na}{c}\right)V_{n1}^{'}(u,b) - \frac{\beta na}{c}V_{n1}(u,b) = 0 \tag{3.26}$$

ii. For $-\frac{c}{\delta} < u < 0$

$$(\delta u + c)V_{n2}^{''}(u,b) + \left[\beta\left(\delta u + c\right) + \delta - (\lambda + na)\right]V_{n2}^{'}(u,b) - \beta naV_{n2}(u,b) = 0 \tag{3.27}$$

Proof.

i. When $0 < u < b$, substituting $dF(x) = f(x)dx = \beta e^{-\beta x}dx$ in (3.11) yields

$$cV_{n1}'(u,b) = (\lambda + na)V_{n1}(u,b) - \lambda\left[\int_0^u V_{n1}(u-x,b)\beta e^{-\beta x}dx + \int_u^{u+\frac{c}{\delta}} V_{n2}(u-x,b)\beta e^{-\beta x}dx\right]$$

Replacing $y = u - x,\ dy = -dx$, the boundaries of integration are converted into $y \to u$ when $x \to 0$, $y \to 0$ when $x \to u$ and $y \to -\frac{c}{\delta}$ when $x \to u + \frac{c}{\delta}$. Hence, we obtain

$$cV_{n1}'(u,b) = (\lambda + na)V_{n1}(u,b) - \lambda\left[-\int_u^0 V_{n1}(y,b)\beta e^{-\beta(u-y)}dy - \int_0^{-\frac{c}{\delta}} V_{n2}(y,b)\beta e^{-\beta(u-y)}dy\right] \implies$$

$$cV_{n1}'(u,b) = (\lambda + na)V_{n1}(u,b) - \lambda\beta e^{-\beta u}\left[\int_0^u V_{n1}(y,b)e^{\beta y}dy + \int_{-\frac{c}{\delta}}^0 V_{n2}(y,b)e^{\beta y}dy\right] \tag{3.28}$$

Differentiating (3.28) with respect to u leads to

$$cV_{n1}''(u,b) = (\lambda + na)V_{n1}'(u,b) + \lambda\beta^2 e^{-\beta u}\left[\int_0^u V_{n1}(y,b)e^{\beta y}dy + \int_{-\frac{c}{\delta}}^0 V_{n2}(y,b)e^{\beta y}dy\right] - \lambda\beta V_{n1}(u,b) \tag{3.29}$$

Multiplying both sides of (3.28) by β yields

$$\beta cV_{n1}'(u,b) = \beta(\lambda + na)V_{n1}(u,b) - \lambda\beta^2 e^{-\beta u}\left[\int_0^u V_{n1}(y,b)e^{\beta y}dy + \int_{-\frac{c}{\delta}}^0 V_{n2}(y,b)e^{\beta y}dy\right] \tag{3.30}$$

Adding down (3.29) and (3.30) and making the cancellations needed, we obtain

$$cV_{n1}''(u,b) + \beta cV_{n1}'(u,b) = (\lambda + na)V_{n1}'(u,b) - \lambda\beta V_{n1}(u,b) + \beta(\lambda + na)V_{n1}(u,b)$$

Dividing by c leads to

$$V_{n1}^{''}(u,b) + \left(\beta - \frac{\lambda + na}{c}\right) V_{n1}^{'}(u,b) - \frac{\beta na}{c} V_{n1}(u,b) = 0$$

ii. When $-\frac{c}{\delta} < u < 0$, substituting $dF(x) = f(x)dx = \beta e^{-\beta x}dx$ in (3.12) yields

$$(\delta u + c)V_{n2}^{'}(u,b) = (\lambda + na)V_{n2}(u,b) - \lambda \int_0^{u+\frac{c}{\delta}} V_{n2}(u-x,b)\beta e^{-\beta x}dx$$

Setting $y = u - x$, $dy = -dx$, the boundaries of integration are converted into $y \to u$ when $x \to 0$ and $y \to -\frac{c}{\delta}$ when $x \to u + \frac{c}{\delta}$. Thus, we obtain

$$(\delta u + c)V_{n2}^{'}(u,b) = (\lambda + na)V_{n2}(u,b) - \lambda\beta e^{-\beta u} \int_{-\frac{c}{\delta}}^{u} V_{n2}(y,b)e^{\beta y}dy \quad (3.31)$$

Differentiating (3.31) with respect to u yields

$$\begin{aligned} \delta V_{n2}^{'}(u,b) + (\delta u + c)V_{n2}^{''}(u,b) = \quad & (\lambda + na)V_{n2}^{'}(u,b) + \lambda\beta^2 e^{-\beta u} \int_{-\frac{c}{\delta}}^{u} V_{n2}(y,b)e^{\beta y}dy \\ & -\lambda\beta V_{n2}(u,b) \end{aligned} \quad (3.32)$$

Multiplying both sides of (3.31) by β yields

$$\beta(\delta u + c)V_{n2}^{'}(u,b) = \beta(\lambda + na)V_{n2}(u,b) - \lambda\beta^2 e^{-\beta u} \int_{-\frac{c}{\delta}}^{u} V_{n2}(y,b)e^{\beta y}dy \quad (3.33)$$

Adding down (3.32) and (3.33) and canceling out the terms with opposite signs, we obtain

$$\begin{aligned} \delta V_{n2}^{'}(u,b) + (\delta u + c)V_{n2}^{''}(u,b) + \beta(\delta u + c)V_{n2}^{'}(u,b) = \quad & (\lambda + na)V_{n2}^{'}(u,b) - \lambda\beta V_{n2}(u,b) \\ & +\beta(\lambda + na)V_{n2}(u,b) \qquad \Longrightarrow \end{aligned}$$

$$(\delta u + c)V_{n2}^{''}(u,b) + \left[\beta\left(\delta u + c\right) + \delta - (\lambda + na)\right] V_{n2}^{'}(u,b) - \beta na V_{n2}(u,b) = 0$$

$\diamondsuit$

3.5.2 Solutions of the Differential Equations

Theorem 3.5.2.1. *The equation (3.26) is a second order homogeneous linear differential equation with constant coefficients*

$$p_n = \beta - \frac{\lambda + na}{c} \quad \textit{and} \quad q_n = -\frac{\beta na}{c} \quad (3.34)$$

According to chapter 4.2.2, page 159, of Alikakos and Kalogeropoulos (2003) (see also Appendix A.2), the general solution of (3.26) is given by

$$V_{n1}(u,b) = c_{n1}e^{r_{n1}u} + c_{n2}e^{r_{n2}u}, \qquad 0 < u < b \tag{3.35}$$

where c_{n1}, c_{n2} *are arbitrary constants and* r_{n1}, r_{n2} *are the two distinct real roots of the characteristic equation*

$$r^2 + p_n r + q_n = 0$$

provided that the Discriminant is positive, $D = p_n^2 - 4q_n > 0$*, i.e.*

$$r_{n,1,2} = \frac{-p_n \pm \sqrt{p_n^2 - 4q_n}}{2} \tag{3.36}$$

$\diamondsuit$

As far as the solution of the equation (3.27) is concerned, Wang and Yin (2009), firstly, convert it into a confluent hypergeometric equation. Then, based on Abramowitz and Stegun (1972), they give an explicit solution, by using the initial conditions (3.13), (3.14), (3.15) and (3.16), as well.

Proposition 3.5.2.1. *The equation (3.27) can be reduced to a confluent hypergeometric equation*

$$y g_n^{''}(y) + \left[1 - \frac{\lambda + na}{\delta} - y\right] g_n^{'}(y) + \frac{na}{\delta} g_n(y) = 0, \quad -\frac{\beta c}{\delta} < y < 0 \tag{3.37}$$

Proof. Considering the transforms proposed by Wang and Yin (2009)

$$V_{n2}(u,b) = g_n(y(u)) = g_n(y) \text{ and } y = y(u) = -\frac{\beta(\delta u + c)}{\delta},$$

we evaluate separately the derivatives of the first and second order, by using the chain rule, i.e.

