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ON THE DENSITY OF THE TIME OF RUIN WITH EXPONENTIAL CLAIMS

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Mitrodima N. Evangelia

Abstract

In this dissertation, we study the density function of the time of ruin with exponential claims. More precisely, we present two different formulas, the one that Drekic and Willmot (2003) obtained for the classical model of ruin theory and the one that Borovkov and Dickson (2008) derived for a Sparre Andersen process with exponential claim sizes. After the presentation of the two formulas, we prove that they are equivalent for the special case of exponentially distributed times (classical model) and zero initial surplus. The theoretical presentation is followed by numerical examples, executed in the computational package Mathematica, where the behavior of the density function of the ruin time is studied for different values of its parameters. We also examine, if and how much, is there a good fit of the exponential distribution to the density of ruin time, empirically. Apart from this, we also study the failure rate of the distribution function and if there are cases that this is monotonic. Closing, various numerical examples are given, in order to come to a conclusion for the classification of the distribution function of ruin time, in terms of its failure rate.

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Chapter 1

Introduction

Insurance science has a big history, since its roots are found back in the ancient years, when in Athens, ancient Greeks were paying a kind of a loan for the safety of their boats. Not only the Greeks, but a lot of other nations, has been recorded that had a kind of system in order their exchanges to be done under safety. From then, until nowadays, things have changed enough. Insurance is applied in our everyday life. A lot of products have been created in order to cover the synchronous man's needs. Such products combine insurance coverages with financial ones, for instance, etc.

As far as the private insurance is concerned, this is played mostly by insurance companies, which take the additional risk of the insureds in exchange, of course, to the premium. A very important sector in insurance is the actuarial science. Actuarial science includes models for life insurance (survival tables), non life insurance (utility theory, decision theory etc), social insurance, etc. Although actuarial science deals with a lot of problems, here we will state some from the risk theory, since the subject of this dissertation is a ruin theory problem of risk theory. Risk theory comprises a lot of models, such as the individual risk model, or the collective for one period or for a long period.

In an insurance company, a very important quantity is ruin probability, since it is the probability the surplus to become negative for the first time. This ruin is not legal but a technical one. This is, because an insurance company is an organization, which takes measures to face such problems, for instance the risk management of a company deals with such matters. Up to now ruin probability has been studied extensively. In bibliography there are a lot of results for it, not to mention approximations and fractions.

Here, we are interested to study the time, when the surplus of an insurance company $U(t)$ at time t becomes negative for the very first time, that is the ruin time. Additionally to the existing bibliography for the ruin probability, only few papers have come to light for the specific subject, over the years. We use two of them, here, that study the density function of ruin time. Borovkov and Dickson (2008) studied the ruin time distribution, assuming a renewal model with exponential claim sizes. We use the derived formula to study the behavior of the density function of the defective random variable of ruin time. We also do this for the formula for the proper random variable of ruin time that Drekic and Willmot (2003) derived in their paper for the classical model. However, here, we only present a few folds of the density of ruin time, as it remains a big unexplored problem.

More specifically, in Chapter 2 we introduce some of the basic notions of ruin theory, after we give its theoretical and financial background. We introduce in Section 2.2 some concepts from the theory of stochastic processes, the Poisson process, since, in the classical model, we assume that $N(t)$- the number of the claims, follows the Poisson distribution, with parameter λt and consequently the interarrivals $T_1, T_2, \ldots$ will follow the exponential distribution with parameter λ. We also introduce the renewal process, because in the Sparre Andersen process, which is a generalization of the classical one, $N(t)$ does not follow a Poisson process, but a renewal process, with again T_i which stands for the interclaim time, being independent and identically distributed (iid). This means, that T_i will not follow, in this case, the exponential distribution. Then, we present the reserves an insurance company should keep in order to have a good functioning. In Section 2.4 we demonstrate the surplus process in the classical model and the assumptions that hold for it. After that, an introduction of the random variable of ruin time follows, since extended study will take place in Chapter 4. In Section 2.6 we introduce the classical model of ruin theory, which Drekic and Willmot (2003) assumed in order to obtain a formula for the density function of the time to ruin. In the last three sections of this chapter we give the adjustment coefficient, the ruin probability and the renewal model of ruin theory, respectively.

In Chapter 3, we introduce the reliability theory. After a theoretical background of the reliability theory, which gives historical information, follows Section 3.2, where we talk about the failure or lifetime distributions which - as the name implies- attempt to describe mathematically the length of life of a material, a structure, a device, or a human. In Section 3.3, the definition for the failure rate is given. We will use the failure rate in Chapter 5 to study the reliability classification of the distribution of ruin time. In the next section we introduce monotone failure rates. Before we had studied, in Chapter 5, the failure rate of the ruin time distribution function, only guesses we could make for its monotonicity. We will see, as it turned out that it is non monotonic for both of the cases for $u > 0$ and $u = 0$. These classes (non monotonic) are illustrated in Section 3.5. In the last two sections of this chapter we give the residual lifetime distribution and some other classes of distributions.

In Chapter 4, after a presentation of the formulas, given in the papers under examination, and how they were derived we give some numerical examples computed in Mathematica. More specifically, in Section 4.1 and 4.2 we present Drekic and Willmot's (2003) formula and in Section 4.3 we give the examples for this. In Sections 4.4, 4.5, 4.6, we do the same for the Borovkov and Dickson's (2008) formula. What follows is the proof that these two formulas are equivalent for the case of $u = 0$ and exponentially distributed claims and times, which means that in both cases we talk now about the classical and not the renewal model. In Sections 4.8 and 4.9 we also give examples for the density function of ruin time under the new assumptions.

Closing, in Chapter 5, in Section 5.1, we give the failure rate of the distribution function of ruin time for $u > 0$ and in Section 5.2, we give some examples of this failure rate, for different parameter values. What follows, in Section 5.3 and 5.4, is again the failure rate for the case, when $u = 0$ and the corresponding plots.

Chapter 2

Introduction to Ruin Theory

In Ruin Theory, the surplus process and ruin probability are introduced to study how possible is for an insurance company to exercise ruin with initial surplus $U(t)$ at time t. These quantities (surplus, ruin probability), as some others like the ruin time, the adjustment coefficient, etc are of central importance in the classical and renewal risk model, as they are applied in insurance and actuarial science. In particular, here, these quantities are needed among other things, in order the density of the ruin time to be studied. Drekic and Willmot (2003) showed that the probability density function of the time to ruin can be obtained for the classical model and Borovkov and Dickson (2008) did it for the Sparre- Andersen model with exponentially distributed claims. So in this chapter, the concepts of the classical and renewal risk model are reviewed.

Firstly, the theoretical and financial background of an insurance company is given and the reasons why, the above quantities, are so important for the survival of an insurance company (Section 2.1). Apart from that, concepts from stochastic processes are given in Section 2.2, which are crucial for the understanding of the following subject of the density of ruin time. Section 2.3 has a more theoretical character, again, as it presents the reserves, an insurance company should keep. The presentation of the reserves, is believed that will help in the understanding of the surplus process and in particular how this is related to the reserves and the operations of an insurance company. Of central interest in this regard is the surplus process, something which is discussed in Section 2.4. In Section 2.5, the ruin time is presented. However, a more extended representation will be given in Chapter 4, where two formulas for this subject are discussed. Two important quantities related to ruin time is the ruin and non- ruin probability which are discussed in Section 2.6, where the classical model is presented. In Section 2.7 the adjustment coefficient is presented and apart from this, some interesting results are showed, that give a boundary and an asymptotic formula for the ruin probability. Although these results will not be used here, it is worth being represented, because they are some of the main notions of the classical model of risk theory. It is also presented how the ruin probability satisfies a certain renewal equation (Section 2.8). The renewal model and its main differences from the classical risk model, which is based on some other assumptions is being introduced in Section 2.9 of this chapter. The renewal model (or Sparre- Andersen model) is, as we said earlier, the model that Borovkov and Dickson used in order to study the distribution of time to ruin.

2.1 The theoretical and financial background

In ruin theory, as risk we consider every event which causes financial risk or loss. For example, of high interest is the financial loss, as a result of the event of a fire. It goes without saying that risk is not a predictable event and neither is the financial loss, that causes, so its severity will be random. As a result from all the above, only non- negative variables are being used in ruin theory, so for the random variable that represents the severity of loss, one has:

1. The random variable X, which is referred to the severity of the financial loss, has to be non- negative ($X \geq 0$), in order to be able to describe the risk.

2. The variable X has also to have positive skewness in order to be able to estimate the probability of a big loss, because in practice a portfolio might have a lot of risks with small economic value but a small number of risks with big economic value. As far as the skewness is concerned, it is easy, for someone, to understand that the normal distribution, for example, would not be able to express the above situation, because it is symmetric around its mean. Moreover, it is easy to understand why a portfolio, in practice, might have a big number of small risks and a small number of risks with big economic value, if one thinks the portfolio of cars, for instance. In an insurance company the car accidents are an everyday phenomenon, causing a big number of claims to the insurance

company, meaning a lot of small financial losses, but the big ones that involve deaths or serious accidents, are rare. Another example is the modeling of financial losses due to earthquakes, which can cause a rare, but very expensive, financial loss, and in some cases can be disastrous for the insurance company. So, then a distribution with a long right tail is needed to predict this possible event.

2.2 Some concepts from the theory of Stochastic Processes

As we mentioned in Introduction, ruin theory is one of the subjects that risk theory deals with. Risk theory comprises a lot of models, such as the individual risk model, or the collective for one period (distribution of the aggregate claims, compound Poisson, compound negative binomial etc) or for a long period (stochastic processes, Markov processes, martingales, Wiener etc). It is necessary some of the basics of ruin theory to be introduced and analyzed, before entering deep inside to the subject of the density of the time to ruin, T. For this reason, some definitions, which will be used next, from the theory of stochastic processes should be given:

Definition 2.2.1. *A stochastic process is a family of random variables*

$$\{X_t, t \in T\},$$

where t stands for the time (discrete or continuous parameter). Such a process, has the characteristic that for each fixed t, X_t is a random variable. Apart from the time, which can be discrete or continuous, the number of the values for X_t can also be discrete or continuous, and then one has a stochastic process with either discrete or continuous values, respectively.

In practice there are a lot of cases where stochastic processes can be used to describe a real phenomenon. In ruin theory, a quantity of great importance is $S(t)$, which describes the aggregate claims that occur at an insurance company, at time t. Let us consider a stochastic process of the aggregate claims $S(t)$ and a stochastic process of the number of the aggregate claims, $N(t)$. $N(t)$ is called a counting stochastic process, because firstly is a process stationary in the intervals, with jumps of height of one, at times

$$W_1, W_2, W_3, \ldots,$$

which are called arrival times and describe the random times of the incured loss events and secondly, because it holds that $N(0) = 0$. Let us consider

$$T_i = W_i - W_{i-1}, (i \geq 1 \, and \, T_0 = 0),$$

then

$$S(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i, & N(t) \geq 1 \\ 0, & N(t) = 0 \end{cases} \tag{2.2.1}$$

and represents the aggregate claims that arrive at an insurance company during the interval $[0, t]$.

Poisson process is very often applied in ruin theory and the reason why, is that, such a process is used to show how many times an event- in ruin theory, the arrivals of a claim- happen. Next, we give a definition and two properties for such a process. Poisson process is a counting process with $N(0) = 0$, non- decreasing with integral values. In the classical risk model of ruin theory, an assumption which is being made, is that $N(t)$- the number of the claims, follows the Poisson distribution, with parameter λt. Consequently, as it will be represented later on, the interarrivals $T_1, T_2, \ldots$ will follow the exponential distribution with parameter λ. More specifically, one has the following definition of a Poisson process:

Definition 2.2.2. *A stochastic process* $\{N(t), t \geq 0\}$ *is called Poisson process, when*

1. $N(0) = 0$ *and for* $s<t$*, then* $N(s) \leq N(t)$

2. $N(t)$ *has independent increments, meaning that* $\forall \; t < s$ *the random value* $N(s) - N(t)$ *is independent to the variable* $N(t)$

3. $$Pr[N(t+h) - N(t) = 1] = \lambda h + o(h)$$
   $$Pr[N(t+h) - N(t) \geq 2] = o(h)$$

Here, we mention two properties of a Poisson process

1. For every fixed t, the random variable $N(t)$ follows Poisson $N(t) \sim P(\lambda t)$.

2. For every $i \neq j$ the random variables T_i, T_j, representing the times between successive occurrences of an event, are independent and follow the exponential distribution with parameter λ, which is called the rate of the Poisson process.

Apart from a Poisson process, which is used in the classical model of ruin theory, another type of a counting process, which is used to model the claim arrivals at an insurance company, is a renewal process. This is defined next.

Definition 2.2.3. *The renewal process is a generalization of the Poisson process denoted also by $\{N(t), t \geq 0\}$. More precisely, it is a counting process in which the interarrival times are iid (independent and identically distributed), and they do not follow necessarily the exponential distribution as they did in the Poisson process, but an arbitrary distribution F.*

Of great importance is the renewal function

$$m(t) = E(N(t)).$$

It is mentioned that $m(t)$ satisfies the following equation, called renewal equation:

$$m(t) = F(t) + \int_0^t m(t-x)dF(x).$$

In general, a renewal equation is an equation of the form:

$$\mu(t) = g(t) + \phi \int_0^t \mu(t-x)dF(x), \ x \geq 0, \tag{2.2.2}$$

see Grimmett and Stirzaker (2001), where F is the cumulative distribution function in $[0, \infty)$, with $F(0) = 0$, ϕ is a constant in $0 < \phi \leq 1$, $g(t)$ is a bounded function and μ is the unknown function. The above equation will be called proper (non- defective) renewal equation for $\phi = 1$ and defective renewal equation for $\phi < 1$ and has general solution:

$$\mu(t) = g(t) + \int_0^t g(t-x)dM(x), \tag{2.2.3}$$

where $M(t) = \sum_{k=1}^{\infty} \phi^k F^{*k}(t)$. It is worth mentioning that with probability 1, for $t \to \infty$ one has that

$$\frac{N(t)}{t} \to \frac{1}{E(X)}$$

and for $t \to \infty$ one has that

$$\frac{m(t)}{t} \to \frac{1}{E(X)},$$

where $E(X)$ is the mean of the distribution F.

2.3 The Reserves

The assessment of the reserves of an insurance company is a procedure which requires special judgment of a lot of sectors, such as the credibility of the historical data and the credibility of the used models. The calculation of the reserves requires analysis of the responsibilities and the collection of the desired quality and quantity of the data. The process consists of:

- collection and data analysis
- deciding the hypothesis under which the assessment of the reserves will be executed
- modeling, model parameterization and implementation
- evaluation, estimation and checking

The above procedure is being executed by a person of high actuarial and economical mathematical skills. For an insurance or reinsurance company the sufficient reserve is very important for the good and effective exercise of its activities, in order to face unexpected events that can jeopardize its exercise and in some cases put an end to its survival. Experience has shown that insurance companies shut down when they do not have the proper amount of surplus and this is a "hot potato" for the management and the supervisory authorities, too. For further information on the subject of the reserves and how the European authorities face it, the reader can consult the text of the Solvency II or the Advice for Level 2 Implementing Measures on Solvency II (2009) by the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS). CEIOPS, as the name implies, consists of the European Union's insurance and pension fund supervisory authorities and through the above text makes an attempt to give advice and solutions to the problem of implementation of Solvency 2.

As a company, an insurance organization faces a lot of risks, therefore the need for a sufficient reserve is a priority for an insurance company, which takes the additional risk of the insureds in exchange, of course, to the premium. These reserves called **Surplus**, when the capital is used for the dealing with the fluctuation between the expected actuarial liabilities and the assets. $U(0) = u$ stands for the value of surplus at time $t = 0$, that is at the beginning of a company, meaning the initial surplus is u. Therefore,

surplus=assets-expected actuarial liabilities

is a relation which gives a general expression for the evolution of the surplus.

2.4 The Surplus Process

The last expression shows how a company's surplus is expected to evolve through time. However, even in the simplified framework that will be considered in the present dissertation (where assets are represented only by the premium income and the only future liabilities are the payments of the claims), it is unrealistic to think that the evolution of surplus through time is deterministic. More specifically, in the present context of ruin theory, the surplus process $\{U(t) : t \geq 0\}$ is modeled by a stochastic process, where :

$$U(t) = u + P(t) - S(t), \quad t \geq 0, \tag{2.4.1}$$

where

- u is the initial surplus
- $P(t)$ is the total premiums that are being paid during the interval $(0, t]$
- $S(t)$ is the stochastic process of the aggregate claims, which is a compound Poisson process, with Poisson parameter λ. It is further assumed that λ is the Poisson rate for arrivals.

Let us consider that X_i is the random variable for the severity of compensations for the i- risk- we talked in Section 2.1 for the required characteristics of X_i- and let us consider the stochastic process $N(t)$ for the number of risks in $(0, t]$, then the equation (2.2.1) is written as follows:

$$S(t) = X_1 + X_2 + X_3 + \ldots + X_{N(t)} = \sum_{i=1}^{N(t)} X_i. \tag{2.4.2}$$

The surplus process for the classical risk model is based on some assumptions, that follow. In the classical risk model it is assumed that $P(t)$ is deterministic, but in practice something like that does not happen, it has a random character. Apart from that, $P(t)$ and $S(t)$ are not independent, as it is assumed in the classical model, especially in cases of financial crisis (2008), when a big fall in premiums is usually recorded to be followed by an increase in claims. So if the assumption of the deterministic character did not hold then $P(t)$, would be equal to $P(t) = p(t) + Q(t)$, where $p(t)$ is the common function and $Q(t)$ is the stochastic process for the random fluctuations in the long run payment of the premiums, then one would take:

$$U(t) = u + p(t) - [S(t) - Q(t)] = u + p(t) - S'(t),$$

where $S'(t) = S(t) - Q(t)$ and $S'(t)$ is the new stochastic process of the aggregate claims, which now takes into account the fluctuations of the premiums. However, in the classical risk model we assume that

$$p(t) = ct, \forall t \geq 0,$$

which implies that

$$U(t) = u + ct - S(t),\ t \geq 0,\ U(0) = u,\ c > 0, \tag{2.4.3}$$

where c is the rate with which the premium is being paid.

In the classical model the influence of the interest is ignored, whereas something like that does not happen in the real world. In addition, an insurance's gain is fully dependent on the premiums and the income gained from the company's investments. Again here the investment's random fluctuations are highly related to $S(t)$, and as a result in periods of high depression the increase in the claims usually follows a downfall in investments and the exact opposite in periods of high development.

Closing, as far as the plot of surplus is concerned, we should note that $U(t)$ shows jumps (downwards) at times W_i when the loss events occur. These jumps are all of the same length with those (upwards) of $S(t)$ ($S(t)$ is not depicted in the next figure). In the diagrammatic presentation of $U(t)$, which is given next, the upward movement of $U(t)$ has a slope c while there are negative jumps at times $W_1, W_2, \ldots$

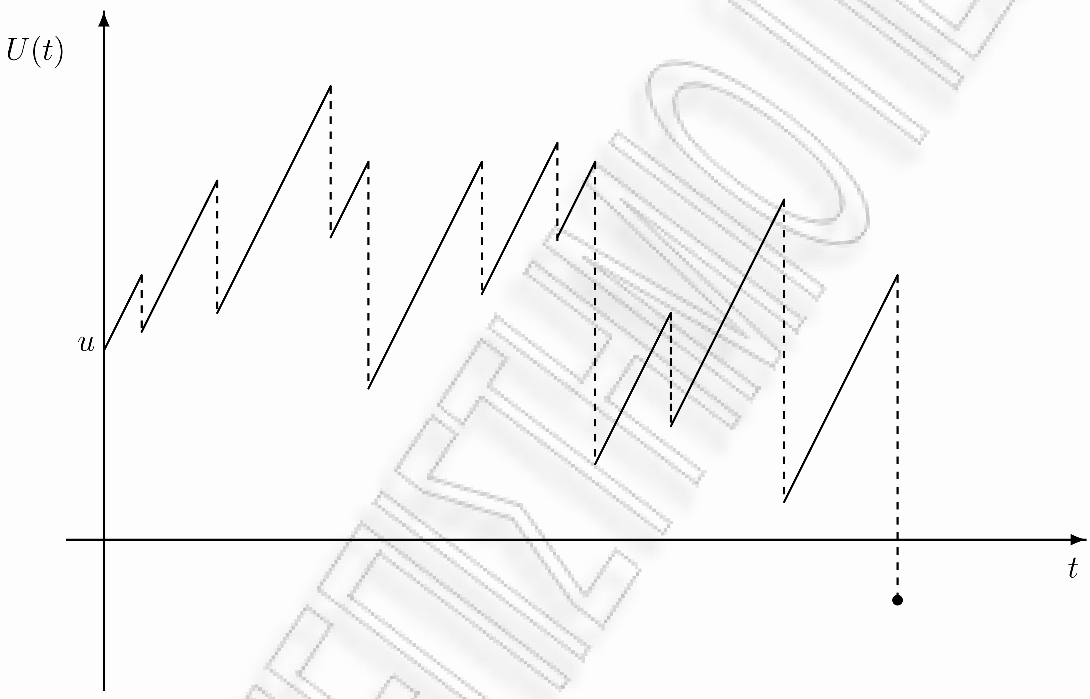

Figure 2.1: The Surplus process in the classical model

2.5 The Ruin Time

In this section we will introduce the random variable T, which is the ruin time and is defined by

$$T = \begin{cases} \inf \ \{t : U(t) < 0\} \\ \infty \ if \ U(t) \geq 0 \ for \ all \ t > 0. \end{cases} \tag{2.5.1}$$

More precisely, the above tells us that we have ruin at time T, when the surplus becomes negative for the first time. We will also present T_c, which is the proper random variable, conditional on ruin occurring, defined by

$$T_c = T|T < \infty.$$

We give both the defective and the proper random variable of ruin time, because in Chapter 4 we will give formulas for the density function for both of them.

Although the ruin time, is an interesting subject in ruin theory, only the recent years, has been made an effort the density function to be assessed. In the past years, some efforts were made in order some asymptotic results to be given for the distribution of T_c and conditions under these asymptotic results give a reasonable approximation to the distribution of T_c.

Here, the results of Borovkov and Dickson (2008), and Drekic and Willmot (2003) will be used. Borovkov and Dickson (2008), derived a closed-form representation for the distribution of the ruin time for the Sparre Andersen process. The derivation is based on transforming the original boundary crossing problem to an equivalent one on linear lower boundary crossing. Drekic and Willmot (2003), obtained the probability density function of the time of ruin in the classical model directly by inversion of the associated Laplace transform and then used it to obtain closed- form expressions for the moments.

2.6 A review of the Classical Model of Risk Theory

As already has been said in Section 2.5, at time T, ruin occurs, so T, is the time when, for the very first time, the surplus becomes negative. This ruin is a technical one and not legal or else. This is obvious from what has already been said, as an insurance company has the means to overcome the negative surplus. Some of these means are the income from the investments, the reinsurance, the reserves or even the Risk Management. In the classical risk model, ruin can only happen as a result of the incured claims. However, in practice, there are a lot of other factors that can cause ruin to an insurance company, such as bad management or operational risk etc. So ruin here means, that at time T, one has, for the first time, a drop of surplus $U(T^+)$ under zero which implies that

$$U(t) \geq 0 \ for \ t < T$$

and

$$U(T) < 0$$

and also

$$U(T^+) = U(T^-) - X_T,$$

where X_T is the size of the claim (loss event) causing ruin, and $U(T-)$ is the surplus immediately before ruin. The ruin probability is the probability of $U(t)$ to become smaller than zero at some time t, given that the initial surplus is u, that is:

$$\psi(u) = Pr(U(t) < 0, \ for \ some \ t \ |U(0) = u). \qquad (2.6.1)$$

Ruin probability shows how possible is for ruin to happen with an initial surplus $U(0) = u$. So the ruin probability $\psi(u)$ is a function of $U(0) = u$ and can also be given by the following expression:

$$\psi(u) = Pr(T < \infty \mid U(0) = u),$$

where T is given in (2.5.1). The probability of non- ruin is given by

$$\delta(u) = 1 - \psi(u)$$

and

$$\delta(u) = Pr(T = \infty \mid U(0) = u). \qquad (2.6.2)$$

An assumption which is made, in order that ruin is not certain ($\psi(u) \neq 1, \ \forall u \geq 0$), is that the insurer's premium incomes are bigger or equal to the expenses, which means that

$$ct > E[S(t)] \ \forall t,$$

which implies that

$$c > \lambda E(X) \tag{2.6.3}$$

or

$$c = (1+\theta)\lambda E(X), \ \theta > 0 \ and \ u > 0, \tag{2.6.4}$$

where θ is the premium loading factor, defined by

$$\theta = \frac{c}{\lambda E(X)} - 1.$$

Since we have introduced θ we should add an interesting result for the ruin probability when $u = 0$, it holds that

$$\psi(0) = \frac{1}{1+\theta}, \tag{2.6.5}$$

(Bowers et al, 2007) which expresses the ruin probability with initial surplus 0. From (2.6.5) we can see the relation between $\psi(0)$ and θ. We can see that in order to minimize the ruin probability $\psi(0)$, θ has to be too big, something impossible in the competition field of the insurance market.