- $\dfrac{dy}{du} = -\beta$ and $\dfrac{d^2y}{du^2} = 0$
- $\dfrac{dV_{n2}(u,b)}{du} = \dfrac{dg_n(y)}{du} = \dfrac{dg_n(y)}{dy} \cdot \dfrac{dy}{du} = -\beta g_n^{'}(y)$
- $\dfrac{d^2V_{n2}(u,b)}{du^2} = \dfrac{d^2g_n(y)}{du^2} = \dfrac{d^2g_n(y)}{dy^2} \cdot \dfrac{dy}{du} \cdot \dfrac{dy}{du} + \dfrac{dg_n(y)}{dy} \cdot \left[\dfrac{d^2y}{du^2}\right] = \beta^2 g_n^{''}(y)$

Substituting them in (3.27) yields

$$(\delta u + c)\beta^2 g_n^{''}(y) + [\beta(\delta u + c) + \delta - (\lambda + na)]\,(-\beta g_n^{'}(y)) - \beta na g_n(y) = 0$$

Multiplying by $-\frac{1}{\beta\delta}$ leads to

$$-\frac{\beta(\delta u+c)}{\delta}g_n^{''}(y)+\left[\frac{\delta}{\delta}-\frac{\lambda+na}{\delta}+\frac{\beta(\delta u+c)}{\delta}\right]g_n^{'}(y)+\frac{na}{\delta}g_n(y)=0$$

Replacing $y=-\frac{\beta(\delta u+c)}{\delta}$, we obtain

$$yg_n^{''}(y)+\left[1-\frac{\lambda+na}{\delta}-y\right]g_n^{'}(y)+\frac{na}{\delta}g_n(y)=0$$

Moreover, when it holds $-\frac{c}{\delta}<u<0$, we have that

$$0<\delta u+c<c\Longrightarrow 0>-\frac{\beta(\delta u+c)}{\delta}>-\frac{\beta c}{\delta}\Longrightarrow -\frac{\beta c}{\delta}<y<0.$$

$\diamondsuit$

Theorem 3.5.2.2. *If $\frac{\lambda+na}{\delta}$ is not integer, the moments $V_{n2}(u,b)$, for $-\frac{c}{\delta}<u<0$, are given by*

$$V_{n2}(u,b)=c_{n4}h_{n4}(u),\qquad -\frac{c}{\delta}<u<0 \tag{3.38}$$

with

$$h_{n4}(u)=\left[\frac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+na}{\delta}}e^{-\frac{\beta(\delta u+c)}{\delta}}M\left(1+\frac{na}{\delta},1+\frac{\lambda+na}{\delta},\frac{\beta(\delta u+c)}{\delta}\right) \tag{3.39}$$

where, $M(x,y,z)$ is the confluent hypergeometric function of the first kind and c_{n4} is an arbitrary constant.

Proof. If $\frac{\lambda+na}{\delta}$ is not integer and $M(x,y,z)$, $U(x,y,z)$ are the confluent hypergeometric functions of the first and second kinds, respectively, according to (13.1.15) and (13.1.18) of Abramowitz and Stegun (1972), the general solution of (3.37) is given by

$$g_n(y)=c_{n3}e^yU\left(1-\frac{\lambda}{\delta},1-\frac{\lambda+na}{\delta},-y\right)+c_{n4}(-y)^{\frac{\lambda+na}{\delta}}e^yM\left(1+\frac{na}{\delta},1+\frac{\lambda+na}{\delta},-y\right)$$

for $-\frac{\beta c}{\delta}<y<0$, where $c_{n3},\ c_{n4}$ are arbitrary constants. Consequently, we obtain

$$V_{n2}(u,b)=g_n(-\frac{\beta(\delta u+c)}{\delta})=c_{n3}h_{n3}(u)+c_{n4}h_{n4}(u) \tag{3.40}$$

for $-\frac{c}{\delta}<u<0$, where

$$h_{n3}(u) = e^{-\frac{\beta(\delta u+c)}{\delta}} U\left(1 - \frac{\lambda}{\delta}, 1 - \frac{\lambda + na}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)$$

$$h_{n4}(u) = \left[\frac{\beta(\delta u + c)}{\delta}\right]^{\frac{\lambda+na}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}} M\left(1 + \frac{na}{\delta}, 1 + \frac{\lambda + na}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)$$

Regarding Wang and Yin (2009), who use properties of the confluent hypergeometric functions of the first and second kinds (see Appendix A.4, Corollary A.4.1), for $\delta \neq \lambda + na$, it holds

$$\lim_{u \to -\frac{c}{\delta}^+} h_{n3}(u) = \frac{\Gamma\left(\frac{\lambda + na}{\delta}\right)}{\Gamma\left(\frac{\delta + na}{\delta}\right)} \quad \text{and} \quad \lim_{u \to -\frac{c}{\delta}^+} h_{n4}(u) = 0 \tag{3.41}$$

Letting $u \to -\frac{c}{\delta}^+$ in (3.40), we have

$$\lim_{u \to -\frac{c}{\delta}^+} V_{n2}(u, b) = c_{n3} \lim_{u \to -\frac{c}{\delta}^+} h_{n3}(u) + c_{n4} \lim_{u \to -\frac{c}{\delta}^+} h_{n4}(u)$$

Substituting (3.14) and (3.41), we obtain

$$0 = c_{n3} \frac{\Gamma\left(\frac{\lambda + na}{\delta}\right)}{\Gamma\left(\frac{\delta + na}{\delta}\right)} \Longrightarrow c_{n3} = 0$$

As a result, (3.40) is reduced to

$$V_{n2}(u, b) = c_{n4} h_{n4}(u), \qquad -\frac{c}{\delta} < u < 0$$

$\diamondsuit$

Lemma 3.5.2.1. *The derivative of $h_{n4}(u)$, $-\frac{c}{\delta} < u < 0$, is equal to*

$$\begin{aligned} h_{n4}^{'}(u) = \ & \beta \left[\tfrac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+na}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}} \left[\tfrac{\lambda+na-\beta(\delta u+c)}{\beta(\delta u+c)} M\left(1 + \tfrac{na}{\delta}, 1 + \tfrac{\lambda+na}{\delta}, \tfrac{\beta(\delta u+c)}{\delta}\right)\right. \\ & \left. + \tfrac{\delta+na}{\lambda+\delta+na} M\left(2 + \tfrac{na}{\delta}, 2 + \tfrac{\lambda+na}{\delta}, \tfrac{\beta(\delta u+c)}{\delta}\right)\right] \end{aligned} \tag{3.42}$$

Proof. Differentiating (3.39) with respect to u and using the property $\frac{d}{dz}M(x, y, z) = \frac{x}{y}M(x + 1, y + 1, z)$, we obtain

$$\begin{aligned}
h'_{n4}(u) &= \beta\tfrac{\lambda+na}{\delta}\left[\tfrac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+na}{\delta}-1} e^{-\frac{\beta(\delta u+c)}{\delta}} M\left(1+\tfrac{na}{\delta},1+\tfrac{\lambda+na}{\delta},\tfrac{\beta(\delta u+c)}{\delta}\right)\\
&\quad -\beta\left[\tfrac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+na}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}} M\left(1+\tfrac{na}{\delta},1+\tfrac{\lambda+na}{\delta},\tfrac{\beta(\delta u+c)}{\delta}\right)\\
&\quad +\beta\left[\tfrac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+na}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}} \tfrac{1+\frac{na}{\delta}}{1+\frac{\lambda+na}{\delta}} M\left(2+\tfrac{na}{\delta},2+\tfrac{\lambda+na}{\delta},\tfrac{\beta(\delta u+c)}{\delta}\right)\\
&= \beta\left[\tfrac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+na}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}} \left[\tfrac{\lambda+na}{\delta}\tfrac{\delta}{\beta(\delta u+c)}-1\right] M\left(1+\tfrac{na}{\delta},1+\tfrac{\lambda+na}{\delta},\tfrac{\beta(\delta u+c)}{\delta}\right)\\
&\quad +\beta\left[\tfrac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+na}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}} \tfrac{\delta+na}{\lambda+\delta+na} M\left(2+\tfrac{na}{\delta},2+\tfrac{\lambda+na}{\delta},\tfrac{\beta(\delta u+c)}{\delta}\right)\\
&= \beta\left[\tfrac{\beta(\delta u+c)}{\delta}\right]^{\frac{\lambda+na}{\delta}} e^{-\frac{\beta(\delta u+c)}{\delta}} \left[\tfrac{\lambda+na-\beta(\delta u+c)}{\beta(\delta u+c)} M\left(1+\tfrac{na}{\delta},1+\tfrac{\lambda+na}{\delta},\tfrac{\beta(\delta u+c)}{\delta}\right)\right.\\
&\quad \left.+\tfrac{\delta+na}{\lambda+\delta+na} M\left(2+\tfrac{na}{\delta},2+\tfrac{\lambda+na}{\delta},\tfrac{\beta(\delta u+c)}{\delta}\right)\right]
\end{aligned}$$

$\diamondsuit$

Corollary 3.5.2.1. *If $\frac{\lambda+na}{\delta}$ is not integer, the moment-generating function of $D_{u,b}$, $M(u,y;b)$, satisfies*

$$M_1(u,y;b) = 1+\sum_{n=1}^{\infty}\frac{y^n}{n!}\left(c_{n1}e^{r_{n1}u}+c_{n2}e^{r_{n2}u}\right), \qquad 0<u<b$$

and

$$M_2(u,y;b) = 1+\sum_{n=1}^{\infty}\frac{y^n}{n!}c_{n4}h_{n4}(u), \qquad -\frac{c}{\delta}<u<0$$

Proof. It is a direct result of substituting the solutions for $V_n(u,b)$, (3.35) and (3.38), in formula (3.10). Indeed,

$$\begin{aligned}
M(u,y;b) &= 1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_n(u,b)\\
&= \begin{cases} 1+\displaystyle\sum_{n=1}^{\infty}\frac{y^n}{n!}\left(c_{n1}e^{r_{n1}u}+c_{n2}e^{r_{n2}u}\right), & 0<u<b\\[2ex] 1+\displaystyle\sum_{n=1}^{\infty}\frac{y^n}{n!}c_{n4}h_{n4}(u), & -\frac{c}{\delta}<u<0\end{cases}
\end{aligned}$$

$\diamondsuit$

Explicit Results for $n = 1$

Step 1. Differentiating (3.35) and (3.38) with respect to u yields

$$V_{11}^{'}(u,b) = c_{11}r_{11}e^{r_{11}u} + c_{12}r_{12}e^{r_{12}u}, \quad 0 < u < b \tag{3.43}$$

and

$$V_{12}^{'}(u,b) = c_{14}h_{14}^{'}(u), \quad -\frac{c}{\delta} < u < 0 \tag{3.44}$$

Step 2. Substituting (3.43) in (3.13) and considering that $V_0(u,b) = 1$ $\forall\, u \in (-\frac{c}{\delta}, b]$, we obtain

$$\begin{aligned} V_{n1}^{'}(u,b)\Big|_{u=b} &= nV_{n-1,1}(b,b) \Longrightarrow \\ V_{11}^{'}(b,b) &= V_{01}(b,b) \Longrightarrow \\ c_{11}r_{11}e^{r_{11}b} + c_{12}r_{12}e^{r_{12}b} &= 1 \end{aligned} \tag{3.45}$$

Step 3. Substituting (3.43) and (3.44) in (3.16) implies

$$\begin{aligned} V_{n1}^{'}(o^+,b) &= V_{n2}^{'}(0^-,b) \Longrightarrow \\ V_{11}^{'}(o^+,b) &= V_{12}^{'}(0^-,b) \Longrightarrow \\ c_{11}r_{11} + c_{12}r_{12} &= c_{14}h_{14}^{'}(0) \end{aligned} \tag{3.46}$$

Step 4. Substituting (3.35) and (3.38) in (3.15) leads to

$$\begin{aligned} V_{n1}(o,b) &= V_{n2}(0^-,b) \Longrightarrow \\ V_{11}(o,b) &= V_{12}(0^-,b) \Longrightarrow \\ c_{11} + c_{12} &= c_{14}h_{14}(0) \end{aligned} \tag{3.47}$$

Step 5. As a result, (3.45)-(3.46)-(3.47) compose a system of three equations with three unknowns, namely

$$(\Sigma) \begin{cases} c_{11}r_{11}e^{r_{11}b} + c_{12}r_{12}e^{r_{12}b} = 1 \\ c_{11}r_{11} + c_{12}r_{12} = c_{14}h_{14}^{'}(0) \\ c_{11} + c_{12} = c_{14}h_{14}(0) \end{cases}$$

where $r_{11},\ r_{12}$ are given by (3.36), $h_{14}(0)$ and $h_{14}^{'}(0)$ are given by (3.39) and (3.42), respectively. Solving the system (Σ), we obtain

$$c_{11} = \frac{h_{14}^{'}(0) - r_{12}h_{14}(0)}{r_{11}e^{r_{11}b}\left[h_{14}^{'}(0) - r_{12}h_{14}(0)\right] - r_{12}e^{r_{12}b}\left[h_{14}^{'}(0) - r_{11}h_{14}(0)\right]}$$

$$c_{12} = \frac{r_{11}h_{14}(0) - h_{14}^{'}(0)}{r_{11}e^{r_{11}b}\left[h_{14}^{'}(0) - r_{12}h_{14}(0)\right] - r_{12}e^{r_{12}b}\left[h_{14}^{'}(0) - r_{11}h_{14}(0)\right]}$$

$$c_{14} = \frac{r_{11} - r_{12}}{r_{11}e^{r_{11}b}\left[h_{14}^{'}(0) - r_{12}h_{14}(0)\right] - r_{12}e^{r_{12}b}\left[h_{14}^{'}(0) - r_{11}h_{14}(0)\right]}$$

$\diamondsuit$

Finally, we conclude through the above procedure to the following proposition for the 1st moment, which is mentioned by Wang and Yin (2009).

Proposition 3.5.2.2. *When $\frac{\lambda+na}{\delta}$ is not integer, the first moment $V_1(u,b)$, of $D_{u,b}$, satisfies*
i. For $0 < u < b$,

$$V_{11}(u,b) = \frac{\left[h_{14}^{'}(0) - r_{12}h_{14}(0)\right] e^{r_{11}u} - \left[h_{14}^{'}(0) - r_{11}h_{14}(0)\right] e^{r_{12}u}}{r_{11}e^{r_{11}b}\left[h_{14}^{'}(0) - r_{12}h_{14}(0)\right] - r_{12}e^{r_{12}b}\left[h_{14}^{'}(0) - r_{11}h_{14}(0)\right]}$$

ii. For $-\frac{c}{\delta} < u < 0$,

$$V_{12}(u,b) = \frac{\left[r_{11} - r_{12}\right] h_{14}(u)}{r_{11}e^{r_{11}b}\left[h_{14}^{'}(0) - r_{12}h_{14}(0)\right] - r_{12}e^{r_{12}b}\left[h_{14}^{'}(0) - r_{11}h_{14}(0)\right]}$$

$\diamondsuit$

Remark 3.5.2.1. *Regarding Cai (2000) and Wang and Yin (2009), it is a common assumption that $\frac{\lambda+na}{\delta}$ is greater than 1 and not integer. This is because the debit interest force δ is usually less than one while the Poisson parameter λ is usually larger than one.*

3.5.3 Numerical Example

Example 3.5.3. *We assume that the intensity of the Poisson process is $\lambda = 4$, the premium rate per unit time is $c = 2$, the claim sizes X_i obey an Exponential distribution with parameter $\beta = 3$ and the dividend barrier is $b = 15$.*
(1) We will study the progress of $V_{11}(u,b)$, $0 < u < b$ (i.e. the expected present value of all dividends paid until the absolute ruin time T_b), in the following cases:

i. Maintaining the discounting interest force constant and equal to $a = 0.025$ and letting the debit interest force receive variable values, namely $\delta = 0.08,\ 0.5$ and 0.9

ii. By contrast, maintaining the debit interest force constant and equal to $\delta = 0.2$, we let the discounting interest force receive the values, $a = 0.025,\ 0.05$ and 0.08

(2) For $a = 0.025$ and $\delta = 0.08$, we will depict the progress of $V_{11}(u,b)$ and $V_{12}(u,b)$ with respect to b, by setting fixed values in the initial surplus u, for instance $u = -2,\ -1,\ 1,\ 4$.

Note. *In the following calculations we set $n = 1$ in any formula we use, as we want to find the first moment of $D_{u,b}$.*

Solution. (1) Firstly, using the expressions (3.34) and (3.36), we estimate the roots r_{11} and r_{12}, i.e.

$$p_1 = \beta - \frac{\lambda + a}{c} \quad \text{and} \quad q_1 = -\frac{\beta a}{c}$$

and

$$r_{11} = \frac{-p_1 + \sqrt{p_1^2 - 4q_1}}{2}, \qquad r_{12} = \frac{-p_1 - \sqrt{p_1^2 - 4q_1}}{2}$$

Then, we evalute at $u = 0$ the $h_{14}(u)$ and its first derivative $h'_{14}(u)$, via the formulas (3.39) and (3.42). So, we obtain

$$h_{14}(0) = \left[\frac{\beta c}{\delta}\right]^{\frac{\lambda+a}{\delta}} e^{-\frac{\beta c}{\delta}} M\left(1 + \frac{a}{\delta}, 1 + \frac{\lambda + a}{\delta}, \frac{\beta c}{\delta}\right)$$

and

$$\begin{aligned} h'_{14}(0) = \ & \beta \left[\tfrac{\beta c}{\delta}\right]^{\frac{\lambda+a}{\delta}} e^{-\frac{\beta c}{\delta}} \left[\tfrac{\lambda+a-\beta c}{\beta c} M\left(1 + \tfrac{a}{\delta}, 1 + \tfrac{\lambda+a}{\delta}, \tfrac{\beta c}{\delta}\right)\right. \\ & \left. + \tfrac{\delta+a}{\lambda+\delta+a} M\left(2 + \tfrac{a}{\delta}, 2 + \tfrac{\lambda+a}{\delta}, \tfrac{\beta c}{\delta}\right)\right] \end{aligned}$$

The values of the arbitrary constants c_{11} and c_{12} are given by

$$c_{11} = \frac{h'_{14}(0) - r_{12}h_{14}(0)}{r_{11}e^{r_{11}b}\left[h'_{14}(0) - r_{12}h_{14}(0)\right] - r_{12}e^{r_{12}b}\left[h'_{14}(0) - r_{11}h_{14}(0)\right]}$$

$$c_{12} = \frac{r_{11}h_{14}(0) - h'_{14}(0)}{r_{11}e^{r_{11}b}\left[h'_{14}(0) - r_{12}h_{14}(0)\right] - r_{12}e^{r_{12}b}\left[h'_{14}(0) - r_{11}h_{14}(0)\right]}$$

Finally, all the terms in (3.35) have been evaluated and we can obtain the values of the first moment

$$V_{11}(u,b) = c_{11}e^{r_{11}u} + c_{12}e^{r_{12}u}, \qquad 0 < u < b$$

Note. All the above calculations have been conducted in Mathematica. It is worth mentioning that the confluent hypergeometric function of the first kind, M(x,y,z), has been estimated with the command