Some other relations that should be obvious from above are

$$\lim_{u\to\infty} \psi(u) = 0$$

and

$$\lim_{u\to\infty} \delta(u) = 1,$$

also because of (2.6.5)

$$\delta(0) = \frac{\theta}{1+\theta}.$$

2.7 The Adjustment Coefficient

The adjustment coefficient (Lundberg's coefficient) is used in results for ruin probability inequalities and asymptotic formulas, among others, in ruin theory. We have chosen to illustrate Lundberg's inequality and an asymptotic formula, which follow after the presentation of the adjustment coefficient. The adjustment coefficient is the unique positive solution, if one exists, of the equation

$$1 + (1+\theta)E(X)r = M_X(r), \tag{2.7.1}$$

where

$$M_X(r) = E(e^{rX}) = \int_0^\infty e^{RX} f(x)dx$$

is the moment generator function of the distribution function of F of the claims, which has to be finite in an interval in order R to exist. There are distributions that do not have moment generator function and more precisely the moment generator function is infinite for all $r > 0$, such as Pareto, Lognormal, etc. So for such distributions, the adjustment coefficient of the surplus process does not exist. As it was mentioned, R satisfies the (2.7.1) and this is its only solution. This happens, because relation (2.6.4) holds, where $\theta \geq 0$, so it follows that

$$(1+\theta)E(X) = \frac{c}{\lambda}$$

then one has that

$$1 + \frac{c}{\lambda} r = M_X(r)$$

or, equivalently,

$$\lambda + cr = \lambda M_X(r).$$

Let us call

$$\psi_1(r) = 1 + (1+\theta)E(X)r$$

and

$$\psi_2(r) = M_X(r) = E(e^{rX}) = \int_0^\infty e^{RX} f(x)dx,$$

so if we differentiate $\psi_2(r)$ once and twice, respectively, we will have that

$$\psi_2'(r) = E(Xe^{rX}) > 0$$

$$\psi_2''(r) = E(X^2 e^{rX}) > 0$$

and

$$\psi_2'(0) = E(X)$$
$$\psi_1'(0) = (1+\theta)E(X)$$

which implies that

$$\psi_2'(0) \leq \psi_1'(0).$$

If (2.7.1) has more than one solutions then each solution satisfies

$$\psi_1(r) = \psi_2(r).$$

If (2.7.1) has n positive solutions

$$\rho_1, \rho_2, \rho_3, \ldots, \rho_n, \; then \; R = \min\{\rho_i : 1 \leq i \leq n\}.$$

The exponential is, probably, the only distribution that has a simple expression for the calculation of R. For most of the other claim size distributions, R is calculated by the use of numerical methods (e.g. Newton- Rampson). It is easy to show that, if $X \sim E(\mu)$, then R is equal to:

$$R = \frac{\mu\theta}{1+\theta}, \tag{2.7.2}$$

see Bowers et al (2007). Now we will give Lundberg's inequality, for which it holds

$$\psi(u) \leq e^{-Ru}, \quad \forall u \geq 0,$$

see Bowers et al (2007). Also, if $R > 0$ then one has the following asymptotic formula

$$\psi(u) \sim Ce^{-Ru} \, (u \to \infty),$$

where

$$C = \frac{\theta E(X)}{E[Xe^{RX}] - (1+\theta) + E(X)}$$

and is the Cramer- Lundberg's asymptotic formula, which is obvious that for exponential severity of risk, $X \sim E(\mu)$ then one has that

$$\psi(u) = \frac{1}{1+\theta} e^{-\frac{\mu\theta}{1+\theta}u}, \tag{2.7.3}$$

see Bowers et al (2007).

2.8 Renewal equations and Ruin Probability

The general formula for a renewal equation was given in (2.2.2). In the classical model for $u > 0$ one has that:

$$\delta'(u) = \frac{\lambda}{c}\delta(u) - \frac{\lambda}{c}\int_0^u \delta(u-x) f(x) dx,$$

see Bowers et al (2007). This means that the probability of non- ruin satisfies one integro- differential equation. From this, it can be showed that $\delta(u)$ satisfies the following defective renewal equation

$$\delta(u) = \delta(0) + \frac{\lambda}{c}\int_0^u \delta(u-x)\overline{F}(x)dx,$$

where $\overline{F}(x) = 1 - F(X)$ is the tail of the distribution function F. Similar results can be shown for $\psi(u)$. More explicitly,

$$\psi'(u) = \frac{\lambda}{c}\psi(u) - \frac{\lambda}{c}\int_0^u \psi(u-x)f(x)dx - \frac{\lambda}{c}\overline{F}(u)$$

and

$$\psi(u) = \psi(0) + \frac{\lambda}{c}\int_0^u \psi(u-x)\overline{F}(x)dx - \frac{\lambda}{c}\int_0^u \overline{F}(x)dx$$

or

$$\psi(u) = \frac{\lambda}{c}\int_0^u \psi(u-x)\overline{F}(x)dx + \frac{\lambda}{c}\int_u^\infty \overline{F}(x)dx.$$

So both the ruin and non- ruin probability can be written as solutions of a renewal equation. Finding a general analytical solution for the ruin probability is quite difficult, especially if the hypotheses of the classical model are not in effect, then typically, only bounds are possible to be found. Of interest is also for the ruin probability the result which one has if the equilibrium density of X, $f_e(x)$ is used. More specifically, the equilibrium density is defined by

$$f_e(x) = \frac{\overline{F}(x)}{E(X)}, \tag{2.8.1}$$

so $f_e(x) \geq 0$ and it is easy enough to conclude the next result

$$\psi(u) = \psi(0)\int_0^u \psi(u-x)f_e(x)dx + \psi(0)\int_u^\infty f_e(x)dx,$$

so it is easy to see that $\psi(u)$ satisfies the following equation

$$\begin{aligned}\psi(u) &= \frac{1}{1+\theta}\int_0^u \psi(u-x)f_e(x)dx + \frac{1}{1+\theta}\overline{F}_e(x)\\ &= \frac{\lambda E(X)}{c}\int_0^u \psi(u-x)f_e(x)dx + \frac{\lambda E(X)}{c}\overline{F}_e(x).\end{aligned}$$

2.9 The Renewal Model of Risk Theory

So far, the classical model of ruin theory has been introduced. According to the above representation, in this model the surplus process, is given by:

$$U(t) = u + ct - S(t), \quad t \geq 0,\, U(0) = u, \quad c > 0,$$

where $S(t)$ stands for the aggregate claims of risk X_i, that occur to an insurance company, in a period of time. These claims are independent and identically distributed (iid) to each other, and to the number of claims of the same period. Moreover, $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ so that the interclaim times of the incured claims follow the exponential distribution with parameter λ and are iid too.

In this paragraph the renewal model- or the Sparre Andersen model, from the name of Sparre Andersen, the mathematician who introduced the specific model- will be presented. The renewal model (Sparre Andersen model) with surplus process

$$U(t) = u + ct - S(t)$$

is actually a generalization of the classical one and can be formulated as a random walk, since here $N(t)$ does not follow a Poisson process, but a renewal process, with again T_i which stands for the interclaim time, being independent and identically distributed (iid). This means, that T_i will not follow, in this case, the exponential distribution. In the classical model of ruin theory, one of the assumptions was that c had to be bigger than $\lambda E(X)$ in order the incomes to be bigger than the expenses, something very logical. So here the assumption which is made, so that ruin is not certain, is

$$c > \frac{1}{E(T_i)}E(X),$$

where $E(T_i)$ is the mean of the random variable of the interclaim times and $E(X)$ is the mean of the random variable of the severity of the claims. So now θ will be:

$$\theta = \frac{cE(T_i)}{E(X)} - 1.$$

Since the assumptions here differ to those of the classical model, the results given in the previous sections will differ, too. One of those results was Lundberg's equation, so we need to reformulate it under the new assumptions. X_i still stands for the positive random variable of the size of the i- claim and is iid to T_i, which stands for the random variable of the time between the $(i-1)$ and i- claims. Let us consider the difference

$$Y_i = X_i - cT_i \textit{ with } E(Y_i) < 0,$$

where c still is the premium rate per unit time and S_n the random variable of the net decrease in surplus up to and including the n- claim, then S_n is defined to be the random walk

$$S_n = \begin{cases} \sum_{i=1}^{n} Y_i, \ n = 1, 2, \ldots \\ 0, \ n = 0. \end{cases}$$

Finally, let us consider the df of the Y_i, which is denoted by G. R was introduced as the only solution of (2.7.1), then here R has to be the smallest solution (positive) of the equation of the moment generator function of the density function of G, then

$$M_G(R) = 1 = \int_{-\infty}^{\infty} e^{Rx} dG(x).$$

Let the moments of X_i, T_i be $M_{X_i}(R), M_{T_i}(R)$ respectively, one has that

$$M_G(R) = M_{T_i}(-cR) M_{X_i}(R),$$

so in the examined case, Lundberg's equation, becomes

$$M_{T_i}(-cR) M_{X_i}(R) = 1,$$

see Willmot and Lin (2001) and Grandell (1991). As far as ruin probability with initial surplus u is concerned, it holds that:

$$\psi(u) = Pr\left\{\{S_1 > u\} \cup \{S_2 > u\} \cup \{\ldots\}\right\} = Pr(\max\{S_1, S_2, \ldots\} > u)$$

and according to the law of large numbers

$$E(\frac{S_n}{n}) = E(Y_i) < 0.$$

So one has

$$\lim_{n\to\infty} S_n = -\infty$$

and

$$\max\{S_1, S_2, \ldots\}$$

is infinite with probability 1 and so

$$\psi(u) < 1 \quad \forall u \geq 0.$$

If one puts

$$M = max_{n\geq 0} \sum_{i=1}^{n} (X_i - cT_i)$$

then it is concluded that

$$\psi(u) = Pr(M > u),$$

suggesting the compound geometric tail nature of $\psi(u)$.

Chapter 3

Introduction to the Mathematical Theory of Reliability

The Mathematical theory of reliability, as a whole, consists of ideas, mathematical models and methods directed toward the solution of problems of predicting, estimating, or optimizing the probability of survival, mean life, or more generally, life distributions of components of systems. Moreover, there are other problems considered in reliability theory, which deal with the probability of proper functioning of the system at either a specified or an arbitrary time or the proportion of time the system is functioning properly. It is obvious that reliability theory plays an important role in decision making over maintenance policies. In this chapter, the reliability background is introduced. One of the main tools in reliability theory are the classes of distributions which are used to study lifetimes of systems, devices or components, as already has been said. Apart from the above fields, various applications of classification of life distributions have been found in insurance and actuarial science, where they have been used in insurance portfolio management and in risk theory. In Chapter 5, these results will be used for the classification of the distribution of time to ruin.

Life distribution functions are characterized in terms of failure rates or conditional distributions of residual lifetimes. There is a big number of proposed classes of life distributions, however here, only few of them will be introduced. In particular, the increasing failure rate (IFR) class, the decreasing failure rate (DFR) class, the bathtub failure rate (BFR) class, the hump failure rate (HFR) or the upside down hump failure rate (UBFR) class, the new better than used (NBU) class, the new worse than used (NWU) class, the new better than used in expectation (NBUE) class, and the new worse than used in expectation (NWUE) class. The interested reader in more classes of distributions, should be referred to Willmot and Lin (2000), Barlow and Proschan (1965, 1975).

Section 3.1 presents the theoretical background of the theory of reliability. An attempt is being made to present some of the main areas of reliability research and its history, such as the needs that gave rose to its study over the years. In Section 3.2 we present the failure distribution functions. In Section 3.3 we give the definition of the failure rate and the example of the exponential distribution, which is the only continuous life distribution, as we will prove later on, with constant failure rate. In this section we introduce, also, the mixture of exponentials, the gamma and the Erlang distribution. For all these we give, in Section 3.4, examples of their failure rates, after we give the definitions of IFR and DFR. The following section (Section 3.5) gives the definition of the BFR and HFR and an example of an HFR (Log- normal distribution) and we depict all the above classes together. However, we do not give an example of a BFR, since none of the commonly used distributions belongs to this category. This is a very important section as its results will be used in Chapter 5, where we will study the failure rate of the ruin time. In the last two sections (Section 3.6 and Section 3.7) we give the residual lifetime distribution and some other classes of distributions.

3.1 The Theoretical Background

Some of the areas of reliability research are life testing, structural reliability (including redundancy considerations), machine maintenance problems (a part of queuing theory), replacement problems, quality control, extreme value theory, order statistics, censorship in sampling. Reliability research has grown out of the demands of modern technology and experiences during the World War II, see Barlow and Proschan (1965) and since then a lot of research has taken place over the subject. It is worth mentioning that Weibull (1951) proposed the- famous now- distribution named after him as an appropriate distribution to describe life length of materials, devices, etc, which long after was used in extreme value theory, too. Also it should be mentioned that the exponential distribution gained popularity as it was applied in life- testing research, because it leads to simple results of failure rates, see Barlow and Proschan (1965). In the following sections a number of analytic results for the exponential distribution will be given. Another distribution of interest that will be presented in this chapter is the gamma, which was suggested for life lengths of structures under dynamic loading and made it possible to express the probability distribution of life length in terms of the load given as a function of time and of deterioration occurring in time independently of loading, see Barlow and Proschan (1965).

3.2 Failure (or Lifetime) Distributions

A failure distribution represents an attempt to describe mathematically the length of life of a material, a structure, a device, or a human. So if Y is the random variable, which stands for the age of a newborn human, for instance, then

$$F(y) = Pr(Y \leq y)$$

is the distribution function and

$$\overline{F}(y) = Pr(Y > y)$$

is the survival distribution (the right tail of the distribution), so one has that

$$\overline{F}(y) = 1 - F(y).$$

For a lifetime distribution, we assume that $F(0) = 0$, which implies $\overline{F}(0) = 1$ and

$$\overline{F}(y) = \begin{cases} 1, \ y = 0 \\ \textit{right continuous and decreasing}, 0 < y < \omega, \\ 0, \ y \geq \omega \end{cases}$$

where $\omega = \min\left\{y : \overline{F}(y) = 0\right\}$ is the maximum life length that can be attained.

For $F(y)$ is assumed that

$$F(0^{-}) = 0$$

and

$$F(+\infty) = 1$$

and F is right continuous.

The failure distribution is highly affected by the models of possible failure for the item in question. However, the decision of the most appropriate among the various non- symmetrical probability functions is a difficult procedure. This is because, it is highly difficult to choose among physical considerations such as environment, manufacturer, material, use, etc. To add to this, it remains a difficult procedure, too, on the basis of actual observations of times to failure, since the differences among the known distributions (non-symmetrical probability functions such as the gamma etc) become only obvious (meaning statistically significant) in the tails of the distributions. It has already been mentioned in the previous chapter that distributions with a long right tail, for example, are needed in predicting the probability of a small number of risks with big economic value in practice. That is because the actual observations are sparse in the tails because of limited sample sizes. So in order to overcome this, it is necessary to appeal to a concept that permits the differentiation among the above distribution functions on some physical considerations. The concept is this of the failure rate function, the analysis of which follows in the next section.

3.3 The Failure Rate

Definition 3.3.1. *If the failure distribution has density f, the failure rate function $r(t)$ is defined for those values of t for which $F(t) < 1$ by*

$$r(t) = \frac{f(t)}{\overline{F}(t)} \tag{3.3.1}$$

and $r(t)dt$ represents the probability that an object of age t will fail in the interval $[t, t + dt]$.

Up to now we have talked in terms of reliability theory, but this quantity is of great importance also for actuarial science. Its name varies depending on by whom is being used. It is used by actuaries under the name of "force of mortality", $\mu(t)$, in order to compute the mortality tables. In statistics it is known as "Mill's ratio" for the normal distribution, in extreme value theory is called "intensity function" and in reliability theory "hazard rate".

In many applications it is logical to believe that the function $r(t)$ beyond a certain age does not increase because of the fact of the inevitable deterioration. This can be seen from the side of a device, after some years of constant use. But, there are also cases, when this function is initially decreasing. All these cases follow in the next sections in detail.

Failure rate can be interpreted as the conditional density function of the random variable T at t, given that the human, device, etc has survived, or generally speaking, has functioned until t. So for small values $r(t)$ is probability and consequently r(t)dt is the probability of failure in the interval [t, t+dt]. Since (3.3.1) holds, then one has that

$$r(t) = -\frac{d}{dt} \ln \overline{F}(t), \; t \geq 0,$$

because $f(t) = F'(t)$ and consequently it holds that

$$\int_0^t r(x)dx = -\ln \overline{F}(t),$$

in other words

$$\overline{F}(t) = e^{-\int_0^t r(x)dx}, \; t \geq 0.$$

It should be mentioned that small values of $r(t)$ indicate a thick right tail, whereas large values a thin right tail. It has been said that there is a wide variety of families of distributions which have been used for the failure of materials and the life length of devices, components, etc. Here some of the most well- known are given

- The exponential:

  $$f(t) = \lambda e^{-\lambda t}, \tag{3.3.2}$$

  with $\lambda > 0$ is the parameter and the survival function is given by

  $$\overline{F}(t) = e^{-\lambda t} \tag{3.3.3}$$

  and from (3.3.1)

  $$r(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda, \qquad \lambda > 0, \; t \geq 0. \tag{3.3.4}$$

- Mixture of exponentials :

  $$f(t) = A_1\lambda_1 e^{-\lambda_1 t} + A_2\lambda_2 e^{-\lambda_2 t} + \ldots + A_n\lambda_n e^{-\lambda_n t}, \tag{3.3.5}$$

  where $\{A_1, A_2, \ldots, A_n\}$ is a probability distribution with $\sum_{k=1}^{n} A_k = 1$ and $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ are the parameters.

- The gamma :

$$f(t) = \frac{(\lambda t)^{\alpha-1}\lambda e^{-\lambda t}}{\Gamma(\alpha)}, \qquad t > 0, \tag{3.3.6}$$

  where $\alpha, \lambda > 0$ are the parameters. In general, $F(x)$ is the Erlang distribution, when α is a positive integer, the case of which follows.

- The Erlang:

$$f(t) = \frac{(\lambda t)^{n-1}\lambda e^{-\lambda t}}{(n-1)!)}, \qquad t > 0, n \in N. \tag{3.3.7}$$

The exponential distribution has constant failure rate. The family of gamma distributions is skewed to the right and may seem appropriate to depict the life length. On the other hand, the family of exponential distributions is very famous and very explored, since has a lot of desirable mathematical properties. However, its applicability is limited because of the "memoryless property", which does not take into account the previous use of a device, or the current age of an individual etc.

In particular, it is easy to prove that if the quantity $Pr(X > t)$ is the probability for a new device, a newborn human, etc, to survive more than t (life length T of a device), then previous use does not affect the future life length, Feller (1957), because it holds that

$$Pr(X > t) = 1 - Pr(X \leq t) = 1 - F(t) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$$
$$and\ Pr(X > s + t | X > s) = \frac{Pr(X > s + t, X > s)}{Pr(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}},$$

which means that $Pr(X > s + t | X > s) = e^{-\lambda t} = Pr(X > t)$.

3.4 Monotone Failure Rates

There are a lot of situations, where the failure rate $r(t)$, is strictly monotone non- increasing (nondecreasing) in t, which is typically associated with the case of a thick (thin) right tail of the distribution. These situations are of high practical interest. The reason why is that a lot of materials, devices etc wear out with time and this is reflected by increasing $r(t)$. On the other hand, some other materials, devices etc harden through use, making the class of distributions with decreasing failure rate also interesting. Below the definitions for the classes of distributions with monotone failure rates are given

Definition 3.4.1. *F is an increasing failure rate (IFR) distribution if F satisfies*

$$\overline{F}(x|t) = \frac{\overline{F}(x+t)}{\overline{F}(t)}$$

is decreasing in $0 < t < \infty$ for each $x \geq 0$

Definition 3.4.2. *F is an decreasing failure rate (DFR) distribution if F satisfies*

$$\overline{F}(x|t) = \frac{\overline{F}(x+t)}{\overline{F}(t)}$$

is increasing in $t \geq 0$ for each $x \geq 0$.

DFR distributions may arise in many ways, see Barlow and Proschan (1965).

The above definitions hold for both the continuous and discrete random variables, although the discrete case will not be used here. However, there are also other expressions for these definitions in the bibliography, which sometimes hold only for the continuous case for example, or the opposite.

Lemma 3.4.1. *Assume F has density f, with $F(0-) = 0$. Then is IFR (DFR) if and only if $r(t)$ is increasing (decreasing).*

We should also mention that here we will, loosely, use the word "decreasing"("increasing") to mean "non- increasing"("non- decreasing"); this usage will be followed throughout the dissertation.

Example 3.4.1. *The Exponential Distribution*

Properties of the exponential distribution were discussed in Section 3.3. In this section we mentioned that the exponential life distribution provides a good description of the life length of a unit which does not age with time, we gave the density function in (3.3.2)*, its survival function and its failure rate, which is a constant, that is both IFR and DFR, see* (3.3.4)*. So one has the following theorem.*

Theorem 3.4.1. *The exponential distribution is the only life distribution with constant failure rate function.*

Proof of Theorem 3.4.1 First we proved in (3.3.4) that the exponential has constant failure rate, and here it is easy to prove that if a distribution has constant failure rate, then it is the exponential distribution, so if

$$r(t) = \frac{f(t)}{\overline{F}(t)} = c$$

then one has that

$$\frac{-dln\overline{F}(t)}{dt} = c$$

which yields

$$\overline{F}(t) = e^{-ct}$$

and so the exponential is given with parameter c

$$F(t) = 1 - e^{-ct}.$$

□

Example 3.4.2. *Mixture of Exponentials*

If a random variable T follows a mixture of exponentials, then it has the probability function (3.3.5) *and if $F(t)$ is the df of the mixture of exponentials, it may be expressed as*

$$F(t) = \int_0^\infty (1 - e^{-\lambda t})dH(\lambda),\ t \geq 0,$$

see Willmot and Lin (2001), where $H(\lambda)$ is the df of a positive random variable. Then the failure rate is given by

$$r(t) = -\frac{d}{dt} ln\overline{F}(t) = \frac{\int_0^\infty \lambda e^{-\lambda t} dH(\lambda)}{\int_0^\infty e^{-\lambda t} dH(\lambda)}$$

and if $r(t)$ is differentiated, then yields

$$r'(t) = -\frac{\int_0^\infty \lambda^2 e^{-\lambda t} dH(\lambda)}{\int_0^\infty e^{-\lambda t} dH(\lambda)} + \left\{ \frac{\int_0^\infty \lambda e^{-\lambda t} dH(\lambda)}{\int_0^\infty e^{-\lambda t} dH(\lambda)} \right\}^2$$

so

$$r'(t) = -E(\Lambda^2) + \{E(\Lambda)\}^2$$

and consequently

$$r'(t) = -Var(\Lambda).$$

Since $r'(t) \leq 0$*, the mixed exponential df is DFR. This is depicted in the figure bellow, where there are being showed five cases of mixed exponentials with weights of* 0.5*.*

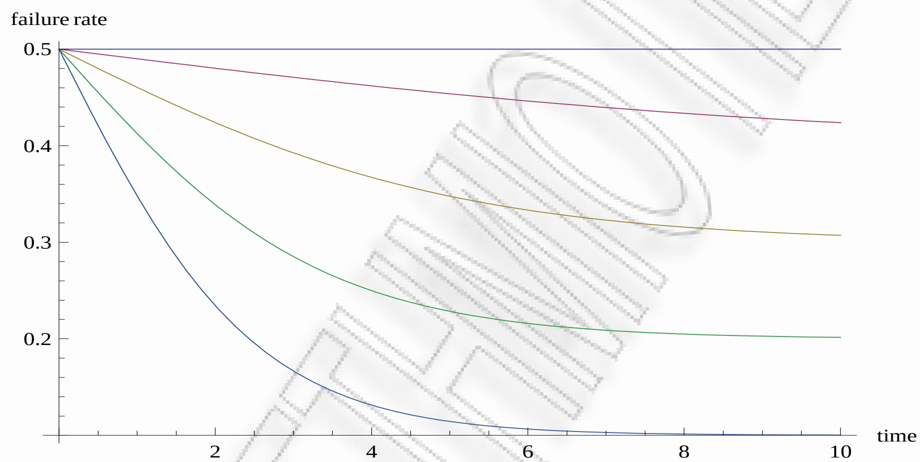

Figure 3.1: Failure rate of five cases of mixture of exponentials, with density function which was given in (3.3.5) for $n = 2$ and weights of 0.5. In the first case (purple) $\lambda_1 = 0.5, \lambda_2 = 0.5$, in the second case (pink) $\lambda_1 = 0.4, \lambda_2 = 0.6$, in the third case (yellow) $\lambda_1 = 0.3, \lambda_2 = 0.7$, in the fourth case (green) $\lambda_1 = 0.2, \lambda_2 = 0.8$ and in the last case (blue) $\lambda_1 = 0.1, \lambda_2 = 0.9$.