Hypergeometric1F1[x, y, z]

i. In Figure 3.5 we can observe that for greater values of the debit interest force δ, the corresponding expected present values of all dividends, $V_{11}(u,b)$, are lower. Moreover, we observe that $V_{11}(u,b) \le \frac{c}{a} = \frac{2}{0.025} = 80$, $\forall\, 0 < u < b$, as we have mentioned in Remark 3.1.3.

ii. Additionally, by calculating the arbitrary constant c_{14},

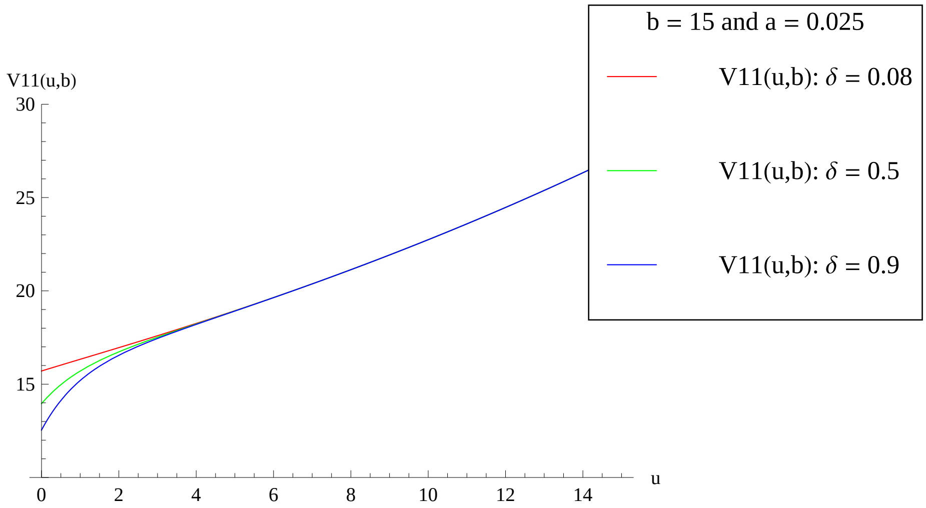


Figure 3.5: Variable Debit Interest Force δ

$$c_{14} = \frac{r_{11} - r_{12}}{r_{11}e^{r_{11}b}\left[h'_{14}(0) - r_{12}h_{14}(0)\right] - r_{12}e^{r_{12}b}\left[h'_{14}(0) - r_{11}h_{14}(0)\right]}$$

we can use the formula (3.38)

$$V_{12}(u,b) = c_{14}h_{14}(u), \quad -\frac{c}{\delta} < u < 0$$

Finally, we observe that a similar case is depicted by Figures 3.6 and 3.7. More specifically, the higher the discounting interest force a, the less the first moment $V_{11}(u,b)$ and $V_{12}(u,b)$.

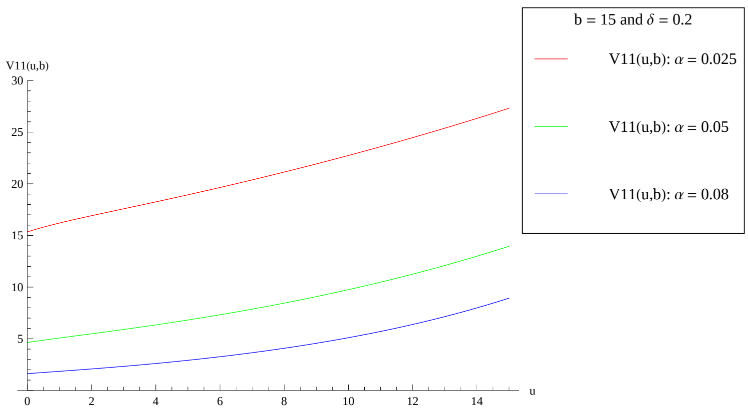


Figure 3.6: Variable Discounting Interest Force a, $0 < u < b$

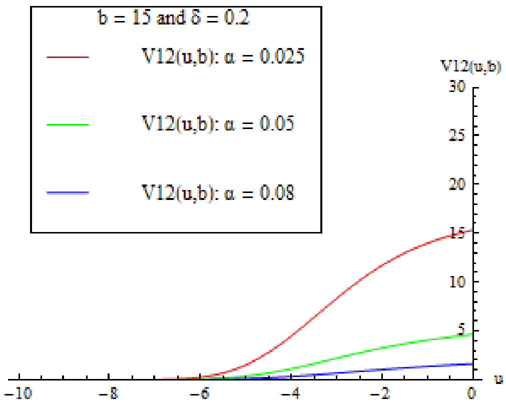


Figure 3.7: Variable Discounting Interest Force a, $-\frac{c}{\delta} < u < 0$

(2) Now, the arbitrary constants c_{11}, c_{12} and c_{14} are considered as functions of b. Thus, by giving specific values to u, $V_{11}(u,b)$ and $V_{12}(u,b)$ behave as functions of b, as well. Figure 3.8 depicts some expected results to us. Firstly, from one value of b and after, $V_{11}(u,b)$ and $V_{12}(u,b)$ are decreasing functions of b (this is because, when the dividend barrier b is high, the surplus process attains the level of b less times, so the dividend payments are less, as well). Moreover, for greater values of the initial surplus u, the corresponding curves of $V_{11}(u,b)$ or $V_{12}(u,b)$ are higher. The aforementioned value of b, where the $V_{11}(u,b)$ and $V_{12}(u,b)$ start decreasing, is the optimal dividend barrier (i.e. that value which maximises the expected present value of all dividend payments).

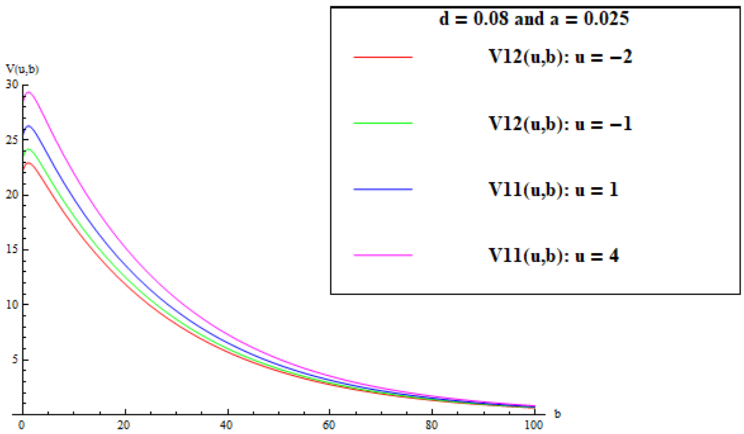


Figure 3.8: Variable Dividend Barrier b

3.5.4 Code of Mathematica

```
c = 2
l = 4
f[x_] = 3 * Exp[-3 * x]
m1 = Integrate[x * f[x], {x, 0, Infinity}]
m = 1 / m1
c > l * m1
d = 0.08
d1 = 0.5
d2 = 0.9
a = 0.025
b = 15

(l + a) / d
(l + a) / d1
(l + a) / d2

50.3125

8.05

4.47222

p1 = m - ((l + a) / c)
q1 = -(m * a) / c

0.9875

-0.0375

Discr = p1^2 - 4 * q1

1.12516

r11 = (-p1 + Sqrt[Discr]) / 2
r12 = (-p1 - Sqrt[Discr]) / 2

0.0366169

-1.02412

h14[u_] = ((m * (d * u + c)) / d)^((l + a) / d) * Exp[-((m * (d * u + c)) / d)] *
  Hypergeometric1F1[1 + (a / d), 1 + ((l + a) / d), ((m * (d * u + c)) / d)]
h14A[u_] = ((m * (d1 * u + c)) / d1)^((l + a) / d1) * Exp[-((m * (d1 * u + c)) / d1)] *
  Hypergeometric1F1[1 + (a / d1), 1 + ((l + a) / d1), ((m * (d1 * u + c)) / d1)]
h14B[u_] = ((m * (d2 * u + c)) / d2)^((l + a) / d2) * Exp[-((m * (d2 * u + c)) / d2)] *
  Hypergeometric1F1[1 + (a / d2), 1 + ((l + a) / d2), ((m * (d2 * u + c)) / d2)]

Dh14[u_] = Derivative[1][h14][u]
Dh14A[u_] = Derivative[1][h14A][u]
Dh14B[u_] = Derivative[1][h14B][u]
```

Figure 3.9: Example 3.5.3 (1/i), Code 1/2

```
c11 = (Dh14[0] - r12 * h14[0]) / (r11 * Exp[r11 * b] * (Dh14[0] - r12 * h14[0]) -
    r12 * Exp[r12 * b] * (Dh14[0] - r11 * h14[0]))
c12 = (r11 * h14[0] - Dh14[0]) / (r11 * Exp[r11 * b] * (Dh14[0] - r12 * h14[0]) -
    r12 * Exp[r12 * b] * (Dh14[0] - r11 * h14[0]))
c14 = (r11 - r12) / (r11 * Exp[r11 * b] * (Dh14[0] - r12 * h14[0]) -
    r12 * Exp[r12 * b] * (Dh14[0] - r11 * h14[0]))
c11A = (Dh14A[0] - r12 * h14A[0]) / (r11 * Exp[r11 * b] * (Dh14A[0] - r12 * h14A[0]) -
    r12 * Exp[r12 * b] * (Dh14A[0] - r11 * h14A[0]))
c12A = (r11 * h14A[0] - Dh14A[0]) / (r11 * Exp[r11 * b] * (Dh14A[0] - r12 * h14A[0]) -
    r12 * Exp[r12 * b] * (Dh14A[0] - r11 * h14A[0]))
c14A = (r11 - r12) / (r11 * Exp[r11 * b] * (Dh14A[0] - r12 * h14A[0]) -
    r12 * Exp[r12 * b] * (Dh14A[0] - r11 * h14A[0]))
c11B = (Dh14B[0] - r12 * h14B[0]) / (r11 * Exp[r11 * b] * (Dh14B[0] - r12 * h14B[0]) -
    r12 * Exp[r12 * b] * (Dh14B[0] - r11 * h14B[0]))
c12B = (r11 * h14B[0] - Dh14B[0]) / (r11 * Exp[r11 * b] * (Dh14B[0] - r12 * h14B[0]) -
    r12 * Exp[r12 * b] * (Dh14B[0] - r11 * h14B[0]))
c14B = (r11 - r12) / (r11 * Exp[r11 * b] * (Dh14B[0] - r12 * h14B[0]) -
    r12 * Exp[r12 * b] * (Dh14B[0] - r11 * h14B[0]))

V11[u_] = c11 * Exp[r11 * u] + c12 * Exp[r12 * u]
V11A[u_] = c11A * Exp[r11 * u] + c12A * Exp[r12 * u]
V11B[u_] = c11B * Exp[r11 * u] + c12B * Exp[r12 * u]
```

$-0.0618444\, e^{-1.02412\, u} + 15.7681\, e^{0.0366169\, u}$

$-1.81263\, e^{-1.02412\, u} + 15.7681\, e^{0.0366169\, u}$

$-3.22624\, e^{-1.02412\, u} + 15.7681\, e^{0.0366169\, u}$

```
<< PlotLegends`;

Plot[{V11[u], V11A[u], V11B[u]}, {u, 0, b},
 PlotRange → {10, 30}, AxesLabel → {"u", "V11(u,b)"},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]},
 PlotLegend → {Style["V11(u,b): δ = 0.08", 12], Style["V11(u,b): δ = 0.5", 12],
   Style["V11(u,b): δ = 0.9", 12]}, LegendPosition → {.9, 0},
 LegendTextSpace → 2, LegendLabel → Style["b = 15 and a = 0.025", 12],
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```

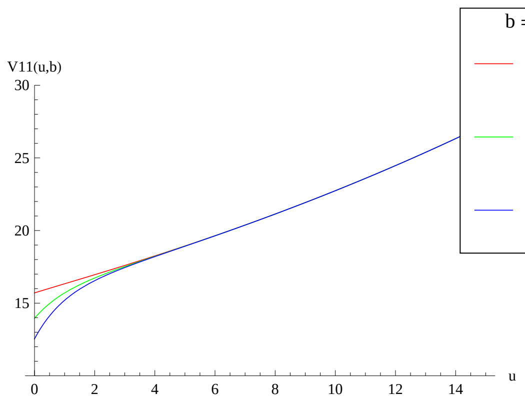


Figure 3.10: Example 3.5.3 (1/i), Code 2/2

```
c = 2
l = 4
f[x_] = 3 * Exp[-3 * x]
m1 = Integrate[x * f[x], {x, 0, Infinity}]
m = 1 / m1
c > l * m1
d = 0.2
a = 0.025
a1 = 0.05
a2 = 0.08
b = 15

(l + a) / d
(l + a1) / d
(l + a2) / d

20.125

20.25

20.4

p1 = m - ((l + a) / c)
q1 = -(m * a) / c
p1A = m - ((l + a1) / c)
q1A = -(m * a1) / c
p1B = m - ((l + a2) / c)
q1B = -(m * a2) / c

Discr = p1^2 - 4 * q1
DiscrA = p1A^2 - 4 * q1A
DiscrB = p1B^2 - 4 * q1B

r11 = (-p1 + Sqrt[Discr]) / 2
r12 = (-p1 - Sqrt[Discr]) / 2
r11A = (-p1A + Sqrt[DiscrA]) / 2
r12A = (-p1A - Sqrt[DiscrA]) / 2
r11B = (-p1B + Sqrt[DiscrB]) / 2
r12B = (-p1B - Sqrt[DiscrB]) / 2
```

Figure 3.11: Example 3.5.3 (1/ii), Code 1/3

```
h14[u_] = ((m * (d * u + c)) / d) ^ ((l + a) / d) * Exp[- ((m * (d * u + c)) / d)] *
  Hypergeometric1F1[1 + (a / d), 1 + ((l + a) / d), ((m * (d * u + c)) / d)]
h14A[u_] = ((m * (d * u + c)) / d) ^ ((l + a1) / d) * Exp[- ((m * (d * u + c)) / d)] *
  Hypergeometric1F1[1 + (a1 / d), 1 + ((l + a1) / d), ((m * (d * u + c)) / d)]
h14B[u_] = ((m * (d * u + c)) / d) ^ ((l + a2) / d) * Exp[- ((m * (d * u + c)) / d)] *
  Hypergeometric1F1[1 + (a2 / d), 1 + ((l + a2) / d), ((m * (d * u + c)) / d)]
Dh14[u_] = Derivative[1][h14][u]
Dh14A[u_] = Derivative[1][h14A][u]
Dh14B[u_] = Derivative[1][h14B][u]
c11 = (Dh14[0] - r12 * h14[0]) / (r11 * Exp[r11 * b] * (Dh14[0] - r12 * h14[0]) -
    r12 * Exp[r12 * b] * (Dh14[0] - r11 * h14[0]))
c12 = (r11 * h14[0] - Dh14[0]) / (r11 * Exp[r11 * b] * (Dh14[0] - r12 * h14[0]) -
    r12 * Exp[r12 * b] * (Dh14[0] - r11 * h14[0]))
c14 = (r11 - r12) / (r11 * Exp[r11 * b] * (Dh14[0] - r12 * h14[0]) -
    r12 * Exp[r12 * b] * (Dh14[0] - r11 * h14[0]))
c11A = (Dh14A[0] - r12A * h14A[0]) / (r11A * Exp[r11A * b] * (Dh14A[0] - r12A * h14A[0]) -
    r12A * Exp[r12A * b] * (Dh14A[0] - r11A * h14A[0]))
c12A = (r11A * h14A[0] - Dh14A[0]) / (r11A * Exp[r11A * b] * (Dh14A[0] - r12A * h14A[0]) -
    r12A * Exp[r12A * b] * (Dh14A[0] - r11A * h14A[0]))
c14A = (r11A - r12A) / (r11A * Exp[r11A * b] * (Dh14A[0] - r12A * h14A[0]) -
    r12A * Exp[r12A * b] * (Dh14A[0] - r11A * h14A[0]))
c11B = (Dh14B[0] - r12B * h14B[0]) / (r11B * Exp[r11B * b] * (Dh14B[0] - r12B * h14B[0]) -
    r12B * Exp[r12B * b] * (Dh14B[0] - r11B * h14B[0]))
c12B = (r11B * h14B[0] - Dh14B[0]) / (r11B * Exp[r11B * b] * (Dh14B[0] - r12B * h14B[0]) -
    r12B * Exp[r12B * b] * (Dh14B[0] - r11B * h14B[0]))
c14B = (r11B - r12B) / (r11B * Exp[r11B * b] * (Dh14B[0] - r12B * h14B[0]) -
    r12B * Exp[r12B * b] * (Dh14B[0] - r11B * h14B[0]))
V11[u_] = c11 * Exp[r11 * u] + c12 * Exp[r12 * u]
V11A[u_] = c11A * Exp[r11A * u] + c12A * Exp[r12A * u]
V11B[u_] = c11B * Exp[r11B * u] + c12B * Exp[r12B * u]
```

$-0.426084\, e^{-1.02412\, u} + 15.7681\, e^{0.0366169\, u}$

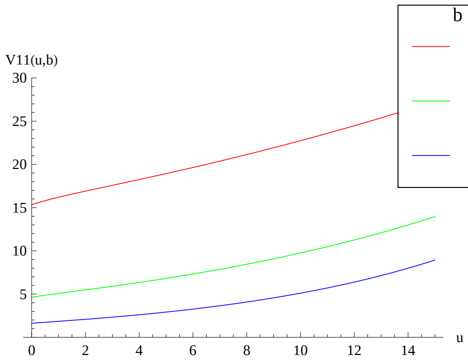
$-0.127868\, e^{-1.04666\, u} + 4.76366\, e^{0.0716567\, u}$

$-0.0437596\, e^{-1.07195\, u} + 1.66621\, e^{0.111946\, u}$