Example 3.4.3. *The Gamma Distribution*

In the previous section, the probability density function of a gamma was given in (3.3.6)*. If random variable* T *follows gamma then its df will be of the form*

$$F(t) = 1 - e^{-\lambda t} \sum_{k=0}^{\alpha-1} \frac{(\lambda t)^k}{k!}$$

and $r(t)$ is not easily expressed in simple analytic form unless $\alpha = 1, 2, 3, \ldots$. Since $r(\infty) = \lim_{t\to\infty} r(t)$ and

$$r(\infty) = \lim_{t\to\infty} \frac{f'(t)}{f(t)}$$

and since

$$f'(t) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \left\{(\alpha - 1)t^{\alpha-2} - \lambda t^{\alpha-1}\right\} e^{-\lambda t},$$

then one has that

$$r(\infty) = \lambda - \lim_{t\to\infty} \frac{\alpha - 1)}{t} = \lambda.$$

So, $f(t)$ is log- convex (log- concave) if $\alpha \leq (\geq)1$ and as a consequence $F(t)$ is DFR(IFR). When $\alpha = 1$ the exponential distribution holds, which is both IFR and DFR. This is depicted in the figure that follows, where are being showed five cases of gamma with parameter $\lambda = 1$ and different values of α.

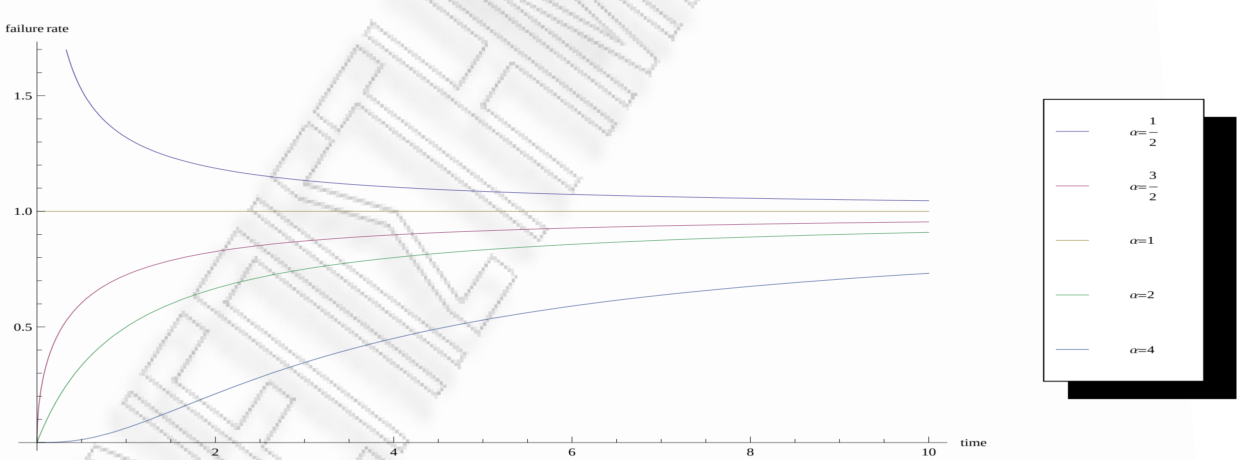

Figure 3.2: Failure rate of gamma distributions with $\lambda = 1$ and various values of α. In the first case (purple) $\alpha = 0.5$, in the second case (red) $\alpha = 2/3$, in the third case (yellow) $\alpha = 1$, in the fourth case (green) $\alpha = 2$ and in the last case (blue) $\alpha = 4$.

Example 3.4.4. *The Erlang Distribution*

In Section 3.3 the probability function of an Erlang was given, in (3.3.7). *If a random variable T follows Erlang then its df will be of the form*

$$F(t) = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}$$

Which implies that $F(t)$ is IFR for $n \geq 2$.

3.5 The Bathtub and Hump Failure Rate

So far we have introduced only the IFR(DFR) classes of distribution, where the failure rate is monotone. However, there are two more categories of failure rate, the bathtub and the hump failure rate or the upside down bathtub failure rate, because physical phenomena exhibit, also, failure rates functions that are non- monotonic. We have mentioned that in practice we have IFR, when there is "damage" or "aging" and DFR if some materials, devices etc harden, instead of wearing out, through use. DFR are very rare and hold when there is a big possibility of failure in the beginning, such as the functioning time of some specific electronic devices, which has decreasing failure rate during the first period of use.

As far as BFR is concerned some examples that can be given are the case of the functioning time, again, for a brand new device or the lifetime of a person from the moment that is born and for a long period of time. There is a phase where $r(t)$ is decreasing up to a point and then increasing, such $r(t)$ is BFR. In the beginning, the failures (damage, death, etc) can be a result of a bad design or infant diseases, respectively. What follows this, is a period when $r(t)$ is almost stable and failures are a result of random factors. This is the useful period for a machine or the age of 30 for a person. At the last (third) period the failure rate is increasing and depicts the damage or the aging as time passes on. More precisely, a df is said to be BFR if there exists a point t_0, called "turning point" of the distribution, such that the failure rate is decreasing in $[0, t_0)$ and increasing in $[t_0, \infty)$. We should mention that none of the commonly used distributions belongs to this category. Bellow we give the definitions for the BFR and UBFR, respectively.

Definition 3.5.1. *A life df F having support on $[0, \infty)$ is said to be a bathtub failure rate (BFR) distribution if there exists a point $t_0 (\geq 0)$ such that $R(t) =$*

$-\log_e \overline{F}(t)$ *is concave in* $[0, t_0)$ *and convex in* $[t_0, \infty)$. *The point* t_0 *is referred to as a turning point of the df* F.

It is apparent that if strict concavity or convexity is not insisted upon in the above definition, then a BFR df may have more than one turning point, see Mitra and Basu (1996).

F is a hump failure rate (HFR) distribution if $r(t)$ is increasing in the beginning and then decreasing. If we want to give a physical interpretation of this case, a good example would be the study of the time to failure, after a successful surgery, where there was increased death risk because of complications, that follow the surgery and wear out as time passes on.

Up to now we have introduced some of the well known distributions that were IFR(DFR) or both IFR and DFR, that is the exponential distribution. Here, we have presented the BFR and HFR, too. So, before proceeding, we think that this is a proper point to give a plot that will depict all of the four introduced classes of distribution of failure rate together. This follows next.

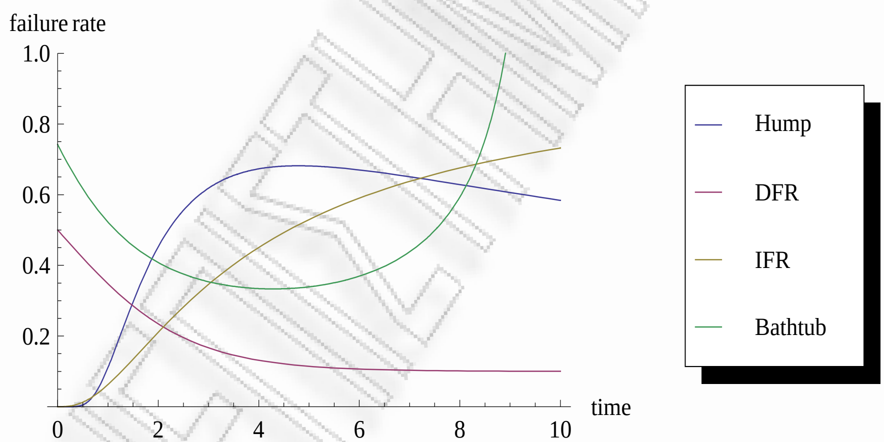

Figure 3.3: The four classes of the failure rate.

As we have mentioned there are no commonly used distributions that are bathtubs, we will only present the Log- Normal in the example that follows, which is one of the distributions that belong to the upside down bathtub class. There are also other known distributions that are UBFR, such as the Logistic, etc, see Antzoulakos(2003).

Example 3.5.1. *The Log- Normal Distribution*

For the Log- Normal distribution with parameters μ *and* σ^2*, we have that*

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma t} e^{-\frac{1}{2}(\frac{\log - \mu}{\sigma})^2}, \quad t \geq 0, \sigma > 0$$

and the survival function is given by

$$\overline{F}(t) = 1 - \Phi(\frac{\log - \mu}{\sigma}).$$

The Log- Normal distribution is called after the fact that the distribution of the random variable $Y = \log T$*, where* T *follows* $LN(\mu, \sigma^2)$*, is the Normal with parameters* μ, σ^2 $(T = e^Y)$*. The survival function and the failure rate are expressed as functions of* $\Phi(t)$ *of the standard normal distribution. It holds that* $r(0) = 0$*, is increasing in* $[0, t\max]$*, decreasing in* $[t\max, \infty]$ *and* $\lim_{t\to\infty} r(t) = 0$*.*

Here we show the failure rate of the Log- Normal distribution.

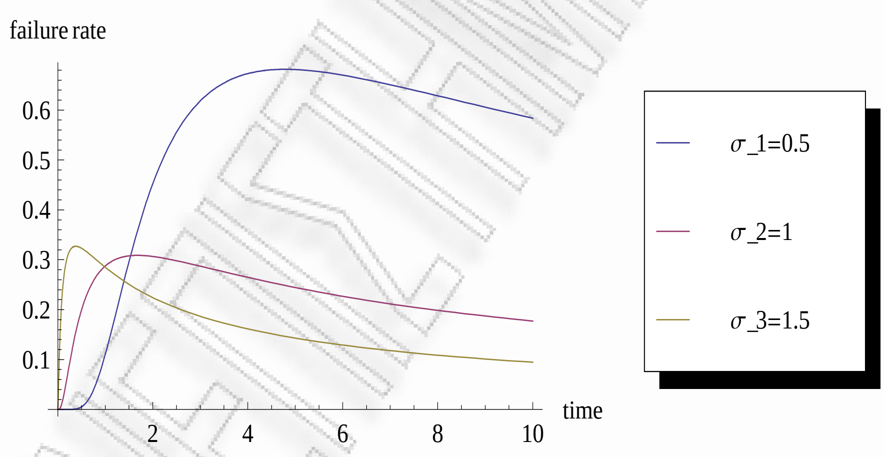

Figure 3.4: The failure rate of LN with $\mu = 1$ and different values of σ.

3.6 The Residual Lifetime Distribution

In this section, the residual lifetime will be introduced. It plays an important role not only in actuarial science and in the computation of mortality tables, but also in the classification of distributions, something that will be shown in this section. Let X be a random variable and for each $x \geq 0$, let a new variable $T(x)$ be defined by:

$$T(x) = X - x | X > x.$$

This represents the remaining lifetime of a person, who has survived until time x. The distribution function of this will be then:

$$Pr(T(x) \leq t) = Pr(X - x \leq t | X > x),$$

$$\textit{which implies that } Pr(T(x) \leq t) = \frac{\overline{F}(x) - \overline{F}(x+t)}{\overline{F}(x)}$$

$$\textit{or } Pr(T(x) \leq t) = 1 - \frac{\overline{F}(x+t)}{\overline{F}(x)}, \quad t \geq 0.$$

It is worth mentioning that in an actuarial context the above quantity represents the probability a person of age of x to die in t years and is symbolized by ${}_tq_x$. If one rewrites the above equation as

$${}_tq_x = 1 - \frac{\overline{F}(x+t)}{\overline{F}(x)},$$

then

$$\frac{\overline{F}(x+t)}{\overline{F}(x)} = 1 -_t q_x$$

and then the probability of survival of a person of age of x for t years ${}_tp_x$ is produced. It is easy to imply the survival function of the residual lifetime. It is given next:

$$Pr(T(x) > t) = \frac{\overline{F}(x+t)}{\overline{F}(x)}, \quad t \geq 0.$$

The expected value of $T(x)$ is called mean residual lifetime MRL and it holds that :

$$e_x^0 = E[T(x)] = \int_0^\infty {}_tp_x dt,$$

so one has the following equation

$$e_x^0 = E\left[T(x)\right] = \frac{\int_x^\infty (t-x) dF(t)}{\overline{F}(x)}, \quad x \geq 0. \tag{3.6.1}$$

It holds that $E(X) = \int_0^\infty \overline{F}(x)dx$ so one has that:

$$e_x^0 = E\left[T(x)\right] = \int_0^\infty Pr(T(x) > t)dt = \int_0^\infty \frac{\overline{F}(x+t)}{\overline{F}(x)}dt.$$

MRL is useful for analysis of tail thickness- some remarks on this have been made up to now. In particular, large values of e_x^0 imply a thick tail for the examined distribution. To add to this, MRL is closely related to the failure rate and the equilibrium distribution $F_e(x)$, relation (2.8.1). This relation gives the density of equilibrium distribution and here will be given a relation between the equilibrium distribution and the MRL. From (3.6.1) one has that:

$$e_x^0 = \frac{\int_x^\infty \overline{F}(t)dt}{\overline{F}(x)} = \frac{E(X)\overline{F}_e(x)}{\overline{F}(x)}.$$

It is obvious that for x=0, then $e(0) = E(X)$.

A distribution function is called increasing (decreasing) mean residual lifetime IMRL(DMRL), when e_x^0 is nondecreasing (non- increasing) in x. It is easy to prove that DFR(IFR) class of distributions is contained in the IMRL(DMRL) class, as $F(x)$ is DFR(IFR) when

$$\frac{\overline{F}(x+t)}{\overline{F}(x)}$$

is nondecreasing (non- increasing) in x for fixed $t \geq 0$ and

$$e_x^0 = \int_0^\infty Pr(T(x) > t)dt = \int_0^\infty \frac{\overline{F}(x+t)}{\overline{F}(x)}dt.$$

3.7 Other Classes of Distributions

In this section some other classes of distributions will be presented. These distributions are not classified in terms of failure rate as did the IFR(DFR), BFR(HFR) classes, but in terms of their survival function or survival function of their equilibrium distributions. First of all, there is the new worse(better) than used NWU(NBU) class of distribution, which owes its name to the fact that the inequality is a restatement of $Pr(T(x) > x) \geq (\leq) Pr(X > x)$, so it is to imply that the DFR(IFR) class of distributions is contained in the NWU(NBU) class.

An equivalent definition is the following: a distribution function is said to be in the NWU(NBU) class if $\overline{F}(x+t) \geq (\leq)\overline{F}(x)\overline{F}(t),\ for\ all\ x \geq 0\ and\ t \geq 0$. To add to this, there are also the 2-NWU(2-NBU) and the 3-NWU(3-NBU) classes, but in this context will not be presented, so for the interested reader it is recommended to see Willmot and Lin (2001).

Closing, there are also the new worse(better) than used in expectation NWUE(NBUE) classes. A distribution function is called NWUE(NBUE) class if the MRL satisfies $e_x^0 \geq (\leq)e_0$ or in other words $E(T(x)) \geq (\leq)E(X)$ and the residual lifetime has a larger(smaller) expectation that the original or new lifetime X. So, generally speaking it holds that if F(x) is NWU(NBU), then:

$$e_x^0 = \int_0^\infty \frac{\overline{F}(x+t)}{\overline{F}(x)}dt \geq (\leq) \int_0^\infty \overline{F}(t)dt = E(X) = e_0,$$

so the NWU(NBU) class is a subclass of the NWUE(NBUE) class. It goes without saying that properties of the NWUE(NBUE) class will also hold for its subclasses.

Chapter 4

The Density of the Ruin Time

Up to now, an attempt has been made to represent the risk and reliability theory background. More specifically, as far as risk theory is concerned, the surplus process, the classical and renewal model and ruin probability were introduced, because these notions are needed for the analysis of the procedures that were followed in order the expressions for the density of time to ruin to be derived.

The presentation of the subject under examination is mostly based on two papers, see Borovkov and Dickson (2008) and Drekic and Willmot (2003). In Borovkov and Dickson's paper, a study for the ruin time for a Sparre Andersen process with exponential claims, is taking place. Whereas, in the Drekic and Willmot's paper, the density and the moments of the time of ruin with exponential claims are being studied. So in this chapter, after the presentation and the introduction of the two expressions, a comparison will take place which should result in the equality of the two forms, in the case of exponential times and claims. We will, also, assume that $u = 0$, meaning that the initial surplus of the insurance company will be zero, something that does not hold normally in practice, see Section 2.3. Nevertheless, it gives interesting and useful results from a theoretical point of view.

In this chapter, in Section 4.1, the theoretical background for the derivation of the density of ruin time with exponential claims will be introduced and in the following section we will give the derivation of the formula. After the representation and the study of the formula of the density function for the classical model, some numerical examples will follow in Section 4.3 assuming $u > 0$. Then the distribution of ruin time for the case of the renewal model will follow.

So in Section 4.4 the theoretical background for the derivation of the distribution of ruin time for a Sparre- Andersen process with exponential claims will be introduced and in the following section it will be given the derivation of the formula followed by some numerical examples, executed in Mathematica, as were those in the classical model. These examples will help us to see the behavior of the distribution of ruin time for various values of its parameters. In section 4.7 of this chapter we will make a comparison between the two alternative expressions for the density of the time to ruin and we will show their equivalence for initial surplus zero and exponential claims and times. We have checked this equality numerically, too, in Mathematica. Finally, in the last two sections some plots will be given for the special case of initial surplus zero and exponential claims and times, which will show how $g(t)$ and $p(t)$ are affected by $u = 0$ and various values of the other parameters. We should note that if we assume exponentially distributed claims and times, then we have the classical model in both cases and we do not talk then, for the renewal model.

4.1 The theoretical background of the study of the density of ruin time with exponential claims

Beginning with the theoretical background of the Drekic and Willmot's paper, it worths to be mentioned that the main expression for the classical model with exponential claims is obtained directly by inversion of the associated Laplace transform and then is used for various parameter choices. In Chapter 2 the classical risk process was introduced. According to this, as already has been said, the number of claims process that occur in an insurance company is a Poisson process with rate λ. Apart from that, it was said that the individual claims are an iid sequence of positive random variables and premiums are paid continuously at rate c per unit time, where relation (2.6.4) holds. Closing the short review, it should be recalled that the surplus process was given with initial surplus u and was defined by relation (2.4.3). The time to ruin is a defective random variable $T = \inf\{t : U_t < 0\}$. The Laplace transform of the defective distribution of the time to ruin satisfies the Gerber- Shiu defective renewal equation and, as mentioned in Theorem

4.1.1 below, is the tail of a compound geometric distribution. That is,

$$\widehat{L}(\delta) = E\left\{e^{-\delta T}1(T<\infty)\right\} = \sum_{n=1}^{\infty}(1-\phi)\phi^n \overline{F_\delta}^{*n}(u),$$

$$\phi = \frac{\int_0^\infty e^{-\rho x}\overline{F}(x)dx}{(1+\theta)E(X)},$$

with $\rho = \rho(\delta)$ the unique non- negative root of Lundberg's equation

$$\lambda \int_0^\infty e^{-\rho x} dF(x) = \lambda + \delta - c\rho,$$

where $0 < \phi < 1$ and $\overline{F_\delta}^{*n}(u)$ is the tail of the n- fold convolution of the df $F_\delta(u) = 1 - \overline{F_\delta}(u)$ satisfying

$$\overline{F_\delta}(u) = \frac{\int_0^\infty e^{-\rho x}\overline{F}(u+x)dx}{\int_0^\infty e^{-\rho x}\overline{F}(x)dx},$$

with $\rho = \rho(\delta)$ the unique non-negative root of generalized Lundberg's equation and $\overline{F}$ the right tail of the density of claims, see Drekic and Willmot (2003) and $1(A)$ denotes the indicator function of the event A, which is equal to 1 if the event occurs and is 0 if the event does not occur.

Moreover, in the present context we mention, that if the negative surplus at the time of ruin is denoted by $|U_T|$ and $w(|U_T|)$ is a nonnegative function of the deficit (negative surplus), then the function $w(u)$ can be viewed as a penalty for the case when $|U_T| = u$, see Gerber- Shiu (1998). Then the expected discount penalty for $\delta \geq 0$ and for the indicator function 1(A) will be

$$m(u) = E\left\{e^{-\delta T}w(|U_T|)I(T<\infty)\right\},$$

where δ is the force of interest. The function $m(u)$ contains as special cases many important quantities in the classical risk model. For special cases where, for instance, $\delta = 0$ and $w(x) = 1$, one reproduces the probability of ruin. The function $m(u)$ can also be seen in terms of Laplace transforms with δ being the argument. Thus, it holds that $\widehat{L}_\delta(t) = E(e^{-\delta t}) = M_\delta(-t)$. It follows a theorem which depicts the connection between the associated compound geometric tail $\overline{G}_\delta(u) = \sum_{n=1}^{\infty}(1-\phi)\phi^n\overline{F_\delta}^{*n}(u),\ u \geq 0$ with a mass point to zero $G_\delta(0) = 1-\phi$ and a special expected discounted penalty when $w(x) = 1$.

Theorem 4.1.1. *Define for $\delta \geq 0$ the function*

$$\overline{G}_\delta(u) = E\left\{e^{-\delta T} I(T < \infty)\right\}.$$

Then $\overline{G}_\delta(u)$ is the compound geometric tail given by $\overline{G}_\delta(u) = \sum_{n=1}^{\infty}(1-\phi)\phi^n \overline{F_\delta}^{*n}(u)$ where $0 < \phi < 1$.

For a proof see Willmot and Lin (2001). In the special case of exponential claims, one has that

$$\widehat{L}(\delta) = \phi e^{-\mu(1-\phi)u}. \tag{4.1.1}$$

If the above Laplace transform is inverted directly, one gets the density function $p(t)$ satisfying $\widehat{L}(\delta) = \int_0^\infty e^{-\delta t} p(t) dt$. The probability density function $g(t)$ of the ruin time $T_c = T|T < \infty$, which means that is conditional to the event of ruin and is a proper random variable instead of the defective T, is then given if we invert the Laplace transform. So the density function of T_c will be

$$g(t) = \frac{p(t)}{\psi(u)}.$$

Since, we have assumed that the claims are exponentially distributed, (2.7.3) holds for the ruin probability and (2.7.2) for Ludberg's adjustment coefficient, where μ is the parameter of the exponential distribution.

4.2 Derivation of the density function of ruin time with exponential claims

In this section, we present the main steps for the derivation of the density function with exponential claims, as in Drekic and Willmot (2003). The goal is to invert the (4.1.1), so one has that

$$\widehat{L}(\delta) = \phi e^{-\mu(1-\phi)u},$$

which implies

$$\widehat{L}(\delta) = \phi e^{-\mu u} e^{\mu\phi u},$$

which yields that

$$\widehat{L}(\delta) = e^{-\mu u}(\phi e^{\mu\phi u})$$

and since

$$e^{\mu u}=\sum_{n=0}^{\infty}\frac{(\mu u)^n}{n!},$$

then it holds that

$$\widehat{L}(\delta)=e^{-\mu u}\sum_{n=0}^{\infty}\phi^{n+1}\frac{(\mu u)^n}{n!}, \tag{4.2.1}$$

because claims are exponentially distributed then it will hold that $\overline{F}(u) = e^{-\mu u}$, relation (3.3.3). Therefore, since (2.6.4) then $c\mu = \lambda(1+\theta)$. This permits to yield an explicit representation in terms of δ for ϕ which is

$$\phi=\frac{\lambda+\delta+c\mu-\sqrt{(\lambda+\delta+c\mu)^2-4\lambda c\mu}}{2c\mu}. \tag{4.2.2}$$

So if (4.2.2) is substituted in (4.2.1) then

$$\begin{aligned}\widehat{L}(\delta) &= e^{-\mu u}\sum_{n=0}^{\infty}\frac{(\mu u)^n}{n!}\left\{\frac{\lambda+\delta+c\mu-\sqrt{(\lambda+\delta+c\mu)^2-4\lambda c\mu}}{2c\mu}\right\}^{n+1}\\ &= \frac{e^{-\mu u}}{2c\mu}\sum_{n=0}^{\infty}\frac{(\frac{u}{2c})^n}{n!}\left\{s-\sqrt{s^2-\alpha^2}\right\}^{n-1},\end{aligned} \tag{4.2.3}$$

where $s=\lambda+\delta+c\mu$ and $\alpha=2\sqrt{\lambda c\mu}$. Taking inverse Laplace transform $p(t)=L^{-1}[\widehat{L}(\delta)]$ of $\widehat{L}(\delta)$ it yields $p(t)$ for $t>0$

$$p(t)=\frac{e^{-\mu u}}{2c\mu}\sum_{n=0}^{\infty}\frac{(\frac{u}{2c})^n}{n!}L^{-1}\left[\left\{s-\sqrt{s^2-\alpha^2}\right\}^{n-1}\right],$$

and after the use of transform results and the Bessel function of the first kind of order ν, see Drekic and Willmot (2003), one has that

$$p(t)=\frac{e^{-\mu u}e^{-\lambda(2+\theta)t}}{t\sqrt{1+\theta}}\sum_{n=0}^{\infty}\frac{(n+1)(\frac{\mu u}{\sqrt{1+\theta}})^n I_{n+1}(2\lambda t\sqrt{1+\theta})}{n!},\ t>0.$$

The definition of the Bessel function of the first kind is given by

$$I_\nu(x)=\sum_{k=0}^{\infty}\frac{(\frac{x}{2})^{2k+\nu}}{k!(k+\nu)!}, \tag{4.2.4}$$

then it is easy to obtain the density function of T_c. So one has that

$$g(t) = \frac{\sqrt{1+\theta}e^{\frac{-\mu u}{1+\theta}}e^{-\lambda(2+\theta)t}}{t}\sum_{n=0}^{\infty}\frac{(n+1)(\frac{\mu u}{\sqrt{1+\theta}})^n I_{n+1}(2\lambda t\sqrt{1+\theta})}{n!},\ t>0. \tag{4.2.5}$$

The relation (4.2.5) is the main expression for the density of ruin time with exponential claims. For the df $g(t)$ several plots will be given in the following section. Apart from that, an attempt will be made next, the above results to be compared with the results that Borovkov and Dickson produced for the special case that claim sizes are exponentially distributed, given that it folds

$$g(t) = p(t)/\psi(u),$$

where $g(t)$ is the df of T_c and $p(t)$ the df of T.

4.3 Numerical examples for the density function of T_c, with exponential claims

After the derivation of the formula for the ruin time density function, we show the behavior of the density function, which was given in (4.2.5), with some plots. Let us assume that $\mu = 1$ and $u > 0$ and examine the behavior of $g(t)$ for various values of its other parameters (λ, θ, u) and compare our results with those, Drekic and Willmot (2003) gave in their paper. Plots of $g(t)$ in the next six figures were generated using the computational package Mathematica.

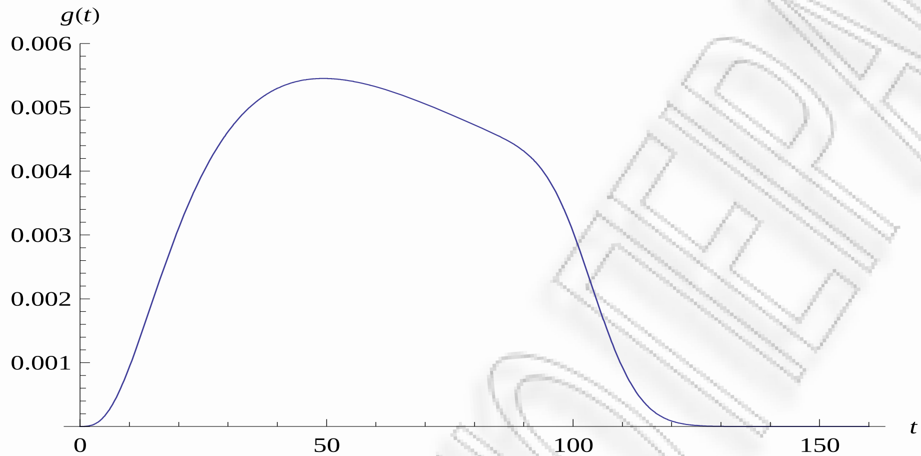

Figure 4.1: Density function with exponential claims $\mu = 1, \lambda = 1, \theta = 0.1, u = 20$.

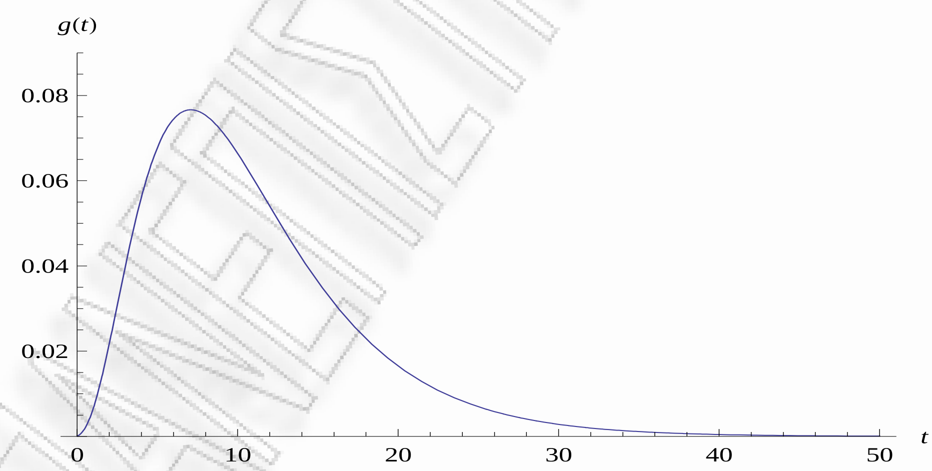

Figure 4.2: Density function with exponential claims $\mu = 1, \lambda = 1, \theta = 1, u = 20$.

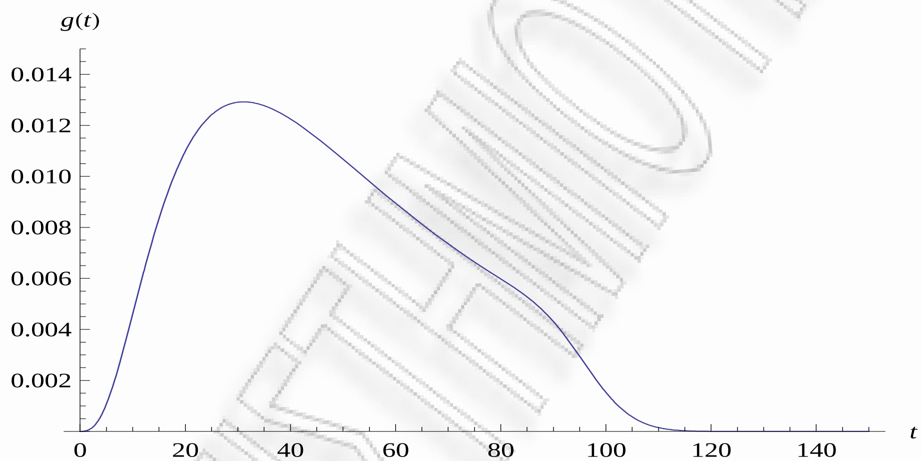

Figure 4.3: Density function with exponential claims $\mu = 1, \lambda = 1, \theta = 0.25, u = 20$.

In the above three figures we have tried to show how different values of the loading factor ($\theta = 0.1, 1, 0.25$, respectively) affect the behavior of the density function of ruin time, keeping all the other parameters the same. It has already been mentioned, in previous chapter, that θ can not be too big, because of the competition in the insurance market. Generally speaking, it is difficult to determine its value and this is obvious from the above figures, where changes in the value of the loading factor affect the behavior of the df a lot, resulting in quite different curves.

More specifically, Figures 4.1 and 4.3 are different to all those, which are presented here and those, in Drekic and Willmot's paper (2003). These two plots have both, roughly, local maximum at the same point, as we can see. They are increasing and then decreasing up to a point, where they become, again for a while, increasing until they decrease again. A remark is that, as θ, becomes bigger but still takes values close to zero, (0.1,0.25), the second point, where the monotonicity of $g(t)$ is changing is negligible. For the specific parameter values μ, λ, u, $g(t)$ is more bell- shaped, comparing to all the others.

We should also note that all the other plots are increasing up to a point and then decreasing, they do not have two points, where the monotonicity is changing, as Figures 4.1 and 4.3 did. A general remark is that, all the plots are skewed to the right, showing positive asymmetry. It would be very interesting the coefficient of skewness, for $g(t)$, to be studied in a future paper. We should also note, that Figures 4.2, 4.5 look like Figures A and D in Drekic and Willmot's paper (2003). In their paper, Drekic and Willmot make some assumptions for the monotonicity of the failure rate of the distribution of ruin time, using CV, since these two plots have both $CV < 1$, the coefficient of variation is given by

$$\frac{s}{\overline{x}},$$

where s is the standard deviation and $\overline{x}$ is the mean. Here we do not give values for the descriptive statistics of $g(t)$. However, in the next chapter we will examine the monotonicity of its failure rate, directly.

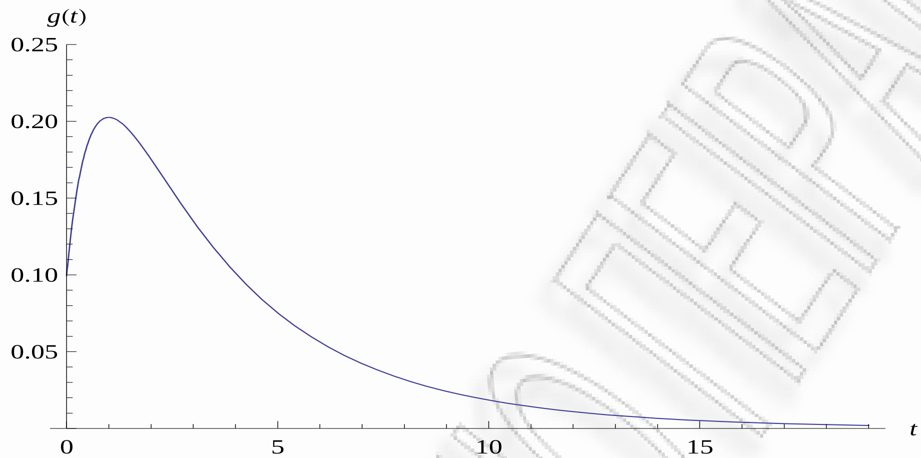

Figure 4.4: Density function with exponential claims $\mu = 1, \lambda = 1, \theta = 1, u = 6$.

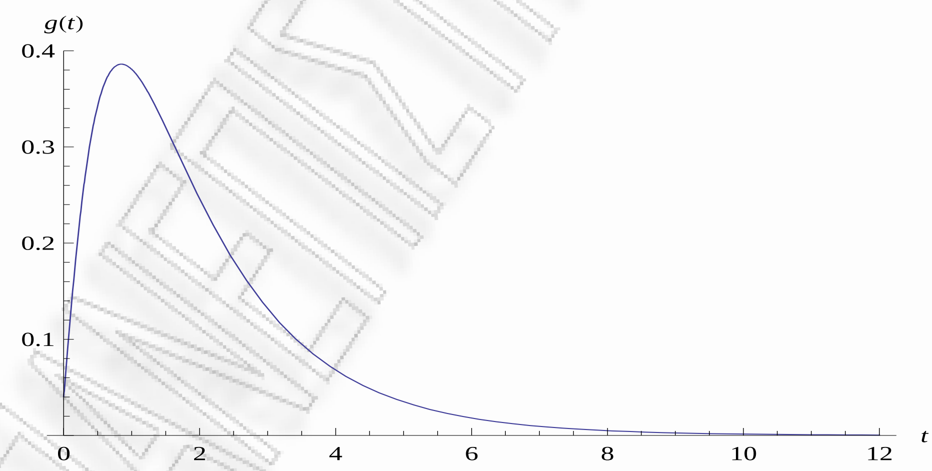

Figure 4.5: Density function with exponential claims $\mu = 1, \lambda = 3, \theta = 1, u = 10$.

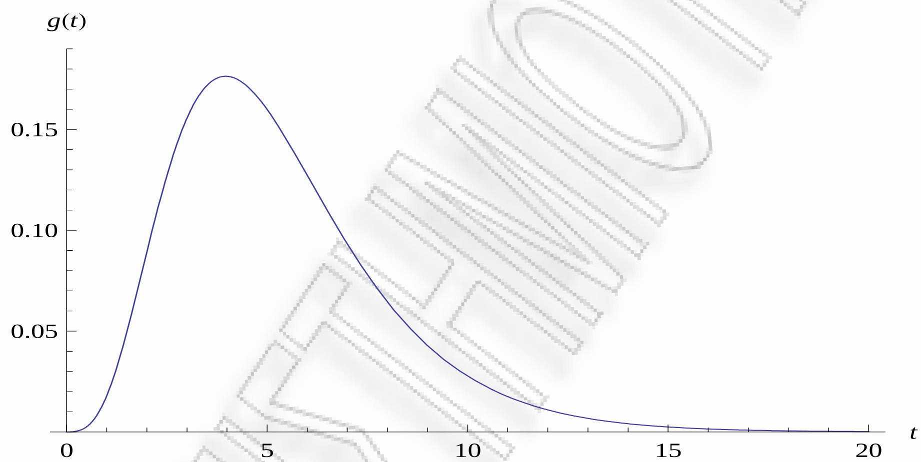

Figure 4.6: Density function with exponential claims $\mu = 1, \lambda = 3, \theta = 1, u = 30$.

Another parameter of great interest, as has been mentioned, too, is the initial surplus u. We can conclude that important role is played by the initial surplus. This is obvious if we compare Figures 4.1 and A in Drekic and Willmot's paper (2003), where the only parameter that do not have in common is u and yet their behavior is completely different. This is more obvious from the above three figures, where we have kept all the parameters the same and changed u, only.

All of the figures have positive asymmetry and Figures 4.4,4.5 seem to have $g(0) \neq 0$. In Drekic and Willmot's paper (2003), there, also, figures with $g(0) \neq 0$, in both of the cases, there and here, $\theta = 1$ and u takes various values (5,6,10). So we can conclude, that as the initial surplus decreases and $\theta = 1$, $g(0)$ takes values different to zero.

For the above three figures, we can note that $u = 6$ and then $u = 10$, $u = 30$, meaning that the initial surplus in Figure 4.4 is smaller than these in Figures 4.5 and 4.6, so there is a difference in the behavior of $g(t)$. The fact, that big changes in the value of the initial surplus affect the plot of $g(t)$, is more clear if we compare Figures B and C from the Drekic and Willmot's paper (2003). Figure B ($u = 5$) looks like Figure 4.4, where $u = 6$, but for $u = 2$ (Figure C) $g(t)$ has a completely different behavior, resembling the pdf of an exponential distribution with mean 1/6, see Drekic and Willmot (2003). We will try to examine empirically if there is a good fit of the exponential distribution to $g(t)$, in the next section for $u = 0$.

The table below shows some values of the density of T_c, when $u > 0$ and μ takes the values $1, 2, 3, 4$, respectively. These values have been found by computing the df using, again, Mathematica. We have assumed that $\lambda = 3, \theta = 0.1, u = 5$.

t	$\mu = 1, c = 3.3$	$\mu = 2, c = 6.6$	$\mu = 3, c = 9.9$	$\mu = 4, c = 13.2$
1	0.120491	0.0187466	0.0017428	0.000120968
2	0.101924	0.0336832	0.00623486	0.000817018
3	0.081014	0.0404721	0.0113201	0.00221474
4	0.0650363	0.0423473	0.0156947	0.00407913
5	0.101924	0.0417177	0.0189955	0.00611563
6	0.0444011	0.0399105	0.0212769	0.00810349
7	0.0376491	0.0376152	0.0227216	0.00991218
8	0.0323759	0.0351828	0.0235233	0.0114802
9	0.0281767	0.0327886	0.0238482	0.0127906
10	0.0247749	0.0305159	0.0238282	0.0138516
20	0.00968005	0.0158035	0.0178231	0.0159662
30	0.00475583	0.00896905	0.0119474	0.0129411
40	7.97506×10^{-6}	0.000038213	0.000114997	0.000256894
50	2.4405×10^{-13}	3.92107×10^{-12}	3.54325×10^{-11}	2.18623×10^{-10}
60	2.72809×10^{-24}	1.45583×10^{-22}	3.88902×10^{-12}	6.5059×10^{-20}
70	1.15449×10^{-27}	1.98976×10^{-35}	1.51822×10^{-12}	6.63227×10^{-35}
80	7.99007×10^{-52}	4.32004×10^{-50}	9.09709×10^{-12}	1.00037×10^{-45}
90	2.35794×10^{-69}	3.88994×10^{-66}	2.18935×10^{-12}	5.86009×10^{-61}
100	5.73557×10^{-87}	5.73557×10^{-87}	4.10819×10^{-12}	2.59606×10^{-77}

4.4 The theoretical background of the study of the ruin time distribution for a Sparre Andersen process with exponential claims

The derivation of the expression due to Borovkov and Dickson (2008) differs from the above. They did not try to invert the associated Laplace transform. Borovkov and Dickson based their derivation, for the ruin time distribution for a Sparre Andersen process with exponential claims, on transforming the original boundary crossing problem to an equivalent one on linear lower boundary crossing by a spectrally positive Lévy process. See Borovkov and Dickson (2008). Then, they used their main result in the cases of gamma, mixed exponential and inverse Gaussian inter- claim time distributions. But before entering deep inside to the main result, there is need to recall again some results from the previous chapters about the Sparre Andersen model. The renewal model (Sparre Andersen model) with surplus process

$$U(t) = u + ct - S(t)$$

is actually a generalization- as already has been said- of the classical one and can be formulated as a random walk, since here N(t) does not follow a Poisson process, but a renewal process, with again T_i which stand for the interclaim times, being independent and identically distributed (iid). This means, that T_i may not follow, in this case, the exponential distribution, but some other. In their paper, for instance, Borovkov and Dickson used the cases of gamma, mixed exponential and inverse Gaussian. To add to this, in the specific note, it was assumed that the claims were exponentially distributed. It worths also to recall that in order the income to be bigger than the expenses- something very logical- so that ruin is not certain, it has to hold

$$c > \frac{1}{E(T_i)} E(X),$$

where $E(T_i)$ is the mean of the random variable of the interclaim times and E(X) is the mean of the random variable of the severity of the claims, which here as it is exponential with parameter μ, will be

$$\frac{1}{\mu}.$$

So the safety loading θ will defined by:

$$\theta = \frac{cE(T_i)}{E(X)} - 1.$$

These are the most important elements of the renewal model, which are needed for the presentation of the formula of the ruin time distribution, which follows.

4.5 Derivation of the ruin time distribution for a Sparre Andersen process with exponential claims

Here, the goal is to present the derivation of a precise formula for the df of ruin time $T = \inf\{t > 0 : U(t) < 0\}$. The basic idea for the derivation of the examined expression was that one can express the probability of interest in terms of a linear lower boundary crossing probability for a compound Poisson process with positive jumps, for which there exists a well- known explicit formula. So, since X_i has been assumed that follows the exponential distribution with parameter $\mu > 0$, then one has for its right tail that

$$P(X_j > x) = e^{-\mu x}, \quad x \geq 0.$$

Borovkov and Dickson (2008) consider a delayed renewal risk model, where the time until the first claim, has density f_0, while subsequent interclaim times have a common density f. In Theorem 4.5.1 below, we give the main result as stated by Borovkov and Dickson (2008), although later on we focus on the case, where $f_0 = f$, which corresponds to the ordinary renewal risk model.

If the convolution of the functions g, h on $(0, \infty)$ is represented by $g * h$, then:

$$(g * h)(t) = \int_0^\infty g(t - v)h(v)dv,$$

and the n- fold convolution of g with itself is defined by $g^{*n} = g^{*n-1} * g, n \geq 2$.

Theorem 4.5.1. *Under the above assumptions, the ruin time T has a (defective) density $p(t)$ given by*

$$p(t) = e^{-\mu(u+ct)} \left\{ f_0(t) + \sum_{n=1}^{\infty} \frac{\mu^n (u + ct)^{n-1}}{n!} [u(f^{*n} * f_0)(t) + c(f^{*n} * f_1)(t)] \right\},$$

where $f_1(t) = tf_0(t)$. *In the special case, when* $f_0 = f$ *one has that*

$$p(t) = e^{-\mu(u+ct)} \sum_{n=0}^{\infty} \frac{\mu^n (u+ct)^{n-1}}{n!} * (u + \frac{ct}{n+1}) f^{*(n+1)}(t). \tag{4.5.1}$$

Here, (4.5.1) will be used and examined. A question that arises from the above is what holds for $p(t)$ and $g(t)$ for the two cases (Borovkov- Dickson and Drekic- Willmot), when the interclaims follow the exponential distribution.

The proof of (4.5.1) is based on the crossing of a linear boundary by the pure jump process and then the swapping of the roles of the time and space coordinates, which means that (v, u) will swap in the new 'time' $s = u - x$ and the new 'space' $y = t - v$, since t, x were representing the original time and space respectively. In the new system of coordinates, the $\{U^0(t)\}$ is still a pure jump step function. See Borovkov and Dickson (2008).

Because of the assumption that X_i follows the exponential with parameter $\mu > 0$ and the swapping of the coordinates (now the jumps of sizes are reflected by $T_1, T_2, ...$ and the times by $X_1, X_1 + X_2, ...$), one has that

$$Z^0(s) = \sum_{k \le M(s)} T_k$$

and that

$$P_{Z^0(s)}(y) = e^{-\mu s} \sum_{n=1}^{\infty} \frac{(\mu s)^n}{n!} f^{*n}(y), \; y > 0.$$

So, there exists a crossing of the lower linear boundary $x = -ct$ by the process $\{U^0(t)\}$ at time τ which corresponds to a continuous crossing of the lower linear boundary $y = s/c - (v + u/c), \; s > 0$ by the process $\{Z^0(s)\}$ at time $\sigma = u + c\tau$, so that $\tau = (\sigma - u)/c$.

After some observations for the crossing time of the lower level, the conditional density for the stopping time τ can be given and from the observation that always $\tau \ge T_0$ one has that

$$P(T \le t) = P(T_0 = T \le t) + P(T_0 < T \le t),$$

which after differentiating both sides and substituting the representation for the conditional density, yields the density of ruin time. By the use of the Laplace transform of the function g and setting $\phi(s) = \widehat{L}(s)$, one takes the relation (4.5.1).