```
<< PlotLegends`;
```

Figure 3.12: Example 3.5.3 (1/ii), Code 2/3

```
Plot[{V11[u], V11A[u], V11B[u]}, {u, 0, b},
 PlotRange → {0, 30}, AxesLabel → {"u", "V11(u,b)"},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]},
 PlotLegend → {Style["V11(u,b): α = 0.025", 12], Style["V11(u,b): α = 0.05", 12],
   Style["V11(u,b): α = 0.08", 12]}, LegendPosition → {.9, 0},
 LegendTextSpace → 2, LegendLabel → Style["b = 15 and δ = 0.2", 12],
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```




```
V12[u_] = c14 * h14[u]
V12A[u_] = c14A * h14A[u]
```

$1.46924 \times 10^{6}\, e^{-15.\,(2+0.2\,u)}\,(2+0.2\,u)^{20.125}$ Hypergeometric1F1[1.125, 21.125, 15. (2 + 0.2 u)]

$309264.\, e^{-15.\,(2+0.2\,u)}\,(2+0.2\,u)^{20.25}$ Hypergeometric1F1[1.25, 21.25, 15. (2 + 0.2 u)]

```
V12B[u_] = c14B * h14B[u]
```

$71437.5\, e^{-15.\,(2+0.2\,u)}\,(2+0.2\,u)^{20.4}$ Hypergeometric1F1[1.4, 21.4, 15. (2 + 0.2 u)]

```
Plot[{V12[u], V12A[u], V12B[u]}, {u, -(c / d) + 0.001, 0 - 0.001},
 PlotRange → {0, 30}, AxesLabel → {"u", "V12(u,b)"},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]},
 PlotLegend → {Style["V12(u,b): α = 0.025", 12], Style["V12(u,b): α = 0.05", 12],
   Style["V12(u,b): α = 0.08", 12]}, LegendPosition → {.9, 0},
 LegendTextSpace → 2, LegendLabel → Style["b = 15 and δ = 0.2", 12],
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```

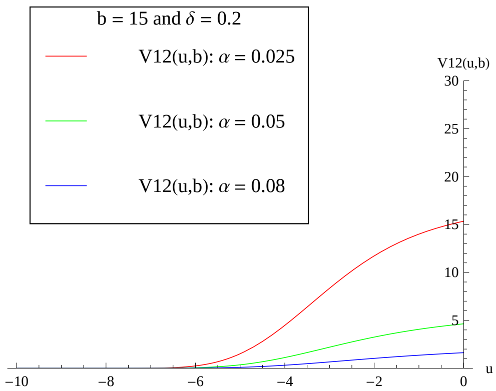


Figure 3.13: Example 3.5.3 (1/ii), Code 3/3

```
l = 4
c = 2
f[x_] = 3 * Exp[-3 * x]
m1 = Integrate[x * f[x], {x, 0, Infinity}]
m = 1 / m1
a = 0.025
d = 0.08
(l + a) / d
```

50.3125

```
p = m - ((l + a) / c)
q = -(m * a) / c
Disc = p^2 - (4 * q)
r11 = (-p + Sqrt[Disc]) / 2
r12 = (-p - Sqrt[Disc]) / 2
h14[u_] = (((m * (d * u + c)) / d)^((l + a) / d)) * Exp[-((m * (d * u + c)) / d)] *
  Hypergeometric1F1[1 + (a / d), 1 + ((l + a) / d), ((m * (d * u + c)) / d)]
Derivh14[u_] = Derivative[1][h14][u]
c11[b_] =
 (Derivh14[0] - (r12 * h14[0])) / (r11 * Exp[r11 * b] * (Derivh14[0] - r12 * h14[0]) -
    r12 * Exp[r12 * b] * (Derivh14[0] - r11 * h14[0]))
c12[b_] = ((r11 * h14[0]) - Derivh14[0]) /
  (r11 * Exp[r11 * b] * (Derivh14[0] - r12 * h14[0]) -
    r12 * Exp[r12 * b] * (Derivh14[0] - r11 * h14[0]))
c14[b_] = (r11 - r12) / (r11 * Exp[r11 * b] * (Derivh14[0] - r12 * h14[0]) -
    r12 * Exp[r12 * b] * (Derivh14[0] - r11 * h14[0]))
```

$$\frac{3.33537\times 10^{65}}{1.33972\times 10^{63}\, e^{-1.02412\, b}+1.22131\times 10^{64}\, e^{0.0366169\, b}}$$

$$-\frac{1.30817\times 10^{63}}{1.33972\times 10^{63}\, e^{-1.02412\, b}+1.22131\times 10^{64}\, e^{0.0366169\, b}}$$

$$\frac{1.06073}{1.33972\times 10^{63}\, e^{-1.02412\, b}+1.22131\times 10^{64}\, e^{0.0366169\, b}}$$

Figure 3.14: Example 3.5.3 (2), Code 1/2

```
V11[u_, b_] = c11[b] * Exp[r11 * u] + c12[b] * Exp[r12 * u]
V12[u_, b_] = c14[b] * h14[u]
```

$$-\frac{1.30817\times 10^{63}\, e^{-1.02412\, u}}{1.33972\times 10^{63}\, e^{-1.02412\, b}+1.22131\times 10^{64}\, e^{0.0366169\, b}}+\frac{3.33537\times 10^{65}\, e^{0.0366169\, u}}{1.33972\times 10^{63}\, e^{-1.02412\, b}+1.22131\times 10^{64}\, e^{0.0366169\, b}}$$

$$\left(1.65598\times 10^{79}\, e^{-37.5\,(2+0.08\,u)}\,(2+0.08\,u)^{50.3125}\ \text{Hypergeometric1F1}[1.3125,\ 51.3125,\ 37.5\,(2+0.08\,u)]\right)\Big/\left(1.33972\times 10^{63}\, e^{-1.02412\, b}+1.22131\times 10^{64}\, e^{0.0366169\, b}\right)$$

```
<< PlotLegends`;

Plot[{V12[-2, b], V12[-1, b], V11[1, b], V11[4, b]}, {b, 0, 100},
 PlotRange → {0, 30}, AxesLabel → {"b", "V(u,b)"}, PlotStyle →
  {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1], RGBColor[1, 0, 1]},
 PlotLegend → {Style["V12(u,b): u = -2", 12], Style["V12(u,b): u = -1", 12],
   Style["V11(u,b): u = 1", 12], Style[ "V11(u,b): u = 4", 12]},
 LegendPosition → {.9, 0}, LegendTextSpace → 2,
 LegendLabel → Style["d = 0.08 and a = 0.025", 12],
 LegendLabelSpace → .2, LegendOrientation → Vertical,
 LegendBackground → GrayLevel[1], LegendShadow → None, Background → None]
```

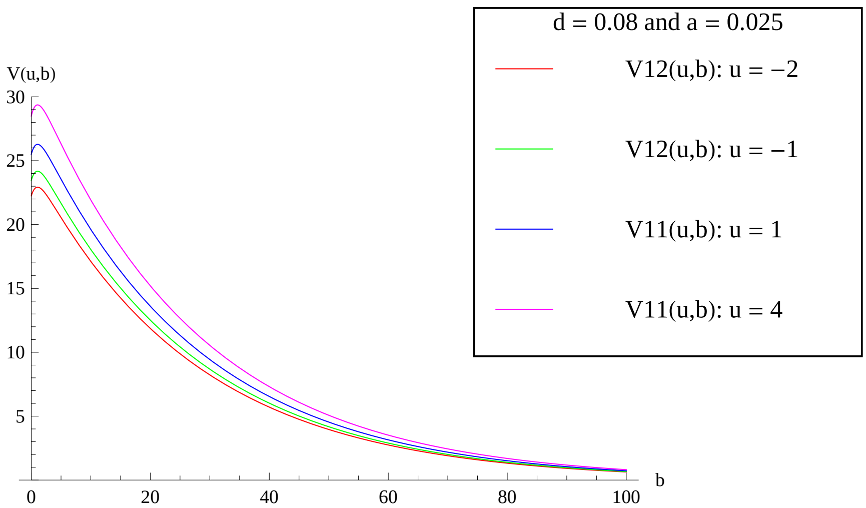


Figure 3.15: Example 3.5.3 (2), Code 2/2

Chapter 4

Conclusions

In Chapter 1, having set up all the necessary notions for the compound Poisson surplus process, we define the Gerber-Shiu function. This function can be reduced to many essential quantities for risk theory, for instance the probability of ruin, the ruin time, the surplus just before ruin time, the deficit at ruin time, etc. Then, using the renewal argument and the law of total probability, we obtain the integro-differential equations satisfied by Gerber-Shiu function. Using the Laplace transform and the Dickson-Hipp operator, we are led to a defective renewal equation. Its solution has been based on a compound geometric distribution. At the end of this chapter, explicit results for the ruin probability and the Laplace transform of the ruin time have been presented, when the claims are exponentially distributed.

In Chapter 2, we have followed a similar methodology to Chapter 1. The main difference is that the compound Poisson surplus process has been enhanced with one more property. When the surplus drops below zero, the insurer can borrow money equal to the deficit, at a debit interest force. Meanwhile, the debt is paid back by the premium income. The surplus may become positive again. On the other hand, if the surplus attains or falls below a critical value, it will not become positive again and then absolute ruin is said to happen. Under this scope, having followed the methodology in Chapter 1, we define the Gerber-Shiu function at absolute ruin and reach the explicit results for the absolute ruin probability and the Laplace transform of the absolute ruin time, when the claims are exponentially distributed.

In Chapter 3, we have added to the compound Poisson surplus process the property of dividend payments according to a barrier strategy. This means that whenever the surplus reaches a threshold, the insurer will pay dividends to shareholders at a rate equal to the premium rate, until the next claim occurs. Moreover, similar to Chapter 2, there is a debit interest force for the money borrowed, when the surplus falls below zero. Absolute ruin happens when the surplus is equal or less than a critical value. Firstly, we have denoted the moment-generating function and the moments for the present value of all dividends paid until the absolute ruin time. Then, having used the renewal argument, we obtain the integro-differential equations for the moment-generating function of the discounted dividend payments. Through them, we are led to the integro-differential equations satisfied by the corresponding moments. When we have exponential claims, the latter equations can be converted into differential equations and by solving them properly, we obtain explicit expressions for the moments of the discounted dividend payments. Especially, we have focused on the first moment which denotes the expected present value of all dividends paid until the absolute ruin time. Finally, there are a brief definition of Gerber-Shiu function and the integro-differential equations satisfied by it, under this modified surplus process.

Appendix A

A.1 2nd Order Linear Differential Equations

Let

$$\phi^{''}(u) + p(u)\phi^{'}(u) = g(u) \tag{A.1}$$

be a second order linear differential equation where $p(u)$ and $g(u)$ are functions of u. If we set

$$w(u) = \phi^{'}(u) \tag{A.2}$$

(A.1) is reduced to a first order linear differential equation, i.e.

$$w^{'}(u) + p(u)w(u) = g(u) \tag{A.3}$$

The corresponding homogeneous linear differential equation is

$$w^{'}(u) + p(u)w(u) = 0$$

and the general solution of it can be found as below

$$w^{'}(u) = -p(u)w(u) \Longrightarrow \frac{w^{'}(u)}{w(u)} = -p(u) \Longrightarrow \int \frac{w^{'}(u)}{w(u)} du = -\int p(u)du + c_1 \Longrightarrow$$

$$\ln|w(u)| = -\int p(u)du + c_1 \Longrightarrow w(u) = e^{-\int p(u)du + c_1} \Longrightarrow$$

$$w(u) = c\, e^{-\int p(u)du} \qquad \text{where} \quad c = e^{c_1} \text{ is an arbitrary constant.}$$

The solution of the non homogeneous linear differential equation (A.3) is described by Alikakos and Kalogeropoulos (2003), Chapter 1.3 "Linear Equations", page 9. According to them, the general solution of (A.3) is similar to the general solution of the homogeneous, with the difference that c will be a function of u. Thus, we obtain

$$w(u) = c(u)\, e^{-\int p(u)du} \tag{A.4}$$

Substituting (A.4) in (A.3), implies

$$c'(u)e^{-\int p(u)du} - c(u)p(u)e^{-\int p(u)du} + p(u)c(u)e^{-\int p(u)du} = g(u) \Longrightarrow$$

$$c'(u) = g(u)e^{\int p(u)du} \Longrightarrow \int c'(u)du = \int e^{\int p(u)du} g(u)du + c_2 \Longrightarrow$$

$$c(u) = \int e^{\int p(u)du} g(u)du + c_2 \qquad \text{where } c_2 \text{ is an arbitrary constant.}$$

As a result, (A.4) can be written as

$$w(u) = e^{-\int p(u)du}\left[\int e^{\int p(u)du} g(u)du + c_2\right] \tag{A.5}$$

Substituting (A.5) in (A.2), we obtain the general solution of (A.1)

$$\begin{aligned} \phi'(u) &= e^{-\int p(u)du}\left[\int e^{\int p(u)du} g(u)du + c_2\right] \Longrightarrow \\ \phi(u) &= c_1 + \int e^{-\int p(u)du}\left[\int e^{\int p(u)du} g(u)du + c_2\right] du \end{aligned}$$

A.2 2nd Order Linear Homogeneous Differential Equations with Constant Coefficients

Alikakos and Kalogeropoulos (2003), in Chapter 4.2.2, page 159, give the general solution of a second order linear homogeneous differential equation with constant coefficients. In this paragraph we present briefly the results of their work.

Consider the second order linear homogeneous differential equation

$$y'' + a_1 y' + a_2 y = 0 \tag{A.6}$$

where y is a function of t, $y = y(t)$, and $a_1,\ a_2$ are constant coefficients. Let $y = e^{rt}$ be a solution of (A.6) for an unknown constant r. Then, substituting $y = e^{rt}$ in (A.6), we obtain

$$\begin{aligned} r^2 e^{rt} + a_1 r e^{rt} + a_2 e^{rt} &= 0 \Longrightarrow \\ \left(r^2 + a_1 r + a_2\right) e^{rt} &= 0 \Longrightarrow \\ r^2 + a_1 r + a_2 &= 0 \quad \left(\text{because } e^{rt} > 0 \ \forall\, t\right) \end{aligned}$$

As a result, $y = e^{rt}$ is a solution of (A.6) if and only if $r^2 + a_1 r + a_2 = 0$. The polynomial

$$p(r) = r^2 + a_1 r + a_2$$

is called the characteristic polynomial of (A.6) and the counterpart equation

$$r^2 + a_1 r + a_2 = 0$$

is called the characteristic equation of (A.6). This is a quadratic equation with unknown r and it is always solvable, with roots given by the quadratic formula

$$r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Consequently, any second order linear homogeneous differential equation with constant coefficients is always solvable, as well.

We point out the following cases regarding the values of the Discriminant $D = a_1^2 - 4a_2$:

i. If $a_1^2 - 4a_2 > 0$, then there are two distinct real roots $r_1 \neq r_2$. The general solution of (A.6) is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

ii. If $a_1^2 - 4a_2 < 0$, then there are two complex conjugate roots $r_{1,2} = \sigma \pm iw$. The general solution of (A.6) is

$$y(t) = e^{\sigma t} \left(c_1 \cos wt + c_2 \sin wt\right)$$

iii. If $a_1^2 - 4a_2 = 0$, then there is one repeated real root $r_1 = r_2 = r$. The general solution of (A.6) is

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$

where $c_1,\ c_2$ are arbitrary coefficients in all cases. To find a particular solution of (A.6), i.e. two specific values of c_1 and c_2, requires two initial conditions.

A.3 Property for $\gamma_\delta(u)$

When the penalty function $w(x, y) = 1$ and the claim sizes X_i follow an Exponential distribution with mean $\frac{1}{\beta}$, we have seen that $\gamma_\delta(u) = e^{-\beta u} e^{-\beta \frac{c}{\delta}}$. Under those assumptions, it always holds that $\gamma_\delta(u)$ satisfies the (2.25), i.e.

$$L \; = \lim_{u \to -\left(\frac{c}{\delta}\right)^+} \int_u^0 (\delta y + c)^{-1-\frac{\lambda+a}{\delta}} \gamma_\delta(y) dy = \infty$$

Indeed,

$$
\begin{aligned}
L &= \lim_{u \to -\left(\frac{c}{\delta}\right)^+} \int_u^0 (\delta y + c)^{-1-\frac{\lambda+a}{\delta}} \gamma_\delta(y) dy = \lim_{u \to -\left(\frac{c}{\delta}\right)^+} \int_u^0 (\delta y + c)^{-1-\frac{\lambda+a}{\delta}} e^{-\beta y} e^{-\beta \frac{c}{\delta}} dy \\