Below some examples are given for the ruin time density. There are given two situations where the inter- claim times are gamma and mixed exponential respectively.

Example 4.5.1. *The Gamma Distribution*

If the occurred claims follow an ordinary renewal process then T_j will be a gamma density with

$$f(t) = f_0(t) = \frac{\beta^n t^{n-1} e^{-\beta t}}{\Gamma(n)}, \quad n, \beta > 0.$$

In this situation the $m+1$- fold convolution is

$$f^{*(m+1)}(t) = \frac{\beta^{n(m+1)} t^{n(m+1)-1} e^{-\beta t}}{\Gamma(n(m+1))}$$

and then formula (4.5.1) *gives*

$$p(t) = (\beta t)^{n-1} \frac{u\beta e^{-\lambda(u+ct)-\beta t}}{u+ct} \sum_{m=0}^{\infty} \frac{\lambda^m (u+ct)^m}{m!} \frac{(\beta t)^{nm}}{\Gamma(n(m+1)}$$

$$+(\beta t)^n \frac{cne^{-\lambda(u+ct)-\beta t}}{u+ct} \sum_{m=0}^{\infty} \frac{\lambda^m (u+ct)^m}{m!} \frac{(\beta t)^n m}{\Gamma(n(m+1)+1}.$$

Example 4.5.2. *Mixed exponential*

If the T_j is mixture of two exponentials with

$$f(t) = f_0(t) = p\alpha e^{-\alpha t} + q\beta e^{-\beta t}, \quad p+q=1, \ \beta > \alpha > 0,$$

then the m- fold convolution of f with itself will be

$$f^{*m}(t) = \sum_{j=0}^{\infty} q^m (1 - \frac{\alpha}{\beta})^j \sum_{r=0}^{m} \binom{m}{r} \frac{(r)_j}{j!} \left(\frac{\alpha p}{\beta q}\right)^r e(m+j, \beta; t),$$

where $e(m+j, \beta; t)$ denotes the Erlang(m) density with scale parameter β. See Borovkov and Dickson (2008). So if we replace the above m- fold convolution, we can take the corresponding formula from (4.5.1).

4.6 Numerical examples for the density function of T, with exponential claims

Here, we will give some plots for the defective density $p(t)$, with exponential claims, which was given in (4.5.1). Since we have assumed, exponentially distributed claims and times, we take the classical model, instead of the Sparre Andersen process with exponential claims. This will help us to compare the two expressions, in the following section. Let us assume, that $\mu = 1$, $u > 0$ and $\lambda = 6$ and examine the behavior of the density $p(t)$ for various values of its other parameters (θ, u). The following plots were, also, generated using Mathematica.

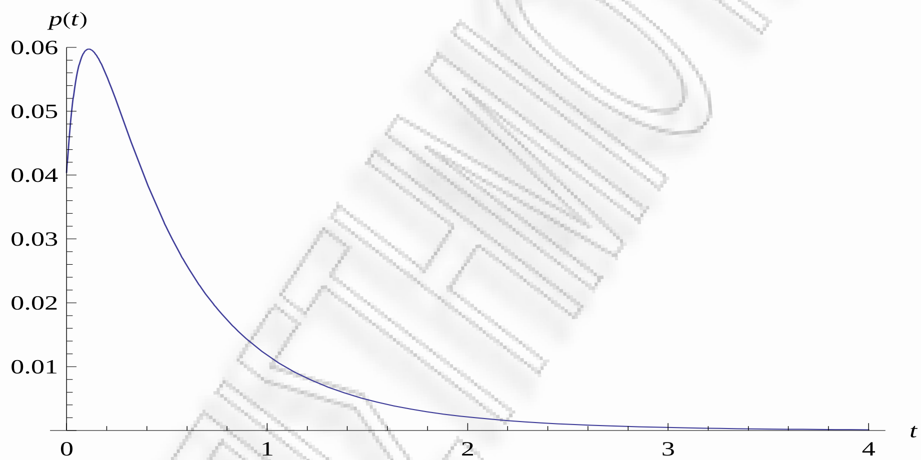

Figure 4.7: Density of ruin time for the classical model with exponential claims and $\mu = 1, \lambda = 6, c = 12, u = 5, \theta = 1$.

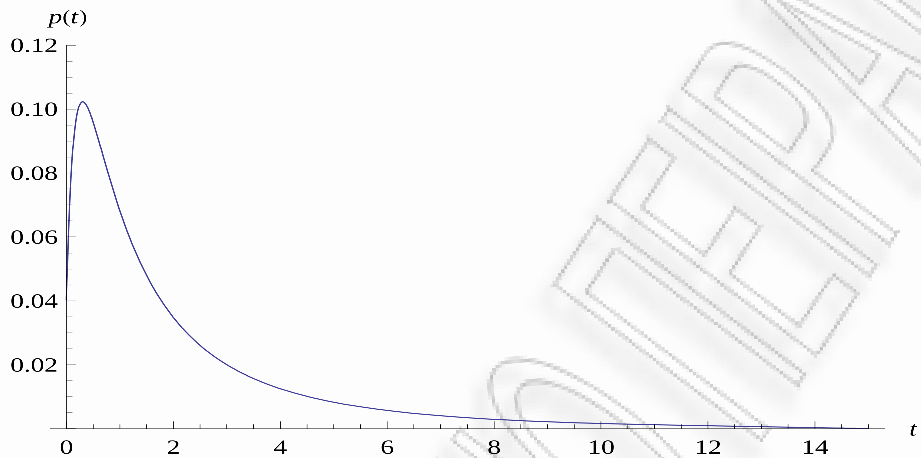

Figure 4.8: Density of ruin time for the classical model with exponential claims and $\mu = 1, \lambda = 6, c = 8, u = 5, \theta = 1/3$.

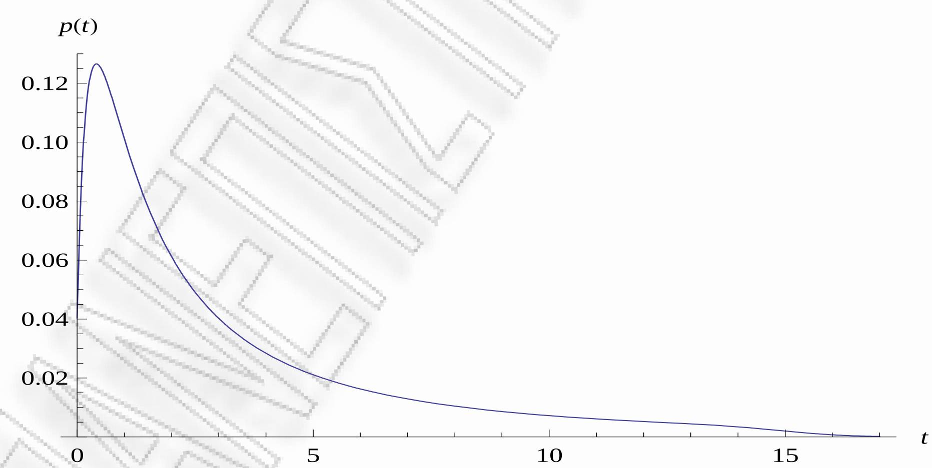

Figure 4.9: Density of ruin time for the classical model with exponential claims and $\mu = 1, \lambda = 6, c = 7, u = 5, \theta = 1/6$.

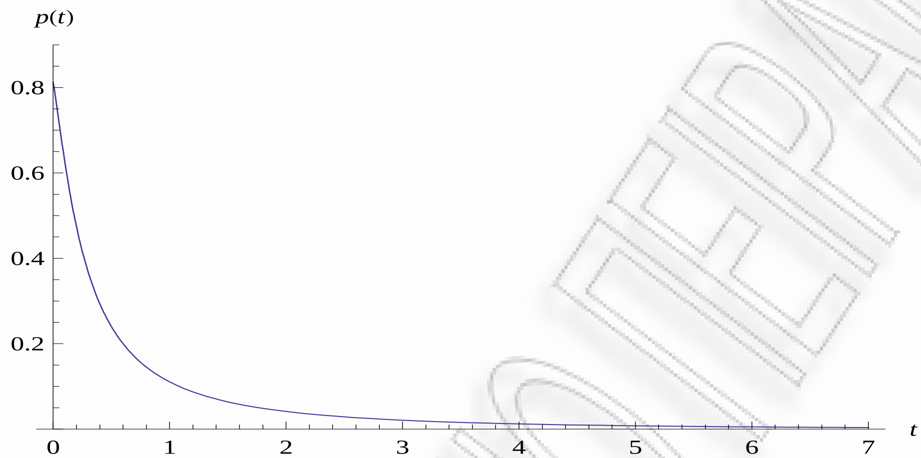

Figure 4.10: Density of ruin time for the classical model with exponential claims and $\mu = 1, \lambda = 6, c = 8, u = 2, \theta = 1/3$.

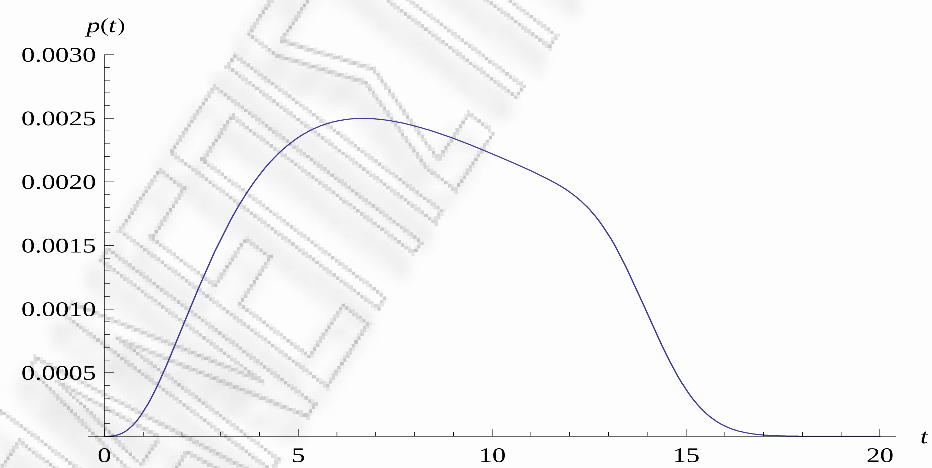

Figure 4.11: Density of ruin time for the classical model with exponential claims and $\mu = 1, \lambda = 6, c = 7, u = 20, \theta = 1/6$.

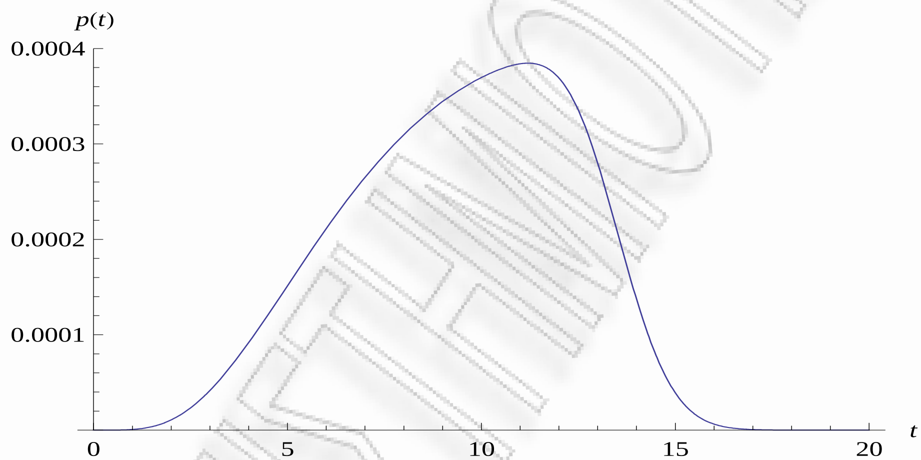

Figure 4.12: Density of ruin time for the classical model with exponential claims and $\mu = 1, \lambda = 6, c = 7, u = 30, \theta = 1/6$.

Firstly, we should remark that the three plots, 4.7, 4.8, 4.9 have the same behavior as far as $p(t)$ is concerned. We can see that they are increasing up to a point and then become decreasing. They all characterized by positive asymmetry and by the fact that $p(0) \neq 0$, just like Figure 4.4, where the density function with exponential claims was depicted for the proper random variable of ruin time and we had assumed that $\mu = 1, \lambda = 1, \theta = 1, u = 6$. Whereas in Figures 4.7, 4.8, 4.9 we have assumed $u = 5$ and $\theta = 1, 1/3, 1/6$, respectively.

Figure 4.10, where $\theta = 1/3, u = 2$, is decreasing and resembles to the exponential distribution and consequently Figure C, see Drekic and Willmot (2003). This is, because of the fact, that the initial surplus takes very small value.

In Figure 4.11, we can see that is the only case, that the monotonicity changes twice and is more bell- shaped, as we noted in Figure 4.1. In both of them, we assumed big initial surplus. In the beginning $p(t)$ is increasing up to a point, where it becomes decreasing. Then becomes again increasing and at the end, again, decreasing.

In the last figure, the density of the defective ruin time is increasing and then becomes decreasing. This figure is more bell- shaped. This might be, because of the assumed big initial surplus.

Closing, we can make some notes for the above plots, in general. So, a general conclusion from the two last plots, it seems to be that, for very big values of u and c (so that the safety loading factor θ is close to zero), $p(t)$ is more bell- shaped and in some cases its monotonicity changes twice, depending on the values of its other parameters. To add to this, when the initial surplus takes very small values and c takes big ones (so that the safety loading factor θ is close to zero), the density seems to look like the exponential distribution. When u is small or θ is not close to zero, the density exhibits a positive skewness of the underlying distribution.

The table below shows some values of the distribution of T, when $u > 0$ and μ takes the values $1, 2, 3, 4$, respectively. These values have been found using again Mathematica. We have assumed that $u = 20, \lambda = 2$.

t	$\mu = 1, \theta = 0.5$	$\mu = 2, \theta = 1$	$\mu = 3, \theta = 0.5$	$\mu = 4, \theta = 1$
1	2.09914×10^{-6}	2.52295×10^{-13}	5.37341×10^{-20}	1.22056×10^{-27}
2	9.49089×10^{-6}	3.64531×10^{-12}	6.6258×10^{-18}	2.61952×10^{-25}
3	0.0000201132	0.0000201132	0.0000201132	0.0000201132
4	0.000030786	3.47958×10^{-11}	1.22864×10^{-14}	7.23443×10^{-23}
5	0.0000395591	5.83426×10^{-11}	6.50007×10^{-15}	4.01398×10^{-22}
6	0.0000457298	7.95857×10^{-11}	2.43764×10^{-14}	1.49406×10^{-21}
7	0.0000493448	9.43184×10^{-11}	7.15142×10^{-14}	4.18259×10^{-21}
8	0.0000507896	1.01067×10^{-10}	1.74652×10^{-13}	9.46456×10^{-21}
9	0.0000505474	1.00492×10^{-10}	3.70198×10^{-13}	1.81699×10^{-20}
10	0.0000490797	9.4366×10^{-11}	7.01348×10^{-13}	3.06192×10^{-20}
20	0.0000206718	1.14076×10^{-11}	1.68722×10^{-11}	1.35661×10^{-19}
30	6.51091×10^{-6}	3.50638×10^{-13}	3.00235×10^{-11}	4.34983×10^{-21}
40	4.85363×10^{-7}	7.84442×10^{-18}	1.96303×10^{-13}	3.6196×10^{-27}
50	4.1549×10^{-11}	1.56644×10^{-26}	4.92636×10^{-19}	5.02175×10^{-27}
60	4.733117×10^{-18}	4.06578×10^{-38}	3.53067×10^{-27}	1.75887×10^{-49}
70	3.98008×10^{-27}	8.88081×10^{-52}	3.49505×10^{-27}	8.20316×10^{-64}
80	8.32127×10^{-38}	5.389×10^{-67}	1.3443×10^{-48}	1.46397×10^{-79}
90	9.88059×10^{-50}	2.01985×10^{-83}	4.05319×10^{-61}	2.03234×10^{-96}
100	1.18988×10^{-62}	8.17951×10^{-101}	1.57379×10^{-74}	3.61753×10^{-114}

4.7 Equivalence of the two formulas for the ruin density with $u = 0$ and exponential claims and times

In the previous sections, we were mentioned to two papers, that have as subject the study of the density of T and T_c, in two different cases. The first case was that of the classical model of ruin theory and the second was that of the Sparre Andersen process with exponential claims. A very logical assumption is that these two density functions should be equivalent in the special case of exponentially distributed times and claim sizes. In order to examine the validity of this assumption it would be right to study the ruin time distribution for a Sparre Andersen process with u=0 and exponential claims and times and then compare the result with the one that Drekic and Willmot derived in their paper.

So let us first consider the situation when the distribution of claims is exponential with density function

$$\mu e^{-\mu x}, \ \mu > 0$$

and consequently (4.5.1) holds. Let us also assume that the initial surplus of the insurance company is $u = 0$. Then one has that

$$p(t) = e^{-\mu(0+ct)} \sum_{n=0}^{\infty} \frac{\mu^n (0+ct)^{n-1}}{n!} (0 + \frac{ct}{n+1}) f^{*(n+1)}(t),$$

which gives that

$$p(t) = e^{-\mu 0} e^{-\mu ct} \sum_{n=0}^{\infty} \frac{\mu^n (0+ct)^{n-1}}{n!} 0 f^{*(n+1)}(t) + e^{-\mu 0} e^{-\mu ct} \sum_{n=0}^{\infty} \frac{\mu^n (0+ct)^{n-1}}{n!} \frac{ct}{n+1} f^{*(n+1)}(t)$$

and since the first part is zero we have that

$$p(t) = 1 \cdot e^{-\mu ct} \sum_{n=0}^{\infty} \frac{\mu^n (0+ct)^{n-1}}{n!} \frac{ct}{n+1} f^{*(n+1)}(t)$$

and finally

$$p(t) = e^{-\mu ct} \sum_{n=0}^{\infty} \frac{(\mu ct)^n}{n!(n+1)} f^{*(n+1)}(t),$$

where since it is assumed that the distribution of each $T_j, j = 0, 1, 2, ...$ is exponential then the $n+1$- fold convolution of f with itself can be given. Generally speaking for the n- fold convolution for continuous random variables, one has that

$$f^{*n}(t) = \int_0^t f(y) f^{*n-1}(t-y)dy, \; n \geq 1$$

and it holds that $f^{*1}(t) = p(t)$ and $F^{*1}(t) = P(t)$.

For the specific case of exponentially distributed random variables with parameter λ, it holds that for $n = 2$

$$f^{*2}(t) = \int_0^t f(y) f^{*n-1}(t-y)dy = \int_0^t \lambda e^{-\lambda t} \lambda e^{-\lambda(t-y)} dy$$

which yields

$$f^{*2(t)} = \lambda^2 t e^{-\lambda t},$$

which is a gamma density with parameters $n = 2$ and λ.

For $n = 3$, one has that

$$f^{*3}(t) = \int_0^t f^{*2}(y) f(t-y)dy = \int_0^t \lambda^2 y e^{-\lambda y} \lambda e^{-\lambda(t-y)} dy$$

which gives

$$f^{*3}(t) = \lambda^3 t e^{-\lambda t} \int_0^t y dy = \frac{\lambda^3}{2} t^2 e^{-\lambda t},$$

which is a gamma density with parameters $n = 3$ and λ.

So for the $n+1$- fold convolution one has that

$$f^{*n+1}(t) = \frac{\lambda^{n+1}}{\Gamma(n+1)} t^n e^{-\lambda t},$$

which is a gamma density with parameters n+1 and λ.

So formula (4.5.1) gives

$$p(t) = e^{-\mu ct} \sum_{n=0}^{\infty} \frac{(\mu ct)^n}{n!(n+1)} \frac{\lambda^{n+1}}{\Gamma(n+1)} t^n e^{-\lambda t} = e^{-\mu ct} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\mu c)^n \lambda^{n+1} t^{2n}}{n!(n+1)!},$$

which is the same as

$$p(t) = e^{-(\mu c + \lambda)t} \sum_{n=0}^{\infty} \frac{(\mu c)^n \lambda^{n+1} t^{2n}}{n!(n+1)!}, \tag{4.7.1}$$

because of the property of Gamma distribution for which it folds that

$$\Gamma(n+1) = n\Gamma(n) = n!.$$

In the previous sections the formula for the pdf of the time of ruin with exponential claims was introduced in relation (4.2.5). The question is how this formula becomes for the case of initial surplus $u = 0$. So it is given that

$$g(t) = \frac{\sqrt{1+\theta}e^{\frac{-\mu 0}{1+\theta}}e^{-\lambda(2+\theta)t}}{t}\sum_{n=0}^{\infty}\frac{(n+1)(\frac{\mu 0}{\sqrt{1+\theta}})^n I_{n+1}(2\lambda t\sqrt{1+\theta})}{n!},\ t>0,$$

which is the same as

$$g(t) = \frac{\sqrt{1+\theta}e^{-\lambda(2+\theta)t}}{t}\sum_{n=0}^{\infty}\frac{(n+1)(\frac{0}{\sqrt{1+\theta}})^n I_{n+1}(2\lambda t\sqrt{1+\theta})}{n!},\ t>0$$

and then

$$g(t) = \frac{\sqrt{1+\theta}e^{-\lambda(2+\theta)t}}{t}I_1(2\lambda t\sqrt{1+\theta}),\ t>0$$

and for $u = 0$ it holds that

$$\sum_{n=0}^{\infty}(\frac{\mu u}{\sqrt{1+\theta}})^n = \sum_{n=0}^{\infty}0^n = 0^0 = 1,$$

where since the definition of the Bessel function of the first kind has been given in (4.2.4), then

$$I_1(2\lambda t\sqrt{1+\theta}) = \sum_{n=0}^{\infty}\frac{(\frac{2\lambda t\sqrt{1+\theta}}{2})^{2n+1}}{n!(n+1)!},$$

so we have that

$$g(t) = \frac{\sqrt{1+\theta}e^{-\lambda(2+\theta)t}}{t}\sum_{n=0}^{\infty}\frac{(\lambda t\sqrt{1+\theta})^{2n+1}}{n!(n+1)!},\ t>0. \tag{4.7.2}$$

So now we have to prove that the above special cases for zero initial surplus are equivalent. We know that $c = (1+\theta)\lambda E(X)$ and we have supposed that claims are exponential so $E(X) = 1/\mu$, which means that

$$c = \frac{(1+\theta)\lambda}{\mu}.$$

So if we put this in (4.7.1), then we will have

$$p(t) = e^{-(\mu\frac{(1+\theta)\lambda}{\mu}+\lambda)t}\sum_{n=0}^{\infty}\frac{(\mu\frac{(1+\theta)\lambda}{\mu})^n\lambda^{n+1}t^{2n}}{n!(n+1)!}$$

which is equivalent to

$$p(t) = e^{-((1+\theta)\lambda+\lambda)t}\sum_{n=0}^{\infty}\frac{((1+\theta)\lambda)^n\lambda^{n+1}t^{2n}}{n!(n+1)!}$$

and

$$p(t) = e^{-((1+\theta)+1)\lambda t}\sum_{n=0}^{\infty}\frac{(1+\theta)^n\lambda^{2n+1}t^{2n}}{n!(n+1)!},$$

which at last gives that

$$p(t) = e^{-(2+\theta)\lambda t}\sum_{n=0}^{\infty}\frac{(1+\theta)^n\lambda^{2n+1}t^{2n}}{n!(n+1)!}.$$

Now if we recall that

$$g(t) = \frac{p(t)}{\psi(0)} = \frac{p(t)}{\frac{1}{1+\theta}} = p(t)(1+\theta),$$

and substitute (4.7.1) in this, then one has that

$$g(t) = (1+\theta)e^{-\lambda(2+\theta)t}\sum_{n=0}^{\infty}\frac{(1+\theta)^n\lambda^{2n+1}t^{2n}}{n!(n+1)!}.$$

So if we substitute in the above relation, (4.7.2), we have to prove that

$$\frac{\sqrt{1+\theta}e^{-\lambda(2+\theta)t}}{t}\sum_{n=0}^{\infty}\frac{(\lambda t\sqrt{1+\theta})^{2n+1}}{n!(n+1)!} = (1+\theta)e^{-\lambda(2+\theta)t}\sum_{n=0}^{\infty}\frac{(1+\theta)^n\lambda^{2n+1}t^{2n}}{n!(n+1)!}$$

and

$$\frac{\sqrt{1+\theta}}{t}\sum_{n=0}^{\infty}\frac{(\lambda t\sqrt{1+\theta})^{2n+1}}{n!(n+1)!} = (1+\theta)\sum_{n=0}^{\infty}\frac{(1+\theta)^n\lambda^{2n+1}t^{2n}}{n!(n+1)!}$$

and

$$\frac{\sqrt{1+\theta}}{t}\sum_{n=0}^{\infty}\frac{(\lambda t)^{2n+1}(1+\theta)^{n+1/2}}{n!(n+1)!} = (1+\theta)\sum_{n=0}^{\infty}\frac{(1+\theta)^n\lambda^{2n+1}t^{2n}}{n!(n+1)!},$$

which gives

$$\frac{\sqrt{1+\theta}}{t}\lambda t(1+\theta)^{1/2}\sum_{n=0}^{\infty}\frac{(\lambda t)^{2n}(1+\theta)^n}{n!(n+1)!} = (1+\theta)\lambda\sum_{n=0}^{\infty}\frac{(1+\theta)^n\lambda^{2n}t^{2n}}{n!(n+1)!}$$

and

$$(1+\theta)\lambda\sum_{n=0}^{\infty}\frac{(\lambda t)^{2n}(1+\theta)^n}{n!(n+1)!} = (1+\theta)\lambda\sum_{n=0}^{\infty}\frac{(1+\theta)^n\lambda^{2n}t^{2n}}{n!(n+1)!},$$

so we have proved that the two formulas are equivalent under the assumptions of exponentially distributed claims and times and initial surplus zero.

4.8 Numerical examples for the ruin time density function of T_c, with $u = 0$ and exponential claims and times

In this section, some plots are given for the ruin time density function under the assumptions of $u = 0$ and exponentially distributed claims and times. As we did in previous sections we will examine the behavior of $g(t)$ under the specific assumptions for various values of its other parameters. First of all, we can see that $g(t)$ is decreasing in all the examined cases and differs a lot from the case of $u > 0$, where in most of the cases was increasing up to a point and then decreasing. This could be expected for the case where $u = 0$, since we have already noted in Section 4.3, that u affects the behavior of the df of ruin time, a lot. It remains a question how will behave its failure rate under the same assumptions. If $g(t)$ is decreasing, here, in all cases, what will be $r(t)$, will be answered in the next chapter.

We can also note that Figures 4.13,4.16,4.18 have all the same value for θ. We should also remark that all of the figures resemble to Figure C, see Drekic and Willmot (2003), where $u = 2$, a quite small value, which, as Drekic and Willmot note in their paper, resembles the pdf of an $Exp(6)$. For this reason we have added three plots, Figures 4.14,4.17,4.19, where in each plot we depict $g(t)$ and the exponential distribution with $\lambda = 1, 0.5, 0.5$, respectively. The parameters of the exponential distribution were selected empirically. In all these three cases, the exponential distribution seems to offer a reasonable satisfactory fit to the density of the time to ruin.

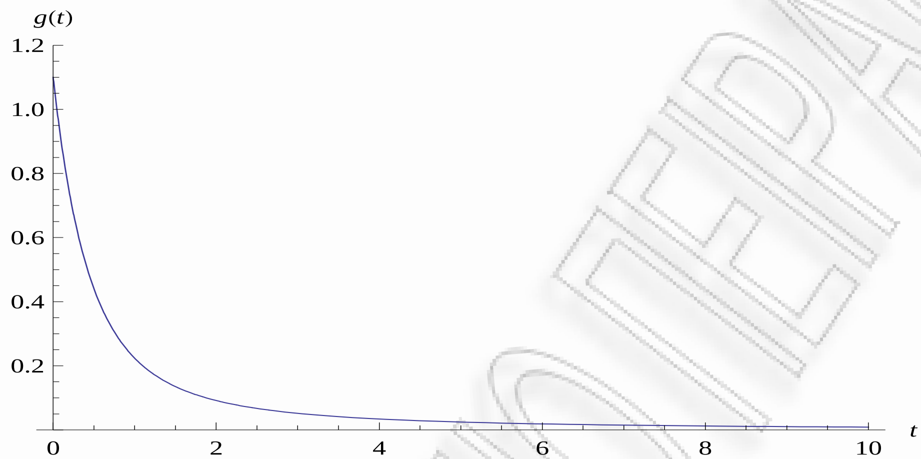

Figure 4.13: Density function with exponential claims $\lambda = 1, \theta = 0.1$.

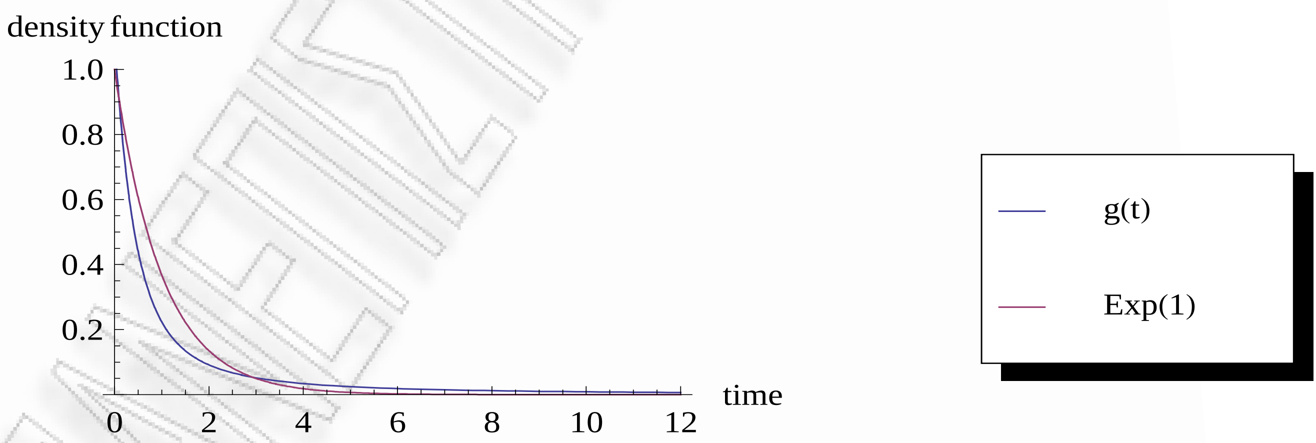

Figure 4.14: Density function with exponential claims $\lambda = 1, \theta = 0.1$ and the pdf of the exponential distribution with parameter $\lambda = 1$.

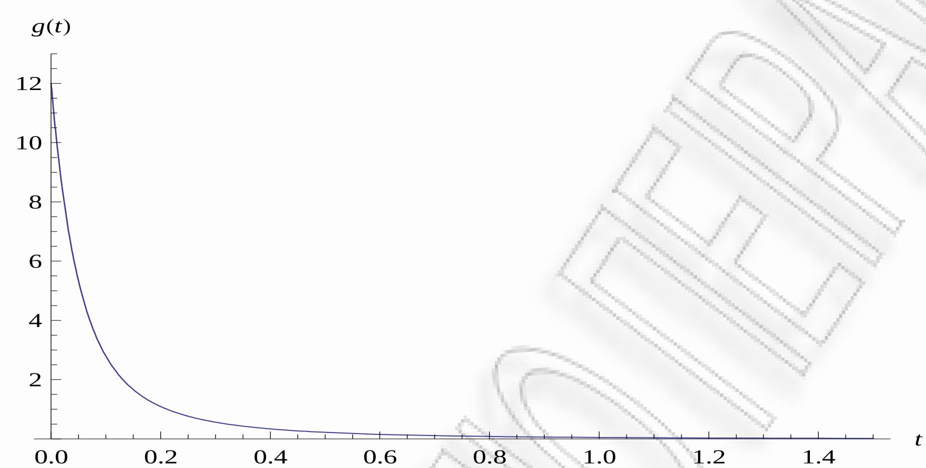

Figure 4.15: Density function with exponential claims $\lambda = 6, \theta = 1$.

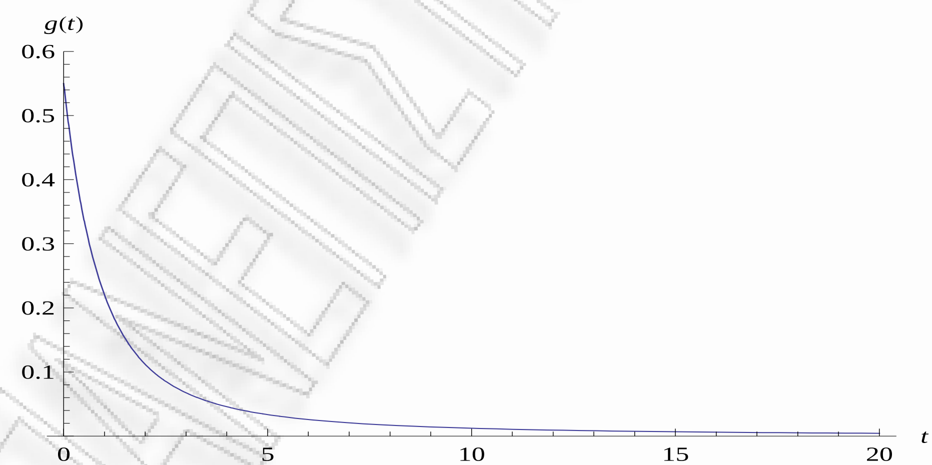

Figure 4.16: Density function with exponential claims $\lambda = 0.5, \theta = 0.1$.

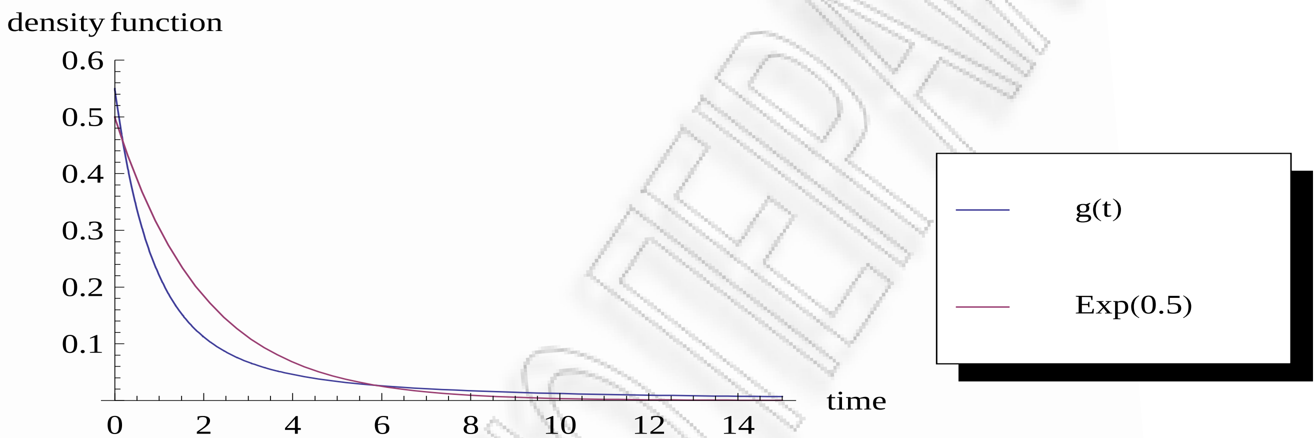

Figure 4.17: Density function with exponential claims $\lambda = 0.5, \theta = 0.1$ and the pdf of the exponential distribution with parameter $\lambda = 0.5$.

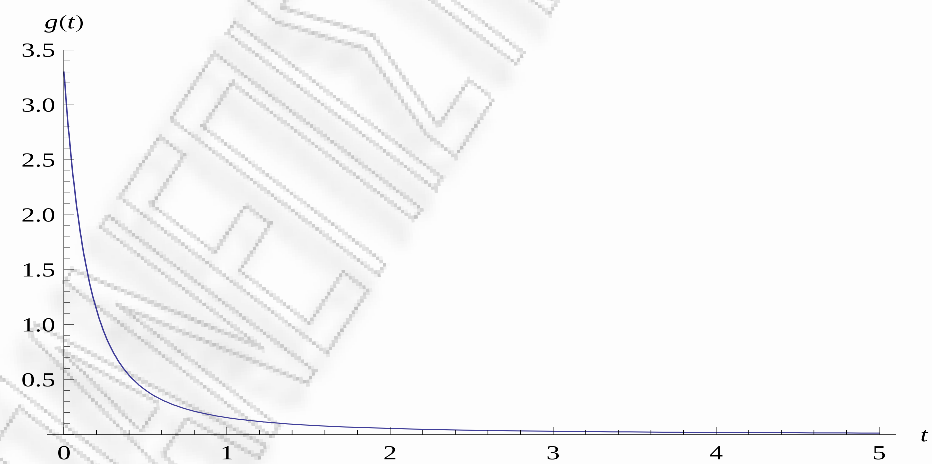

Figure 4.18: Density function with exponential claims $\lambda = 3, \theta = 0.1$.

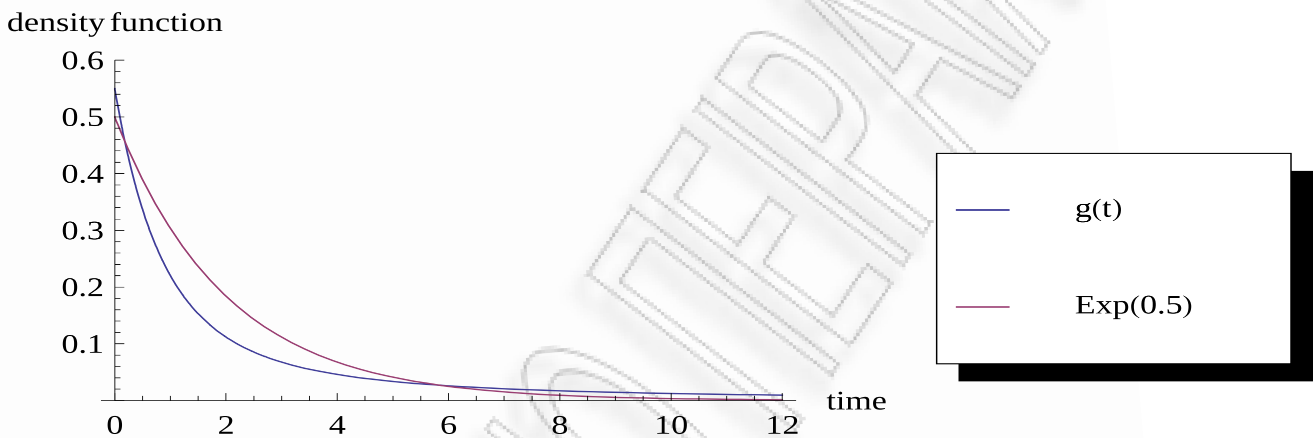

Figure 4.19: Density function with exponential claims $\lambda = 3, \theta = 0.1$ and the pdf of the exponential distribution with parameter $\lambda = 0.5$.

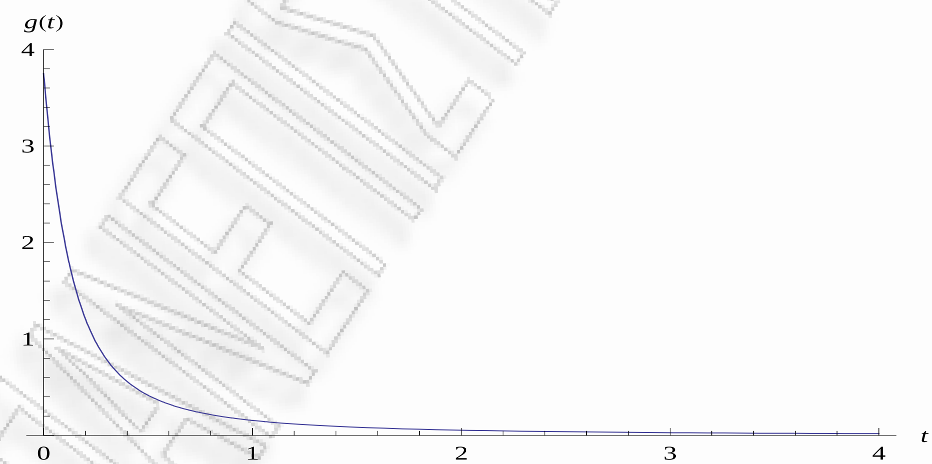

Figure 4.20: Density function with exponential claims $\lambda = 3, \theta = 0.25$.

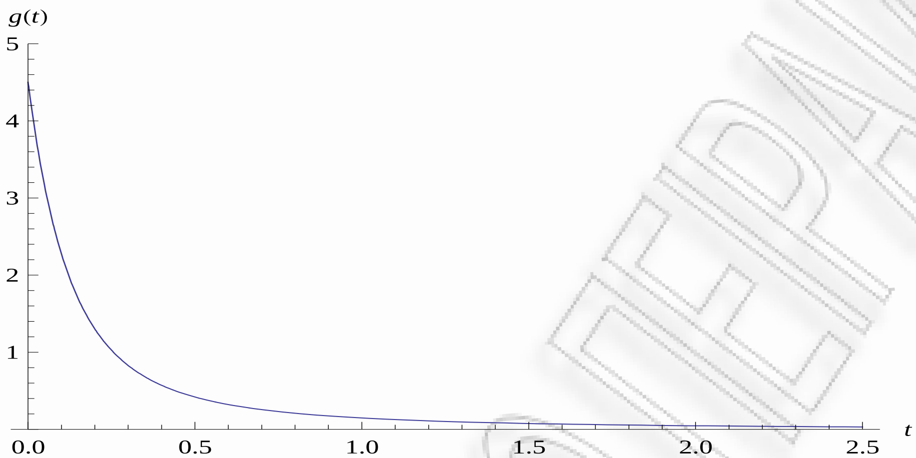

Figure 4.21: Density function with exponential claims $\lambda = 3, \theta = 0.5$.

The table below shows some values of the density of T_c, when $u = 0$, λ takes the values $1, 2, 3, 4$, respectively and $\mu = 1, \theta = 0.1$. These values have been found by computing the df using, again, Mathematica.

t	$\lambda = 1, c = 1.1$	$\lambda = 2, c = 2.2$	$\lambda = 3, c = 3.3$	$\lambda = 4, c = 4.4$
1	0.223687	0.183303	0.155156	0.136415
2	0.0916517	0.0682077	0.0563554	0.0489644
3	0.0517187	0.0375703	0.0307838	0.026608
4	0.0341038	0.0244822	0.019956	0.0171847
5	0.024591	0.0175174	0.0142216	0.0122078
6	0.0187851	0.013304	0.0107637	0.00921272
7	0.0149403	0.0105306	0.00849314	0.00724936
8	0.0122411	0.00859236	0.00690954	0.00588201
9	0.0102613	0.00717579	0.00575417	0.00488576
10	0.0087587	0.00610389	0.0048813	0.00413405
20	0.00305194	0.00206702	0.00161161	0.00132567
30	0.0016271	0.00107441	0.000687635	4.47959×10^{-7}
40	0.00103351	0.000662837	3.35969×10^{-7}	8.507×10^{-19}
50	0.000722765	2.23979×10^{-7}	2.25324×10^{-15}	5.66517×10^{-36}
60	0.000537205	2.23181×10^{-12}	5.59819×10^{-27}	1.1785×10^{-56}
70	0.000416444	4.2535×10^{-19}	5.42311×10^{-41}	9.09471×10^{-80}
80	0.000331418	3.73213×10^{-27}	8.83876×10^{-57}	1.15668×10^{-104}
90	0.000229212	$0.0000623994 \times 10^{-9}$	6.30264×10^{-74}	6.36982×10^{-131}
100	0.000070663	2.83259×10^{-36}	3.79144×10^{-92}	2.94212×10^{-158}

4.9 Numerical examples for the ruin time density function of T, with $u = 0$ and exponential claims and times

Closing this chapter, we should also show the behavior of the defective ruin time density function with $u = 0$ and exponential claims and times, that is the classical model. So we give some plots for the defective density $p(t)$, which was given in relation (4.7.1), assuming $u = 0$. Let us assume that $\mu = 4$ in the first three plots and $\mu = 1$ in the following three. So we can examine the behavior of the ruin time distribution changing the values of the other parameters that affect it.

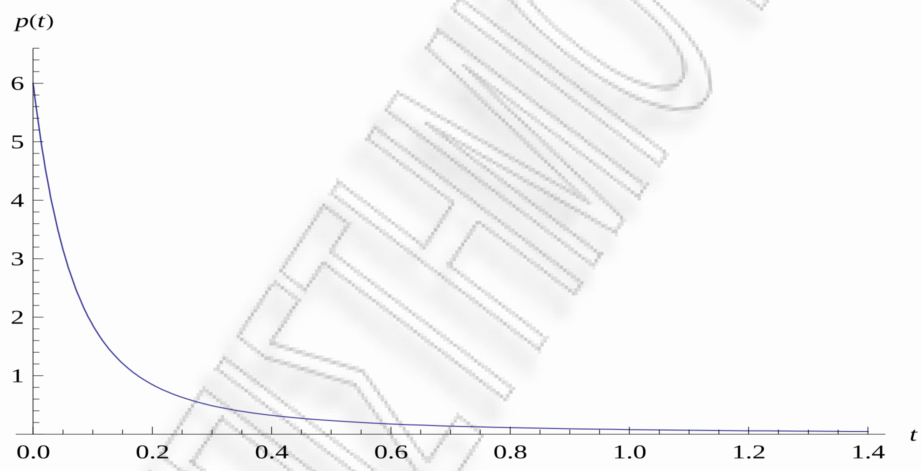

Figure 4.22: Ruin time density with u=0 and exponential claims and times for the classical model and $\mu = 4, c = 2, \lambda = 6, \theta = 1/3$.

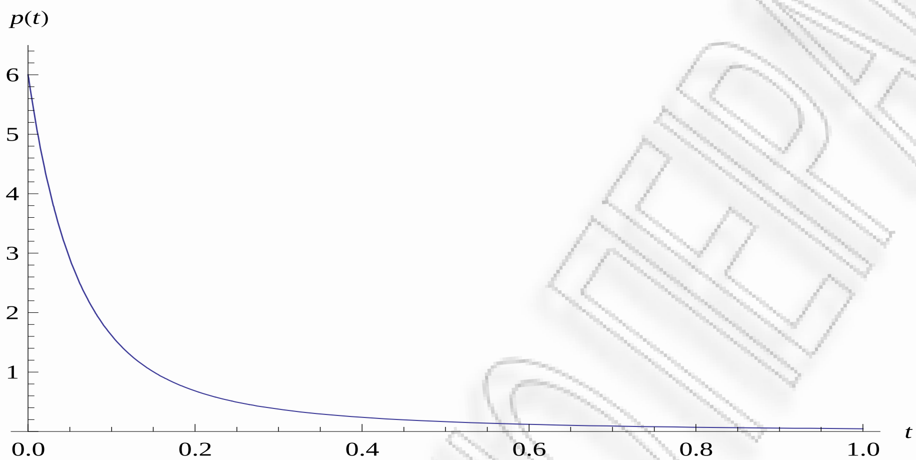

Figure 4.23: Ruin time density with u=0 and exponential claims and times for the classical model and $\mu = 4, c = 2.5, \lambda = 6, \theta = 0.666667$.