&= \int_{-\left(\frac{c}{\delta}\right)^+}^0 (\delta y + c)^{-1-\frac{\lambda+a}{\delta}} e^{-\beta y} e^{-\beta \frac{c}{\delta}} dy
\end{aligned}
$$

and we observe that

$$
\lim_{y \to -\left(\frac{c}{\delta}\right)^+} (\delta y + c)^{-1-\frac{\lambda+a}{\delta}} e^{-\beta y} e^{-\beta \frac{c}{\delta}} = \frac{\lim_{y \to -\left(\frac{c}{\delta}\right)^+} e^{-\beta y} e^{-\beta \frac{c}{\delta}}}{\lim_{y \to -\left(\frac{c}{\delta}\right)^+} (\delta y + c)^{1+\frac{\lambda+a}{\delta}}} = \frac{1}{0} = \infty
$$

So, the initial limit L converges to $+\infty$, as well.

A.4 Confluent Hypergeometric Functions

Definition A.4.1. *According to Abramowitz and Stegun (1972), page 505, the confluent hypergeometric function of the first kind, $M(a, b, z)$, has an integral representation*

$$
M(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt
$$

whereas, the confluent hypergeometric function of the second kind, which is denoted by $U(a, b, z)$, has an integral representation

$$
U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt
$$

where

$$
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt
$$

is the Gamma function (Abramowitz and Stegun (1972), page 255).

Proposition A.4.1. *Based on the aforementioned definitions, we can easily prove the following*

i. $\dfrac{\Gamma(z+1)}{\Gamma(z)} = z \Longleftrightarrow \Gamma(z+1) = z\Gamma(z)$

ii. $\dfrac{d}{dz} M(a, b, z) = \dfrac{a}{b} M(a+1, b+1, z)$

iii. $\dfrac{d}{dz} U(a, b, z) = -aU(a+1, b+1, z)$

Proof.

i.
$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty t^z e^{-t}dt = \left[-t^z e^{-t}\right]_0^\infty + \int_0^\infty z t^{z-1} e^{-t}dt \\ &= z\int_0^\infty t^{z-1}e^{-t}dt = z\Gamma(z)\end{aligned}$$

ii.
$$\begin{aligned}\frac{d}{dz}M(a,b,z) &= \frac{d}{dz}\left[\frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt\right] \\ &= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 te^{zt}t^{a-1}(1-t)^{b-a-1}dt \\ &= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^{zt}t^{(a+1)-1}(1-t)^{(b+1)-(a+1)-1}dt \\ &= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\cdot\frac{\Gamma\left[(b+1)-(a+1)\right]\Gamma(a+1)}{\Gamma(b+1)}\cdot M(a+1,b+1,z) \\ &= \frac{\frac{\Gamma(a+1)}{\Gamma(a)}}{\frac{\Gamma(b+1)}{\Gamma(b)}}M(a+1,b+1,z) \\ &\overset{(i)}{=} \frac{a}{b}M(a+1,b+1,z)\end{aligned}$$

iii.
$$\begin{aligned}\frac{d}{dz}U(a,b,z) &= \frac{d}{dz}\left[\frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}dt\right] \\ &= \frac{1}{\Gamma(a)}\int_0^\infty -te^{-zt}t^{a-1}(1+t)^{b-a-1}dt \\ &= -\frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{(a+1)-1}(1+t)^{(b+1)-(a+1)-1}dt \\ &= -\frac{\Gamma(a+1)}{\Gamma(a)}U(a+1,b+1,z) \\ &\overset{(i)}{=} -aU(a+1,b+1,z)\end{aligned}$$

$\diamondsuit$

Proposition A.4.2. *By (13.5.5) of Abramowitz and Stegun (1972), page 508, we have*

$$\lim_{z\to 0} M(a,b,z) = 1 \tag{A.7}$$

whereas, by (13.5.12) of Abramowitz and Stegun (1972), page 508, for $b < 0$ *it holds*

$$\lim_{z \to 0} U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \tag{A.8}$$

$\diamondsuit$

Corollary A.4.1. *In practice, from Cai (2000), we know that the intensity* λ *of the Poisson process is larger than one, while the debit interest force* δ *is usually less than one. Thus,*

$$1 < \frac{\lambda + na}{\delta} \Longrightarrow 1 - \frac{\lambda + na}{\delta} < 0, \qquad n \in \mathcal{N}$$

Now, we are able to give the following limits appeared in the main body of this project, by using the properties (A.7) and (A.8), whenever it is necessary.

$$(a) \qquad \lim_{u \to -\frac{c}{\delta}^+} h_1(u) = \lim_{u \to -\frac{c}{\delta}^+} e^{-\frac{\beta(\delta u + c)}{\delta}} U\left(1 - \frac{\lambda}{\delta}, 1 - \frac{\lambda + na}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)$$

$$\stackrel{(A.8)}{=} \frac{\Gamma\left[1 - \left(1 - \frac{\lambda + na}{\delta}\right)\right]}{\Gamma\left[1 + \left(1 - \frac{\lambda}{\delta}\right) - \left(1 - \frac{\lambda + na}{\delta}\right)\right]} = \frac{\Gamma\left(\frac{\lambda + na}{\delta}\right)}{\Gamma\left(\frac{\delta + na}{\delta}\right)}$$

$$(\beta) \qquad \lim_{u \to -\frac{c}{\delta}^+} h_2(u) = \lim_{u \to -\frac{c}{\delta}^+} \left[\frac{\beta(\delta u + c)}{\delta}\right]^{\frac{\lambda + na}{\delta}} e^{-\frac{\beta(\delta u + c)}{\delta}} \cdot M\left(1 + \frac{na}{\delta}, 1 + \frac{\lambda + na}{\delta}, \frac{\beta(\delta u + c)}{\delta}\right)$$

$$\stackrel{(A.7)}{=} 0 \cdot 1 = 0$$

$\diamondsuit$

Bibliography

Foreign Bibliography

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Function: With Formulas, Graphs and Mathematical Tables, U.S. Government Printing Office: Washington, DC, 1972

[2] H.U. Gerber, An Introduction to Mathematical Risk Theory, Heubner Foundation Monograph Series, vol.8. Richard D. Irwin: Homewood, 1979

[3] A. Dassios, P. Embrechts, Martingales and Insurance Risk, Stochastic Models 1989; 5:149-166

[4] P. Embrechts, H. Schmidli, Ruin Estimations for a general Insurance Risk Model, Advances in Applied Probability 1994; 26:404-422

[5] B. Sundt, J.L. Teugels, Ruin Estimates under Interest Force, Insurance: Mathematics and Economics 1995; 16, 7-22

[6] D.C.M. Dickson , A.D. Egidio dos Reis, The Effect of Interest on Negative Surplus, Insurance: Mathematics and Economics 1997; 21(1):1-16

[7] H.U. Gerber, E.S.W. Shiu, On the Time Value of Ruin, Actuarial Research Clearing House 1997 vol.1

[8] H.U. Gerber, E.S.W. Shiu, On the Time Value of Ruin, North American Actuarial Journal 1998; 2(1):48-78

[9] C.S. Zhang, R. Wu, On the Distribution of the Surplus of the D-E Model prior to and at ruin, Insurance: Mathematics and Economics 1999;24(3):309-321

[10] X. S. Lin, G.E. Willmot, Analysis of a defective renewal equation arising in ruin theory, Insurance: Mathematics and Economics 25(1999) 63 - 84

[11] J. Cai, On the time value of Absolute Ruin with Debit Interest, 2000 Mathematics Subject Classification: Primary 91B30, Secondary 60K05; 91B70

[12] S. Asmussen, Ruin Probabilities, World Scientific, Singapore, 2000

[13] J. Cai, D.C.M. Dickson, On the Expected Discounted Penalty Function at Ruin of a Surplus Process with Interest, Insurance: Mathematics and Economics 2002; 30, 389-404

[14] S.X. Lin, G.E. Willmot, S. Drekic, The Classical Risk Model with a Constant Dividend Barrier; Analysis of the Gerber-Shiu Discounted Penalty Function, Insurance: Mathematics and Economics 2003; 33(3):551 - 566

[15] D.C.M Dickson, H. Waters, Some Optimal Dividend Problems, ASTIN Bulletin 2004; 33(1):49-74

[16] H.U. Gerber, E.S.W. Shiu, On Optimal Dividends; Analysis with Brownian Motion, North American Actuarial Journal 2004; 8(1):1-20

[17] J. Cai, Ruin probabilities and penalty functions with stochastic rates of interest, Stochastic Processes and their Applications 2004; 112 53-78

[18] J. Cai, H.U. Gerber, H.L. Yang, Optimal Dividends in an Ornstein-Uhlenbeck Type Model with Credit and Debit Interest, North American Actuarial Journal 2006; 10(2):94-108

[19] S.M. Li, The Distribution of the Dividend Payments in the Compound Poisson Risk Model Perturbed by diffusion, Scandinavian Actuarial Journal 2006; 2:73-85

[20] H.U. Gerber, H.L. Yang, Absolute Ruin Probabilities in a Jump Diffusion Risk Model with Investment, North American Actuarial Journal 2007; 11(3):159-169

[21] J. Cai, On the Time Value of Absolute Ruin with Debit Interest, Advances in Applied Probability 2007; 39:343-359

[22] K.C. Yuen, G.J. Wang, W.K. Li, The Gerber-Shiu Expected Discounted Penalty Function for Risk Processes with Interest and a Constant Dividend Barrier, Insurance: Mathematics and Economics 2007; 40(1):104-112

[23] H.U. Gerber, E.S.W. Shiu, N. Smith, Methods for Estimating the Optimal Dividend Barrier and the Probability of Ruin, Insurance: Mathematics and Economics 2008; 42(1):243-254

[24] K.C. Yuen, M. Zhou, J.Y. Guo, On a Risk Model with Debit Interest and Dividend Payments, Statistics and Probability Letters 2008, DOI:10.1016/j.spl.2008.02.021

[25] C. Wang, C. Yin, Dividend Payments in the classical Risk Model under Absolute Ruin with Debit Interest, Applied Stochastic Models in Business and Industry 2009; 25:247-262

[26] G.E. Willmot, J.K Woo, Surplus Analysis of Sparre Andersen Insurance Risk Processes, Springer International Publishing AG 2017

Greek Bibliography

[27] N. D. Alikakos, G. Kalogeropoulos, Differential Equations, Sychroni Ekdotiki EPE, Athens 2003 (Greek Edition)

[28] K. Politis, Ruin Theory, academic notes, 2013-2014 (Greek digital teaching notes),
http://www.unipi.gr/faculty/kpolitis/ruin/ruin_notes.pdf

[29] E. Chadjikonstantinidis, academic notes for the course Risk Theory I, 2016,
http://stat.unipi.gr/eclass/modules/document/?course=ACT134

[30] M. Koutras, Probabilities Theory and Applications, Ekdosis Stamouli AE, Athens, October 2016 (Greek Edition)