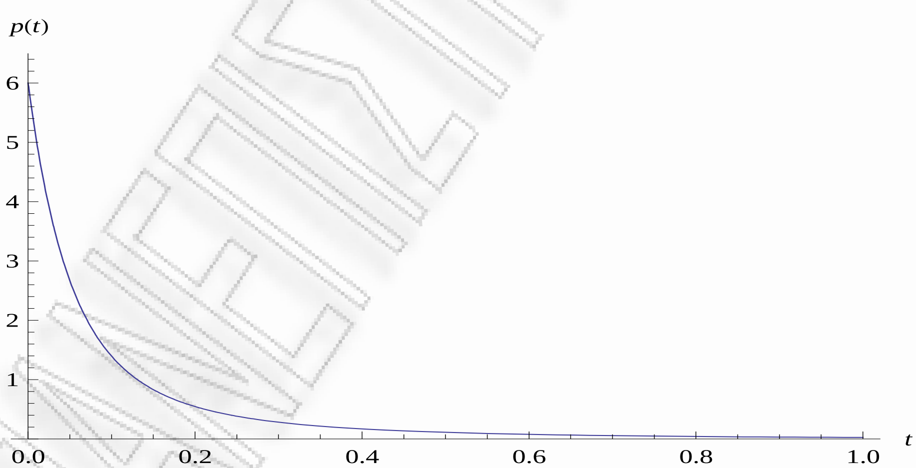

Figure 4.24: Ruin time density with u=0 and exponential claims and times for the classical model and $\mu = 4, c = 3, \lambda = 6, \theta = 1$.

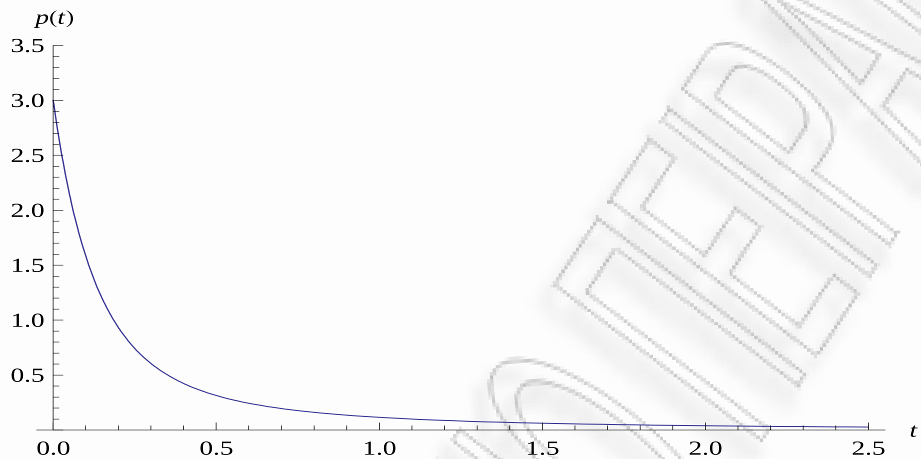

Figure 4.25: Ruin time density with u=0 and exponential claims and times for the classical model and $\mu = 1, c = 4, \lambda = 3, \theta = 1/3$.

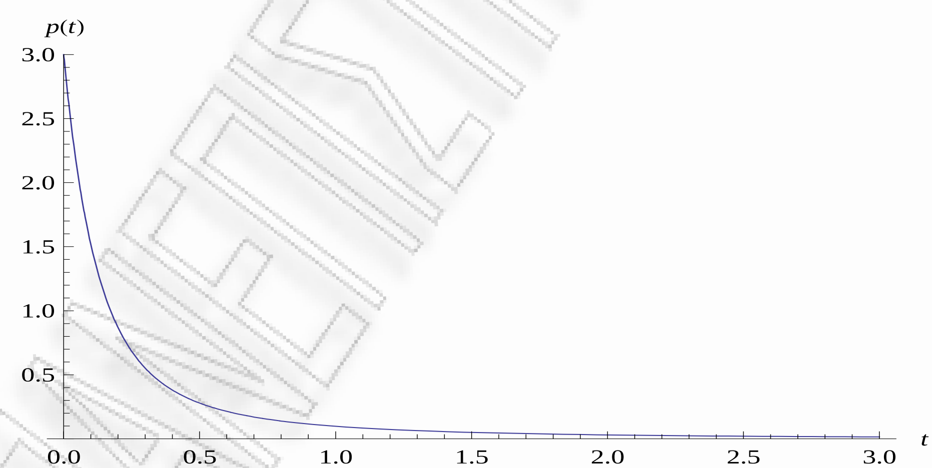

Figure 4.26: Ruin time density with u=0 and exponential claims and times for the classical model and $\mu = 1, c = 4.5, \lambda = 3, \theta = 0.5$.

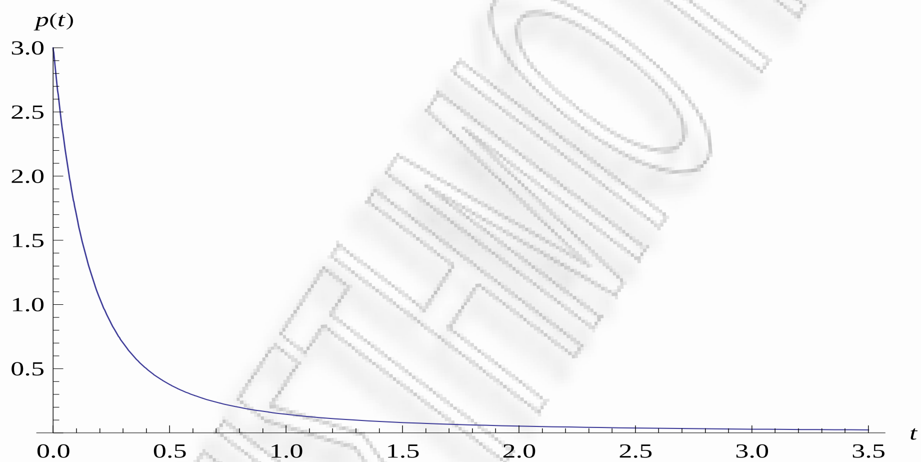

Figure 4.27: Ruin time density with u=0 and exponential claims and times for the classical model and $\mu = 1, c = 3.2, \lambda = 3, \theta = 0.0666667$.

A first observation, again, from the above plots is that they all are decreasing, something that did not hold in the case of $u > 0$, where $p(t)$ was, in most cases, increasing and then decreasing, with the exception of Figure 4.10, which was decreasing, as here. Here, we believe that $p(t)$ is decreasing because of our assumption of $u = 0$. In the first three plots we have kept $\mu = 4$ and $\lambda = 6$ and we have given different values to c, in order to examine, how this parameter affects the behavior of $p(t)$. We can see from the plots, that for different values of c, $p(t)$ is always decreasing. This might be because of relation (2.6.4). We have showed that u affects the behavior of $p(t)$ a lot, here since $u = 0$, all have a similar behavior. In the last three figures ($\lambda = 3$), we can see, that there is also a common behavior. If $u > 0$ the behavior would be completely different as showed in previous section.

The table below shows some values of the density of T, when $u = 0$ and μ takes the values $1, 2, 3, 4$, respectively. We have assumed that $\lambda = 6$.

t	$\theta = 1/6, c = 7$	$\theta = 1/3, c = 4$	$\theta = 1, c = 4$	$\theta = 0.666667, c = 2.5$
1	0.0957844	0.0781711	0.0239096	0.0460691
2	0.0330902	0.0242862	0.00305473	0.00992529
3	0.017418	0.0115049	0.000596208	0.00326405
4	0.0109127	0.00648803	0.000138588	0.00127818
5	0.00752453	0.00402701	0.0000354623	0.000550946
6	0.00551321	0.00265611	9.6436×10^{-6}	0.000252372
7	0.00421268	0.00182704	2.73508×10^{-6}	0.000120566
8	0.00331947	0.00129601	7.99975×10^{-7}	0.0000593983
9	0.00267786	0.000941203	2.3954×10^{-7}	0.00002996
10	0.00220072	0.000696333	7.25663×10^{-10}	0.0000153928
20	6.77118×10^{-8}	7.51938×10^{-11}	4.09446×10^{-28}	1.29904×10^{-18}
30	2.19811×10^{-29}	1.2399×10^{-36}	3.51237×10^{-71}	5.02722×10^{-53}
40	6.42815×10^{-61}	1.68296×10^{-72}	2.12234×10^{-124}	1.44756×10^{-97}
50	5.07184×10^{-98}	6.08025×10^{-114}	3.31962×10^{-183}	1.09042×10^{-147}
60	1.17297×10^{-138}	6.41143×10^{-159}	1.50151×10^{-245}	2.38381×10^{-201}
70	9.82229×10^{-182}	2.44347×10^{-206}	2.44478×10^{-310}	1.87891×10^{-257}
80	1.34001×10^{-226}	1.51577×10^{-255}	6.46601×10^{-377}	$2.407563 \times 10^{-315}$
90	7.86666×10^{-273}	4.04414×10^{-306}	7.34679×10^{-445}	7.34679×10^{-445}
100	3.85915×10^{-320}	9.01378×10^{-358}	6.96849×10^{-514}	$6.968485 \times 10^{-514}$

Chapter 5

The reliability classification of the distribution of ruin time

In this chapter, we are interested to show how the failure rate of the distribution function of ruin time, the density function of which was given in (4.2.5), behaves in terms of reliability classification. As far as the theory of reliability is concerned, the failure (or lifetime) distributions, the monotone failure rate and some other classes of distributions were presented, because they are needed for the study of the monotonicity of the corresponding failure rate, which will give interesting results and will be the subject of this chapter. The most important notions of reliability theory, concerning this dissertation, were introduced in Chapter 3.

So in this chapter, after introducing the failure rate of the distribution of $g(t)$, we will give some numerical examples for it, using Mathematica, in order to examine if $r(t)$ is IFR(DFR) or BFR(UBFR). Here, firstly, we will assume, that the initial surplus $u > 0$ and then $u = 0$, so we will use $g(t)$, relation (4.2.5) and then the simplified formula of $g(t)$, which was given in relation (4.7.2). The monotonicity of the distribution of $g(t)$- which, as we will see, here, turns out to be non- monotonic- is a very interesting subject, which has not been studied, yet, in bibliography. Here, we make an effort to begin a study for it, but yet a lot of subjects remain to be examined, for example for different distributions other than the exponential.

5.1 The Failure Rate of the distribution function of $g(t)$, for $u > 0$

In order to study the monotonicity of $r(t)$ we should first give its failure rate. The formula of the failure rate was given in (3.3.1) for a distribution F and was defined for those values of t for which $F(t) < 1$. Here, we are interested to study the failure rate of the distribution of $g(t)$, which is the proper density function of ruin time T_c, for $u > 0$. This df was given in (4.2.5). If we assume that $\overline{G}(t)$ is the right tail of $g(t)$, then we have that

$$\overline{G}(t) = \int_t^\infty g(y)dy,$$

so

$$\overline{G}(t) = \int_t^\infty \frac{\sqrt{1+\theta}e^{\frac{-\mu u}{1+\theta}}e^{-\lambda(2+\theta)y}}{y} \sum_{n=0}^\infty \frac{(n+1)(\frac{\mu u}{\sqrt{1+\theta}})^n I_{n+1}(2\lambda y\sqrt{1+\theta})}{n!} dy,$$

so for the failure rate of $G(t)$ for $u > 0$, we have that

$$r(t) = \frac{g(t)}{\overline{G}(t)}$$

and

$$r(t) = \frac{g(t)}{\int_t^\infty \frac{\sqrt{1+\theta}e^{\frac{-\mu u}{1+\theta}}e^{-\lambda(2+\theta)y}}{y} \sum_{n=0}^\infty \frac{(n+1)(\frac{\mu u}{\sqrt{1+\theta}})^n I_{n+1}(2\lambda y\sqrt{1+\theta})}{n!}} dy. \tag{5.1.1}$$

In the following section, we will study the classification of the failure rate of the distribution of $g(t)$.

5.2 Numerical examples for the failure rate of the distribution function of $g(t)$, when $u > 0$

In this section, we study the failure rate of the distribution, $G(t)$. To do this, we have computed in Mathematica the formula for $r(t)$, which was given in (5.1.1) and the following plots were generated.

According to the theory, in order a distribution, to be IFR, $r(t)$ has to be increasing in all the interval. If there is one point, where the failure rate is not increasing, that is the failure rate increases and then decreases, then we talk about a BFR. Bathtub classes of failure rate, were introduced in Chapter 3, Section 3.5 and according to the presentation, that took place there, the bathtub and the hump failure rate or the upside down bathtub failure rate are non- monotonic. Here, there are no cases of upside down bathtub failure rate.

Here, in all the following plots, we can see that at the beginning $r(t)$ is almost stable, without being clear what exactly happens for values close to zero. Then $r(t)$ becomes increasing. More precisely, it seems that the rate, with which $r(t)$ decreases at the beginning, is intense for small u. Whereas, as the initial surplus gets bigger, the rate is negligible, and shows decreasing route in a very small interval.

In Chapter 3, we presented the classes of failure rate in a reliability context, so the analysis of the behavior of them, was done on this basis. Here, in the context of ruin theory, we can say that, as time passes on and ruin does not happen, then the probability of ruin to happen becomes smaller in the first interval, close to zero. But then, as time passes, the probability of ruin to happen becomes bigger in the next moment, something very logical, if we bear in mind that we talk about the proper random variable of time to ruin.

Here, we have assumed that, $\lambda = 1, \mu = 1$ and we have given various values to the remaining parameters, u, θ. More precisely, we can see, that Figures 5.1, 5.2, 5.3, 5.5, look like to each other, whereas Figure 5.4 resembles to Figure 5.6. In these two plots, we have assumed small values for u, $u = 5$ and $u = 2$, respectively. They, also, have the same value for the loading factor, θ. What makes them look different to the other plots, is the fact that all the other have $r(0) = 0$, while these two do not. So, as mentioned before, here is more obvious that there is an interval, where rate seems to decrease more intensively and then increases, abruptly.

So, closing, a conclusion is that for big initial surplus, $r(t)$, except from a small interval, increases. Which means that if ruin does not happen at the beginning, then increases the probability ruin to happen in the next moment.

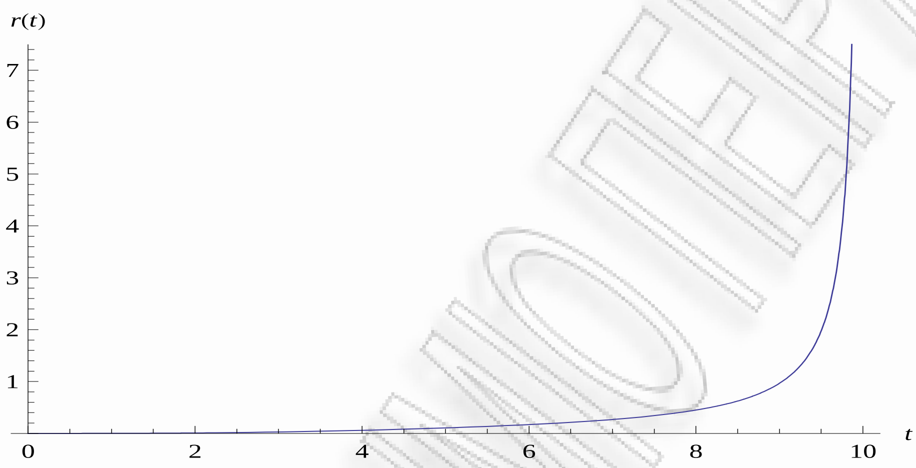

Figure 5.1: The failure rate of $G(t)$ with $u = 20$, $\lambda = 1$,$\mu = 1$ and $\theta = 0.1$.

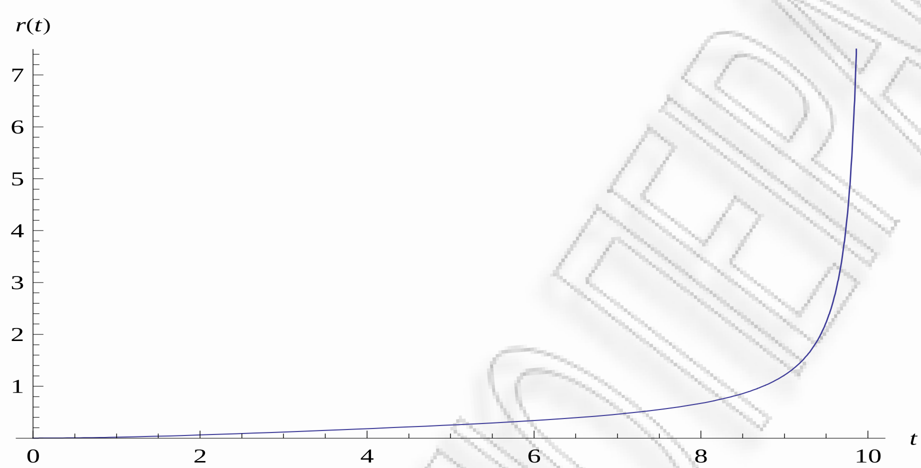

Figure 5.2: The failure rate of $G(t)$ with $u = 20$, $\lambda = 1$,$\mu = 1$ and $\theta = 1$.

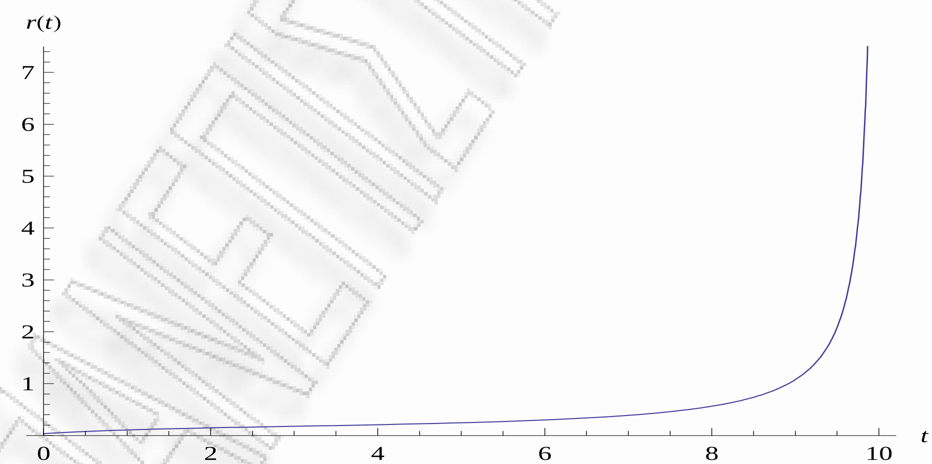

Figure 5.3: The failure rate of $G(t)$ with $u = 5$, $\lambda = 1$,$\mu = 1$ and $\theta = 0.1$.

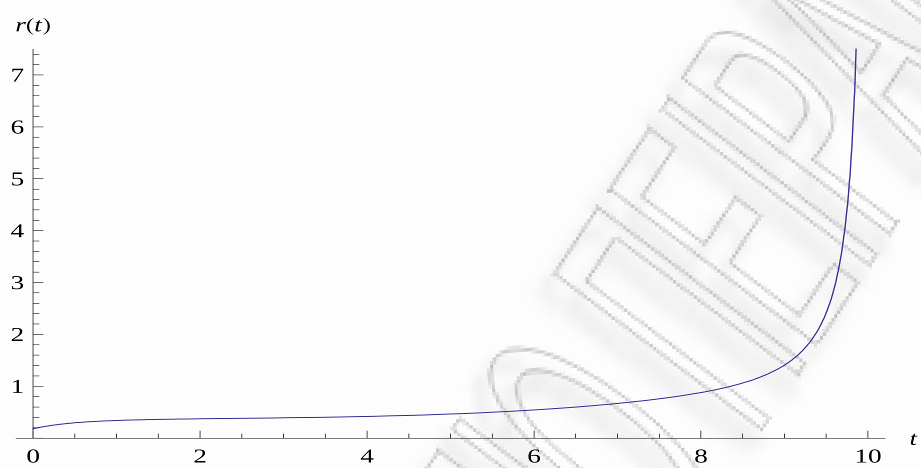

Figure 5.4: The failure rate of $G(t)$ with $u = 5$, $\lambda = 1$,$\mu = 1$ and $\theta = 1$.

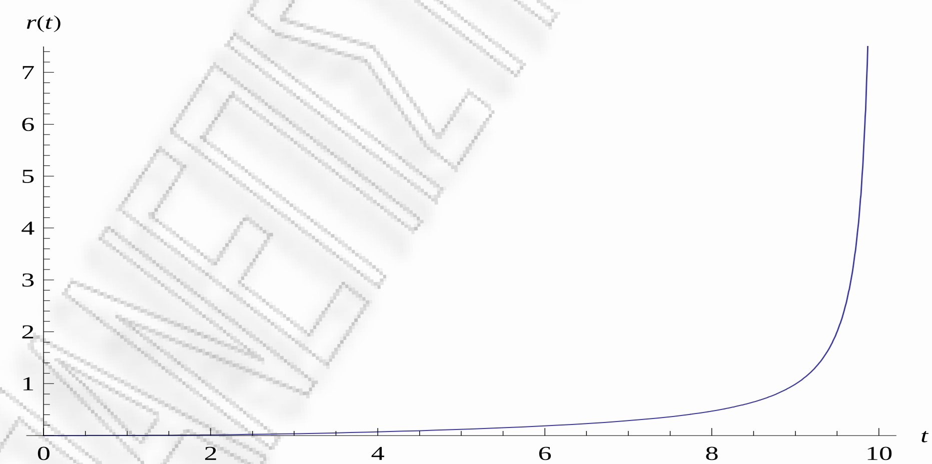

Figure 5.5: The failure rate of $G(t)$ with $u = 20$, $\lambda = 1$,$\mu = 1$ and $\theta = 0.25$.

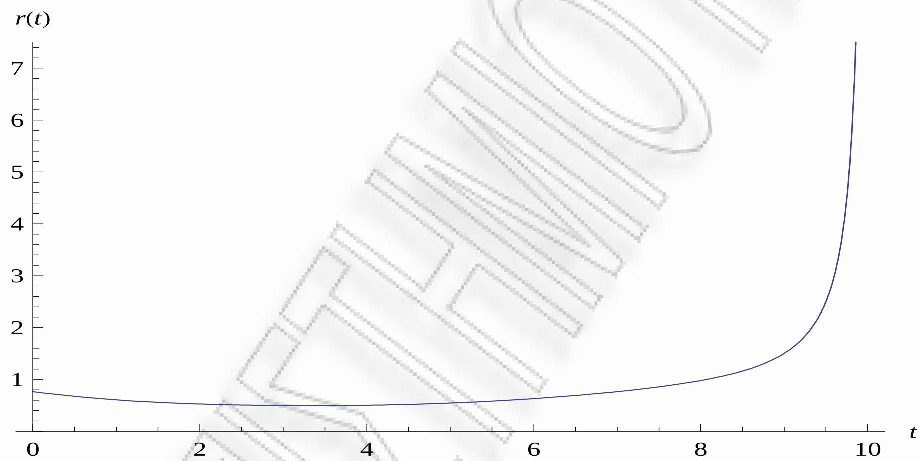

Figure 5.6: The failure rate of $G(t)$ with $u = 2$, $\lambda = 1$,$\mu = 1$ and $\theta = 1$.

5.3 The Failure Rate of the distribution function of $g(t)$, for $u = 0$

In order to study the monotonicity of $r(t)$ for $u = 0$. We will use the formula for $g(t)$, which was given in relation (4.7.2). We have assumed that $\overline{G}(t)$ is the right tail of $g(t)$, then we have that

$$\overline{G}(t) = \int_t^{\infty} \frac{\sqrt{1+\theta}e^{-\lambda(2+\theta)y}}{y} \sum_{n=0}^{\infty} \frac{\left(\lambda y\sqrt{1+\theta}\right)^{2n+1}}{n!(n+1)!} dy,$$

so for the failure rate of $G(t)$ for $u = 0$, we have that

$$r(t) = \frac{g(t)}{\overline{G}(t)}$$

and

$$r(t) = \frac{g(t)}{\int_t^{\infty} \frac{\sqrt{1+\theta}e^{-\lambda(2+\theta)y}}{y} \sum_{n=0}^{\infty} \frac{\lambda y\sqrt{1+\theta}^{2n+1}}{n!(n+1)!}} dy. \tag{5.3.1}$$

As it appears in the next section the distribution of T_c is BFR, in all cases and despite the values of its other parameters.

5.4 Numerical examples for the failure rate of the distribution function of $g(t)$, when $u = 0$

In this section, our goal is to study the monotonicity of the failure rate of $G(t)$ of the random variable T_c, which we have assumed to be the proper ruin time. However, as the plots that follow show this is non- monotonic. We have computed $r(t)$, relation (5.3.1), in Mathematica for initial surplus zero and different values of its other parameters (λ, θ). We have introduced the theory of reliability in Chapter 3, where among others we presented the classes of failure rate (IFR,DFR,BFR,HFR). If $r(t)$ is increasing monotonically, we have IFR and if is decreasing, we have DFR. However, there are cases, where $r(t)$ is non- monotonic. According to the theory discussed earlier, a df is said to be BFR, if there exists a point t_0, called "turning point" of the distribution, such that the failure rate is decreasing in $[0, t_0)$ and increasing in $[t_0, \infty)$, but there are also cases, where there might be more than one turning point, see Mitra and Basu (1995).

In the previous chapter, Chapter 4, we introduced the density function of ruin time with exponential claims, (4.2.5) and we gave some plots, where depending on the values of its parameters, $g(t)$ was increasing up to a point and then decreasing. But here, we are interested to see the behavior of $r(t)$ for $u = 0$. For this, apart from plots for $g(t)$ and $u > 0$, in the previous chapter, we also showed plots for $g(t)$ assuming $u = 0$, relation (4.7.2). In these plots, in all the cases and for different values of its other parameters, $g(t)$ was decreasing. According to the theory if $r(t)$ is decreasing then

$$\frac{g(t)}{\overline{G}(t)} \text{ is decreasing}$$

so

$$\frac{g(t)}{\overline{G}(t)}\overline{G}(t) \text{ is decreasing}$$

and then $g(t)$ is decreasing. So if density is not decreasing, then G is not DFR.

Here we can see that $r(t)$ is bathtub, in all the following plots. Something we can not say, for sure, for the case of $u > 0$. The only difference, that is noticed among the following plots is that some of them (Figures 5.8, 5.10, 5.15) seem to have two "turning points", meaning that in the first phase

$r(t)$ is decreasing and after a point becomes increasing and then decreasing, before it becomes again increasing. In the rest of the plots (Figures 5.9, 5.12, 5.13) we can see that at the beginning $g(t)$ is decreasing up to a point and then becomes increasing. So a conclusion is that, the combination of the parameters θ, λ affects the behavior of $r(t)$.

Some notes, will be that, Figure 5.10 has a big loading factor and two turning points, while Figures 5.8, 5.9, 5.12, 5.13, 5.15, have a small loading factor, close to zero, but they do not have all of them two turning points. It seems that two turning points, have those plots, that have small loading factor and $\lambda = 2$. Apart from that, one turning point, have those plots with very small θ and $\lambda = 1$.

A general observation, again, here is that, as time passes on, the probability of ruin to happen in the next time interval reduces, up to a point and then increases, if there is one turning point. If there are two turning points, this happens twice, that is, after the decrease follows an increase until we have a decrease, again, which is followed by an increase, at the end. Again, here, one can say that, as ruin does not occur at the beginning, it is more probable to happen in the next moment, with a decreasing rate. And after a point, if ruin has not happen it is more probable to happen as time passes.

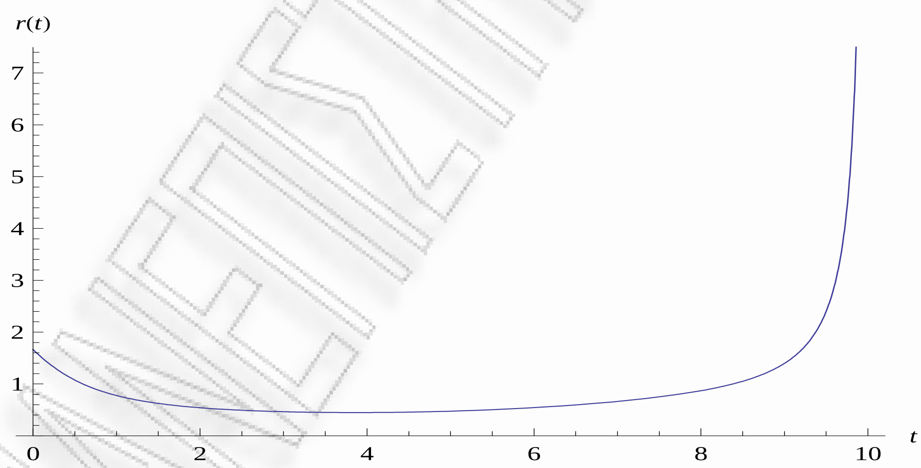

Figure 5.7: The failure rate of $G(t)$ with $u = 0$, $\lambda = 1$ and $\theta = 0.6$.

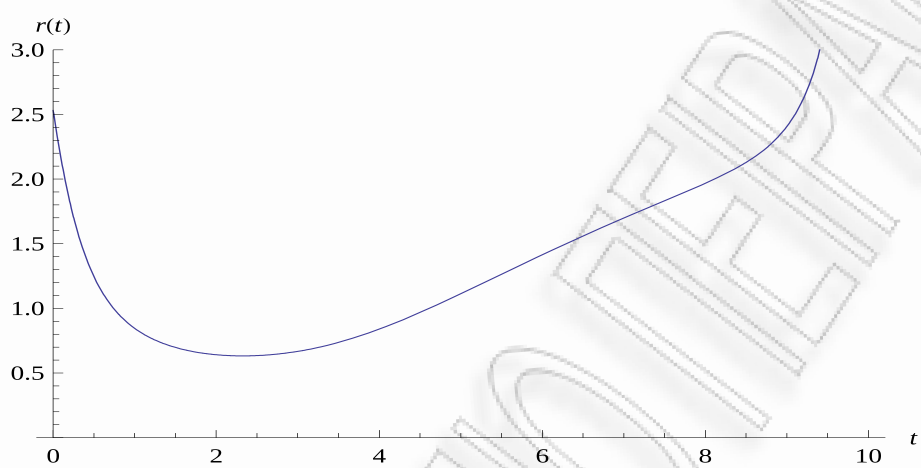

Figure 5.8: The failure rate of $G(t)$ with $u = 0$, $\lambda = 2$ and $\theta = 0.1$.

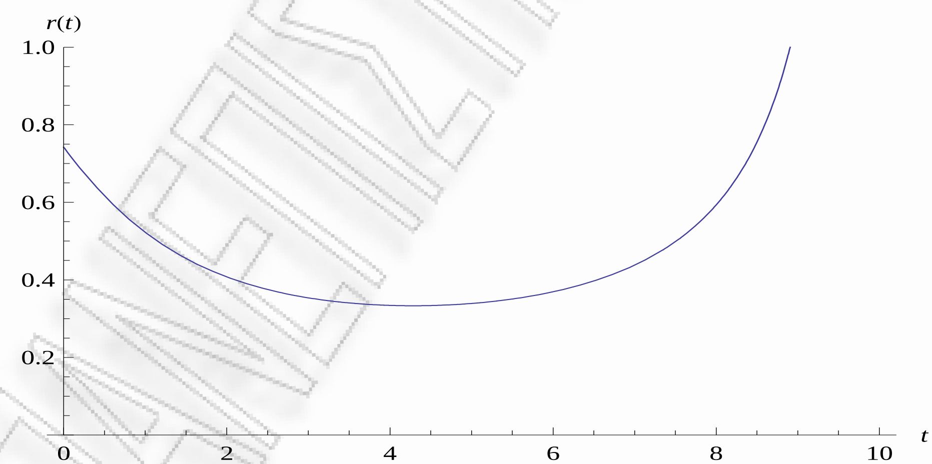

Figure 5.9: The failure rate of $G(t)$ with $u = 0$, $\lambda = 0.5$ and $\theta = 0.25$.

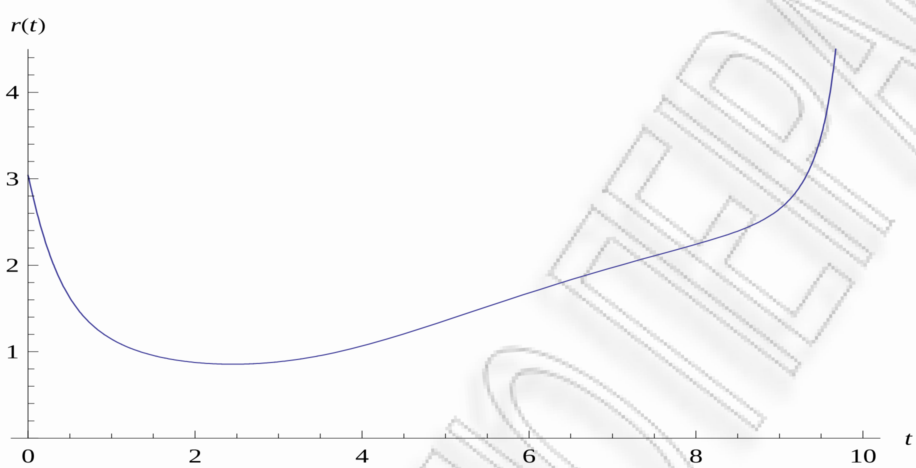

Figure 5.10: The failure rate of $G(t)$ with $u = 0$, $\lambda = 1.5$ and θ =1.

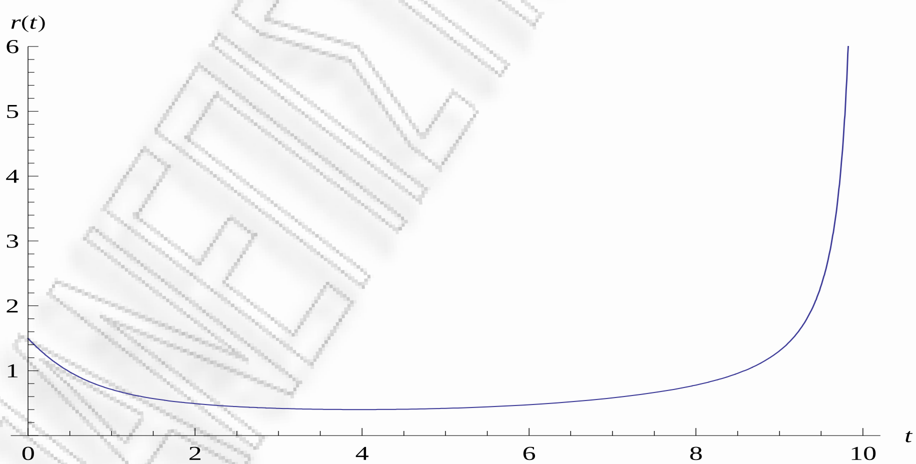

Figure 5.11: The failure rate of $G(t)$ with $u = 0$, $\lambda = 1$ and $\theta = 0.4$.

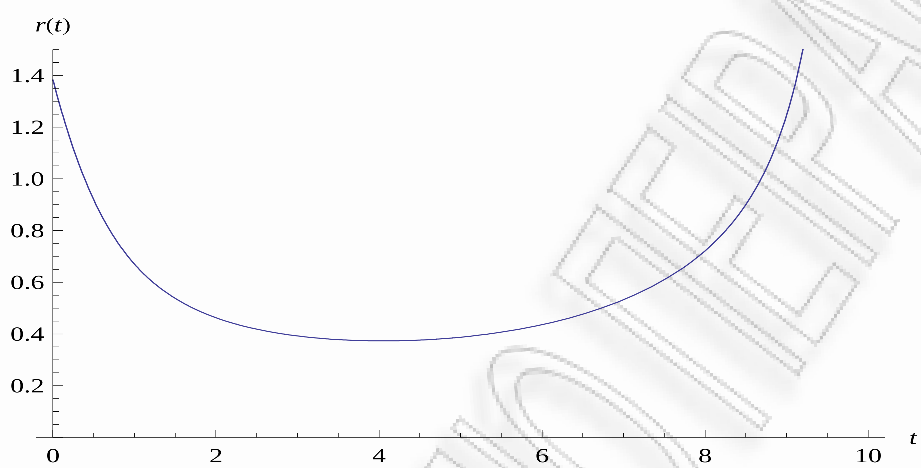

Figure 5.12: The failure rate of $G(t)$ with $u = 0$, $\lambda = 1$ and $\theta = 0.1$.

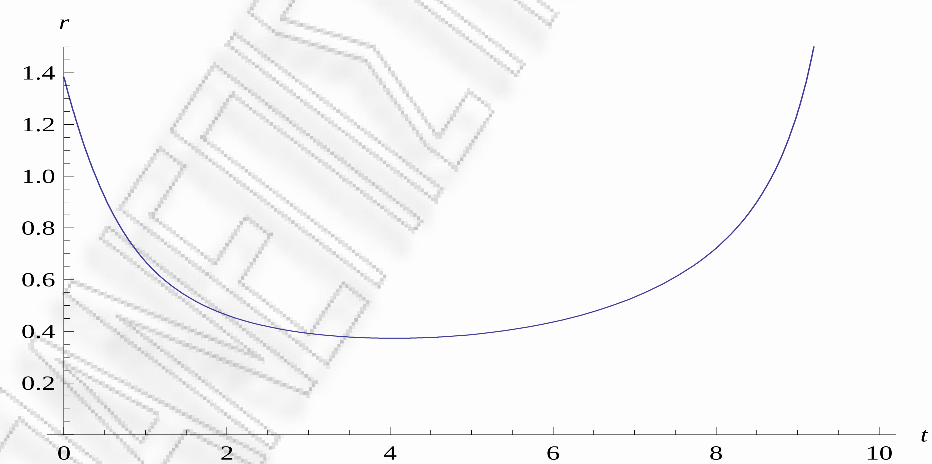

Figure 5.13: The failure rate of $G(t)$ with $u = 0$, $\lambda = 1$ and $\theta = 0.25$.

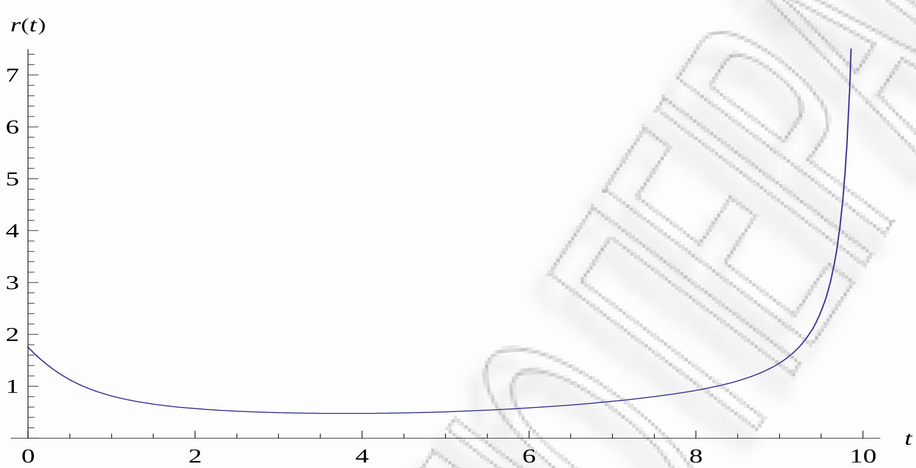

Figure 5.14: The failure rate of $G(t)$ with $u = 0$, $\lambda = 1$ and $\theta = 0.7$.

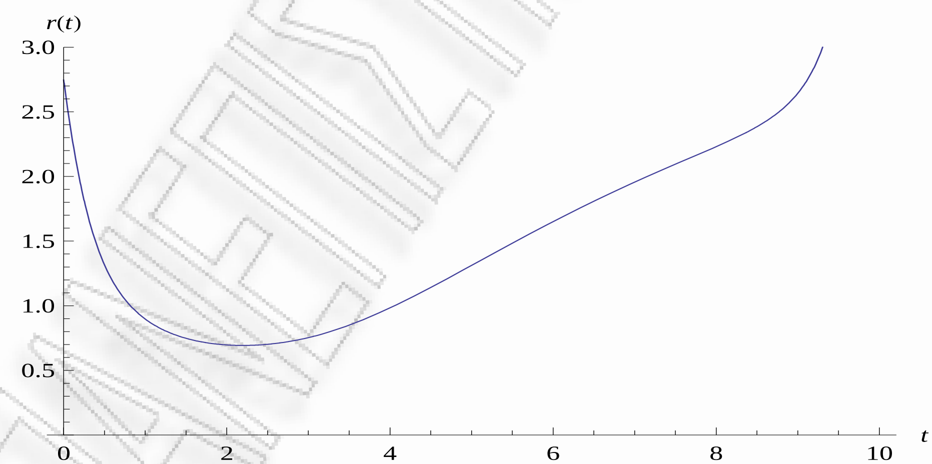

Figure 5.15: The failure rate of $G(t)$ with $u = 0$, $\lambda = 2$ and $\theta = 0.25$.

Appendix A

The Commands in the computational package Mathematica

Here, we give some of the commands, that we executed in Mathematica, in order to give the above examples. Not all the examples, that were presented in this dissertation, are given, because in some cases the commands are repeated. Before representing the specific examples, we should note that in Mathematica, we define a function f of a random variable, e.g. t, by $f[t] :=$. In the beginning of a program we use the command "ClearAll", in order to clean our file and delete previous results, then we define the values of our parametrs.

More precisely, in Chapter 3, as far as the example of the mixture of exponentials is concerned, we wanted to study the classification of its failure rate, so we defined five cases of mixtures of exponentials, choosing weights of 0.5 and different values of λ_1, λ_2, relation (3.3.5). After this, we also defined their distribution and survival functions, in the same way, so we could use (3.3.1). The procedure in Mathematica for the first case is following. The other 4 mixtures of exponential were created in the same way, changing the parameter values.

- The probability density function for the mixture of exponentials with weights of 0.5 and $\lambda_1 = 0.5, \lambda_2 = 0.5$

$$fexp1[t] := 1/2PDF[ExponentialDistribution[0.5], t]+1/2PDF[ExponentialDistribution[0.5], t]$$

- The distribution function

$$Fexp1[t] := Integrate[fexp1[x], x, 0, t]$$

- The distribution function

$$Sexp1[t] := Integrate[fexp1[x], x, t, Infinity]$$

- The failure rate

$$rexp1[t] := fexp1[t]/Sexp1[t]$$

- The plot for all the mixture of exponentials together is given

$$Plot[rexp1[t], rexp2[t], rexp3[t], rexp4[t], rexp5[t], t, 0, 10]$$

with the command "AxesLabel→ time, failure rate", we label the axes of our plot.

In the next example, we gave the gamma distribution. We wanted to show how different values of the parameter α affect its failure, so we took, respectively, $\alpha >=< 1$ and we kept $\lambda = 1$. We took the same steps as above, using (3.3.6). We also used the command *Needs["PloLegends"]*, in order to depict on the plot, the different values of the parameter α, that corresponded to each of the gamma distributions.

In Section 3.5, we gave a plot that depicted all of the four introduced classes of distribution of failure rate, together. To produce this plot, we used two distributions, used in Section 3.4, an example for the IFR (the gamma with $\alpha > 1$) and DFR (a mixture of exponentials) and an example for BFR (the distribution of the ruin time), UBFR (the Log- normal), which were used in the same section and their presentation in Mathematica follows.

In the same section, in example 3.5.1, we showed the failure of a Log-Normal distribution. What we did there, resembles to the example of the mixture of exponentials. So we defined the pdf by

$$f1[t] := PDF[LogNormalDistribution[1, 0.5], t]$$

for $\mu = 1$ and $\sigma = 0.5$. There we gave two other cases for $\mu = 1$ and $\sigma = 1$ and $\mu = 1$ and $\sigma = 1.5$, respectively. Then we defined the survival function by

$$S1[t] := Integrate[f1[x], x, t, Infinity]$$

and then we computed the failure rate

$$r1[t] := f1[t]/S1[t]$$

so we could produce its plot

$$Plot[r1[t], t, 0, 10, AxesLabel- > time, failurerate.$$

In Chapter 4, we studied the density function of the defective and the proper random variable of ruin time. So here, the commands that we executed in Mathematica, for the proper density follow, using (4.2.5), in order to make the plot of the density function of $g(t)$. The same hold for $p(t)$.

The basic ideas, that were used above in the mixture of exponentials and gamma, hold here, too. In the beginning, we give values to the parameters of $g(t)$, which are separated by ; and then we define four functions in order to study $g(t)$

- $q = Sqrt[1 + v]$
- $h = 1 + v$, where $v = \theta$
- $j[t] := Sum[(l * t * q)^(2 * k + n + 1)/(k! * (k + n + 1)!), k, 0, 100]$, which is the Bessel function
- $g[t] := ((q * Exp[(-m * u - l * h * (1 + h) * t)/h])/t) * Sum[((n + 1) * (m * u/q)^(n) * j[t])/n!, n, 0, 100]$

so now we can do for the plot

$$Plot[g1[t], t, 0, 160, AxesLabel- > t, g[t], PlotRange- > 0, 0.006]$$

the command "PlotRange" defines the range of the plot in order not to lose the information for our density.

Next, we did the same for the density of the defective ruin time T.

In Section 4.8 and 4.9, we gave examples for the case of zero initial surplus., using (4.7.2) and (4.7.1), respectively. The procedure, that was followed is similar to the above. Here, we give our computations in Mathematica, in order to study the equivalence of the two densities, numerically. Since the relation between them is

$$g(t) = \frac{p(t)}{\psi(0)} = \frac{p(t)}{\frac{1}{1+\theta}} = p(t)(1+\theta),$$

in Mathematica

$$h = 1 + v,$$

where $v = \theta$ and

$$g[100] = h * p[100],$$

100 is a randomly selected value for time, we tested the equality between the two densities for different values of t. Finally, in Chapter 5, we computed the failure rate of $G(t)$, for $u > \ and = 0$ and gave its plots. So we used (4.2.5), for the case of $u > 0$ and the corresponding for $u = 0$ as defined above and then we defined its survival function as

$$S[t] := Integrate[g[x], x, t, 10]$$

and

$$r1[t] := g[t]/S[t].$$

Bibliography

A. ENGLISH

[1] Barlow R.E., Proscan Fr. (1996) *Mathematical Theory of Reliability,* L.A.S.S.I.C.S In Applied Mathematics

[2] Borovkov K.A., Dickson D.C.M (2008) *On the ruin time for a Sparre Andersen process with exponential claim sizes,* Insurance: Mathematics and Economics 42, 104-1108

[3] Bowers N.L., Jones D.A., Gerber H.U., Nesbitt C.J., Hickman J.C. (2007) *Actuarial Mathematics,* Society of Actuaries, Ithaca

[4] Drekic St., Willmot G.E. (2003) *On the density and moments of the time of ruin with exponential claims,* Astin Bulletin, Volume 33, Number 1, pp. 11-21

[5] Gerber H.U. and Shiu E. (1998) *On the time value or ruin,* North American Actuarial Journal 2 (1998) 48-78, North America

[6] Grandell J. (1991) *Aspects of risk theory,* Springer Series in Statistics, Probability and its Applications, Springer-Verlag, New York

[7] Grimmett G.R., Stirzaker D.R. (2001) *Probability and Random Processes,* Oxford Science Publications

[8] Mitra M. and Basu S.K. (1995) *On some properties of the bath failure rate family of life distributions,* Microelectron. Reliab., Vol. 36, No. 5, pp. 679-684. 1996, Pergamon, Great Britain

[9] Willmot G.E.,Lin X.Sh., Cai J. (2001) *Lundberg inequalities for renewal equations,* Applied Probability Trust, Volume 33, Number 3, 674-689

[10] Willmot G.E. and Lin X.Sh. (2001) *Lundberg Approximations for Compound Distributions with Insurance Applications,* Springer Series, New York

[11] *Advice for Level 2 Implementing Measures on Solvency II: Technical Provisions*, Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS) (2009)

[12] *Solvency II* (2000), European Commission

B. GREEK

[13] Atzoulakos D. (2003) *Notes on Survival Analisis* Master of Science in Applied Statistics, Department of Statistics and Insurance Science, University of Piraeus

[14] Polits K. (2007) *Notes on Ruin Theory* BSc in Statistics and Insurance Science, Department of Statistics and Insurance Science, University of Piraeus

[15] Koutsopoulos K.I. (1999) *Actuarial Mathematics Part I, Risk Theory,* Athens

[16] Chadjiconstantinidis St. (2008) *Notes on Risk Theory Iand II*, Master of Actuarial Science and Risk Management, Department of Statistics and Insurance Science, University of Piraeus